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ERRATA

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Page 278, beginning with line 14, "The rule he gives is seemingly rational," expunge the succeeding forty-one lines (including Fig. 36), ending "* * * its value, if any, is greatly overrated."

Page 964, lines 18 and 19, change to read: "energy fed into the bridge can be dissipated, and by which resulting amplitudes can be controlled, it becomes of utmost importance."

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AMERICAN SOCIETY OF CIVIL ENGINEERS
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Paper No. 2388

THE MUNICIPAL ASPHALT PLANT, BOROUGH OF MANHATTAN

By L. Csanyi,[1] M. ASCE

With Discussion by Messrs. Clarkson H. Oglesby, Clark D. Henderson, and Oscar C. Holmes, Jr.; J. H. Creed, and L. Csanyi.

Synopsis

The Municipal Asphalt Plant of the Borough of Manhattan (New York, N. Y.) is situated in a modern, fashionable, residential area. Why such a site was selected is discussed briefly. The various factors involved in the final acceptance of the plant as a good neighbor—such as the character and operation of the equipment and machinery installed, the new operational procedures developed, the adaptation of unique machinery, and the use of large 4-ton batch mixers—are separately developed. Manufacturing costs covering 1 year of operation during 1944 and 1945 are compiled and presented.

The conclusions indicate that the foresight and courage of the planners of the plant, in selecting the site and utilizing the modern equipment, are amply repaid by the efficient and economical operation of the plant.

Introduction

On the Franklin D. Roosevelt Drive (East River Drive) at East 90th Street in New York, facing the East River and Hell Gate Bridge, stands a building which compels attention by its unusual design (Fig. 1). Overlooking it is Gracie Mansion, residence of the mayor of the City of New York, beside it on two sides are towering apartments with gardens and penthouses, and on the other side is a park playground. This unique building, in this beautiful setting, is the Municipal Asphalt Plant of the Borough of Manhattan, dubbed the "Cathedral of Asphalt" by Robert Moses, Commissioner of the Department of Parks.

On realizing its purpose many questions arise. Why was an asphalt plant built in such a highly developed area? How does it overcome the objectionable characteristics that make other asphalt plants unacceptable in similar areas?

Note.—Published in November, 1948, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] With Edwards and Kelcey, Cons. Engrs., New York, N. Y.

Why was this plant built in an area of such high development? The plant is a manufacturing enterprise, and must consequently meet the basic principles of every manufacturing enterprise if it is to be successful. These basic principles require that a ready access to an abundant supply of raw materials be available, that the manufacturing process be as simple and as economical as possible, and that an easy method of distribution of its products to a broad market be provided. Having provided for these fundamentals, it is also necessary for them to be maintained in economic equilibrium as ever changing conditions require. The choice of any plant site, therefore, is predicated almost entirely on a balance between the source of raw materials and the distribution of the finished products—with all the ramifications of transportation, handling, and storing of materials involved.

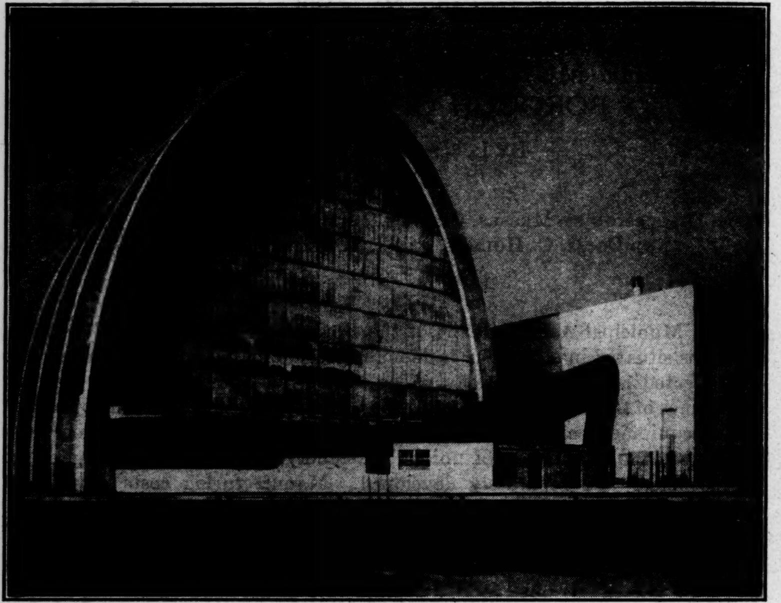

Fig. 1.—New Municipal Asphalt Plant, Borough of Manhattan, New York, N. Y.

The Plant Site

In the Borough of Manhattan there prevail certain conditions which must be weighed if the balance is to be achieved. There, all the raw materials are predominantly available by barge delivery, making a waterfront site essential. Furthermore, because asphaltic paving mixtures of the hot plant-mix type cannot be stored, it is essential that rapid, comparatively short hauls to the site of their application be provided. In addition, the output of this plant is generally utilized for maintenance and repair work, so that the need for complete, economical coverage of the entire area serviced must also be considered.

Bearing these conditions in mind while glancing at a map of Manhattan, which shows an island some 15 miles in length and varying in width from about ½ mile to 2½ miles, it becomes quite evident that a site on either the East River or the North River between 72d Street and 110th Street would be most satisfactory. Closer examination of these areas discloses that the North River side is unavailable, and that the East River side may be utilized under certain conditions. The North River side cannot be used because of the absence of commercial waterfront areas offshore the Henry Hudson Drive. On the East River side the extension of the East River Drive is proposed through this area to complete another link in the circumferential highway system of the island.

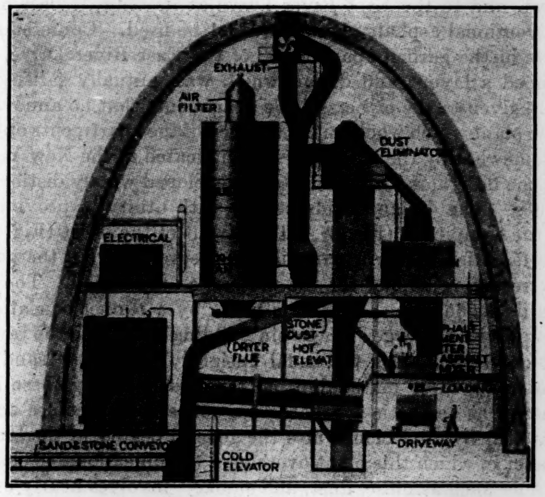

FIG. 2.—INTERIOR OF PLANT

However, between 92d Street and 98th Street a commercial area is provided wherein the plant may be located. Furthermore, rezoning of areas along the drive is contemplated to reclaim the area into highly restricted districts. The dividing line runs through 90th Street, with provision for adjustment in the unrestricted zone to the north. The old asphalt plant (since razed) was also situated at 90th Street. Thus, this site was retained for the new plant as first choice.

Even though this area would be favorable to most efficient operation, other areas were studied where adjustment of the plant to a highly residential district would not be necessary, and where perhaps some efficiency in distribution could be forfeited to less expensive accommodations than those required at the 90th Street site. In line with this program, sites at 132d Street on the Harlem River and at 32d Street on the East River were studied. These sites were within the proposed commercial areas provided in conjunction with the layouts of the

East River Drive and the Harlem River Drive. After exhaustive studies these sites were rejected for several reasons, among which were the curtailment and the combining of commercial areas along the river front to enhance the beauty of the drive and to yield better efficiency to waterfront operations. To infringe upon already limited waterfront operations of private industry would offset the desired efficiency and further limit barge delivery of materials, not only to private plants, but also to the asphalt plant. Thus, the 90th Street site on the East River was finally selected as the site of the new asphalt plant because of the ideal relationship between the source and the distribution of its materials.

Realizing the anticipated change in the vicinity of the selected site, particularly with regard to the construction of garden apartments on the south side of 90th Street, and to the East River Drive at its door, it became apparent that the conventional asphalt plant could not be used. Conferences between the architects of the garden apartments, of the East River Drive, and of the plant developed a layout and design which was mutually satisfactory. Assurances were given to the owners of the apartments that no annoyance would arise from the plant which would make tenancy of the apartments objectionable. Influenced somewhat by the modern trends indicated at the New York World's Fair (1939), the design of the building was developed wholly distinct from conventional lines. The design blends beauty with utility in providing efficient housing for the equipment (Fig. 2). The layout of the plant (Fig. 3) provides for dockside facilities within a park-like enclosure, wherein the gantry crane and the low receiving hopper for the materials are located. The connection between the dockside and the plant is made by a tunnel beneath the drive. There is also provided a large storage-bin building to enclose the raw materials, thereby not only eliminating the usual unsightly mounds of sand and stone, but also protecting these materials from the weather. By these innovations the plant, as far as aesthetic standards are concerned, becomes acceptable to the residents near by, and blends into the general beauty of the landscape.

The manner in which this plant overcomes the objectionable characteristics of other plants—such as noise, dust, and smoke—involves nearly every step of the manufacturing process. To determine how this is accomplished, each step of the process is reviewed separately.

RAW MATERIALS

Sand and Stone.—The raw materials used by the plant are sand, stone, mineral filler, asphaltic cement, and fuel oil. The sand is obtained under rigid specifications[2] for character and gradation from sources around Port Washington (Long Island) or South River, N. J. The crushed stone, usually a limestone, is similarly secured under specifications[3] from the upper Hudson River Valley. Both materials are delivered alongside the dock area in barges of from 600-cu-yd capacity to 1,000-cu-yd capacity.

The storage-bin building, wherein the materials are stored, contains 5 bins—1 bin of 3,000-cu-yd capacity for asphalt sand; 1 of 2,000 cu-yd capacity for broken stone; 2 of 1,000-cu-yd capacity for concrete, sand, and stone; and 1 coal bin of 200-ton capacity.

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[2] "Specification for Sand," *24-S-35-S*, Board of Standardization, Dept. of Purchase, City of New York, New York, N. Y., 1945.

[3] "Specification for Aggregate Coarse," *24-A-20-S*, Board of Standardization, Dept. of Purchase, City of New York, New York, N. Y., 1945.

The various materials are transferred by a gantry crane from the barge to a hopper, which feeds the main conveyor belt. This belt is located in a tunnel under the East River Drive and carries the material up an enclosed incline to the top of the bins, where it discharges upon a distribution system consisting of reversible cross and transfer conveyors, and three movable, reversible shuttle conveyors that discharge the materials into their respective bins. The entire system has a capacity of 100 tons per hr and is operated by five men—two barge trimmers, one craneman, and two belt operators.

With the exception of the crane, the entire system is enclosed. The only part that is visible is the transfer from the barge to the hopper, where the material seemingly disappears. Inasmuch as the entire system is electrically driven it is virtually noiseless.

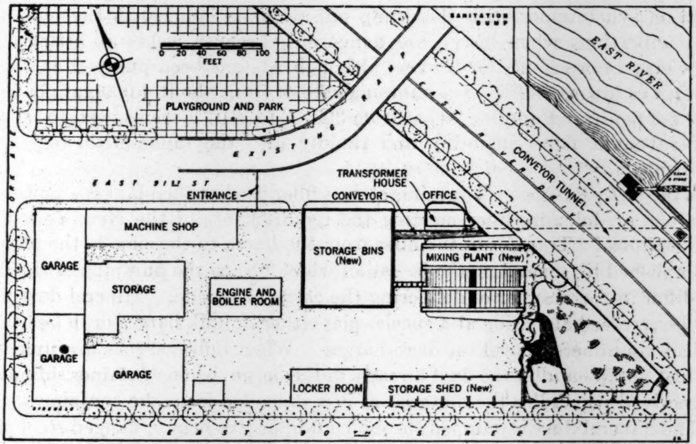
0 20 40 60 80 100
FEET
PLAYGROUND AND PARK
SANITATION DUMP
EAST RIVER
EXISTING RAMP
EAST RIVER DRIVE
CONVEYOR TUNNEL
EAST 91ST ST
ENTRANCE
TRANSFORMER HOUSE
CONVEYOR
OFFICE
MACHINE SHOP
GARAGE
STORAGE
ENGINE AND BOILER ROOM
STORAGE BINS (New)
MIXING PLANT (New)
GARAGE
GARAGE
LOCKER ROOM
STORAGE SHED (New)
EAST 90TH STREET

FIG. 3.—LAYOUT PLAN OF PLANT

Asphaltic Cement.—The asphaltic cement (AC) is received alongside the dock in barges with capacities of from 50,000 gal to 60,000 gal. The dockside is of sufficient length, and the hopper and AC intake are so situated that the unloading of barges of both aggregate and AC may be carried on simultaneously. Steam pumps on the barge, to which steam is furnished by a tug, pump the AC through electrically heated lines to the 5 storage and service tanks (total capacity 120,000 gal), which are situated in the main plant building (Fig. 2).

From the intake the pipe line runs in a pipe gallery underground along the dock to the conveyor tunnel, thence along the wall of the tunnel to the plant building, wherein valving is provided to convey the AC to any one, or several, of the tanks.

The AC pipe lines are constructed of 4-in., continuous, welded steel pipe. Instead of the usual steam jacketing, the pipes are fitted with "calrod" heating units strapped to the underside of the pipe. The pipe and the heating units are covered by a sheet metal covering and a 2-in. asbestos insulation. Fittings, valves, and other pipe appurtenances are suitably fitted with "calrod" heaters and insulation. Electrical thermostats (manually adjusted) are installed in the line at strategic points for automatic temperature control. Details of these heating elements will be found in the original manuscript on which this paper is based, which is on file in the Engineering Societies Library.

With the lines heated, the connections between barge and line completed, and the tank distribution valves set, pumping begins. The AC is usually delivered at a temperature[4] ranging from 280° F to 330° F. Within this temperature range the pumping rate varies from 14,000 gal per hr to 15,000 gal per hr for a 60–70 penetration AC. When the temperature falls below 270° F, the pumping rate falls rapidly, a heavy overload is placed on the pump, and a heavy vibration synchronous with the pump pulsations is induced in the pipe line. These vibrations when heavy are dangerous, because welds at fittings and bends may be weakened or broken, although AC has been pumped at a temperature as low as 250° F at a pumping rate of from about 4,000 gal per hr to 5,000 gal per hr. Pumping AC at from 300° F to 310° F seems best, for at this temperature it flows smoothly and rapidly into the tanks with very little vaporization.

Mineral Dust Filler.—The mineral dust filler used at the plant is a powdered limestone or dolomite filler meeting the requirements of the New York City specifications.[5] Transfer of the filler from the barge to the silos in the plant is accomplished by several methods—all of which utilize the pumping of the dust but differ in the manner of delivering the cargo alongside. Mineral dust filler may be received alongside at a special pier either in bulk barges or in less-than-carload containers placed on deck barges. When bulk cargoes are unloaded the dust pump handles the dust directly in the barge; when container shipments are received it handles the material as it is unloaded from the containers.

The mineral dust is pumped through a 6-in., extra heavy, flanged steel pipe, extending from the pump to the silo, a distance of 450 lin ft (including an 85-ft lift). At the heads of the 3 silos a series of valves distributes the dust into the proper silo.

Each silo, about 18 ft in diameter, 30 ft high, and constructed of concrete block, has a capacity of 225 tons as pumped and about 275 tons after settlement. Each is provided with a bag dust collector at the point where the pumping air is exhausted.

In the less-than-carload container type of delivery, 20 to 21 containers are loaded onto a deck barge in such fashion that air and dust discharge hoses may be readily attached. Each container carries about 10 tons of dust. In this method, the discharge of the dust from the containers to the silos is accomplished in two successive steps. In the first step, the dust is transferred from the container to an auxiliary dust tank; in the second, it is fed from the tank to a box enclosing the pump in a stationary position, from which the pump blows the dust to the silos. The function of the auxiliary tank is temporarily to collect the dust and control its flow to the pump.

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[4] "Specification for Asphalt Cement," *24-A-7-S*, Board of Standardization, Dept. of Purchase, City of New York, New York, N. Y., 1945.

[5] "Specification for Mineral Dust," *24-M-17-S*, Board of Standardization, Dept. of Purchase, City of New York, New York, N. Y., 1945.

Comparing the two methods under existing plant conditions, the container system of delivery was found to be the more efficient. In bulk deliveries quantities of from less than 400 tons to 500 tons per shipment are uneconomical. Inasmuch as the storage capacity at the plant is limited to 700 tons of dust as pumped and 800 tons of dust as settled, the reserve storage supplies must be drawn very low to provide space for these large shipments. Furthermore, the time lost in making displacement measurements, putting the pump into the barge, removing doors (which invariably jam), permitting dust to settle in silos, unloading the pump, and completing measurements adds up so rapidly that from 6 days to 8 days are required to unload a 500-ton barge. At full plant production the material would be used almost as fast as it was pumped, thereby leaving no reserves. The maintenance of sufficient reserve supplies is extremely difficult, not only at full plant capacity, but also at low storage reserve in the event of delayed shipments. With the container method—wherein shipments as low as 200 tons are economical, the quantity of shipment is accepted by certified weights on bills of lading, and the time lost by the various phases of the other method are eliminated—a shipment can be unloaded in 9 hours. Under such conditions the hazards of maintaining adequate reserves disappear, and even a time lapse for shipping delays may be allowed.

Again, in comparing the two methods, in the bulk barge method men must enter and work in the dust. This is far from satisfactory, and inexperienced operation of the pump can create great clouds of dust which are carried far and wide by the winds, creating a nuisance throughout the neighborhood. In the container method, the dust is almost fully enclosed from the containers to the silo, making this system almost dustless. From the standpoint of economy, under present plant conditions, the container method provides a material saving in labor costs by eliminating the cumbersome measurements and the handling of the pump and doors in the barge.

A third method under study provides complete pneumatic operation, having a larger capacity (70 tons per hr) and a longer effective pumping range (about 700 ft). Although this method can be used only with container shipments, the operating power costs compare favorably with both the other methods. In addition, it is extremely efficient, requiring only three men; it is completely dustless; its capacity is more than double the other methods; and, since it requires no moving parts nor electrical controls, it is almost ideal from the standpoint of maintenance and repair.

MANUFACTURE OF PAVING MATERIALS

The machinery within the plant is divided into two complete asphalt plant units, which may be operated singly or simultaneously. The units, each having an output capacity of about 350 tons per day, are designated as No. 2 and No. 3. Space, foundations, and storage have been provided for a third or No. 1 unit in the event of future expansion. The units are alined side by side in an east-west direction, this alinement extending through to the storage bins so that materials may be fed to either unit in the event of a breakdown.

The operation of the units in the manufacturing process may be divided into three integral steps, successive in operation but not continuous. In the first step the materials are handled and processed in passing from the bins to the mixers; in the second, the processed materials are proportioned and mixed; and, in the third, the paving mixture is discharged into vehicles which convey it to the paving operations. In these three steps are included all the ramifications of operating, designing, and controlling the plant and a check test of the materials.

PROCESSING OF MATERIALS

Transfer of Aggregates from Storage Bins to Driers.—The methods used in handling aggregates from the storage bins to the driers may vary with the special conditions of different plants. In this plant, where enclosed and protected bins are provided for the aggregates in order to assure year round operation, the system used is somewhat different from the usual plant installations.

In the storage-bin building, the bins for the sand and stone lie one behind the other, divided by a wall. Discharge openings are provided in the floor of the bins directly beneath the shuttle belts and over a tunnel alined with its corresponding unit in the plant. The openings are about 4 ft square and are fitted underneath with truncated pyramidal chutes and adjustable discharge gates. To assure steady flow of material through the gates, heavy-duty, portable vibrators are attached to the chutes. Experience shows that the stone moves freely through the gates without assistance, whereas the sand can be readily moved by the vibrators.

During the design stage, a drainage system for the bottom of the storage bins was studied. However, it was not adopted and its elimination has created several operating problems. Accumulated moisture drained from the materials, at first, ran through the discharge gates onto the belts, which in turn carried it into the rest of the machinery, causing corrosion and flooding. It was necessary to construct detachable scuppers on the gates to divert the flow from the belts to the tunnel gutters, which connect with a sewer. Although this method works well, it is not wholly satisfactory because the tunnels are continually damp. Based on this experience, it is believed that all-enclosed bins should be provided with drainage systems to divert the drainage from the machinery.

The discharge gates feed directly upon belt conveyors operating in the tunnels. The conveyors, of the conventional rubber belt type with 18-in. belting and operating at a capacity of 80 tons per hr, are 170 ft in length and will carry sand or stone separately or together, depending on the requirements of the plant. The belts discharge into surge hoppers, each of about 5-cu-yd capacity, which act as equalizers between the belt feed and the feed to the elevators. Because uniform, continuous feed to the elevator is desirable, and because such flow cannot be secured in the operation of the belts, the surge hoppers regulate the flow by discharging into the elevator through a rheostat-controlled vibrating feeder. Furthermore, this system gives the drier operator finger-tip control of the material feed.

The material is raised about 40 ft from the surge hoppers to the drier feed by the cold elevator, which has a capacity of 75 tons per hr.

Driers.—The driers perform the function of drying and heating the sand and stone in preparation for their use in the paving mixtures. They are of the double-fired type, each having a capacity of 60 tons per hr. The drier drum is 28 ft in length, 6 ft in diameter, and is rotated in the usual manner at 8 rpm by a 40-hp constant speed motor. The drum is fired directly at the discharge end by a rotary oil burner operating in a cylindrical firebox 6 ft in length and 5 ft in diameter. The hot gases of this fire pass through the entire length of the drier. The middle of the drum is completely enclosed in a secondary rectangular firebox, also fired by a rotary oil burner (Fig. 4). The hot gases of the secondary fire enter the interior of the rotating drum through 6 elbows fastened to the shell of the drum in the interior. The elbows face in the direction of the flow of the materials through the drier and are capped with hoods to prevent leakage of the material into the firebox. This permits entrance of the gases through the 2-in. space provided between the hood skirt and the elbow. Cascading vanes on the inside of the drum move the materials forward through the drum and at the same time drop them through the hot gas flow. The object of the secondary fire is to dry the materials at a point where the moisture may be removed quickly, thereby eliminating the zone of corrosion that appears in

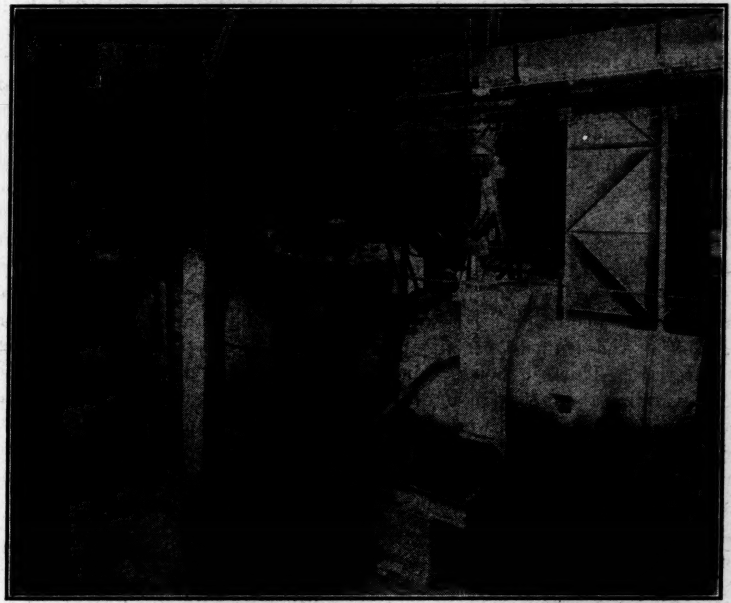

FIG. 4.—MATERIAL DRIER

driers of the single-fired type. The primary fire aids in drying, but functions predominately in heating the materials. By this combination, longer effective life for the drier and higher fuel economy are secured.

Acceptance of the driers was predicated on a test performance wherein the drier operating at a capacity of 60 tons per hr was required to process asphalt sand (New York City specification type 1) with a moisture content of 6% from a temperature of from 40° F to 400° F at a maximum consumption of 3 gal of No. 6 fuel oil per ton. These requirements were met in a series of tests on each drier. In normal operation, however, several problems developed. A heavy leakage of flue gases around the concentric rings of the secondary firebox in which the drum rotates caused extremely disagreeable conditions in the plant. Furthermore, it was found that the secondary fire could not be operated at full power without considerable smoking. Analysis of the conditions indicated that the draft reaching the secondary fire was insufficient to maintain a negative pressure in the firebox and prevent escape of gases, or to draw in sufficient air through the dampers to provide full combustion of the oil. As it was imperative that the plant be properly ventilated in order to provide proper working conditions, hoods connected to the main draft of the drier were installed over the rings of the firebox. This expedient trapped the gases and slightly improved the operation of the fire; it also diverted some of the heat. It was apparent that sufficient draft was not available through the elbows, and adjustments had to be made at that point. Further examination disclosed that the annular bent between the elbow and its hood could not be increased without leakage of materials into the firebox. It also disclosed that sufficient draft velocities were available in the drum to permit openings in the hood itself without leakage of materials. Thirteen holes, 1 in. in diameter, were drilled in each hood, thereby adding about 50 sq in. of draft vent to the secondary fire. The number and the size of the holes were arbitrarily selected as an initial trial. With the added vent, results indicated that a negative pressure of ⅜ in. of water was created in the firebox, and that the fire could be operated at full intensity without smoke. By manipulation of dampers and fuel supply a wide range of efficient control over the fire was attained. After the correction was made, the dampers on the concentric ring hoods used to trap escaping gases could be closed and the full effectiveness of the fire directed into the drum.

At the primary fire a very unusual condition was noted. In spite of the heavy draft through the drier, a back draft persisted in the firebox around the flame. Because of this back draft the sand that was being discharged from the drier was sucked into the firebox, where it settled, accumulated, and vitrified. Coatings of this vitrified sand as much as 6 in. thick accumulated in as little as 10 days of operation. To maintain the efficiency of the fire it was necessary that the coatings be removed periodically. Inasmuch as the coatings were so hard that small power chipping hammers were required, numerous cleanings were injurious to the firebox brickwork. In fact, relining of these fireboxes was necessary after 18 months of operation. Studies and tests give no results other than that sand with a high iron content builds up the coatings very rapidly.

The driers were equipped with automatic controls to coordinate the intensity of the fires with the desired temperature settings for processed materials under constant uniform feed conditions. Operation of the controls centers upon a recording pyrometer, at which the desired temperature of the processed materials may be set, and which records the temperature of the stack gases and the processed materials discharged. Through complicated electrical controls the relationship between stack and material temperatures is translated into the operation of the stack fans (through speed adjustments made by a fluid drive), the quantity of oil fed to the burners (by electrically actuated cocks), and the position of the fire dampers (on the rotary burners). Actually, it was found that the narrowest range in which the system could operate was within 50° F, plus or minus, of a desired temperature. Thus, instead of a steady temperature there was an unsatisfactory fluctuation of 100° F. Tests, adjustments, and changes failed to reduce this range to a suitable value. Studies disclosed that the variables of a wide range in moisture content and the combination of materials processed could not be handled on a uniform flow basis by the system. The time lag in correcting the fires was so great that overrun and underrun of the limits could not be avoided. As it requires 20 min for a particle of the material to pass through the drier, short period adjustments at the discharge end could not be made for temperature. Better results could be attained by varying the feed, but oil efficiencies began to decline.

Since the equipment was at hand, and since its operation required manual control of feed, it was decided to modify the system to one of simplified manual control. Fires are set at an intensity which, from experience, is relative to the average moisture content and the combination of materials, and also to the flow of the materials to maintain the temperature within desired limits. This conforms to plant operation wherein heavy flows are desired during a certain periodic output, and lighter flows, during replenishment of storage in hot bins in off periods.

Substantial fuel oil economies were secured by the modifications made in the drier and in its controls and by the operators as they become experienced. Skilled operators can maintain the discharge temperature of the materials within 25° F of the desired temperature. During full scale operation when 50 tons per hr of material are processed steadily for 5 hours or 6 hours, the fuel oil consumption has varied between 1.8 gal per ton and 2 gal per ton. Over a 1-year period, including winter months and rainy days when daily output was as little as 50 tons, the fuel oil consumption averaged about 2¼ gal per ton. Compared to specification requirements under steady operations at full output of a maximum of 3 gal per ton, a saving of 33% at full production and about 25% on over-all production has been realized.

After 18 months of operation the driers have performed well—with the exception of the points described, there has been slight warping due to overheating and slight buckling of the main support straps in the secondary firebox. The drum, rotating bands, girth and pinion gears, supporting rollers, and the interior shell, vanes, and elbows show very little wear.

Hot Elevator.—The hot elevator, which raises the heated processed materials from the drier discharge about 75 ft to a point of gravity feed to the screens, is of the vertical continuous bucket type and has a capacity of 75 tons per hr. The elevator is driven by a 15-hp, constant speed, gear-reduced motor.

Difficulties were encountered almost immediately in the operation of the elevators. The conditions under which they operate are severe, in that they must handle, while still cold, materials at 400° F, and then, after reaching operating temperature, they are shut down at the end of the day to cool rapidly. Expansion and contraction in the chain linkage, due to the operating temperature conditions, amount to about 2 in. at the take-up boxes. Operating the elevator with the take-ups set so that the chain was taut when cold created so much slack when the elevator warmed up that the buckets swayed, slapping the casing and hooking in the intake and discharge chutes. Operation with the chain taut when warmed up, and not released during cooling, bent and snapped the shaft. An intermediate setting at a point where slapping of the buckets and strain on the shaft were reduced and balanced proved unsatisfactory. Thus, it became apparent that adjustment during heating and cooling was necessary, and a counterweighted automatic take-up was therefore installed. This, however, required manual take-up on the screws to dampen the oscillations of the counterweight. Because continual attendance during the warm-up period was required in any case, it was decided to take up the slack only to a snug position, periodically, until maximum expansion was attained, and to release take-ups fully on shutdown. This procedure is not foolproof, but it has worked satisfactorily. Based on these experiences with the vertical type elevator, it is believed that a continuous bucket elevator of the inclined type, wherein slack is provided on the unloaded side of the chain, would eliminate the difficulties encountered.

Vibrating Screens.—The hot elevator discharges its processed material into the vibrating screens through a gravity chute sloping downward at an angle of 60° with the horizontal. The vibrating screens, situated atop the hot storage bins, separate the combined sand and stone and deposit each into its respective compartment of the hot bins. They also screen out oversize particles into a tailing chute. The vibrating screens used at this plant are of the horizontal type and have a capacity of 75 tons per hr. A motor drive imparts to the spring mounted screen carriage an elliptical motion (⅜ in. horizontally about the long axis and ⅛ in. vertically about the short axis). The screens themselves are mounted in the vibrating carriage in two layers, the upper layer being composed of a ½-in. wire screen extending 8 ft ahead of a 4-ft bar screen with 2½-in. openings. The lower layer is composed of a No. 4 screen extending the full 12-ft length. Between the two layers of screening and directly beneath the 2½-in. bar screen a steel plate is introduced. Material entering the machine falls upon the ½-in. screen, where the passing portion is dropped on the No. 4 screen and the retained portion is carried forward to the 2½-in. screen. The portion passing the 2½-in. screen falls upon the steel plate, which in turn deposits its charge into the stone compartment, the retained portion being carried forward into the tailing chute. Material passed by the ½-in. screen onto the No. 4 screen is sifted into the sand compartment; that retained on the No. 4 screen is dropped into the stone compartment

Although these vibrating screens operate quietly and efficiently, like the hot elevators they are subject to the same severe temperature changes, which similarly create trouble.

Hot Storage Bins.—Directly beneath and receiving the discharge from the vibrating screens are the hot storage bins, which in turn are directly over the mixers. The hot storage bins are compartmented into two equal sections, each of 50-ton capacity, for the storage of sand in both sections, or of sand and stone separately, in respective compartments. The hot storage bins are unique in that they are fully insulated by a 2-in. blanket and a ½-in. plastic insulation covered by a heavy canvas jacket.

The plant, in supplying paving materials to field repair crews, must furnish nearly 50% of its total daily output during the first few hours of its operation. Output then eases off until just before noon when the second peak load must be met. To meet the initial peak load effectively, the plant must either be started extremely early in the morning, or make provision for carrying suitable materials over from the preceding day. In the design of the plant the latter method was selected and the hot storage bins suitably insulated to meet this need.

Under normal operation the materials are delivered to these bins at temperatures between 400° F and 425° F. When the bins are being filled for overnight storage the temperature of the materials is raised to between 500° F and 550° F. In this manner, sufficient materials are made available for the production of more than 130 tons of paving materials the first thing on the following morning without even starting the driers. Nevertheless, the driers are started at the same time as the mixers, and within 20 min begin replenishing the bins. Thus, nearly 200 tons of paving materials may be produced by a single unit of the plant between 7:30 a. m. and 9:30 a. m. Because the production falls off after the initial surge, the driers which operate under constant load catch up and replenish the bins again ready for the second peak between 11 a. m. and 1 p. m., after which the temperature of the materials is raised and the bins are filled for the following day.

Utilizing self-reading pyrometers permanently installed in the hot bins extensive tests on the heat retention characteristics of materials stored in the bins indicate that fully loaded bins will lose from 50° F to 80° F in 18 hours during the summer, and from 100° F to 125° F in the winter. With the bins partly filled, temperature losses are considerably higher. Generally the loss in temperature is dependent upon the atmospheric temperature around the bin, and the quantity stored therein. On several occasions during the summer full bins were carried over from Thursday to Monday morning with the materials still warm enough for a satisfactory mix.

At times, long periods of inclement weather are encountered. On these occasions the materials in the bins fall below usable temperatures, necessitating their removal from the bins. The bins are unloaded, and the material is either returned to the main storage bins or used for other purposes. An installation of auxiliary equipment that would return this material to the driers, thereby saving fuel and handling, is under study. However, correlation of bin filling activities with weather forecasts helps materially in avoiding this situation. The bins have functioned most satisfactorily and have met plant operational requirements ideally.

Limestone Dust.—Limestone dust stored in the silos is not processed in any way, and is fed directly (cold) to the mixers. Dust is drawn from the hopper bottoms of the silos through circular valves and conveyed to the mixers by rotary screw conveyors which are operated by motors controlled by push buttons at the mixer. Silo No. 3 feeds directly to mixer No. 3. Mixer No. 2 can draw dust from either silo No. 2 or silo No. 1.

Anyone who handles mineral fillers in bulk quickly becomes acquainted with their extremely contrary flow characteristics. Limestone dust when air activated flows like water, but when deaired and settled it packs so firmly that little short of brute force can budge it again. In silos 30 ft in height, like those in the plant, almost anything can happen—and usually does—while drawing the dust from them. The one essential, uniform flow in desired quantities, seldom occurs unless the silos are full. With the silo full, the dust flows smoothly through the conveyor. As the material is drawn out of the silo a crater of ever increasing depth forms, and things begin to happen. The crater continues to increase in depth until the angle of repose (as determined by existing conditions) is exceeded. Then the entire lip of the crater caves in, in a huge avalanche. The angle of repose is seldom the same as it varies with the atmospheric humidity, the amount of entrained air, the degree of compaction, the degree of absorbed moisture, the grain size, and other conditions. As neither the formation of the crater nor its estimated angle of repose may be observed during operation, no warning of immediate trouble is sounded. As the avalanche fills the crater, the dust pours through closed valves, through stationary conveyors, and through every hole and crevice of the system out into the plant. At one time, 800 lb of dust poured into the weigh box of the mixer, through a closed discharge valve and 40 ft of stationary conveyor, on a day the plant did not operate. Another time, it passed the closed valve and poured into the conveyor, jamming the conveyor. By contrast, at times the avalanche will jam the dust at the pit of the crater and form an arch over the gate.

From these and other experiences it appears that uniform flow necessitates the elimination of the formation of craters which cause subsequent avalanches in the silos. Anticipating some difficulty, air jets were installed in each hopper leg of the silo during construction. The action of the jets was as unpredictable as the dust itself. Sometimes the high-pressure air would blow a small hole in the dust with no effect upon the formation of the crater. Then, after the crater formed, it would cause an avalanche. Low-pressure air was no better. Experience proved that single or double air jets were unsatisfactory. Multiple jets, as in the skirt of a blast furnace, were considered, but their adaptation was not feasible with the silos.

The problem of maintaining the dust level in the silos became a serious problem. In the spirit of "last hope" a light-duty electric vibrator mounted on the hopper of the silo was tried. Tests showed that constant operation of the vibrator packed the dust more solidly, whereas haphazard intermittent operation caused avalanching. However, interlocking of vibrator operation with the conveyor feed showed promise. Momentary operation just prior to conveyor feed seemed to solve the problem, so a temporary installation of the vibrator was made with controls mounted at the mixer for extended operational tests. After 8 months of testing, during which satisfactory flow was maintained under all conditions, it was decided to make permanent vibrator installations.

Asphaltic Cement.—AC is stored in a composite storage tank of 125,000-gal capacity situated in the main plant building (Fig. 2). This tank is composed of five sections, all rectangular in shape, set in two tiers—two tanks in the lower, and three tanks in the upper tier. The two tanks in the lower tier, designated as tanks No. 1 and No. 2, each have a capacity of 30,000 gal. The three tanks in the upper tier, designated as tanks No. 3, No. 4, and No. 5, have a capacity of 20,000 gal, 25,000 gal, and 20,000 gal, respectively. The upper tanks, resting directly on the tops of the lower ones, are separated by two walkways—giving access to fill lines, inspection manholes, and vents required in the two lower tanks. Catwalks over the upper tanks serve a similar purpose. The entire composite tank and its upper sections are fully insulated by a 3-in. asbestos blanket and ½ in. of plastic insulating material covered by an 8-oz canvas jacket. The quantities of AC contained in the tanks are measured by a self-reading, multiple-liquid, level gage.

Each of the tanks in the composite tank was designed and is used for a specific purpose. Tanks No. 1, No. 2, and No. 3 are for main storage, wherein the AC is maintained at 270° F. Tanks No. 4 and No. 5 are used for service, in these the temperature of the AC is raised and maintained at 330° F.

A complete system of piping provides for the flow of AC from the barges to any one or all of the tanks; also, for the transfer of AC from one tank to another, or from either one or both service tanks to either surge tank, and from either surge tank to the mixers, together with returns from the surge tanks. Two motor driven asphalt pumps, each having a capacity of 120 gal per min, may be used in transferring AC within the composite tank or from service tanks to surge tanks. Two air compressors transfer the AC at relatively uniform pressure from the surge tank to the mixers.

All the storage tanks conveying pipe lines, transfer pumps, and meters handling AC, as well as the paving material mixers, are electrically heated—thereby completely replacing the conventional method of high-pressure steam for this purpose. Although this method of heating is not original with this plant, it is, however, the first complete large scale installation ever made for this purpose in an asphalt plant. Little precedent was available either for design, construction, or operation of such an installation.

"Calrod" heating elements of varying lengths and heating intensities are utilized for the heating of piping, tanks, mixers, other equipment, and appurtenances. The heating of AC in tanks is automatically controlled by recording thermometers operating in conjunction with time switches. The heating of pipes, mixers, and other equipment, either directly by the elements, or indirectly by the circulation of heat transfer oils heated by immersion "calrods," is operated manually by push-button switches with temperature control maintained by thermostats. The system has operated effectively and efficiently after its operational characteristics were determined and an operational procedure developed.[6,7]

MIXERS

Because of the peak demands on the plant and the uniform and continuous operation of its processing machinery, it is evident that the mixer, which is the heart of the production equipment, must have a huge capacity if it is to meet the high intermittent demands.

In the design phase some controversy developed concerning the feasibility of large capacity mixers for the manufacture of high-grade sheet asphalt paving mixers. The alternative to employing large mixers was the installation of batteries of smaller mixers operating simultaneously in order to meet the time requirements of the production output. Inasmuch as space was at a premium, and simultaneous operation of dual mixers posed severe complications in operating equipment and necessitated an increase in operating personnel, it was decided to use large 4-ton mixers.

Each mixer of the usual pugmill type has a capacity of 8,000 lb per batch. Two rotary shafts, each fitted with 28 blades and rotating in opposite directions, knead the materials downward against the mixer liners. The blades are fitted with replaceable and adjustable tips, 8 in. square. An 125-hp, 3-phase, 1,150-rpm electric motor drives the shafts through a chain gear drive and reducing gear. Shaft speeds can be adjusted in steps of 5 rpm from 45 rpm to 60 rpm by gear changes in the chain drive. The mixer body is lined with interchangeable, reversible, and replaceable cast manganese steel liners ¾ in. thick. All shaft bearings are of the heavy-duty, self-alining, spherical, roller-bearing type. The gate under the discharge opening in the body of the mixer is actuated by a double-acting air cylinder of special design, operated by air pressure at 100 lb per sq in.

Above the body of the mixer, and directly below the discharge chutes of the hot bin, a weigh box is suspended having a capacity of 8,000 lb of materials. The scale of this weigh box has 10-lb dial graduations to a capacity of 10,000 lb. The gate used in discharging the contents of the weigh box is actuated by a small air ram operating on compressed air at 100 lb per sq in.

Injection of AC into the mixer is accomplished by a spray bar of unusual trapezoidal cross section situated above the mixer blades and extending across the front side of the mixer. This spray bar is fitted with suitable baffles to assure uniform distribution of AC through 16 jets, uniformly spaced and adjustable in size from spray tip to 1¼-in. diameter.

The entire mixer—composed of the body, blades, weigh box, hot bin discharge chutes, limestone dust conveyor chute, and spray bar—is entirely enclosed in a dustproof housing. Air lines, electrical conduits, and levers for the actuation of the discharge gates of the hot bins all pass through sealing gaskets in the housing.

The AC meter, limestone dust feed control, and interlocking solenoids are situated on a bench formed by the body and the housing at the front of the

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[6] "Electric Heating for Asphalt Plants," by Ladis H. Csanyi, *Engineering News-Record*, April 1, 1948 p. 82.

[7] "Asphalt Plant Operating Costs," by Ladis H. Csanyi, *ibid.*, April 15, 1948, p. 90.

mixer. The AC is measured and injected into the spray bar by a meter having a capacity of not less than 100 gal of AC in 30 sec, and operating at a pressure of 30 lb per sq in. Because the accuracy of the meter depends largely on the temperature of the AC, it is essential that this temperature be maintained between the operating limits of 320° F and 330° F.

At this point a discussion of the surge tank is essential to an understanding of its function. The surge tank serves a dual purpose. First, since the AC pumps have a capacity of 120 gal per min, it is evident that they could not furnish sufficient AC to meet the meter requirements of not less than 100 gal in 30 sec, or 200 gal per min. The surge tank balances this discrepancy by storing sufficient AC during the mixing cycle to satisfy the meter requirements. Inasmuch as the pump operates continually during the 90-sec cycle between AC discharges, there is ample capacity to recharge the surge tank. Furthermore, the surge tank is of sufficient size to meet the double load caused by the simultaneous demand of both mixers. Secondly, for operation the AC meter requires an AC fluid pressure of 30 lb per sq in. In filling the surge tank in the morning, an air pocket is trapped at its top. By using pressure regulating valves on the AC return circulating lines, the pressure of the air pocket may be retained at the required 35 lb per sq in. Thus, by the utilization of the surge tank in the AC feed line, smaller pumps are adequate for pumping AC, and a simple method of maintaining the necessary fluid pressure is provided.

The mixer body, spray bar, and AC meter are all either directly or indirectly heated by electricity. Originally, the mixer body and spray bar were heated by electric strip heaters. Because of the high temperatures developed during the heating, the special heat-resistant insulation on the wiring to these heaters failed. Therefore, the strip heaters were removed and replaced with "calrods." In a "calrod" installation all wiring is outside the heated zone, thereby eliminating the need for special wiring. By this modification more rapid and more uniform heating resulted, the heating time being reduced from 5 hours to 2½ hours. The meter and its screen were originally heated by the circulation of a heat transfer oil, electrically heated by an immersion "calrod." This method was unsatisfactory because the heating time required was from 7 hours to 8 hours and a heavy overload developed on the heat transfer oil circulating pump. Examination of this system disclosed that the oil lines in the head of the meter were crimped in many places. Replacement without crimping was impossible, due to the tortuous coils and twists, so the piping was abandoned and "calrods" were installed around the head of the meter and its screen, leaving only the body of the meter to be heated by circulating oil. By this change the overload on the circulating pump was eliminated, and the heating time was reduced from 7 hours to 2 hours.

Semi-automatic operation was intended by the design of the mixer. The weighing of materials was to be performed manually and then, by a system of interlocked controls, the mixing and injection of AC was to be performed automatically by the pressing of a button. Manual discharge of the mixer was to be restricted until the mixing cycle was completed. The system worked, but an analysis disclosed that it was not foolproof. By virtue of its operation, the meter required a resetting after each measurement of AC. If this resetting was overlooked, the next measurement of the meter would be 10,000 gal of AC. Hence, rather than burden the mixer operator with the responsibility of avoiding a flood of searing hot AC, it was decided to modify the system and restrict its function to a safety measure. In this alteration the interlocking device prevents tripping of the weigh box discharge gate before the mixer is emptied and the gate closed, and aids the operator in timing the mixing cycle.

In operating the mixer, the operator weighs the proper quantities of materials into the weigh box. After emptying the contents of the weigh box into the mixer a 20-sec dry mix is made. Injection of AC follows after which a 40-sec wet mix period is provided. The 4 tons of paving mixture are then discharged into a truck below the mixer gate. Signal lights, "klaxons," and a peephole assure the operator that he is discharging his mixer into a truck and not into the truck runway. Thus, the mixing cycle requires from 100 sec to 120 sec. Each mixer produces 4 tons of paving materials every 2 min, or 120 tons per hr. To assure full efficiency the mixers are offset, one over each of two truck lanes in the runway. Hence, with both mixers operating, the plant can produce at least 240 tons per hr while the hot bin supplies last.

During an 18-month period, with the exception of the few minor difficulties, the mixers have operated most satisfactorily. The character of the product indicates that the mixers consistently produce good, uniform mixes. The wear on the blade tips, liner plates, and discharge gates has been negligible. The spray bar jets were varied from spray tips up to full 1¼-in. openings during the period. Orifices smaller than the full opening tended to clog with carbonized AC and were gradually increased in size until full openings are in use with little or no effect upon the uniformity of the mix. As for shaft speed, 45 rpm is the nearest speed available to provide best results, but it is believed that 42 rpm would eliminate the slight splashing of the material noticed at 45 rpm. Based on the knowledge gleaned through experience and the results obtained with these mixers, it appears that the selection of large mixers was a good one and any trepidation concerning their operation and their ability to produce good, uniform mixes has been completely dispelled.

Dust Control

The control of dust expelled from such a plant and escaping from the machinery therein has received increasing attention during the past decade. The extent and the intensity of the interest generated may be traced to the degree of the nuisance committed on the neighborhood, the increasing realization of the cost of dust waste, and the appreciation of the detrimental effects of the dust on operating machinery and on the working conditions of the plant personnel.

As asphalt plants either entered more populated areas, or were hemmed in more closely by other industries, the great clouds of yellowish brown dust billowing from their stacks became more and more objectionable. In some cases the condition was so bad that the plant was confronted with the alternatives of moving or of providing some method of abating the nuisance if harmonious neighborly relations were to be preserved. Increasing costs of materials further emphasized the undesirability of dust clouds by stressing the costly wastefulness of usable materials. With the development of this new conception of dust, fugitive dust within the plant also gained new importance. Increasing costs and difficulty in securing new machinery or parts prematurely worn out by the abrasiveness of the dust, and the loss of trained men to other industries where working conditions were more pleasant and agreeable all aided in crystallizing the appreciation of the true problem of free dust.

At the plant in Manhattan, situated as it is in a high class residential area, the significance of complete dust control was thoroughly appreciated. The design and the operation of the plant were and are consequently directed to the attainment of the highest feasible degree of dust control. The method of control adopted was of the two-stage type, wherein heavy coarse particles are removed in the primary stage and the light fine particles in the secondary phase of the system. The phases of the process and the entire process are parts of the main blower exhaust system of the plant.

Each unit in the plant possesses its individual and separate dust control and main blower exhaust system and each of the latter includes a main duct exhaust fan having a capacity of 30,000 cu ft per min, driven by a 60-hp motor operating through an adjustable speed fluid and a V-belt drive to yield a maximum fan speed of 980 rpm. The exhaust fans have rectangular intake ducts 69 in. by 20 in., and a circular exhaust stack 42 in. in diameter. Under operating conditions of 60 hp, fan speeds of from 860 rpm to 920 rpm are developed; the capacity ranges from 24,000 cu ft per min to 28,000 cu ft per min at a static head of 7 in. of water at the fan intake. Input power and air capacity vary within limits with atmospheric conditions, stack temperature, and moisture content of the materials processed.

The main exhaust fan performs a dual purpose by acting as the main draft for the drier fires and as a secondary draft for the collection and conveyance of the dust from the various machines. A main draft duct 36 in. in diameter conducts flue gases and dust from the drier to a "multiclone" precipitator. Secondary ducts provided with adjustable dampers convey dust from the vibrating screen and the mixer through lines 12 in. in diameter to the hot elevator. From the hot elevator a line 24 in. in diameter conveys the accumulated dust to the precipitator. All these machines are built as dust-tight as possible and operate under negative pressure. The multiclone discharge is connected to the fan intake by a rectangular duct about 10 sq ft in cross section. The dust precipitated in the multiclone is discharged into the hot elevator intake chute by a motor driven rotary valve for re-use in the plant.

At first some difficulty was encountered in securing the desired results from the system, but complete control of fugitive dust in the plant was attained by damper settings whereby about 24,000 cu ft per min of the total intake fan capacity of 28,000 cu ft per min is allocated to the drier exhaust and about 4,000 cu ft per min to fugitive dust control. When the plant processes material having an average moisture content of 6% at the rate of from 45 tons per hr to 50 tons per hr in the drier, dust samplings taken at the intake and discharge of the multiclone indicate a removal of from 80% to 85% of the total dust entering the precipitator. The dust sampling of the stack discharge indicates a dust content of 2.25 g per cu ft of air.

Only the primary phase of the dust control system—namely, the multiclone precipitators—has been installed. Contracts for the installation of the secondary phase (electrical precipitators) are in progress. It is expected that the secondary phase will remove 90% or more of the discharge, giving the entire system an over-all efficiency of about 98%. The air and gases exhausted from the system then will be purer than the atmospheric air as far as dust is concerned.

Operation of the system as installed, although smooth and efficient as far as it goes, is unsatisfactory because the residual dust from the multiclones is being exhausted from the stack. The quantity, although far less than in most other plants, is still objectionable, indicating that a primary system in itself is insufficient and must be augmented by a secondary system to yield satisfactory dust control. Prevailing winds blowing the dust out over the East River have so far favored the plant, and the objectionable dust has as yet created no great degree of nuisance.

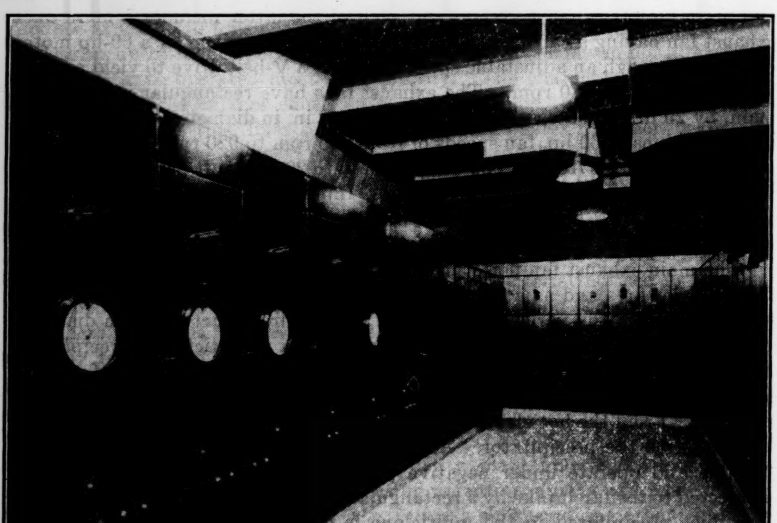

Fig. 5.—Main Electrical Control Room, Showing Asphalt Tank Recording Thermometers

Electrical Controls

Electrical power used by the plant, repair shops, and other facilities in the plant area is supplied at the rates of $1.75 per kw for demand and ¾¢ per kw-hr for consumption. A special building contains the necessary equipment to transform the power in the main supply feeders to the requirements of the plant. This equipment is owned and serviced by the utility company. In another section of this building are located the plant's main power circuit breakers, which are owned, operated, and serviced by New York City. Power furnished to the plant is 208-volt, three-phase, 60-cycle current for power, and 120–208 volt, three-phase, 60-cycle current for lighting.

In the control room (Fig. 5) the power is distributed to the various machines through a U-shaped control panel board. Separate panels of this board contain all the necessary electrical devices and controls for each piece of machinery—such as the driers, mixers, exhaust fans, etc.—for simplicity in tracing breaks and in electrical maintenance and repair. Controls for the electrical heating of each storage or service AC tank similarly are contained in separate panels. From the panels the wiring runs to the operating control stations for various machines. In most cases, operating control stations contain the controls for several pieces of machinery whose operations are interrelated or must be started or stopped in proper sequence. In this manner all conveyor belts from dock to bins are operated from one control station, and the heating of all lines, pumps, and other appurtenances is controlled from another. Similarly a single control station operates the reclaiming belt, cold elevator, drier, hot elevator, vibrating screen, fuel oil pumps, oil burners, and exhaust fan of a single plant unit. A similar station controls the other unit, and also contains the controls for the building ventilator fans. Because these machines must be started or stopped in a definite sequence, the controls are mounted in the station in that order. With controls so organized, operating interlocks and personnel safety controls can be readily provided.

Costs

Considerable interest has been displayed in the operating costs of the various phases of production and in the final cost of materials manufactured by this plant. The elegance of the buildings, their location, the intricate controls, the extraordinary dust control, and the other seeming extravagances create grave doubts concerning the ability of the plant to produce paving materials at competitive or even reasonable costs. Visitors are amazed when informed that the plant can manufacture its products economically and even competitively. When apprized of the actual cost of the plant ($1,000,000 exclusive of land value), the perplexity increases, or the certainty of some subterfuge becomes even stronger. To commercial operators, the carrying charges on such an initial investment alone would be staggering, without adding the labor costs of what appears to be a comparatively large personnel. Can a visitor be blamed, after a superficial inspection, for wondering what form of mental gymnastics perpetrates the bland statement that costs are reasonable?

In comparing costs, it must be remembered that costs are relative and should be collated on a basis of real value received. To a city responsible for public safety and convenience, over-all costs and availability of the materials when required are of greater importance than specific costs. A contractor operates an asphalt plant during the paving season at high volume and low specific cost. When volume falls off and specific cost rises above an economical limit he shuts down and lays off his force. A city cannot do this. Public safety and convenience must prevail at all times, and repairs must be made in or out of season. Under such conditions a ton of asphaltic paving material may be worth a great deal, particularly when all the manufacturers of this commodity are hibernating. Again, a contractor builds a plant with an eye to as great a return on investment in as short a time as possible for, by the vicissitudes and competitive nature of the industry, he may be out of business in a year. The city, on the other hand, will remain in business as long as streets and roads exist and require repairs, and can therefore project its returns on investment over a long period of time. Furthermore, the city's borrowings are tax exempt and consequently its bonds are attractive to investors at very low interest rates. In addition, the city sees no need for taxing its own property. It will be seen, therefore, that many factors enter into the costs to make them relative, and dependent almost solely on the value received by the purchaser.

The city, acting in the dual capacity of manufacturer and purchaser of asphaltic materials produced in a city owned plant, is interested in specific costs only in so far as they affect over-all costs. The specific costs of manufacturing a ton of sheet asphalt, for instance, are submerged in the over-all cost of placing this material in a pavement, regardless of season or weather, to maintain public safety and enhance public convenience. Nevertheless, it is interesting to examine the specific costs involved in the production of asphaltic materials at the plant.

Specific costs, again, are relative, for they depend greatly on the relationship between plant capacity and plant output. In a plant owned and operated by the city this relationship will vary from very low output to full capacity from day to day, depending on weather conditions the whole year round. Of course, on days when emergency paving is impossible the plant will not operate. Thus, specific costs will vary between exhorbitant values and very economical values. The extreme costs are not comparable with commercial plants that operate only when the demand in relation to capacity is sufficiently high to assure a return above overhead, operating costs, and material costs. Therefore, the most suitable basis for comparison of tonnage cost may be derived by dividing actual annual or monthly costs by the average daily output and by the average number of operating days per year or month, and then adding the actual cost of materials.

Before applying the formula, the average daily output and the average number of operating days must be ascertained. Although the plant has a maximum average output of 375 tons per day per unit, this quantity is not applicable because actual output may vary from 450 tons per day per unit during the height of the paving season to 50 tons per day during the winter in emergencies. Experience has shown that an average daily output of 300 tons per day per unit can be attained over the operating period A survey of 30 years of operating experience at the old plant discloses that the average operating time prevalent in the New York City area is 212 days per yr. This annual average may be broken down to 21 days per month for 10 months, thereby permitting 1 month for vacations in the summer and 1 month for plant overhaul in the winter.

In applying the formula it is best to reduce separately each factor involved in the costs—such as labor, power, materials, and overhead. By this method each phase may be compared individually, and the whole collectively.

Labor.—Labor at the plant is paid on an annual basis, so that the actual annual labor cost is used. This cost includes full time for supervisory, operating, maintenance, and repair forces comprising the full plant complement of twenty-one men. The total annual payroll, during 1945, for these men was approximately $55,000. By applying the formula and the factors previously discussed, the cost of labor is found to be $260 per operating day, or 87¢ per ton.

Power.—Power is paid for on a monthly basis; therefore, actual average power charges per month are used. This cost includes the operation of one unit, all the material conveyor belts, the dust pump, heating of AC tanks and lines to both units, all other equipment required in operation, and light. The average monthly charge for power is approximately $1,300. By applying the formula and the factors previously determined, the power cost is found to be $62 per operating day, or 21¢ per ton.

TABLE 1.—COSTS OF MATERIALS, IN DOLLARS, FOR VARIOUS PLANT PRODUCTS (1945 PRICES)

Product (1)	INGREDIENTS					Total (per ton) (7)
	Sand (per cu yd) (2)	Mineral dust (per ton)[a] (3)	AC (per gal) (4)	Stone (per cu yd) (5)	Fuel oil No. 6 (per gal) (6)	
Sheet asphalt (top)	0.92	4.50	0.0611	....	0.0439	2.88
Binder	0.92	....	0.0611	1.55	0.0439	1.85
Asphaltic concrete	0.92	4.50	0.0611	1.45	0.0439	2.21

[a] In bulk.

Materials.—The cost of materials (Table 1) is based on 1945 prices; the cost of fuel oil is predicated on the maximum use of 3 gal per ton of product.

Overhead.—It is in the calculation of overhead costs that variations occur. To avoid any stigma of subterfuge, an outline of the costs involved is given (Table 2) so that the reader may combine them in any way he pleases and determine his own results.

The outline indicates a maximum overhead cost of 80¢ per ton. Actual expenses, however, consist of only the interest on capital investment, and repair parts, lubricants, etc.—or a total of 33¢ per ton. As time goes on, the replacement of an entire piece of equipment may be necessary, involving some part of the 31¢ set aside for depreciation of machinery. As a result of preventive maintenance this may not be likely, but it is reasonable to assume that preventive maintenance will double the effective life of the equipment and extend its depreciation to 40 years. By this time some of the equipment will be obsolete, but by proper replacement and gradual modernization it can be maintained in service. Thus, depreciation charges on the machinery may be reduced to 16¢ per ton. Depreciation charges on the structures are not included, for by their character they can far outlast their usefulness. The remaining $100,000 of the original $1,000,000 cost is allocated to the electrical dust precipitators not yet installed. After this installation depreciation on machinery will be based on $500,000 instead of the present $400,000. By this calculation a reasonable overhead of 49¢ per ton is obtained. Adding this overhead cost to that of labor, power, and materials gives the over-all costs as shown in Table 3. These costs compare favorably with prices quoted by contractors in the area for similar materials.

TABLE 2.—OVERHEAD COSTS, IN DOLLARS

Description (1)	Principal (2)	Time (yr) (3)	Costs Annual (4)	Costs Per ton (5)
Interest on capital investment[a]	900,000	....	18,000	0.28
Depreciation on structures	500,000	50	10,000	0.16
Depreciation on machinery	400,000	20	20,000	0.31
Repair parts, lubrication, etc.	....	....	3,000	0.05
Total	....	....	51,000	0.80

[a] At 2%.

TABLE 3.—TOTAL COSTS, IN DOLLARS PER TON

Description (1)	Sheet asphalt (2)	Binder (3)	Asphaltic concrete (4)
Labor	0.87	0.87	0.87
Power	0.21	0.21	0.21
Materials	2.88	1.85	2.21
Overhead	0.49	0.49	0.49
Total	4.45	3.42	3.78

The preceding costs were based on the operation of only one plant unit. When labor conditions improve, and field paving forces are recruited to full strength, and when field requirements increase to 600 tons per day, operation of both units will be required. Under these conditions the costs on a tonnage basis will decrease materially. Only three additional men will be required to operate the second unit, therefore annual labor costs will only increase $7,500 to a total of $63,000. Applying the formula, labor costs decrease to 50¢ per ton. Power costs will also decrease, because the demand costs of material handling will be but one half of their present ratio to operating demand costs, to 16¢ per ton on an average of 600 tons per day. Material costs will obviously remain constant, and overhead costs will decrease to one half of their present costs on fixed charges. On repair parts and lubricants they are expected to double, thus giving an estimated overhead cost of about 32¢ per ton. Applying these savings and compiling the costs on a basis of 600 tons per day gives $3.86 per ton for sheet asphalt, $2.83 per ton for binder, and $3.19 per ton for asphaltic concrete. These total costs of manufacturing a ton of the respective materials not only compare favorably with prices quoted by private contractors, but also are definitely competitive and produce a substantial annual saving to the city in the maintenance and repair of its streets. It is emphasized again that the plant produces materials for maintenance, repair, and resurfacing of streets and highways in the Borough of Manhattan only, and not for new construction or complete repaving of streets, which is done by private contractors.

Conclusions

The Municipal Asphalt Plant of the Borough of Manhattan, after 2 years of operation, has clearly demonstrated that an asphalt plant, properly equipped and suitably housed, can enter the heart of an urban area and be fully accepted as a satisfactory neighbor. By this development a multiple-unit plant can displace the usual practice of utilizing numerous small plants scattered about the outskirts of a city. The concentration of production at a focal point yields not only higher efficiency at reduced cost at the plant, but also greater efficiency at reduced cost in the transportation of materials from the plant to the field.

This plant has firmly established the economy and efficiency of electrical heating of AC, with all lines and equipment appurtaining thereto, for large stationary plants. By the innovation of insulated hot bins it has developed a new operational procedure which lends far greater flexibility to plant production. It has also definitely demonstrated the feasibility of large 4-ton batch mixers in producing high-grade paving materials. To the files of experience it has added a profusion of data concerning the operation and control of modern plant equipment and accessories. Lastly, it is hoped that this treatise may assist other plant designers and operators in utilizing some of the information gained.

DISCUSSION

Clarkson H. Oglesby,[8] **Assoc. M. ASCE, Clark D. Henderson,**[9] **and Oscar C. Holmes, Jr.,**[10] **Jun. ASCE.**—The description of the Municipal Asphalt Plant is interesting in that it records the developments and advances that have been made. The author is to be commended for the careful description of the problems encountered and the measures taken toward their solutions.

As presented, however, the suitability of the cost data for comparison with the prices of competitive producers is open to question. It seems to the writers that the data cannot be used to measure production economy.

In arriving at total annual overhead costs, it is essential that a proper allowance be made for capital recovery with a return commensurate with the risk. The appropriate method of determining annual capital recovery cost is by the exact method recommended by E. L. Grant,[11] M. ASCE:

$$R = (P - L) \frac{i(1 + i)^n}{(1 + i)^n - 1} + L\,i \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

in which R is the uniform annual capital recovery amount; P is the total investment including depreciable assets and land; L represents the salvage value and the land value estimated to be realizable at the end of the useful economic life; n is the expected useful economic life, in years; and i is the interest rate appropriate for the risk involved.

Straight-line depreciation plus average interest may give a satisfactory approximation of R when i and n are small. The author has used straight-line depreciation plus interest on the initial cost of depreciable assets. This method overstates the interest cost by neglecting the declining balance.

Interest on land value is omitted. Although the land was not purchased, it could have been sold or leased and the proceeds could have been used by the local government. Clearly, therefore, the use of this land involves a sacrifice to the city which should be added to the cost of plant operation. In the absence of a basis for decision to the contrary, it is common practice to assume that the present value will remain unchanged and recoverable.

Prospective lives and applicable interest rates can rarely be fixed precisely on the basis of facts. In the typical case, judgment plays an important part. Where genuine reason for doubt exists, conservative practice dictates that the higher interest rate and the shorter life be chosen. The author's estimates seem far too optimistic.

Considering the vast changes in paving practices since the beginning of the twentieth century, recent developments in petroleum refining, and the prospective shortages of petroleum oils, a capital recovery period of 40 years for an asphalt plant cannot be considered as safe. The complete elimination of depreciation on the structures comes from the erroneous assumption that, because a structure is as useful as ever, it is as valuable as ever. Considering the location and the highly specialized nature of this plant, the assumptions of perpetual useful life and constant realizable value are not warranted.

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[11] "Principles of Engineering Economy," by E. L. Grant, The Ronald Press Co., New York, N. Y., 1938, p. 90.

With regard to the choice of length of life for equipment, it is inaccurate to assume a length of life (20 years), compute annual depreciation charges on that basis, and then use all or part of that depreciation for preventive maintenance and replacements, thereby doubling the life of the asset and reducing by half the annual depreciation charge. By traveling around this circle again and again, the depreciation charge soon approaches zero and the life approaches infinity.

Depreciation charges and preventive maintenance expense are separate items of cost. Although a good maintenance policy will tend to prolong the life of the equipment, the original investment must still be recovered through adequate depreciation charges. The inadequacy of the author's combined provision can be demonstrated in this way. If the amount of the depreciation charge ($10,176 per yr) were to be deposited in a sinking fund and compounded annually at 2%, the sum available at the end of 20 years would be slightly less than $250,000, but machinery costing initially $400,000 would be worn out.

The cost of borrowed money (which apparently the author has used) is rarely, if ever, applicable in the determination of cost. The 2% interest rate does not reflect the risk involved in a public asphalt plant, but reflects the investing public's opinion of the general credit of the local government and the worth of the federal income tax exemption on local government bond interest. A suitable interest rate should be determined for this particular venture and it undoubtedly will be considerably greater than 2%.

Certain essential costs are not included in the paper, such as administrative costs, insurance, taxes, maintenance on structures, etc. Administrative services, such as purchasing, cost and pay roll accounting, personnel administration, and a great variety of miscellaneous items, do not appear in the cost estimate presented.

A property risk exists in this plant and, if it is insured, the rate is probably rather high. If it is not insured, the risk is borne by the local government in its self-insurance plan. In either case, the cost should be allocated to the product.

The omission of property taxes is an error of considerable importance. The effect on the taxpayer must be kept in mind for he is the ultimate purchaser of the paving material. If taxes are omitted in computing the cost of the paving material, the equivalent sum must be paid in other ways. Since private manufacturers are among the taxpayers, it is obviously unfair to omit taxes for the public manufacturer but to include them, as is necessary, in competitive prices. Maintenance expense on the structures is also omitted; and this expense is unlikely to be negligible over a long period of time.

Although most of these comments are directed toward understatements of costs, they do not necessarily imply that accurately determined cost would compare unfavorably with the prices of private manufacturers. Furthermore, an unfavorable direct cost comparison must be tempered by consideration of the value of extra services not reducible to money terms.

Summary.—In the opinion of the writers, the cost data developed by the author are unsuited for comparative purposes because of his treatment of capital recovery (depreciation); the omission of the present realizable value of the land and of depreciation on structures; the use of unwarranted long lives; the confusion of depreciation charges and maintenance expense; the adoption of too low an interest rate; and the failure to include administrative costs, insurance, taxes, and maintenance on structures.

J. H. CREED,[12] Assoc. M. ASCE.—The frank discussion of the troubles encountered in a Municipal Asphalt Plant in New York, N. Y., is refreshing. Too often such things are somewhat glossed over or avoided entirely.

The selection of a site for an asphalt plant in any settled area always presents many difficulties, which have been met excellently in this case. One of the major costs of an asphalt pavement is frequently that of hauling the hot mix from the plant to the street. The central location minimizes this cost, and can well offset the extra cost of a location on expensive property. Even the prevailing wind seems to have favored the site, blowing the dust across the river.

The materials seem to be well handled, although the capacity of 100 tons per hr might seem a little small for the rock and sand conveyer. The bulk handling of powdered limestone or other finely ground mineral dust always presents many problems, the principal of which are to keep it aerated and thus to prevent packing. A lower silo would help somewhat, as it would reduce the pressure and minimize avalanching.

The 700-ton storage would give a 20-day run at 5% batching of 700 tons per day—indicating that shallower and smaller bins might suffice. The two mixers are arranged quite conveniently, assuring a sure supply of mix in case of the breakdown of one of the units. Of course, this arrangement increases the cost of the installation, as well as the space occupied—quite an item in a central location, on expensive property.

The setup for aggregate handling is good. The system of feeding the aggregate from several bins onto a belt makes for good control of the grading of the cold feed. The capacity of 80 tons per hr would seem a little small, but with large plant bins and an irregular demand, it may suffice. A pan feeder under the sand bins is very satisfactory, feeding even rather wet, fine sand quite well, and obviating the necessity for vibrators. Ordinarily a direct feed from the belt to the cold elevator pit will give a very uniform feed.

The double-fired type of drier is unfamiliar to the writer. Ordinarily a jet type of burner with a pressure of about 150 lb per sq in. on the fuel oil, and 100 lb of steam or compressed air, is installed. From two to three jets are generally set up, which are used to develop the proper heat with the most efficient operation of the burners. The suction of a fan, with a capacity from 25,000 cu ft to 40,000 cu ft, on the dust collector will draw the fire well into the front end of the drum, with very little, if any, tendency for back draft.

Since coating in the combustion chamber of a drier is almost unavoidable, its removal is considered a routine maintenance operation on most plants. There seems to be no real substitute for a good fireman when it comes to controlling drier temperatures. A system whereby the cold feed can be kept uniform, both as to quantity and as to moisture content, and close correlation between the operation of the cold feed and the fire are essential. A correctly set pyrometer in the drier discharge is helpful, but this instrument has a tendency to lag. The vertical hot elevator is always troublesome for the reasons given by the author, and should be avoided, even at the expense of the extra space required by the inclined type. The choice of vibrating screens is a wise one. Much better screening can be obtained than with revolving screens.

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Insulated hot bins are a great help, particularly in a plant where there is much stand-by time. Big bins, such as those in the Municipal Asphalt Plant, are also invaluable, giving as they do so much surge space. Continuous running of the drier is important in maintaining even mix temperatures, and big bin storage mininizes drier shutdowns. In the writer's experience, it has been found necessary to have at least three bins; and, on coarser mixes, four bins are preferable, to obtain a properly graded mix.

The asphaltic cement (AC) seems to be excellently handled in the plant. In the writer's experience, asphalt has generally been heated with steam coils in the tanks; but, the "calrod" system may well have many advantages.

The size of the mixers is rather startling to one who is accustomed to plants using mixers of from 3,000-lb to 4,000-lb batch capacities. In the writer's experience mixing times of 45 sec for asphaltic concrete and 30 sec for plant mix have always proved adequate to coat the aggregate thoroughly. In a properly set-up plant, 45-sec and 60-sec cycles, respectively—or a capacity of 120 tons per hr for the 4,000 lb plant running asphaltic concrete—are necessary.

The writer has usually found it advisable to weigh the batch of asphalt, using a hanging bucket and a dial scale. The bucket is allowed to fill to the desired weight while the aggregate is being batched into the weigh box. When the mixer is dumped and the discharge gate closed, the aggregate for the new batch is dumped into the mixer and then the asphalt is dumped in, using either a gravity dump or a pump. Generally the gravity dump produces satisfactory results if the distributor pan is properly adjusted, and it has the advantage of not breaking down, which is not always the case with a pump. Asphalt is fed to the weigh bucket with a two-way valve, which by-passes the asphalt back to the storage tank while the feed to the bucket is shut off. Metering has the inherent weakness that the volume of asphalt will vary with temperature changes. Also, meters have a tendency to break down at times.

With only a "multiclone" precipitator, it is remarkable that there has not been more trouble with dust. Numerous operations have been shut down by injunction, and damage suits sustained on the strength of dust from plants in much less populous areas than New York. Electrical precipitation will undoubtedly clean up the situation. It would have been interesting to have included the considerations which led to the adoption of electrical precipitation rather than the more familiar steam or water spray methods.

The costs of producing material seem to be quite reasonable, considering the necessary operating conditions. The crew of twenty one seems rather large for a highly mechanized operation such as this. It would be interesting to see a breakdown of duties. One fallacy which enters the cost picture is the failure to charge an item for taxes. If the city sold the property to a private individual, he would pay taxes to the city. Therefore it is costing the city that much money to own the property and the taxes should be included to obtain a true cost picture.

All in all, Mr. Csanyi has written a very interesting and instructive paper, and merits the thanks of those in the profession whose work involves the use of asphaltic paving.

L. Csanyi,[13] M. ASCE.—Messrs. Oglesby, Henderson, and Holmes have introduced into their discussion an age-old controversy which will range on for many years to come—namely, public methods versus private methods of financing and accounting. Although this cannot be settled here, it is only fair since one side has been portrayed to present the other also.

In public financing, regardless of the method, the primary object must always be: Does the taxpayer receive full value? To assure that he does, therefore, each project must be considered upon its own merits and not by generalities. This is particularly true of operating plants, and holds alike for public or private enterprise. Furthermore, those services rendered which do not reduce readily to a monetary basis must be evaluated and given proper proportion in the final analysis. Thus, general formulas do not readily apply without extensive and proper adjustment.

As a case in point, it would be difficult if not impossible to assign a value for i (risk), in Eq. 1, for a plant whose full output is assured as long as it operates. Also, if this had been a private enterprise, the value would have the same significance, with $i = 0$, $R = 0$. It is apparent, then, that the method used gives a reasonable value for R.

Another omission cited is interest on land. The conditions that permit such an omission are as follows: The land, originally a swamp, was purchased for a very small sum many years ago. Private enterprise was glad to be rid of it; yet when the City of New York (N. Y.) converted it to a convenient dump for the disposal of excavated materials, private enterprise as well as the the city used it gladly to save money. In 1913 the city reclaimed the filled area for the site of the old asphalt plant; 27 years later the construction of the East River Drive enhanced the land values in this area to a point where this site was valued at $3,000,000. The reasons why the land was retained as the site of the new plant are adequately indicated in the paper. Under these conditions interest could not properly be charged on original cost, actual cost (including the large savings made), or present value. A taxpayer required to pay this interest charge in the cost of materials used to pave an emergency water pipe repair to his house would have a legitimate complaint.

Taxes fall into a similar position to interest under these conditions. Considering land alone, there is substantially nothing to tax. The discussers hold that private enterprise would have to pay in the same situation, and that this sum has been removed from the tax rolls. This argument is open to question. Another site equally expensive but less efficient would still reduce tax rolls and cost more. Why build the plant at all? To carry this reasoning to the ridiculous, why should the city build anything? Why not turn back the streets, they are valuable and would greatly increase the tax rolls? The answer is to be found in whether the taxpayer gets a fair return.

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The problem of taxes on the improvement poses similar questions, dependent on specific conditions. This plant operates when the management of private enterprise deems operation uneconomical or inexpedient. Emergencies exist during rainy days and inclement winter weather. These emergencies must be met and damage must be repaired if public life and property are to be protected. It is true that the production costs rise when the plant produces from only 30 tons to 50 tons on these days. Estimated cost of production is $25 a ton; yet this rate is averaged in on an annual basis and the taxpayer is charged at the nominal rate. If this plant were not in existence, and private enterprise were induced to meet the need, the cost per ton would be nearer $50. Since the plant operates about 100 days per yr under these conditions, a saving of 40 tons per day for 100 days at $25 a ton would amount to $100,000 per yr. To the taxpayers paying the specific bill for this material, it will amount to $180,000 per yr. Taxes at an average of $3.00 per hundred for $1,200,000 equal $36,000 on improvement, or $126,000 on land at present value plus improvement. Taking the difference equal to $26,000 in the latter, this is not too high a cost for services rendered in protecting life, limb, and property.

Depreciation is a major dispute not only in public enterprise but also in private enterprise. Again, examine the conditions. The vast changes referred to in the discussion do not apply to that manufacturing where only refinements have been developed. Furthermore, the plant as designed anticipates future modifications to be readily incorporated. Thus, sudden obsolescence is not expected. Therefore, 20 years of life for equipment seems reasonable, particularly when the old plant operated 27 years without major replacement. As to equipment cost, this included such items as bins, electrical controls and wiring, supporting structure and foundations for machinery, tanks, piping, and other items which have a life expectancy in excess of 20 years. The discussion calculated a depreciation reserve of $250,000 in 20 years at 2%. This sum is more than enough to meet the replacement cost of the operating machinery. Furthermore, the large annual costs included under maintenance, and preventive maintenance under labor, provide leeway for long life of both equipment and structures. (Pay roll, purchasing, supervision, etc., have been covered under supervision in labor cost breakdown.)

This is the other side of the argument. The reader should appraise the whole as a taxpayer as well as an economist.

Mr. Creed has indicated an interest in the reasons behind the selection of electrical precipitation for last-stage dust control. Both methods mentioned and many others were studied before final decision. In all these methods the purity of the stack exhaust proved to be below the desired result. Other objectionable problems were introduced also—namely, collection and disposal of sludge produced and loss of dust. Since the paper was prepared, this installation has been placed in operation. The system, the first of its kind used for this type of work, functions excellently under a wide range of stack temperatures, steam content of flue gases, and gas volumes. The combined system removes 99.7% of dust, makes the stack discharge purer than the surrounding atmosphere, and frees the plant from reliance on favorable winds.

Although the original manuscript (on file in the Engineering Societies Library) contains a breakdown of personnel, a brief review here may serve a worthy purpose:

Department	Function	Number of men	
Supervision and office	Engineer in charge	1	
	Foreman	1	
	Clerk	1	
			3
Materials handling	Craneman	1	
	Barge trimmers	2	
	Conveyor operators	2	
			5
Manufacturing paving materials	Conveyor operator	1	
	Drier man	1	
	Mixer man	1	
			3
Maintenance and repair (regular and preventive; also asphalt and dust handling)	Master electrician	1	
	Electrician	1	
	Electrician's helper	1	
	Master mechanic	1	
	Mechanic	1	
	Mechanic's helper	1	
	Oilers	2	
			8
General	Laborers		2
Total			21

From this breakdown, it will be noted that only thirteen men are required to operate the plant. The other eight are used for extensive preventive maintenance and repair of equipment and structures. The cost per ton is slight for the return in long efficient life required of a municipally operated plant.

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THE LOGICAL SYSTEM OF FREIGHT RATES

By Charles L. Hall,[1] M. ASCE

With Discussion by Messrs. Raphael H. Courland, Otto Meyer, Alexander Markowitz, Haywood R. Faison, Malcolm Elliott, and Charles L. Hall.

Synopsis

The effects of the existing illogical system of freight rates are presented, and a plea is made for a new system. The plan calls for a separation, to be reflected in all rates, between terminal charges and actual transportation charges. It further requires (1) a uniform classification of all freight, regardless of its method of carriage; (2) charges for each form of carrier, uniform throughout the United States, fixed by each branch of the transportation industry in terms of the cost of moving the most easily transported product; (3) a ratio between charges by the various types of carrier, to be determined by independent members of the engineering profession; and (4) determination by a public body of the basic rate, to which the foregoing ratios shall apply.

Struck by the difficulty of determining the savings resulting from the introduction of a new form of carrier between any two terminals, the writer has analyzed the existing system of fixing freight rates. This analysis has convinced him that the public can be fully served only if the whole freight rate structure is established on a purely rational basis without regard for what might be called the "particularistic" motives on which the existing system of rates is based.

The United States has been built up by internal free trade. Thus, any commodity can be moved from any point within the country to any other point within the country without consideration of either customs duties or the relative values of money, as there are no customs and the value of money is uniform. Moreover, the cost of the sale or the distribution of an article, once it arrives at its destination, is independent of its place of manufacture. No one claims that the United States could have been built up on any other system. Admittedly, however, the price of the article, when sold to the ultimate consumer, must in-

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Note.—Published in November, 1948, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

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clude the cost of transporting it from its point of production to its point of consumption.

From the standpoint of national prosperity and union, the point to be selected as a manufacturing or processing site should be the place where the necessary raw materials can be most economically assembled and processed, and where the total cost of transporting the finished products to points of consumption is a minimum. This is, more or less, how a manufacturer selects a site. If the cost of assembling raw materials and transporting the products thereof is based entirely on the necessary expenses of transportation, all sections of the United States would be treated fairly. Each area of the country would be engaged in the business it can most economically pursue.

If, as is now the case, the cost of transporting any particular raw materials or manufactured products is due to any plan for putting more than a fair share of transportation cost on a few items, or on items manufactured in one section of the United States, in order to permit less than a fair share of such a cost to be allotted to some other items, or to items from some other part of the country, somebody is being treated unfairly. The economic result to the processor or manufacturer is the same as that he would suffer if duties were charged on imports, or if his dollar were depreciated. The system under which the country has been built up would then have been altered, and the national union weakened.

These truisms have scarcely ever been denied. Discussion of details nearly always has been based on the definition of "fair." Few members of the engineering profession have been heard from. This paper is an attempt to apply engineering principles to the question of freight rates.

Of course commodities to be moved are not identical. Nobody expects transits to be moved by any carrier as cheaply as crushed rock. From the standpoint of pure logic, freight should be classified according to the difficulties involved in loading, transporting, delivering, and protecting it. This latter factor will allow some higher classifications for more expensive products. The obvious first step in regulation of freight rates is the classification of all forms of material into as few groups (or classes) as possible. The classification must be applicable to all forms of carriers, with the exception of some products which are physically unsuited to some carriers—for example, very few types of freight can be carried in pipe lines. Subject to that limitation, however, there should be a general classification of freight, agreed to by representatives of all the carriers in the United States. Normally they should agree. If they do not, the Interstate Commerce Commission will have to do the work.

When a package is moved from point A to point B, four different services are performed. Somebody loads the package into a vehicle at A, moves the vehicle (with the package in it) from A to B, unloads the package at B, and meanwhile protects it from weather and thieves. The costs incurred in loading the package are, in the normal case, independent of its destination and are due solely to the physical characteristics of the package, its volume, its mass, and the number of packages of the same sort, intended for the same vehicle, committed to it at the same time. The cost of transporting the vehicle is due to its weight, the distance it travels, the amount of uphill haul involved, and—in unbalanced traffic—the distance the vehicle must subsequently travel unloaded to secure another revenue producing load. The cost of unloading it depends on the same factors as does the cost of loading. The cost of protection varies with packaging, perishability, and value.

These costs cannot, of course, be isolated for any particular package, because there are too many variables. Certain terminals are more efficient than others. It is usually cheaper to move a vehicle a given number of miles on a much-traveled road than on a poorly-traveled one, because there is a vast amount of road maintenance and central operation partly independent of train density. Taxes vary from state to state. Nevertheless, although the cost of moving any particular package cannot be determined, it is possible to determine—on a national basis—the relative costs of loading (or unloading) a package of one type and a package of another. Similarly, it is possible to compare the cost of transporting a vehicle of one type and gross weight for a given number of miles with the cost of moving a vehicle of another type and gross weight for the same number of miles. It is even possible to compare the relative costs of moving vehicles where there is much uphill haul, and where there is none. The costs of protection can be divided into protection connected with loading and unloading, and protection in transit.

Let an algebraic symbol, a, be assumed as the lowest unit cost. This can perhaps be taken as the cost of transporting 1 ton of coal for 1 mile on a railway of no grade, the coal being moved as part of a carload lot consigned at one point by one consignor to another point for one consignee. It should then be possible to state, as a multiple of a, every other factor connected with every other cost. For example, take a ton of household goods, moving as part of a carload lot. The loading cost might be 450 a per ton; the ton-mile cost, 4 a; the elevation cost, based on the number of feet of increases in elevation on all the uphill grades, 2 a per ton-foot; and the unloading cost, 390 a per ton.

In actual practice, of course, very few cars, except those moving less than carload lot shipments, are loaded or unloaded by the carrier. It is desirable, however, to retain the loading and unloading charge to cover the few cases, like transfers from rail to truck carriers, where it is incurred. Moreover, some part (perhaps 15%) of the loading fee should be charged against the shipper under any circumstances, because the carrier accumulates at terminals some charges which cannot equitably be assessed on any sort of mileage basis. To prevent abuse of terminal facilities there might be a minimum charge per car, regardless of the weight of its contents, to avoid very short carload shipments.

It should be noted that the loading and unloading charges are based on the weight of the packages moved, whereas the ton-mile cost should equitably be based on the weight of the loaded vehicle. This would automatically dispose of arguments about minimum carloadings. If the shipper wanted to pay a rate based on a 50-ton car for moving 1 lb, that would be his privilege. He would, perhaps, have had to pay a minimum terminal fee, as indicated in the preceding paragraph. Naturally, the surcharge on uphill traveling also should be based on tare weight. Carload lot tariffs would thus require only three subheads under each classification of freight: Loading charge per ton (net), transportation per ton-mile (gross), uphill haul per ton-foot (gross). It would probably be necessary to prescribe that the distance (and the uphill grade) of the most advantageously situated carrier should govern all movements between any two given cities. Otherwise, a carrier with a few miles shorter haul would monopolize all the traffic. However, a carrier should be allowed to demand rates based on its actual mileage if it is willing to risk losing traffic.

The fourth factor governing rates (value of the freight) can be handled in several ways. Probably the most convenient is to assume certain maximum values per ton under each classification, to be eventually reflected in the ton-mile rate for that class of freight, and to require any excess risks to be covered by insurance. This is more or less the method of compensating for the liability now.

Shipments of less than carload lots would naturally involve more factors. There presumably would be entries in the freight rate book under each classification for loading rate per hundredweight, loading rate per individual package, corresponding unloading rates, cost per hundredweight-mile, and cost per hundredweight-foot for uphill traffic. Needless to mention, weights used would be net. Presumably there would be a prescribed minimum weight for each shipment, so that the carriers would not be annoyed by small packages; or, if they were so annoyed, they would be paid for their trouble.

The preparation of the rate book, in terms of a and not of dollars, should be committed to representatives of the railroads. Presumably these would vote according to the ton-milage of their respective roads. Once all local controversies about special rates are eliminated, as they will be in this plan, the railroad industry can be relied on to settle these matters better than anyone else. Its chiefs are well aware of the relative costs of moving different kinds of goods. Nobody in the industry should have a motive for unbalancing such data for sordid reasons. If he had, his colleagues could soon find him out.

A similar plan must be followed for inland river commerce. This branch of transportation is in the ridiculous situation of charging at the same rate for upstream and downstream commerce, although everyone knows that the most economic balance is 8 lb downstream to 3 lb upstream. Whereas the economy of bargeload lots should be recognized, it probably would not be necessary to quote any differences between small lot shipments and bulk shipments, so that the significant entries in the water rate book would number eight—classification of freight, loading charge per package, loading charge per ton, corresponding unloading charges, transportation per ton-mile, percentage of reduction in this value for bargeload lots, allowable weight of maximum package, and dimensions of maximum package. The table of distances would merely show water distances between all points. To penalize upstream traffic, distances upstream should count double. Thus, the present unremunerative balance would quickly be destroyed.

The data cited could not of course be absolute. They would have to be quoted in terms of some other algebraic symbol b, which might be defined as the cost of carrying 1 ton of coal 1 mile downstream.

The rates should be based on the expectation that, at rail-to-water transfer points, the shipper would have to pay the rail unloading charge or the water loading charge, whichever were the greater. By application of this rule, joint rail-water rates would not be difficult to determine.

The maritime industry should decide the costs of movement in terms of *b* just as previously suggested for the railroads. There would appear to be no motive either to prescribe an unrealistic schedule or to play false with the public.

Exactly similar arrangements should be made for coastwise and Great Lakes shipping, common carrier trucks, pipe lines, and—if deemed wise—air traffic. The transportation industry would thus finally have a series of values, $a, b, c, \ldots, n$, independent of each other. Obviously, the next problem is to fix a ratio such that $b = p\,a$, $c = q\,a$, and so forth. The evaluation of $p, q, r, \ldots$ is a problem in engineering and should be committed to a group of civil and mechanical engineers, either permanent members of the Civil Service or engineers having no financial interest in any branch of the transportation industry. The object in setting these ratios is to cause traffic to move along the most economical lines, so that the real cost to the citizens of the United States for moving a unit of freight through a unit of distance will be a minimum. The ratios certainly cannot be formulated to scare branches of transportation into reducing individual rates, for which purpose too much competing traffic is now being used.

There still remains the basic problem of determining the value of *a*—essentially a mixed problem of finance and politics, which would have to be accomplished by a first-class body responsible in the last analysis to the people (presumably the Interstate Commerce Commission). The task of this body would be to fix the minimum value under which the American public would be served and which, at the same time, would provide a reasonable return for the capital employed in transportation, taken as a whole. There would be little temptation either to fix the amount too high, for that would bring commerce to a standstill, or to fix it too low, for that would bankrupt the carriers. The quality required is good judgment rather than the ability to resist the pressure of special interests.

In this connection it should be noted that a subsidy to any carrier, or its operation with public assistance at an admitted loss, unbalances the whole setup. For instance, the granting of any form of subsidy to an ocean carrier moving from Sacramento, Calif., to Houston, Tex., through the Panama Canal would result in the movement of certain goods by water between the Pacific Coast and Texas which ought to have moved by rail. Thus, the value of the ratio between rail and ocean carrier rates is reduced artificially. If the carriers are operating at a loss in the first place, the increased traffic by water, caused by the relative rate reductions, either overloads the subsidy and thus harms the taxpayer, or deprives the railroad of freight and thus drives the rail-water ratio further out of balance. Most likely it does both. Only if carriers meet all their own expenses out of their own receipts will the logical system of freight rates work smoothly.

It can thus be noted that there are four successive steps in establishing a logical and noncontroversial system of freight rates:

(1) The classification of freight according to the costs of handling it;

(2) The establishment of appropriate relative costs within each type of transportation, to be accomplished by a body representing the industry;

(3) The establishment of ratios between the basic costs of the various forms of transportation, to be accomplished by a body representing the engineering profession; and

(4) The establishment of a basic unit cost of national transportation, to be accomplished by a body representing the public.

The first task (classification of freight) is easy if the carriers by rail, water, truck, and pipe line can get along with each other. Even if they do not, the authority charged with adjusting differences should not have too many difficulties. It should always be possible to transfer goods from one class to another by unanimous agreement of all carriers. Increase in the number of classifications or contested changes in classification should be made very difficult.

The second task (rate schedules for the various types of carrier) is perhaps the most difficult, but the results should be relatively permanent when once accomplished, as seldom will any invention make it possible to carry some particular form of goods at an unexpectedly low rate. Revision at 10-year intervals should be sufficient.

The third task (fixing the ratios) must be performed from time to time. Inventions effecting economies in one form of transportation, but not in all forms, have been frequent during the life of the United States. Revision at 5-year intervals probably will be required.

The fourth task (fixing the basic transportation cost) will be almost continuous, for the general level of transportation costs is altered almost as fast as the cost of living. The Interstate Commerce Commission will probably have to hold hearings on the subject every year.

The advantages of this scheme, taken as a whole, are obvious. The interminable squabbles between sections of the United States, and between ports of entry, are abolished. The country will be better integrated. Transportation in general is assured a reasonable return. All freight will be moved by the carrier best suited to the task and the nation will be benefited thereby.

The objections to the plan rest basically on its effects on capital already expended. Undoubtedly some lines in every type of transportation will be forced out of business. Certain ports which now have an unfair advantage, based on reasons having nothing to do with engineering facts, will lose commerce. However, the United States has been built up by the ruthless scrapping of outmoded forms of industry. The first railroads to the west ruined both stage coach lines and New England agriculture. Where would the nation be if railroad building had been delayed for any such reason? This plan will put transportation on its mettle. The best argument in its favor is that if the proposed plan were now in force nobody would think of setting up the present illogical rate structure.

DISCUSSION

Raphael H. Courland,[2] Assoc. M. ASCE.—"The Logical System of Freight Rates" and the implied definition of "fair" rates are reasonable as far as they go, but they do not take into consideration certain pertinent factors that bear no relation to an engineering-economic analysis of transportation, as such. These factors, in some instances, are the only determining elements in rate setting; in fact, sometimes they constitute the sole justification for the existence of the carrier. Thus, it is not always true that "The object in setting these ratios is to cause traffic to move along the most economical lines * * * ."

Immediate military expediency, flood control, hydroelectric power development, long-range national research and development of certain natural resources, national policy in support of the exploitation of external natural resources, encouragement of "infant" or young industries and travel facilities (in furtherance of national military potential), use of mail service subsidy—all these have been, are, and will continue to be, factors of major influence in the national transportation system. Because of them, the generalization—"only if carriers meet all their own expenses out of their own receipts will the logical system of freight rates work smoothly"—cannot be considered absolute.

Hypothetically, if a commercial enterprise such as a carrier is to be considered self-sufficient, rate setting must be such that income will amortize the capital investment, pay operating expenses and maintenance, permit normal replacement, and yield attractive profits. When the enterprise is not self-sufficient, and when the operation of the carrier is maintained only by continuous subsidy, then it is evident that some purpose besides that of commercial transportation is being served. Assuming that any one, or any combination of the noncommercial purposes listed in the second paragraph of this discussion is being served by the carrier, then the problem is that of paying for the noncommercial purpose being served. Such a purpose normally has its inception with some agency (federal, local, or a combination of both) which sees to its implementation, and sometimes, its subsidy.

Shall the freight rate on the carrier that is not self-sufficient be such that it would be the "fair" rate on a hypothetical self-sufficient carrier rendering the same commercial service on a competitive basis in a well-developed area, and thus permit mail service subsidy or direct subsidy to satisfy the resulting annual deficit? Or, shall those who use the carrier be required to pay a rate that will make the enterprise solvent on the basis of commercial use only?

The answer to the question seems to lie in the definition of "fair." If the carrier serves a commercial purpose only, and is operated as a monopoly or quasi-monopoly, the concept of "fair" rates as defined by Colonel Hall would appear to be valid. However, if the carrier serves, in part, a noncommercial purpose, rates should be determined by the character of that purpose.

If the noncommercial purpose bears no relation to commerce (for example, military expediency), the agency initiating the action to implement that purpose should be obliged to stand its cost through direct subsidy or service subsidy; and rates should be a matter of maintenance and operating costs only (no amortization or profit included).

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[2] Civ. Engr., Maurice Courland and Son, New York, N. Y.

If the noncommercial purpose is of a long-range commercial venture (such as shifting or encouragement of industry, or travel facilities), the rate should be one that could initially compete with similar existing services; and annual deficits must be satisfied by the initiating agency. In these cases it is assumed that within a reasonable period the development of the areas involved would be adequate to carry the carrier without subsidy. However, it must be noted in this regard that some industries, transportation facilities, and settlements are long-range ventures into military potential. In such cases, as long as the need for the potential is judged to exist, rates should be set so as to attract personnel and equipment that would accomplish the ultimate military objective, with no regard to an engineering-economic analysis. Deficits, of course, should then be met indefinitely by subsidy.

If the nonmilitary purpose is related indirectly to the economic well-being of a region, another factor must be considered. Many of these enterprises, especially related to water, are of dual and sometimes triple purpose. Water transportation is frequently the concomitant benefit of the basic enterprise when flood control and hydroelectric power development alone are considered. At times, river canalization is the primary objective; but the indisputable fact remains that without too basic a modification of design concepts, and with a proper sequence of storage and release, all three benefits may be secured. In general, it is assumed that the rise in land values resulting from flood protection, added power, and reliable transportation could amortize an entire project within a 50-year period. In such cases it would seem reasonable that the increased revenue in the region derived from land taxes, new industries, and commerce should be applied toward amortization of the capital investment, whereas maintenance and operating costs should be satisfied by electric rates (for power) and by lock and canal rates (for river transportation).

OTTO MEYER,[3] ASSOC. M. ASCE.—The proposed scheme for establishing a logical rate structure is well set up, and worthy of most serious consideration by legislative bodies. However, a major constitutional obstacle is the distinction between interstate and purely intrastate transportation.

Under one set of circumstances—namely, when the "fair" rate is more than the traffic can bear—an exception to the system should be made. An example might be a bulk agricultural product grown in a region so far from the market that it cannot be produced and sold at a profit without a favorable freight rate. Such a product might well be a keystone in the economy of the producing region. If it can be shown that the commodity in question will not be produced and shipped at all except with a freight rate concession, a special rate is justified if, and only if, the net earnings of the carrier giving the rate concession will be not less than they would be without giving it. The carrier would then not be subsidizing the commodity, nor would other commodities, as neither the shipping costs of other shippers nor the carrier's earnings would be adversely affected. On the contrary, the entire economy of the region would benefit. The writer believes, however, that special rates should never be allowed if increased shipping costs of other shippers or loss of earnings to the carrier would result.

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[3] Cons. Engr., Corvallis, Ore.

Alexander Markowitz.[4]—The thesis of this very brief paper is that the existing system of freight rates is illogical and that a new system is required. To the writer it seems that one-half dozen pages is too little space for disposing of this complicated subject. Many important factors have thus been omitted from consideration—the complex nature of the nation's industrial life; the geographic pattern of seacoasts, ports, rivers, and harbors; the number and kinds of transportation services available in different parts of the United States which are dependent on the volume of traffic; the costs of handling such traffic; and the effects of competition. In formulating a new system, the author may properly be criticized for oversimplifying a complex, controversial problem, on a mechanical, formulated "template," which is not so readily adapted to the engineering approach.

Density of traffic, the value of the commodity, and the service on which the ability to pay in freight rate making is based, for example, have a direct effect on the level of freight rates. It is thus that commodities of higher value can be required to bear a fair share of the transportation burden, and that raw materials can move cheaply to points of manufacture; yet, these factors receive no consideration in the paper. That there are distinct differences between types of transportation and between carriers of the same type, and that different costs result from their operations, do not enter into Colonel Hall's proposal for a new system.

Actually, all kinds of business are located everywhere, and there are no industrial reservations in which only certain kinds of manufacture are located to the exclusion of others. In a rational approach to any new system the entire effect of geographical distribution of wealth, raw materials, manufacturing skills, agriculture, mining, and other kinds of industrial activity must be taken carefully into account. To the writer, it seems that the only way that the author's proposal could be put in effect would be to make a few individuals or groups of men responsible for rearranging the nation's industry and its commerce to conform to his pre-arranged system, and thereafter administer the freight rates by engineering formula.

The new system proposed would eliminate competition quite effectively; and, if that is the price that the nation must pay to obtain the ultimate in simplicity in rate making, one problem may seemingly be solved only to create many new ones, more dangerous than the present problems. Finally, Colonel Hall proposes a system of engineered differentials to keep each form of transportation in some allocated place. The fundamental fallacies that some body of men represents the entire transportation industry (or that it should) and that it would be possible to leave the element of competition to regulation by the engineering profession are involved in the author's assumption.

According to the author it should always be possible to transfer goods from one prescribed class to another by unanimous agreement of all carriers. To the present time it has been impossible to gain unanimous consent even among one type of carrier. The Interstate Commerce Commission is having a difficult time establishing a uniform classification of freight even between rail carriers. What would be the elements comprising a basic unit cost of transportation by which the public (presumably through the Interstate Commerce Commission) could fix freight rates as outlined in the paper? Rail carriers may be using a 40-ft boxcar, motor carriers a 30-ft trailer, and steamships their sundry volumes of hold and deck space. How would the author suggest that light and bulky articles be classified uniformly so as to obtain the unanimous consent of all, on some weight and measurement basis?

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[4] Gen. Traffic Mgr., New York and New Brunswick Auto Express, Highland Park, N. J.

The author would fix rate schedules so that they would need revision only once every 10 years, as against the present thousand changes or more each day. The engineered differentials would be fixed at 5-year intervals. The proposed basic factor (which is not defined clearly in the paper) would be examined once each year. The dynamic transportation industry, with its numerous facets of commerce, would be placed on a drawing board; it would be examined with dividers, engineers' scales, slide rules, and other engineering implements, until the plan conformed with the assumed dictates of a fixed formula and then it would be set aside marked "not to be opened" for 1, 5, or 10 years, as the case may be. The author shows scant interest in, or sympathy for, the competition for markets and the rivalry between ports. Presumably one or two ports would be selected to move the traffic, and the remainder would be employed for some other, unspecified, use. This would be the purest kind of "planned economy" which, finding itself at odds with reality, and inadequate to meet the needs of a dynamic commerce, would propose to limit commercial freedom to gain its ends. It would stop the pulse of commerce, destroy all initiative and free enterprise, and would substitute the approach of an engineered problem held in check by rigid timed formulas to produce a preconceived result.

The writer can assemble no words strong enough to condemn the philosophy of this paper. It would "stop the clock" and stifle progress in transportation forever. It would be the opening step in establishing a technocratic state, because the same engineered formulas could be applied with equal impunity to every form of enterprise. It would impose a superplanning board to run things as the engineers would have them, and would substitute drawing-board planning for real lifeblood thinking and action. The freight rate problem is a difficult one. Possibly it is overcomplex, and no doubt changes could be made to simplify its structure, equalize opportunity, and create better adjustments. This, however, is not a stagnant need, to be regulated as by a valve and turned on and off at will. It is ever present in the living organism of commerce which transportation serves. To treat it as the author proposes would require control by an iron hand motivated by an iron will. Colonel Hall's theories must be rejected in their entirety. They hold no single suggestion of improvement by natural changes and constructive planning. To accomplish what he proposes, it would first be necessary to destroy the forces that make the present situation seem so illogical. To do so would certainly be a high price to pay for logic applied to one activity in an otherwise illogical world. Even in the engineering profession, the place for the drawing board and slide rule is subordinated to dynamic thinking and planning. These devices are servants—not the master—and so it is in transportation.

Haywood R. Faison,[5] M. ASCE.—At first glance the design of a logical scale of transportation charges might appear to fall outside the province of the engineer, but a close look at the existing rate structure reveals inequalities that add complications to practically every problem that confronts him. Transportation seems to be the only important element in the entire process of production, assembly, manufacture, and distribution that cannot be measured in terms of a standard yardstick of uniform value. In 1947 public carriers transported 995,000,000,000 ton-miles of freight at a cost to the American people of $13,000,000,000,[6] but under those impressive totals lies an almost hopeless tangle of conflicting rate bases. Under a logical scale of charges the concentration of fuels and raw materials and the distribution of the end products could be expected to occur at rates established to minimize the need for transportation, in recognition of the natural accessibility of sources and markets to the milling plants. Under the prevailing system, however, these processes are controlled by rates adjusted by the carriers, in so far as regulation permits, to encourage long hauls, to discourage competitive movements, and to exact as much revenue from the remainder of the freight as the traffic will bear. There is nothing astonishing in this policy of the carriers thus to promote the commodity which they have to sell, as long as legislative and regulatory bodies seem to encourage a view of transportation as an end in itself, rather than as a means to an end. There is room for legitimate question, however, whether this has been the policy best adapted to the development of the United States along sound lines.

Before discussing any shortcomings in the author's formula it may be appropriate to examine the flaws which he seeks to cure. The original system, under which transport agencies were supposed to "take care" of the producers and shippers on their respective lines by independent rate adjustments, was inherited by the Interstate Commerce Commission before 1890 and many of its defects still remain to plague the shipping public. Apparently the majority regard this benevolent dominion over its welfare as just another unnecessary and unpredictable business risk. The commission's efforts to eliminate such discrimination have been hampered not only by the fact that its function has not been to establish rates but rather to pass judgment upon them, but also by the fact that the original, basic rates available for measuring comparative reasonableness may, in themselves, have been unfair and discriminatory.

According to prevailing practice, the selling price of transportation represents a three-way compromise among conflicting requirements—the demand for the service, the revenue needs of the carrier, and the ability of the traffic to pay. Only in the aggregate must earnings cover the full cost of operation, fixed charges, and return on investment. Demand for the service is measured by the difference in price of the goods at origin and destination, and this spread, in turn, is determined by the carrier's test of what the traffic will bear. Within regulatory restrictions, rates are continuously adjusted and readjusted in response to what these agencies believe to be to their best interests. The doubtful economic soundness of this policy is defended on the so-called "out-of-pocket-cost" or "added traffic" principle. This theory recognizes two distinct kinds of expense—(1) those constant, general costs that can be assigned only to the conduct of the enterprise as a whole, and (2) those variable expenses that are directly allocable to specific commodity movements. Costs in the first group do not vary with temporary fluctuations in tonnage; those in the second group are added as the traffic is added and are saved when the freight is not hauled. Advocates of this basis of rate making, chiefly among the railroads, claim that with the transportation plant functioning and the supervisory staffs employed, a few more tons can always be added to a car, or a few more cars added to a train, without incurring a commensurate increase in total carrier expense. In accordance with this philosophy they can successfully justify the establishment of competitive rates covering only a fraction of the fully distributed costs, and yet term them compensatory.

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[5] Chf., Div. of Economics, Board of Engrs. for Rivers and Harbors, Washington, D. C.

[6] "Annual Report of the Interstate Commerce Commission," Washington, D. C., 1948, p. 14.

The theory clearly requires the assumption of ample unused freight carrying capacity and its validity obviously depends upon whether the long-term result is sought. There would be no profit in merely meeting competition on a permanent scale of charges fixed below fully compensatory levels because, in the long run, the maintenance of such a scale would constitute a drain on invested capital. The only logical objective would be the effective discouragement of competition followed by the restoration of monopolistic rates. The flaw in the logic of this policy is readily seen.

Railroad ton-mileage has increased more than fivefold since 1900. In the normal development of each major line there has come a point of growth when one more ton required an additional car; when one more car required an additional train; and when one more train called for double-tracking the main line—each additional step being accompanied by a commensurate expansion of terminal and yard facilities and full overhead. Hence, over the long-term period, out-of-pocket costs have practically coincided with full costs, and economic justification for noncompensatory rates on the "added traffic" basis disappears. Despite its obvious unsoundness, however, the principle still is invoked in support of most rate cases fought before regulatory agencies. Since the burden of meeting constant overhead expenses is merely shifted by such distortion of rates from competitive tonnage to noncompetitive tonnage, the public pays the bill in any case. Notwithstanding the skepticism with which the rate experts view suggestions from the outside, it can be safely stated that a thesis of such spurious validity would not long survive ordinary engineering analysis.

Added to the illogical character of the existing rate system is the companion defect of instability. No sooner have general adjustments been decreed than the jockeying for special advantage begins all over again. In 1947 no less than 109,965 tariffs containing changes or exceptions to freight, express, pipe line, and forwarder rates were published, and this number does not include the thousands of intrastate tariffs.[7] In order for the validity of one of these rates to be challenged successfully, the ponderous machinery of the Interstate Commerce Commission must be set in motion by the complainants. Some of the more comprehensive rate proceedings may grind on for years, until the situations have changed so radically as to render the dockets obsolete. To claim that such cumbersome procedure is in any appreciable degree responsive to current and changing needs of business would be unrealistic. The author proposes to cure these defects with a highly simplified, rigid, mechanical system of fixing rates on the basis of the full cost of providing the transportation service.

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[7] "Annual Report of the Interstate Commerce Commission," Washington, D. C., 1948.

The science of cost finding in transportation has developed to a point where it may be called well nigh perfect. Objection to the use of costs comes, not from the shipping public, but from the firmly established carriers and rate making specialists, learned in the rituals of the existing labyrinthian system. Nevertheless there is a clearly discernible trend not only in rate regulation but in the conduct of modern transportation to rely more and more on cost as a guide to reasonable and profitable rates. Such authorities as J. B. Eastman (Petroleum Products Case (194-I.C.C.-31)), former chairman of the Interstate Commerce Commission and wartime Director of Defense Transportation; G. E. Talmage, Jr.,[8] Director of the Water Carrier Division of the Interstate Commerce Commisssion; *Traffic World*;[9] F. E. Asher,[10] representative of an influential group of midwestern motor carriers; and the Senate Subcommittee on Small Business[11] have expressed themselves as believing that a system of rates for all carriers based on cost plus a reasonable profit would be better for all concerned, including the nation itself and its industries, than any system of competitive rates. As a method of offsetting government subsidies, E. A. Jack,[12] General Traffic Manager of the Aluminum Company of America, suggests the enforcement of rate differentials among the respective forms of transportation, arrived at by using the bare cost of operating each facility, exclusive of roadway expense. This proposal drew a sharp rejoinder from A. Markowitz,[13] Jun. ASCE, spokesman for a New Jersey motor express company, who offered as a countersuggestion the dropping of the controversial subject of subsidies in favor of a general improvement in service under a rate policy, making rates upon the fully allocated costs of each form of transportation.

Despite this evidence of fairly general concurrence on full carrier costs as the appropriate basis for establishing rates, this writer does not agree that it would be practicable or expedient to impose a rigid service cost basis on the transportation system of the United States all at once. He believes that realization of its manifest advantages should be set up as a long-range objective of rate regulation. It would be attained by gradual and cautious advances, avoiding, on the one hand, abrupt dislocations of the industrial pattern that would entail extensive sacrifices of invested capital, and, on the other hand, the letting down of the bars to the usual reactionary forces of disintegration that have always heretofore frustrated efforts toward uniform standardization of the rate structure.

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[8] *Marine News.*, October, 1946.
[9] *Traffic World*, June 15, 1946.
[10] *Ibid.*, September 14, 1946.
[11] *Print No. 14*, 79th Cong., 2d Session, 1946.
[12] *Traffic World*, November 6, 1948.
[13] *Ibid.*, December 11, 1948.

To this critic, the ingenious mechanism designed by Colonel Hall to implement his proposal seems to meet many of the practical requirements for a workable rate formula. Its manifold advantages should not be discounted in emphasizing its defects. The writer must confess that he cannot see all the way through to a happy solution of all the complexities that might arise in the application of the proposed formula. He is not convinced that the element of demand for the service can be safely ignored altogether. He is not certain that the distance scales should not be modified by factors that recognize regional differences in density of population and functional differences in through and local hauls. He believes that certain commodities need milling-in-transit privileges and the the liberalizing of transit arrangements to eliminate preference for particular types of carrier will prove difficult. However, many refinements to take care of special situations undoubtedly could, and would, be made in the gradual adaptation of a uniform policy. The writer is inclined to leave the mechanical adjustment of the system to the traffic experts and to concentrate on what he believes to be the most significant passage in the paper under discussion.

The author puts his finger on the most serious deterrent to the adoption of his proposed system when he recognizes the inconsistency in the existing competitive situation in which some modes of transportation must pay for their roadbeds, whereas the routes of others are partly or wholly subsidized by federal or state governments. He contends that a subsidy to any carrier, or its operation with public assistance at an admitted loss, unbalances the entire setup. Only if carriers meet all their expenses out of their own receipts will the logical system he proposes work smoothly. Since it appears highly improbable that public aids to highway, airway, and waterway development will ever be entirely abandoned as a national policy, the only way that carriers operating over these types of improvements could be brought into line with the author's requirements would be by the assessment of tonnage taxes on cargo or cargo capacity sufficient to pay the carrying charges on the facilities they use. The author infers that until this is done, there can be no just and reasonable application of a uniform service cost basis of rates.

The writer holds that the converse is of equal significance, in that there can be no just and effective application of the tolls principle to motor or water transportation unless, and until, a rigid cost-of-service basis of rates is uniformly imposed on all types of carrier. Because of their ubiquity, railroads, for instance, can handle any commodity movement open to any other mode of transportation in intercity carriage. Under the present system they can and do depress their rates on any block of competitive traffic they choose to select; and they raise rates proportionately on other traffic out of reach of the restricted types of carrier. Barge lines, in particular, have no such blocks of noncompetitive traffic within the limited scope of their transportation activities upon which they could shift the burden by levying higher rates. They have no noncompetitive freight. All their tonnage movements must carry their full share of the total expense if these enterprises are to prosper. If tonnage taxes were imposed on their operations, such taxes would have to be passed along to the shippers in proportional additions to each rate. It thus becomes clearly apparent that these water lines would be at the mercy of carriers with diversified traffic and could be eliminated from the competitive field, one by one, by selective rate cutting. Under a tolls system, their only defense would be a rigidly enforced adherence on the part of all modes of transportation to some such system as that proposed by the author, in which each commodity movement would be charged its commensurate share of full carrier expense.

On certain vital aspects of transportation there is practically unanimous concurrence. It is universally conceded, for instance, that adequate carrier potential must be maintained on a sound and stable footing to enable it to handle the national freight burden in time of war. The transportation agencies not only must be prepared to move the ordinary goods of commerce along established routes in sharply increased volume, but also must be ready for prompt expansion of facilities to handle uncommon types of matériel and munitions between unaccustomed terminals on short notice and with minimum delay and congestion. As indicated by the author there is opportunity here for sane compromise between the needs of peace and of war. There is no justification for setting the controlling scale of rates so high in peacetime as to perpetuate the penalties of unsound conception, faulty design, and extravagant financing that characterize many marginal transportation units. Most of these high cost properties furnish an unnecessarily expensive peacetime surplus of freight carrying capacity, basically incapable of profitable operation on any rational basis and representing their maximum recoverable value as salvage. Under the rate structure proposed by Colonel Hall, these wasteful enterprises would be eliminated from the competitive field and the national transportation system made healthier by such economic surgery.

Any reliable industrial map of the United States shows that the principal manufacturing centers (which, according to Colonel Hall, should be located with a view to their accessibility to raw materials and markets) have long since become almost irrevocably concentrated by other considerations into a relatively few limited areas. Under the existing rate structure, with its low charges on long hauls favoring these concentrations, there is but little economic incentive toward changes in location. Doubtless, had the author's rate making formula been adopted a century ago, development of national resources would have followed a more rational design and the problem of sunk capital would not now count so heavily against dispersal. The principal grounds for hope for a more favorable distribution of processing sites in accord with a balanced transportation pattern seem to lie in the manifest trend of national activity since 1945 toward general decentralization. The writer does not see this trend as wholly accounted for by considerations of security, but deems it reasonable to assume that the ever-increasing proportion of total production costs represented in freight charges is also becoming a compelling force toward dispersal of plant sites. With the gradual establishment of a logical system of freight rates, the writer believes that a progressive migration not only of maunfacturers but of consumers to the areas nearer the sources of supply might well be set in motion to the obvious net benefit of the nation at large. Assuredly, any workable procedure that gives reasonable promise of bringing about simplification of the rate system together with reduction in the net transportation bill of the nation to the practical minimum deserves serious consideration.

Malcolm Elliott,[14] M. ASCE.—Colonel Hall defines the illogical features of the present system of freight rates and makes a convincing plea for a new system. If one were to accept what appears to be the author's fundamental assumption—that freight rates shall be based wholly on cost to carriers of rendering the service—his formula might well be regarded as a logical approach to determination of rates on that basis.

In the writer's opinion, however, the problem is not that simple. Actually the rates for transportation of persons and commodities are based in part on factors other than the cost of rendering service. There are to be considered such factors as competition with other forms of transportation, the necessity (in the interest of national welfare or defense) of subsidies for some or all means of transportation, the effects of changing freight rates on established industries—especially in cases where such changes might drive a heavily capitalized business out of existence—and possibly other complicating factors.

One of the most important factors in determining the price for rendering any kind of service is "what the traffic will bear." Practically all competitive business in the United States—as well as in any other country in which "free enterprise" is the guiding principle—bases prices to some extent on what the public is willing to pay. Otherwise the law of supply and demand would have no place in the economy. In the transportation business, however, the determination of rates on a wholly competitive basis is thought to be impracticable. Freight rate stabilization is required by law even though uniformity of prices is a crime in other industries when accomplished by agreement between competitors. Rate stabilization by government regulation, however, should take some account of competitive situations. It might even go so far as to allow a transportation company to carry some of its cargo at a loss when to do otherwise would force the company to abandon service that might be vitally important to established communities or to public welfare.

The author's desire to establish a factual basis for determining transportation costs is commendable. To a large extent these costs should have an influence on the establishment of rates. Nevertheless, it would appear that these costs—by whatever formulas they are developed into rates for various classification of commodities—cannot be adopted as the sole factor in determining freight rates.

Granting that a more logical system of utilizing costs in determining rates is desirable, one may question whether the preparation of the corresponding rate book would not be attended by many more complications than are foreseen in the paper. What would become of the "processing-in-transit" rates whereby wheat, for example, is shipped to a milling city, processed there into flour, and then shipped out to a final destination under one bill-of-lading whereby the proportionate rate to the milling city is lower than the rate for the same haul for local consumption? Under the author's system of basing rates wholly on costs, the rate from point of origin to the milling city would be the same whether the wheat were intended for local consumption or for processing in transit. It is not the writer's intention to discuss here whether such a change would be desirable, but it would certainly cause some concern to the milling industry.

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[14] Col., Corps of Engrs., U. S. Army (*Retired*), St. Louis, Mo.

With rates based wholly on cost is it intended to require that all rates should be remunerative? Enforcement of such a criterion might wipe some communities "off the map."

These difficulties and others of like character would have to be given very careful consideration in revamping the rate structure along the lines suggested by the author. Of course, there should be no objection to a realistic approach to the subject, as is suggested in this paper; and, even if it is found that costs cannot be the sole basis for fixing rates, such an approach should show what adjustments and subsidies are involved in financing and operating a transportation system.

CHARLES L. HALL,[15] M. ASCE.—That some transportation agencies have been set up, and are maintained, for noncommercial reasons (as suggested by Mr. Courland) has undoubtedly been true of many agencies in the past, and may still be true of air lines. It should not be true now of any other carrier within the continental limits of the United States. The problem of rate making is only slightly affected by the existence of navigable rivers equipped with multiple purpose dams. The amount of commerce carried by the Tennessee, Muskingum, and Savannah rivers in not likely to cause any serious fear in the minds of competing carriers.

In his brief discussion, Mr. Meyer recommends a course of action which it is the object of the writer to prevent at all costs. If a commodity is not produced or shipped in some given area without a freight rate concession, it can be produced elsewhere. It is a pure myth (as Mr. Faison points out) that a freight rate concession can be given to any item, without throwing a burden on some other item, or (as in the case of upper Mississippi barge traffic) on the general taxpayer. Let the regional development be a little slow. The United States has plenty of time.

The writer particularly welcomes Mr. Markowitz's criticisms. They present a direct attack to which a reasoned reply can be made. The value of the commodity, to which he states that insufficient attention is given, is allowed for at the time of classification of freight, which is the first step in the writer's suggested process. His expression, "service on which the ability to pay in freight rate making is based," presumably refers to the type of carrier used by the shipper. Differences between types of transportation are certainly very carefully allowed for in the plan presented.

The basic paper is strongly criticized for presenting a scheme that would prevent competition. It is not clear whether the competition prevented is between carriers of the same type, between carriers of different types, or between communities. Carriers of the same type do not compete now, in price. They all quote identical rates for the same cargo between the same cities and all of them try to offer better service. Under the new plan, they can continue to do so. Carriers of different types—notably the railroad and the river boat—have competed for many years; but this competition has almost invariably resulted in increasing freight rates to or from destinations having access to only one type of carrier. It is this arbitrary shifting of the burden which the writer's plan attempts to prohibit. Competition between communities is created by one or more carriers granting a favored position to some city, increasing their rates to other cities in compensation. No student of economic history truly believes that equalization of rates to competing seaports has been to the nation's advantage. For example, the gift, to Baltimore, Md., of its natural freight rate advantage from the Ohio River valley, instead of an insistence on the present so-called equalized rates to New York, N. Y., would have caused an advantageous decentralization in the United States. A port with freight rates against it would have to develop some advantage in service to keep its commerce, or it would have to specialize in supplying its natural market. It is admittedly intended to destroy competition based on penalizing communities provided with only one carrier, or type of carrier, as well as to injure the ports that have insufficient "pull" to avoid the same kind of involuntary servitude to places of more influence.

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[15] Col., Corps of Engrs., U. S. Army (*Retired*), Washington, D. C.

The last paragraph of Mr. Markowitz's discussion is certainly a vigorous summing up. The writer has spent 30 years arguing about rates with all the traffic men whom he has ever met. He is thoroughly accustomed to being disagreed with; but no traffic man has ever before painted the dark picture of coming evil portrayed in this discussion. Much can be said for free competition between all carriers, but, unfortunately, that method was found not to work. When competition does not work, some kind of public control is inevitable.

The writer desires to thank Mr. Faison, whose discussion greatly increases the value of the paper. In particular, his discussion of the fallacy of the so-called "added traffic formula" is a welcome supplement to the arguments presented by the writer. During the discussion of this paper the writer has thought a great deal about the advisability of increasing ton-mileage charges on routes of small traffic density. It is believed that a rough rule of thumb would be to increase the actual mileage, for computing purposes, in counties having a population density more than 10% below the national average. Population and traffic go quite well together. Under the logical system of rates, the only effect of the milling-in-transit privilege would be to eliminate one loading and one unloading charge. Except in the case of export wheat, it would seem unimportant to the nation whether the privilege is granted or not. It would probably be to the national advantage to avoid concentration of mills at seaports.

The writer agrees with Mr. Faison that there is no fair method of applying a transportation tax to any carrier unless rates on all carriers are based on costs of service. Certainly the immediate effect of a tonnage tax on either motor trucks or river boats, right now, would be merely an increase in competing rail rates. Needless to say, no Congress would adopt such a novel form of taxation for the exclusive benefit of rail carriers. The tonnage tax might arrive after the logical system is under way, but the immediate step is to prevent any such economic absurdities as the extension of government-owned inland water lines operating at a loss, or the payment of a marine subsidy to coastwise ocean traffic.

As to Mr. Faison's warning about the dangers of a too precipitate adoption of this proposed system, it may be stated that undue haste seems the least probable of all contingencies. It is much more likely that the plan will be dismissed as an innocent theoretical exercise. Its only hope of adoption is as a consequence of a regional revolt like the flare-up on southern freight rates in 1947.

In reply to Colonel Elliott, the writer admits that the logical system is based on the theory that all freight rates should be remunerative, and that the disappearance of some carriers and some communities is a probable consequence of the plan's adoption. Any reform in rate making is certain to "step on somebody's toes." If California oranges tend to cost too much for New Englanders to eat, they will have to learn to be satisfied with Florida fruit. There is no particular reason why people in Utah should pay more for their lemons in order to extend an artificial market into the northeast at noncompensatory rates.

The writer is thankful to his critics and is sorry that there were not more of them. He is particularly sorry that there was no discussion of his plan to break all freight rates into separate terminal and actual transportation charges. This seems to him to be the most important single feature of the proposed system of rates.

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TRANSACTIONS

Paper No. 2390

CADASTRAL SURVEYS AND RESURVEYS

By A. C. Horton, Jr.[1]

With Discussion by Messrs. Scott P. Stewart and Clarence S. Jarvis, Roger E. Amidon, William R. Bandy, J. C. Carpenter, and William H. Richards.

Introduction

In accordance with Reorganization Plan No. 3 of President Harry S. Truman, in 1946, the Bureau of Land Management (hereinafter called "the Bureau") was established in the United States Department of the Interior on July 16, 1946, through consolidation of the General Land Office and the Grazing Service. The General Land Office was established in 1812 and the Grazing Service in 1934. By the consolidation, the administration of approximately 400,000,000 acres of public land in the west was brought into one unit to provide for more effective administration at less cost and to insure maximum service in the public interest by the decentralization of activities.

In addition to administering the Taylor Grazing Act of 1934, the Bureau exercises a large measure of primary responsibility in the administration of the public land laws. Its duties range from the identification and description of the public lands by cadastral surveys and resurveys to providing for their acquisition or use in the development of mineral resources—in addition to the disposal of lands under the homestead, townsite, and other laws; the leasing of land; the granting of rights of way; the classification of lands; the adjudication of applications from the standpoint of law and conservation policy; and the many other duties placed upon it by public land statutes.

A reference here to the origin of certain duties of the Bureau should reveal more clearly the true significance of cadastral surveys, records, and legal descriptions and the part they have played and will play in the development of the United States.

History

In 1784 the Continental Congress appointed a committee headed by Thomas Jefferson to draft "an ordinance for ascertaining the mode of locating and disposing of lands in the western territory." The rectangular system of

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[1] Chf., Div. of Eng., Bureau of Land Management, Dept. of the Interior, Washington, D. C.

surveying public lands was inaugurated under the ordinance passed by the Continental Congress on May 20, 1785. In brief, the ordinance provided for townships 6 miles square, each containing thirty-six sections of 1 mile square, "as near as may be." With the location of base lines, standard parallels, and principal and guide meridians (for offsetting the effects of convergency) and the designation of townships, ranges, and sections, it became possible to identify conveniently and with certainty any tract of land anywhere, and to give it a place on a map. The public land area—including, at its maximum extent, the Territory of Alaska, the states of Florida, Alabama, and Mississippi and all states except Texas lying north and west of the Ohio and Mississippi rivers—was subject to survey under this rectangular plan, admittedly the most convenient and simple scheme ever devised by any nation.

Numerous refinements of practice and improvements of methods of far-reaching importance have been made a part of the rectangular system since its beginning. It has been shaped to meet newer aspects of the older problems and has been developed to take care of the intricate practice of resurveying which time and the elements and the frailty of human nature have made necessary; but in principle and in purpose it is the same now as it was then, and is simply the rectangular system of coordinates applied in a practical manner. Rectangular coordinates recognize north in its true position and verify rather than refute Galileo in the age-old theory that the world is round.

Since 1785 the extension of the rectangular system of surveys over more than 2,000,000 sq miles of public land area has created a vast monumented cadastral net. Each square mile is monumented at its four corners, with a monument halfway down each of its four sides in order that crosslines may be run establishing the boundaries of quarter sections. The monuments are the corners for which they were established, even though not exactly where professional care might have placed them in the first instance. Missing corners must be re-established in the identical positions they originally occupied, and when the positions cannot be determined by existing monuments or other verifying evidence, resort must be had to the field notes of the original survey. Law provides that the lengths of the lines as returned in the field notes shall be held as the true lengths, and the distances between identified corner positions given in the field notes constitute proper data from which to determine the position of a lost corner. Hence, the logical and equitable rule is that lost corners are restored at distances proportionate to the original measurements between identified positions.

Since 1910 when the contract system for executing surveys was abolished, the corners of public land surveys have been monumented with a standard metal corner post manufactured from wrought iron or copper bearing steel pipe, filled with a core of concrete. A brass cap riveted to one end of the post is appropriately marked with steel dies. The planted end of the pipe is split and the two halves are spread to form a flange.

SCOPE

The rectangular system of surveys has been extended over approximately 92% of the total area of the public lands in the continental United States; but it has covered less than 1% of the area of the Terriorty of Alaska. The area remaining unsurveyed in the continental United States—comprising approximately 117,150,000 acres—is equal to about the combined area of the states of Arizona and Washington. However, the virgin cadastral survey field of large extent, amounting to about 567,350 sq miles under the jurisdiction of the Bureau, is in Alaska. The resulting vast and comprehensive rectangular survey net provides a simple and certain form of land identification, and anchors in place the land of the public domain.

The rectangular cadastral survey is a prerequisite to the administration and conveyance of the public lands. Under it, area is identified by fixed position on the ground and by descriptive data in the records. Cadastral surveys, unlike scientific surveys of an information character which may be amended with changing conditions or because of error, mark for all time the boundaries of lands on the earth's surface. Early public land surveys may be physically lost, or in error, or fraudulent in whole or in part, but they still define boundaries and area for all time with descriptive fixity of place. In the execution of cadastral resurveys it is the cadastral engineer's problem at times to restore, legally, something that never existed on the ground and to preserve and protect all valid existing rights in the process.

All land titles, land transactions, and considerations of geographical position in the area of the public domain rest upon and are in terms of the Bureau's cadastral public land survey, and its basic, legal descriptions should not be duplicated, confused, and obscured unnecessarily by supplemental records of survey systems alien in character. Cadastral surveys are fundamental factors in the formulation and operation of laws and policies relating to the public lands and, therefore, in a very definite sense, in the orderly administration of the natural resources.

The Bureau is the only congressionally constituted agency charged with the execution of cadastral surveys of the public lands, but the fact is not always appreciated that these surveys do not ascertain boundaries—but instead create them. Cadastral surveys, in contradistinction to information and construction surveys, are made primarily for jurisdictional and proprietary purposes.

Cadastral surveys in the process of execution are divided into two major procedures—namely, the creation and marking of the survey lines upon the ground and the preparation of the official record. The additional step of approval and acceptance by competent authority is necessary under the law to give such surveys their official and legal standing.

The final survey record consists of field notes describing in detail the processes of the survey, the results on the ground, and the plat which is a graphic representation of the surveys made in the field. The primary purpose of the plat is to show the courses and lengths of survey lines and, incidentally, relief, drainage, and culture. Its basic function is to designate and describe areas in specific terms and to serve as the legal basis for all transactions involving the public lands within its borders. The legal significance of the plat in this respect is as important as it would be if incorporated into the patent itself and the same applies to any subsequent deed or transfer.

Although incidental to the primary purpose and basic function of the cadastral survey plat, an adequate and accurate portrayal of relief, drainage, and culture on present and future plats is essential to the proper administration and management of the public lands in conserving natural resources. To meet this demand, the cadastral engineering organization has adopted the use of aerial photographs, resulting in a comprehensive and accurate depiction of relief, drainage, and culture on the cadastral plat.

By application of the system of radial plot, section corners that can be identified on the photographs are selected for ground control. Other corners are resected to the photographs, thereby correlating the lines of the ground survey with the aerial survey. A sufficient number of topography points are selected, and the planimetry is then transferred from the photographs to the plat.

Relief is determined through the measurement of parallax. Vertical control is obtained from clinometer readings taken at the time of survey and tied in, where available, to official bench marks. By measuring parallax, additional spot elevations are found on the photographs and relief is sketched in and transferred to the plat. Normally, the hachure is used to depict relief.

Although the rectangular system remains the basic principle of the modern cadastral survey, the regular extension of townships and ranges is no longer the principal activity. Except in restricted areas within the public land states and in the Territory of Alaska, cadastral engineers are no longer concerned primarily with the mere determination of the length and direction of lines and with the establishment of corner monuments at specified intervals. Such determinations are made, it is true (as are those of time, latitude, and azimuth), with modern instrumental equipment unknown to the pioneer surveyor, and with a degree of precision and refinement of method not approached in early days. Today and in the future, however, the real problems confronting cadastral engineers are those of restoration and the preservation of vested rights.

Cadastral resurveys of large areas of the public domain are being conducted in accordance with the best engineering practice, and with strict attention to the legal and equitable considerations involved in the proper protection of vested rights. Grant and reservation boundaries, often practically obliterated, are being recovered, and their identity established beyond question. Important riparian boundaries which often control rights of great value, and of which practically no physical evidence remains, are being restored and adjusted. Reconnaissance and restoration resurveys are undertaken where information relative to the position of original boundaries is required as evidence in issues pending before the courts.

The cooperative program of cadastral surveys and resurveys, as requested by the various federal bureaus and departments, aggregates 110,000,000 acres, and includes public lands under the jurisdiction or administration of the Bureau of Indian Affairs (United States Department of the Interior), the Forest Service (United States Department of Agriculture), the Bureau of Reclamation (United States Department of the Interior), the Geological Survey (United States Department of the Interior), the National Park Service (United States Department of the Interior), the Federal Power Commission, and others. These bureaus require carefully executed and permanently monumented cadastral surveys for the identification, classification, and administration of the lands. The agencies engaged in mapping these areas find that the cadastral surveys furnish excellent horizontal control for topographic mapping purposes —particularly where aerial photography is being used. An example of a cooperative resurvey follows.

In addition to its regular operations throughout the continental United States and Alaska, the cadastral engineering organization of the Bureau is now (1948) engaged in the identification of more than 10,000 sq miles of land areas for the Bureau of Reclamation within the Missouri Basin Project. On the project cadastral resurveys (with resulting plats) are a prerequisite to planning, land classification, and construction work; to the leasing, disposal, or purchase of land; and to the further subdivision of areas when project units are opened for settlement. The original surveys of the area were made in early days when the prime importance of permanent monumentation of corner positions was not realized as it is today. The ravages of time and the elements, through the many years, have all but obliterated evidence of the wooden stakes and pits that were originally established to identify the corner positions. The practiced eye of the cadastral engineer is required to identify the vague evidence remaining, and his intimate knowledge of intricate resurvey procedure is needed to re-establish, with permanent iron posts, the network of original surveys in their original position, whereby boundary lines between public and private ownership are readily identified.

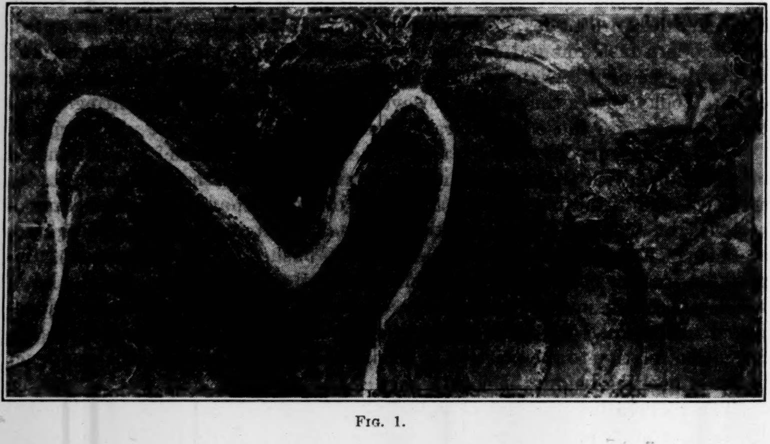

FIG. 1.

The early surveys in the Missouri River Basin were made without the scientific instruments and modern methods in use today, and consequently the length and direction of boundary lines as determined by cadastral engineers are found to vary from that recorded originally. Precise length and direction of section boundaries are required for horizontal control in mapping.

The basic need for land boundary identification and precise horizontal control in mapping operations on the Missouri Basin Project is illustrated by the aerial photograph in Fig. 1. The original is enlarged to a scale of 400 ft to the inch, and mounted on pressed wood, for use as a plane-table sheet. The photograph embraces Sections 9 and 10, T 20 N, R 18 E, Black Hills Meridian, S. Dak., of the Grand River Unit, Missouri Basin Project. The area and boundary lines of the two sections have been identified precisely on the aerial photograph by cadastral engineers of the Bureau. Topographic and soil classification plane-table parties of the Bureau of Reclamation use the aerial photographic sheet, covered with transparent vellum for the mapping requirements. A composite is made of the three, resulting in an accurate, final map. More than 10,000 sq miles of the Missouri Basin Project are to be relocated in the manner indicated by the exhibit.

The Bureau issues a number of publications dealing with the public land surveys and resurveys—the most important and comprehensive of which is the "Manual of Instructions for the Survey of the Public Lands of the United States." The "Standard Field Tables" and an annual publication of the "Ephemeris of the Sun, Polaris and other Selected Stars" are also available. A pamphlet entitled, "Restoration of Lost or Obliterated Corners and Subdivision of Sections," has been issued for the guidance of county and other local surveyors.

Copies of the plats and field notes representing all the public land surveys executed in the United States and in Alaska are on file in the Bureau office in Washington, D. C. The branch offices of the Bureau in the western states, such as the Public Survey Office at Phoenix, Ariz., have the survey records pertaining to the individual state in which they are located.

DISCUSSION

Scott P. Stewart[2] and Clarence S. Jarvis,[3] M. ASCE.—This timely paper presents some of the salient facts related to both the history and the scope of public land surveys in the United States, which have proved to be adequate and satisfactory after a test covering some 164 years of national growth and development. As a practical application of the coordinate system (which, in recent years, has made notable headway among street layouts for metropolitan and other urban areas), the subdivision of public lands into townships, sections, and quarter sections unquestionably facilitated the early settlement following these surveys. The system validated and protected the holdings of those courageous frontiersmen who had established homes in the wilderness prior to such surveys.

The types of corners prescribed and authorized by the "Manual of Instructions for the Survey of the Public Lands of the United States" seemed to cover every possible contingency—a charred stake or a quart of charcoal under a mound of earth; a durable stone about 18 in. long, or a timber post 3 ft long and 5 in. square, buried two thirds in the ground; suitable markings on the exposed edges or on the faces of a marker to indicate its relation to the township boundaries or the distances in miles therefrom; and accessories such as mounds of stone, or pits, or inscriptions on trees, to be placed according to prescribed patterns. Further clues as to distances are afforded by items noted in the survey records, such as crossing ledges, ridges, or stream banks, and as to alinement, by blazing or by removing the bark from restricted parts of tree trunks near the survey lines.

Basic Data Assembled During Survey and Reconnaissance.—The importance of these cadastral surveys seemed to escape well-merited recognition until recent years; and even at this time, full appreciation has not been accorded in some quarters, as richly as deserved. The typical United States deputy surveyor for the General Land Office used some elementary and technical astronomy daily in order to determine solar time, latitude, and true meridians. His field notes revealed some knowledge of geology, botany, and hydrography; otherwise he could not have made adequate detailed descriptions of ledge outcrops, vegetation, or woods through which the survey lines were run, and the springs, brooks, and rivers crossed or located within convenient range of the section boundaries. Even the crude estimates of discharge as recorded during these surveys, together with the dates of such observations, certainly afforded some valuable data for prospective settlers or investors in such regions.

The prevailing stands of native grasses, undergrowth, or shrubs mentioned in the field notes afforded valuable clues to the normal seasonal precipitation depths and distribution. The oft-repeated comment, "Good grass for grazing," together with mention of native wild life including game and also whatever range live stock were encountered or observed, must have wielded a potent influence in the development and utilization of areas suitable for grazing. Likewise, notations giving the precise location of outcroppings, either on the survey lines or adjacent thereto, probably provided the first authentic information concerning unusual mineralized formations, exposed coal strata, oil shales, fissures filled with hydrocarbons such as asphalt and elaterite, and springs or streams covered with iridescent oil slick. Even the estimated or clinometer measured depths of canyons and heights of ridge crests or of beetling cliffs are of practical value for those contemplating journeys or homemaking ventures within the regions to be opened for homesteading, following such surveys.

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[2] Cons. Engr., Salt Lake City, Utah.

[3] Cons. Engr., Salt Lake City, Utah.

The Township Plats.—The writers have in their possession a print of "Township No. 27 South, Range No. 16 East, of the Salt Lake Meridian, Utah," issued as a public service by the Bureau of Land Management (under date of June 5, 1947), which was selected almost at random as representative of rugged, broken country, but also of very recent date. The township is located in southeastern Utah, a few miles west of the Green River. According to the plat, there are notable Indian paintings in sections 17, 18, and 19. Isolated improvements such as a pumping plant and watering troughs evidently have to do with stock watering facilities; ranch houses and other improvements depicted on the plat will insure the owner of first consideration if he but files his claim within the prescribed preferential period.

The various witness corners, designated by the initials "W C," are placed on firm footing to witness the actual corner point whenever it falls on inaccessible ground, or within a river bed, or on the steep and insecure slope of a deep ravine.

The system of surveys devised for the public domain so many years ago undertakes to carve out the maximum number of quarter sections that can be defined within a nearly rectangular tract. For example, a township is 6 miles long from north to south. If they are based on a standard parallel of latitude, the south boundaries of each township have the same length, this dimension decreasing progressively for each township toward the north (but with all the accumulated shortage thrown into the west tier of quarter sections); and the nothernmost boundary is further decreased in length as an adjustment to allow for the convergence of meridians. The ridge crests and drainage channels are connected across the sections for the township plat, thus testing the skill and observation of the United States surveyor.

At costs ranging from about 3¢ per acre to 10¢ per acre, or, say, $9 per lin mile to $27 per lin mile, the public land surveys have assembled and depicted salient basic data of incalculable value to both settlers and government.

Inherent Adventures.—Early survey projects were fraught with the dangers lurking in hostile frontiers and adjacent country. Although these had largely disappeared by the beginning of the twentieth century, it is indisputable that some of the more difficult and rugged terrain still awaited the original surveys.

The writers were associated on the U. S. General Land Office surveys of both the Uncompahgre and the Uinta Indian reservations in eastern Utah, from 1901 to 1904. The first named assignment covered the lofty Book Cliff range with its mammoth outcrops of bituminous coal near Thompsons, and extended northward along Willow and Hill creeks for 50 miles toward Ouray, the western boundary skirting the notorious "Robber's Roost." The "wanted men" of that hideout preferred to pass by unnoticed, to avoid conversation, and to accelerate the forward march of survey parties.

While retracing early survey lines 20 miles westward from the Duchesne Bridge, in June, 1903, for the purpose of locating the initial point for that season's contract surveys, the writers had occasion to question the wisdom of declining the military escort proffered by Captain Mercer, United States Army commandant at the Whiterocks headquarters, some days earlier. A platoon of Indians clothed in gay colors and mounted on their best ponies came charging up the slope and then rode in a circle about the transitman and recorder with threatening gestures and angry shouting. The senior deputy surveyor, keeping his eye at the transit telescope, and apparently oblivious to the imminent danger, merely called, "Get off the line; I can't see through you." The Indians held a powwow, then gradually withdrew toward their camps. Thereafter, the surveyors were under continuous surveillance throughout each day for some weeks. The Indians would lurk in hiding in the vicinity of activities, but occasionally they would peer over a ledge or a ridge or from behind a tree. They demolished the witness mounds of stone, scattering them for some rods from the corner point, but apparently did not recognize the actual corner monuments, mostly buried securely in the ground with notched edges showing above the surface.

Extensive surveys nearer the two-state corner in northeastern Utah, on which the writers were similarly associated, involved the usual meanders for navigable waters along the banks of the Green River in 1906 and included some of the gorges and dam sites now under study for flood control, water conservation, irrigation, and hydroelectric projects. Requisitioned boats and rope ladders facilitated access to some critical points; but the transfer of pack train livestock by swimming from the right bank to the left bank, and of personnel and camp equipment by raft during high water season involved the more colorful adventure. The final loading proved to be an overload; the raft began to submerge in midstream, and thus the saturated equipment added to the overload. With navigation thus made more difficult, the raft was drawn into the rapids instead of landing upstream; the anchor ropes cast to shore and quickly passed about tree trunks, snapped like threads, and tragedy loomed until the raft struck a submerged boulder squarely, and balanced precariously until all the personnel and equipment had been rescued.

Errors of Closure.—Improved surveying equipment such as long steel tapes instead of the four-rod surveyor's chain with its hundreds of wearing surfaces, also the use of clinometers for measurements on slopes, instead of "plumbing" or leveling the chain segments, and employment of engineering students as chainmen helped to reduce the errors of closure about the circuit. Finally, the daily observations on Polaris by the hour-angle method in the gathering twilight out on the survey line (rather than waiting for lantern lighted operations at either culmination or elongation during night hours at the camp site) seemed to make the usual closing limits even in mountainous country more nearly 5 ft or less per section instead of the maximum allowable limit of 2.8 rods per section (46.2 ft per section), and proved to be a morale builder as well.

Roger E. Amidon,[4] Assoc. M. ASCE.—The interesting and educational paper by Mr. Horton is modest in its presentation of the enormous job performed in public land surveys since 1785. The comparative ease of locating, identifying, and establishing land lines in wild areas of the west is immediately apparent to anyone who has encountered similar problems in the undeveloped sections of the eastern United States.

The accurate identification of corners and lines on good aerial photographs permanently records the correct location with respect to land and topographic features. The adaptation of this procedure is certainly a great step forward. The hachured General Land Office plats have been of material aid in assisting field identification. However, the use of contours or form lines instead of hachures, in presenting relief, would facilitate identification in many instances.

During the process of preparing the standard United States quadrangle maps considerable effort is expended locating and identifying land net corners. Usually only a relatively few of the corners are actually identified and located in the field. The remainder are established from General Land Office plats, cultural features such as old fence lines, and information secured from local residents. The inevitable result is that errors occur and lines appear on the final map which are incorrect with respect to topographic and cultural features. The only absolutely dependable corners appearing on the sheets are those shown as "recovered corners." A cooperative arrangement between the Bureau of Land Management and federal mapping agencies whereby land net corners would be recovered or established prior to, or at the same time as, the mapping would result in precise presentation of land lines.

Incorporation of the land net into the geodetic control system throughout the United States would ultimately result in significant economy. Areas will be remapped and surveyed for as long as development takes place and, consequently, a need will always exist for correct geodetic control. The tendency in remapping is toward the adoption of larger scales, with increased detail and precision. This in turn demands a greater intensity of control positions. Township corners tied into the geodetic system with third-order accuracy would provide control for small-scale mapping. Section corners established with this precision would furnish adequate control for all but the most precise surveys. Bulletins showing the coordinate position of township and section corners with specified order of accuracy would furnish private and public agencies a base from which to extend surveys for practically every purpose.

The accomplishment of the ideals suggested herein would require changes in administration of public land surveys; but the need exists and over-all economy would ultimately result. Possibly some thought and study has been given to this problem by officials of the Bureau of Land Management, or others. If so, the results or recommendations would be of interest to the engineering profession.

William R. Bandy,[5] M. ASCE.—Beginning with a brief description of the manner in which the rectangular system for surveying the public lands was devised and inaugurated, this paper describes the magnitude of the job that has been accomplished since its inception and then goes on to show that the job is far from complete. There still remain approximately 116,060,842 acres of unsurveyed land in the United States, and the demand for resurveys is increasing, as a result, to a large extent, of the need for the resurveying of the vast areas scheduled for intensive development in the various river basins throughout the west. The Forest Service of the United States Department of Agriculture has extensive areas to be resurveyed because forest fires have destroyed the wooden corners and bearing trees in certain parts of the forest reserves.

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[5] Regional Chf. of Eng., Region III, Bureau of Land Management, Helena, Mont.

As Mr. Horton very properly states:

> "All land titles, land transactions, and considerations of geographical position in the area of the public domain rest upon and are in terms of the Bureau's cadastral public land survey * * *."

A fact that is not generally known, except to abstractors and realtors, is that the title to every city lot, and parcel of land in the area of the public domain, is based upon the cadastral survey made by the Bureau of Land Management, or its predecessor, the United States General Land Office.

The demand for the remonumentation of the cadastral survey corners on the various irrigation projects that are being developed by the Bureau of Reclamation is explained by the fact that the original surveys of those areas, for the most part, were made in early days when the prime importance of permanent monumentation of corner positions was not realized as it is today, and also because the settlers and subsequent owners of the land failed to appreciate the importance of protecting and preserving the corner monuments. Even today, it is advisable to conduct an educational program to impress upon property owners and road builders the many advantages of protecting the re-established public survey corners.

Mr. Horton gives a clear description of the cooperative cadastral resurveys now being executed for the Bureau of Reclamation. He mentions the fact that in many instances these cadastral surveys serve the Bureau of Reclamation as horizontal control for its mapping activities. This use of the cadastral surveys by the Bureau of Reclamation illustrates the great saving being effected by the cooperation of the various governmental agencies participating in the development of the Missouri River Basin. This dual use of the cadastral survey rectangular net not only supplies the Bureau of Reclamation with an official property boundary survey on its projects, monumented with iron corner posts, but also furnishes a system of horizontal control for its topographic and planning maps, both for the price of one. Formerly it cost as much to establish a triangulation control system as it costs to cover the area with a cadastral survey.

To insure proper coordination of the work being performed by the governmental agencies in the Missouri River Basin is the principal function of the Interior Missouri Basin Field Committee and the Missouri Basin Inter-Agency Committee, both of which are headed by W. G. Sloan, Assoc. M. ASCE, coauthor of the Pick-Sloan Plan. These committees are doing a very worthwhile job.

Mr. Horton calls attention to the improvements in instrumental equipment and methods for making cadastral surveys over those in use in the early days.

In 1948 a test was completed in this region, comparing the lines run by the cadastral engineers with those established by the United States Coast and Geodetic Survey (U.S.C.G.S.) On the Missouri-Souris Unit in North Dakota the Bureau resurveyed 3,262 sq miles, re-established all the section and quarter section corners, and determined the true bearings and distances between the corners. The same area was likewise covered by a control system established by the U.S.C.G.S., its stations tied to the public survey by the cadastral engineers as the work progressed. The cadastral resurvey of this area was executed by field parties headed by eleven cadastral engineers over a period of three seasons—1946, 1947, and 1948.

The tests recorded were taken at random over a large area to include some work of each of the engineers. These tests are made by comparing the measurements of the cadastral survey lines with those made by the U.S.C.G.S. which are considered the ultimate in measurement. The cadastral survey traverses between the triangulation stations were taken from the township plats, using the bearings and distances along the section lines as reported by the cadastral engineers, with ties made to the triangulation stations from the public survey corners. The errors of closure against the U.S.C.G.S. work varied from 1:3,996 to 1:35,000. The best closure was made against the International Boundary, but this may be partly accounted for because it is the largest area closed. However, the average closure for the seventeen separate tests made is found to be 1:10,624.

One interesting feature discovered in computing these closures is that the distances measured by the cadastral engineers are usually shorter than those measured by the U.S.C.G.S. This may have occurred because the chainmen for the cadastral engineers used too much tension on the tape since spring balances were not used to gage the tension. On the other hand, it may be due to failure of the cadastral engineers to take into account corrections for temperature. No allowance for differences in temperature were made by the cadastral engineers, and a large part of the lines was run during the hot summer months.

J. C. CARPENTER,[6] M. ASCE.—The subdivision of 2,000,000 sq miles of public lands was one of the most extensive surveying and mapping projects in the history of the world. It was a major undertaking of the nineteenth century. The work had to be done in advance of settlement in an undeveloped wilderness with few inhabitants except wild animals and Indians, sometimes hostile.

Land values at the time of the survey did not encourage the use of a high degree of accuracy. The magnetic needle for direction and the link chain for distance were considered satisfactory. Prior to 1910 the contract method was employed and this called for speed at the sacrifice of quality.

In spite of these conditions, which adversely influenced the character of the surveys, a remarkably satisfactory final result was attained. The sections were laid off with an accuracy in keeping with the raw nature of the land; section and exterior quarter corners were marked with temporary monuments—usually a charred stake, quart of charcoal, or marked stone, around which was heaped a mound of sod from two or four standard pits. Settlers followed the tracks of the surveyors across the prairies or past blazed trees in the forests. The maps were indispensable in guiding the settlement, and the orderly arrangement of sections and townships has provided a simple self-indexing permanent record and a location guide to every tract in this vast domain.

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[6] Senior Highway Engr., U. S. Public Roads Administration, Washington, D. C.

As Mr. Horton points out, the ravages of time and the elements have all but obliterated the evidence of the primitive monuments. The concrete filled metal pipe markers now being used by the Bureau of Land Management will provide practically permanent monuments for the boundary corners. Local surveyors have perpetuated many of the original corners with stone or concrete monuments and have re-established lost corners in the manner prescribed by the government regulation. In eastern South Dakota 3,000 sq miles were resurveyed from 1900 to 1910 and all exterior section corners were marked with granite monuments 8 in. square and 3 ft long.

The local surveyor searching for a lost corner (or re-establishing it, if it has been obliterated) must follow the footsteps of the original surveyor until he has determined the exact point where the original marker was set. Having accomplished this objective he perpetuates the location by setting a durable monument and establishing adequate references.

A method of reference has been developed which will provide a much more secure and trustworthy identification than has ever been available. This reference medium is the system of standard plane coordinates now established for each state. The United States Coast Survey, early in the nineteenth century, found it necessary to connect the east and west coast surveys and so projected an arc of triangulation across the continent. Other arcs were extended east and west and north and south and these surveys were all adjusted by the United States Coast and Geodetic Survey (U.S.C.G.S.) in 1927 so that a rigid control network of geodetic arcs has been established. In 1933 the U.S.C.G.S. laid out the first system of state plane coordinates, in the State of North Carolina. This provided a method of converting spherical coordinates of latitude and longitude to plane rectangular (X and Y) coordinates on a single grid, expressed in feet. Any surveyor can use these coordinates as he does latitudes and departures in plane surveying. Every state in the Union now has its own standard system of plane coordinates. In twenty-four states the legislatures have officially adopted the standard plane coordinate system for defining and stating the position or location of points on the surface of the earth. Nine of these states (California, Louisiana, Minnesota, Ohio, Oregon, South Dakota, Washington, Alabama, and Nevada) are among the twenty nine in the public lands area.

The use of plane coordinates by the Bureau of Land Management to define the exact location of land boundary corners will have a far-reaching and very valuable result.

In the plane coordinate system the numerical coordinates represent one location only and when accurately determined they are applicable to that particular spot and no other. Until the network of triangulation and the determination of accurate geodetic positions has been extended so that land boundary corners may be tied in with the assurance that they will meet the necessary standards of accuracy, this method must be used with extreme caution. How-

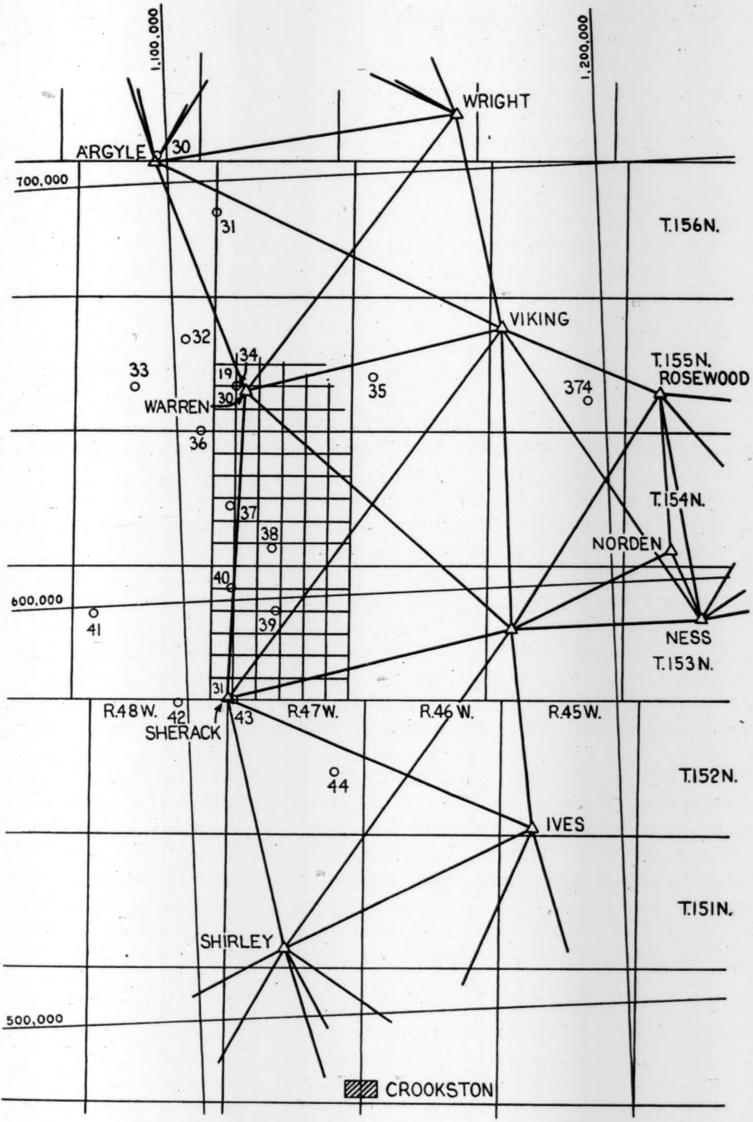
1,100,000
1,200,000
WRIGHT
ARGYLE
30
700,000
31
T.156N.
VIKING
32
33
34
19
30
WARREN
35
374
T.155N.
ROSEWOOD
36
T.154N.
37
38
NORDEN
40
600,000
41
39
NESS
T.153N.
31
R.48W.
42
43
SHERACK
R.47W.
R.46 W.
R.45W.
44
T.152N.
IVES
T.151N.
SHIRLEY
500,000
▨ CROOKSTON

FIG. 2.—PUBLIC LAND SURVEY, MINNESOTA, 98TH MERIDIAN TRIANGULATION AND PLANE COORDINATES (SCALE 1 IN. = 40,000 FT)

ever, the Bureau of Land Management, in cooperation with the U.S.C.G.S., can select the areas where the system may be applied with confidence of its successful use, and these areas can gradually be widened as the system demonstrates its value. Adoption of plane coordinate control for land boundaries will justify an extension of the geodetic network and will result in a closer spacing of monuments, which will provide more control for important surveys. This control will be invaluable in all future mapping. In the recent remapping of the State of Massachusetts the U. S. Geological Survey found that the cost of the topographic work was reduced by using the control placed by the Massachusetts Geodetic Survey in an amount greater than the entire cost of the geodetic work performed by the state, and the topographic maps were completed and ready for use considerably earlier than if the geodetic survey had not been in place.

No geodetic monuments were available for ties when the public land surveys were made. At first the magnetic needle was used to determine direction and more recently the solar transit has become standard equipment. Now that the geodetic network has been extended throughout the entire United States newer surveys and resurveys should certainly be tied in and the plane coordinates accurately established. By this method the accuracy of the surveys can be measured and errors that may be important can be located and corrected.

The endorsement of plane coordinate control by the Bureau of Land Management will stimulate its use by local land surveyors and engineers on public and private projects. It will convince many "doubting Thomases" who have been reluctant about using the control, possibly because it has not been universally adapted by the Bureau of Land Management.

Use of plane coordinate control by the Bureau of Land Management will extend the system into virgin areas ahead of development. It will provide an accurate control system for all kinds of surveys—highway, irrigation, power, municipal, aviation, etc.,—and particularly for topographic mapping. The coordinate system will then be recognized as a truly standard state system.

The location of section corners by plane coordinates will allow accurate determination of their geographic coordinates and will provide for conversion to the military grid and for other uses in accurate military operations.

Some of the original land survey corners were tied to the geodetic network when the triangulation arcs were projected across the United States. To illustrate the theme of this discussion, an example was chosen in Polk County, Minnesota, where two section corners were used as reference marks for principal points on the 98th meridian arc (see Fig. 2).

The public lands survey of this area was made in 1872 and section corners marked with charred stakes, mounds, and pits. In 1906 the U.S.C.G.S. executed its triangulation and established two principal points, "Warren" and "Sherack." Warren was close to the southeast corner of Section 19, T 155 N, R 47 W (Fig. 3), and Sherack, near the southeast corner of Section 31, T 153 N, R 47 W (Fig. 4). The azimuth and distance to these corners were measured. The locations of these two section corners and numerous other supplementary references, such as church spires and elevators, were included in the general readjustment of the first order triangulation in the western part of the United

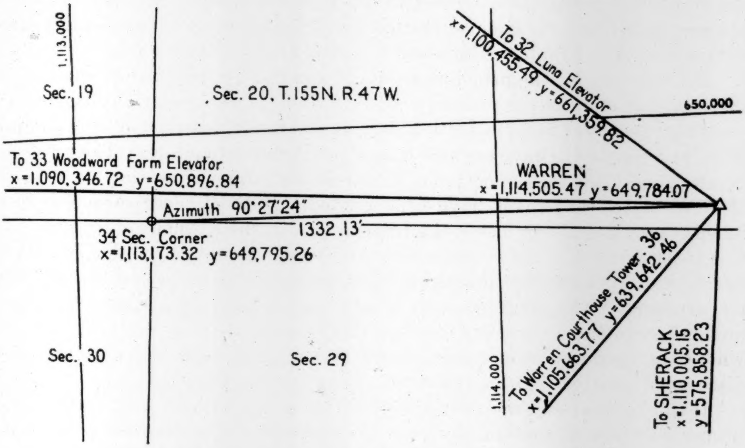
1,113,000
Sec. 19
Sec. 20, T. 155 N. R. 47 W.
To 32 Luna Elevator
x=1,100,455.49 y=661,359.82
650,000
To 33 Woodward Farm Elevator
x=1,090,346.72 y=650,896.84
WARREN
x=1,114,505.47 y=649,784.07
Azimuth 90°27'24"
1332.13'
34 Sec. Corner
x=1,113,173.32 y=649,795.26
To Warren Courthouse Tower 36
x=1,105,663.77 y=639,642.46
To SHERACK
x=1,110,005.15
y=575,858.23
Sec. 30
Sec. 29
1,114,000

FIG. 3.—LOCATION OF SOUTHEAST CORNER OF SECTION 19 WITH REFERENCE TO "WARREN" AND SUPPLEMENTARY POINTS

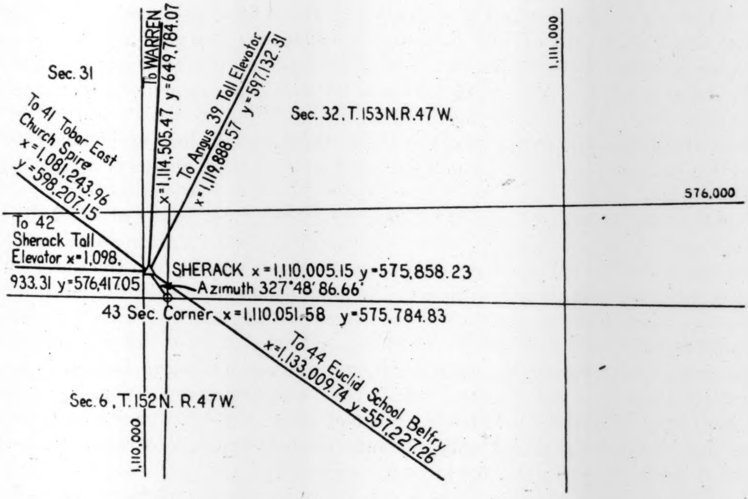
To WARREN
x=1,114,505.47 y=649,784.07
To Angus 39 Tall Elevator
x=1,119,888.57 y=597,132.31
1,111,000
Sec. 31
Sec. 32, T. 153 N. R. 47 W.
To 41 Tobar East Church Spire
x=1,081,243.96
y=598,207.15
576,000
To 42 Sherack Tall Elevator x=1,098,933.31 y=576,417.05
SHERACK x=1,110,005.15 y=575,858.23
Azimuth 327°48' 86.66'
43 Sec. Corner x=1,110,051.58 y=575,784.83
To 44 Euclid School Belfry
x=1,133,009.74 y=557,227.26
Sec. 6, T. 152 N. R. 47 W.
1,110,000

FIG. 4.—LOCATION OF SOUTHEAST CORNER OF SECTION 31, WITH REFERENCE TO "SHERACK" AND SUPPLEMENTARY POINTS, 98TH MERIDIAN TRIANGULATION ARC (SCALE 1 IN. = 400 FT)

States and these stations are now on the North American datum of 1927. The plane coordinates have been computed for all the points and these coordinates, with a wealth of supplemental data, are readily available.[7]

The markers for the principal points Warren and Sherack were surface and underground monuments and they can be found. For the section corner at the southeast corner of Section 19, T 155 N, R 47 W, the description shows that in 1935 it is reported to have been destroyed by road construction. The published description[7] of the section corner at the southeast corner of Section 31, T 153 N, R 47 W, states that roads have been graded. Both of these corners may be relocated accurately by using the information in the special publication. On this arc the record shows seventeen section corners to have been tied in, and the descriptions indicate that practically all have been covered or destroyed by road construction. The measurements from points on the 98th meridian arc triangulation to these corners will allow their locations to be re-established, and where the roadway is in embankment the original stake may sometimes be discovered by excavating to the level of the prairie sod.

If this were a resurvey, or a new survey of areas not previously subdivided, numerous other corners on the 14-mile length should be tied to the network. Plane coordinates for these additional corners would then bring the system into active use for local surveys. In the U.S.C.G.S. description[7] the plane coordinates for the southeast corner of Section 19, T 155 N, R 47 W (station 34), are shown as $X = 1{,}113{,}173.32$ ft and $Y = 649{,}795.26$ ft. For the southeast corner of Section 31, T 153 N, R 47 W (station 43), the plane coordinates are $X = 1{,}110{,}051.58$ ft and $Y = 575{,}784.83$ ft. This develops a right triangle with the X leg 3,121.74 ft and the Y leg 74,010.43 ft. The geodetic distance on the hypotenuse is 74,076.24 ft. The field notes of the 1872 survey in the Washington (D.C.) office of the Bureau of Land Management show compass courses with a needle variation from 13°00′ to 13°30′ on the 14 miles of line between these two corners. Each of the sections is a mile long except the north section in Township 153 N, which shows a length of 80.22 chains or 5,294.52 ft. The chained length on the 1872 survey between these two section corners was 73,934.52 ft. The difference of 141.72 ft gives an average error of 10.12 ft per mile which is an accuracy of 1:522. Considering the circumstances existing at the time of the original survey, this accuracy is reasonable. In the link chain used for linear measurement there is a loop on each end of each link and there are two rings between the links so that there are 588 bearing surfaces to wear and 396 loops and rings that may elongate as the chain is dragged over the ground. The excess length of 10.12 ft per mile means 1.52 in. per chain or a 0.0015-in. average wear or stretch for each bearing surface loop or ring. The chains used were tested as often as was considered necessary, but this element of wear and stretch undoubtedly accounts for part of the almost universal overrun in length of these original measurements.

With the data available it is possible to compute the variation from true north of a line between these two corners. Other valuable checks on linear and angular measurement can be made using the plane coordinates as a method of simplification of the computations. In the surveys now being executed by the

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[7] *Special Publication No. 203*, U. S. Coast and Geodetic Survey, Washington, D. C., 1935.

Bureau of Land Management the use of plane coordinate control for position determination will provide a very valuable basic system, which can be used for all other subsequent surveys. The coordinates will not supplant the section and quarter corners as a description of the tract, but will supplement this description and serve as an accurate engineering location of the position of the corner. For subdivision inside the section the plane coordinates will provide a positive, accurate description of every point. For any such interior point the distance and azimuth from the section and quarter corners can be computed precisely using plane trigonometry.

The methods now specified by the Bureau of Land Management for its surveys may result in better than third order accuracy. The ties to the geodetic network (wherever economically feasible) and the use of plane coordinates in position computation will provide accurate, simple ground control for all kinds of adjacent surveys and particularly for mapping by modern methods. The marker originally used for the section and quarter corners was intended to be temporary but this marker has been established as the true legal boundary position even though the measurements do not agree with the original field notes. The charred stake, quart of charcoal, and durable stone about 18 in. long, are definite and positive identifications of this original position, provided they can be found and legally recognized. Once found, the position must be perpetuated by granite markers or noncorrodible metal pipes filled with concrete and with reference points accurately measured, or other reliable procedures, all legally recorded. In addition, if the corners are tied to the geodetic network, and plane coordinates are used to reference the location to the equator and the North Pole, the best present available method of identification will be utilized.

The value of universal adoption of plane coordinate control for positive identification is so large that it cannot be estimated. It is certain that this value as a method of standard control for land boundaries, for engineering projects of all kinds, and particularly for mapping, is many times the actual monetary cost of developing this control.

WILLIAM H. RICHARDS,[8] M. ASCE.—In the years following the close of the Revolutionary War the Continental Congress was confronted with the problems involved in the administration and disposal of the lands recently ceded to the federal government by the several states. The ordinance of May 20, 1785, provided in principle for the rectangular system of survey of these public lands. It inaugurated a program of land identification and description that continues to the present time. For magnitude and continuity of execution it is without parallel anywhere in the world.

As the public land states were admitted into the Union, the United States continued to hold title to the unappropriated lands. Under the public land laws the Secretary of the Interior was vested with the jurisdiction over the administration, survey, and disposal of these lands. These activities are now the immediate responsibility of the Bureau of Land Management.

The Department of the Interior and the Bureau must consider and determine what are public lands, what lands have been, or should be, surveyed, and what lands have been, or remain to be, disposed of. It has the authority to extend the surveys of the public lands as may be necessary, including the lands that were omitted from earlier surveys.

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[8] Chf., Branch of Surveys, Bureau of Land Management, U. S. Dept. of the Interior, Washington, D.C.

These administrative functions, as well as those having to do with resurveys, involve legal as well as technical questions. The Bureau must establish the fact that the lands in question are within its jurisdiction and must plan the survey procedure in such a manner as to protect, adequately, the rights of adjoining privately owned lands. The cadastral engineer who undertakes the field work must be familiar with the fundamental principles of federal and local law. Finally, the record of the survey must be reviewed and accepted by the Bureau.

As stated by Mr. Horton, an original survey of public lands does not ascertain boundaries; it creates or identifies such lines for the first time. A field execution of the survey and the preparation of the plat and field notes are not sufficient in themselves to constitute an official survey. The public lands are not regarded as surveyed until the surveys are accepted by the duly authorized administrative officer of the Bureau.

Resurveys of large areas of the public lands in the western public land states have constituted an important part of the survey program of the Bureau for many years. However, a general recognition by federal mapping agencies of the importance of complete identification and accurate location of the public land survey network as a basis for precise horizontal control for mapping purposes is a relatively recent development.

In the past, engineers engaged in topographic mapping have endeavored to locate all possible section corners. Frequently, only a limited number are definitely identified with consequent uncertainty as to the correct position of the remaining section lines. Although the results have been fairly satisfactory for general purpose, small-scale topographic maps, considerably more precise results are required for the larger scale maps now in demand. These results can best be obtained by a complete official resurvey involving recovery of all evidence of the original survey combined with proper restorative procedure for missing corners.

The paper makes reference to the fact that the plats and fields notes of all public land surveys are on file in the Interior Department Building in Washington, D. C. An interesting use of these records has developed since 1947. Active exploration for oil and gas in many of the western states has developed a demand for small-scale planimetric maps showing the land net. A number of private mapping organizations engaged in the preparation of such maps have found that a rapid, convenient and space-saving method of securing copies of the official plats is to photograph them on 35 mm microfilm. In this way copies of the plats covering an entire state can be put on eight or ten rolls of film occupying only a few cubic inches of space.

The paper by Mr. Horton fills a definite need in presenting a brief authoritative statement of the activities of the Bureau of Land Management, in the field of cadastral surveys and resurveys. The author has prepared a comprehensive outline of the origin, purpose, scope, and character of the public land rectangular system of surveys in the United States and Alaska.

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Paper No. 2391

INTEGRATING THE EQUATION OF NONUNIFORM FLOW

BY M. E. VON SEGGERN,[1] ASSOC. M. ASCE

WITH DISCUSSION BY MESSRS. PHILLIP Z. KIRPICH, PAUL V. HODGES, A. H. DAVISON, STEPONAS KOLUPAILA, WILLIAM D. MITCHELL, FRANCIS F. ESCOFFIER, GUY C. THATCHER, CHESLEY J. POSEY, AND M. E. VON SEGGERN.

SYNOPSIS

A complete integration of the equation of nonuniform flow would appear to be highly desirable, as an integration method makes it possible to analyze quickly problems that the average engineer avoids because of the mass of tedious computations that would be required by other methods. This paper presents a complete and simple integration method adapted to all types of cross sections ordinarily encountered in practice. The hydraulic exponent, n, which was introduced by Boris A. Bakhmeteff,[2] Hon. M. ASCE, is used in this presentation exactly as it is used by him. In addition, however, a secondary exponent, m, is employed to set up the equation of nonuniform flow in such form that it can be completely evaluated by integration. Also presented is a novel and time saving method of computing the exponents, n and m.

INTRODUCTION

Although various attempts have been made to solve, by integration, hydraulic problems involving nonuniform flow, most of the early efforts have only an historical interest and will not be mentioned. Perhaps the best known of these was that of J. A. C. Bresse,[3] in which the Chézy friction formula is used and the channel is idealized into a rectangular channel of infinite width. More recently, Mr. Bakhmeteff[2] devised an integration method that accommodates all shapes of channels and allows the user to select his own flow formula. The Bakhmeteff method, however, evaluates only a part of the integral, thus making it necessary to apply a factor or a series of factors to account for the

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NOTE.—Published in January, 1949, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Engr., U. S. Bureau of Reclamation, Huron, S. Dak.

[2] "Hydraulics of Open Channels," by Boris A. Bakhmeteff, McGraw-Hill Book Co., Inc., New York, N. Y., 1932.

[3] "Hydraulics of Steady Flow in Open Channels," by S. M. Woodward and C. J. Posey, John Wiley & Sons, Inc., New York, N. Y., 1941, pp. 63–74.

remainder. Nagaho Mononobe[4] presented an integration method that completely evaluates the integral and includes all channel shapes, but this method contains approximations which are deemed inadmissible. A complete integration of the differential equation of nonuniform flow has been developed by S M. Woodward, Hon. M. ASCE, and C. J. Posey,[5] M. ASCE, for channels with horizontal bottom grade, but this integration is not applied to channels with sustaining slopes.

The method presented herein is, however, capable of such extensions through use of the Bakhmeteff exponent n and of a secondary exponent m to set up the equation of nonuniform flow in such form that it can be completely evaluated by integration. Inasmuch as it is presumed that the reader is familiar with the general features of open channel flow, neither the types of surface curves nor the conditions under which they are formed are discussed.

Notation.—The letter symbols used in this paper are defined where they first appear, in the text or in illustrations, and are assembled for convenience of reference in Appendix I.

Equation of Nonuniform Flow

"Nonuniform" flow, sometimes referred to as "varied" flow, is defined briefly as steady flow in a channel where the water surface and the energy gradient are not parallel to the bottom of the channel. All types of backwater curves come under this heading, which is distinguished from "uniform" flow, where the water surface and the energy gradient are parallel to the bottom. Inasmuch as every standard textbook dealing with open channels develops the equation of nonuniform flow, that derivation will not be repeated here.

The equation of nonuniform flow can be written as

$$S_s = \frac{V^2}{C^2\,R} + \frac{d}{dx}\left(\frac{V^2}{2\,g}\right) \qquad (1a)$$

or

$$\frac{dy}{dx} = S_0\,\frac{1 - (K_0/K)^2}{1 - \dfrac{Q^2\,b_w}{g\,A^3}} \qquad (1b)$$

or

$$dx = \left(\frac{1 - \dfrac{Q^2\,b_w}{g\,A^3}}{S_0 - S_f}\right) dy \qquad (1c)$$

—each of which is an identical expression if

$$K = A\,C\,\sqrt{R} \qquad (2a)$$

and

$$S_s = S_0 - \frac{dy}{dx} \qquad (2b)$$

In Eqs. 1 and 2, S_s is the slope of the water surface; V is the velocity of flow; C is the Chezy coefficient of flow; R is the hydraulic radius of the cross-sectional area; g is the acceleration due to gravity; x is the distance along the channel parallel to the bottom; y is the depth of flow; K is the conveyance of a cross section; Q is the discharge or rate of flow; b_w is the width of the water surface; A is the cross-sectional area; S_f is the slope of the energy grade line; and the subscript 0 denotes the value of a term for uniform flow conditions. For any specific value of Q, K_0 is constant and S_0, being the slope corresponding to uniform flow, equals the bottom slope of the channel.

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[4] "Back-Water and Drop-Down Curves for Uniform Channels," by Nagaho Mononobe, *Transactions*, ASCE, Vol. 103, 1938, p. 950.

[5] "Hydraulics of Steady Flow in Open Channels," by S. M. Woodward and C. J. Posey, John Wiley & Sons, Inc., New York, N. Y., 1941.

Borrowing from the notations used by Mr. Bakhmeteff, in which

$$M = \sqrt{A^3/b_w} \ldots\ldots (3a)$$

and[6]

$$M_c = (\sqrt{A^3/b_w})_c = \sqrt{Q^2/g} \ldots\ldots (3b)$$

—in which M is a function of the cross section and the subscript c denotes the value of a quantity at critical depth—Eq. 1b reduces to

$$\frac{dy}{dx} = S_0 \frac{1 - (K_0/K)^2}{1 - (M_c/M)^2} \ldots\ldots (4)$$

Into Eq. 4 Mr. Bakhmeteff introduces the relation:

$$\left(\frac{K_0}{K}\right)^2 = \left(\frac{y_0}{y}\right)^n \ldots\ldots (5a)$$

which as he demonstrates holds remarkably well for most channels, and in which y_0 is the depth for uniform flow and n is a hydraulic exponent.[7] The writer now proposes the relation—

$$\left(\frac{M_c}{M}\right)^2 = \left(\frac{y_c}{y}\right)^m \ldots\ldots (5b)$$

—which is about as accurate as Eq. 5a, and in which y_c is the critical depth and m is an exponent similar to n. A discussion of the values of the hydraulic exponents m and n is presented elsewhere in this paper. It will be noted that these two exponents are the same as the exponents a and b defined by Messrs. Woodward and Posey.[8]

With these two additional substitutions (Eqs. 5), Eq. 4 becomes

$$\frac{dy}{dx} = S_0 \frac{1 - (y_0/y)^n}{1 - (y_c/y)^m} \ldots\ldots (6)$$

By designating

$$u = y_0/y \ldots\ldots (7a)$$

and

$$v = y_c/y \ldots\ldots (7b)$$

so that

$$y = y_0/u = y_c/v \ldots\ldots (7c)$$

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[6] "Hydraulics of Open Channels," by Boris A. Bakhmeteff, McGraw-Hill Book Co., Inc., New York, N. Y., 1932, p. 35.

[7] *Ibid.*, p. 84.

[8] "Hydraulics of Steady Flow in Open Channels," by S. M. Woodward and C. J. Posey, John Wiley & Sons, Inc., New York, N. Y., 1941, p. 80.

it is found that

$$v = u\,(y_c/y_0) \qquad (7d)$$

and

$$dy = -\,(y_0/u^2)\,du \qquad (7e)$$

Substituting Eqs. 7*d* and 7*e*, Eq. 6 gives

$$dx = \frac{1}{S_0}\,\frac{1 - \left(\frac{y_c}{y_0}u\right)^m}{1 - u^n}\left(-\frac{y_0}{u^2}\right)du \qquad (8a)$$

which can be reduced to

$$dx = \frac{y_0}{S_0}\left[\frac{(y_c/y_0)^m\,u^m}{u^2\,(1 - u^n)} - \frac{1}{u^2\,(1 - u^n)}\right]du \qquad (8b)$$

Integration of Eq. 8*b* gives

$$x = \frac{y_0}{S_0}\left[\left(\frac{y_c}{y_0}\right)^m \int \frac{u^m\,du}{u^2\,(1 - u^n)} - \int \frac{du}{u^2\,(1 - u^n)}\right] \qquad (8c)$$

and by designating

$$Y = \int \frac{u^m\,du}{u^2\,(1 - u^n)} \qquad (9a)$$

and

$$Z = -\int \frac{du}{u^2\,(1 - u^n)} \qquad (9b)$$

Eq. 8*c* becomes

$$x = \frac{y_0}{S_0}\left[\left(\frac{y_c}{y_0}\right)^m Y + Z\right] \qquad (10)$$

The integrals represented by Y and Z are both the sums of infinite series, therefore it is not practical to perform the integration for each particular problem. However, since the values of m for most cross sections lie between 3.0 and 5.0 and most of the values of n are between 2.0 and 5.4, tables can be computed giving the values of Y and Z for various values of m, n, and u as explained in Appendix II. For a given problem, m, n, and u are easily computed so that with the aid of such tables as Tables 1 and 2 the solution is readily obtained.

By integrating between the limits u_1 and u_2 the distance between these points is obtained. Thus,

$$x_2 - x_1 = \frac{y_0}{S_0}\left[\left(\frac{y_c}{y_0}\right)^m (Y_2 - Y_1) + (Z_2 - Z_1)\right] \qquad (11)$$

and, by substituting,

$$\Pi = (y_c/y_0)^m\,Y + Z \qquad (12)$$

TABLE 1.—VALUES OF THE *Y*-FUNCTION

u	*n*			
	3.0	3.2	3.4	3.6
		$m = 3.0$		
0.78	0.389	0.379	0.371	0.364
0.79	0.404	0.394	0.385	0.378
0.999	2.183	2.069	1.968	1.878
1.001	2.788	2.475	2.231	2.032
1.40	0.798	0.622	0.497	0.406
1.42	0.782	0.608	0.485	0.394
4.0	0.251	0.158	0.103	0.068
4.5	0.223	0.137	0.087	0.056
5.0	0.200	0.121	0.075	0.048
7.0	0.143	0.081	0.047	0.028
8.0	0.125	0.069	0.039	0.022
		$m = 3.2$		
0.38	0.055	0.055	0.055	0.055
0.40	0.062	0.062	0.062	0.061
0.64	0.193	0.190	0.187	0.185
		$m = 3.4$		
0.32	....	0.027	0.027	0.027
0.34	....	0.032	0.032	0.032
0.48	....	0.075	0.074	0.074
0.50	....	0.083	0.082	0.082
0.72	....	0.226	0.222	0.218
0.73	....	0.236	0.232	0.228
0.84	....	0.385	0.375	0.366
0.85	....	0.404	0.393	0.383
0.89	....	0.497	0.481	0.467
0.90	....	0.525	0.508	0.493
0.92	....	0.593	0.572	0.554
0.93	....	0.633	0.611	0.591
0.97	....	0.894	0.857	0.825
0.975	....	0.951	0.911	0.875
0.999	....	1.954	1.856	1.769
1.001	....	2.936	2.554	2.283
1.02	....	1.998	1.671	1.450
1.03	....	1.870	1.552	1.337
1.26	....	1.178	0.907	0.725
1.28	....	1.154	0.885	0.705
2.1	....	0.704	0.485	0.348
2.2	....	0.676	0.462	0.328
2.6	....	0.588	0.388	0.267
2.7	....	0.570	0.373	0.255

TABLE 2.—VALUES OF THE *Z*-FUNCTION

u	*n*			
	3.0	3.2	3.4	3.6
0.32	3.073	3.088	3.098	3.105
0.34	2.882	2.898	2.910	2.918
0.36	2.712	2.729	2.741	2.750
0.38	2.558	2.576	2.590	2.600
0.40	2.418	2.438	2.453	2.464
0.48	1.963	1.989	2.009	2.024
0.50	1.868	1.896	1.918	1.934
0.64	1.332	1.373	1.405	1.430
0.72	1.078	1.127	1.167	1.199
0.73	1.048	1.098	1.138	1.171
0.78	0.893	0.950	0.995	1.033
0.79	0.862	0.920	0.966	1.005
0.84	0.697	0.762	0.816	0.860
0.85	0.662	0.729	0.784	0.829
0.89	0.506	0.581	0.643	0.695
0.90	0.462	0.539	0.603	0.657
0.92	0.364	0.446	0.514	0.572
0.93	0.308	0.393	0.464	0.525
0.97	−0.019	0.086	0.173	0.249
0.975	−0.085	0.023	0.115	0.193
0.999	−1.182	−1.006	−0.855	−0.724
1.001	−1.932	−1.789	−1.664	−1.555
1.02	−0.882	−0.809	−0.746	−0.691
1.03	−0.747	−0.683	−0.629	−0.581
1.26	−0.144	−0.128	−0.114	−0.102
1.28	−0.132	−0.117	−0.104	−0.093
1.40	−0.096	−0.083	−0.073	−0.064
1.42	−0.090	−0.078	−0.068	−0.059
2.1	−0.017	−0.014	−0.011	−0.009
2.2	−0.014	−0.011	−0.009	−0.007
2.6	−0.007	−0.006	−0.004	−0.003
2.7	−0.006	−0.005	−0.004	−0.003
4.0	−0.001	−0.001	−0.001	−0.001
4.5	−0.001	−0.001	−0.000	−0.000
5.0	−0.001	....	....	....
6.0	−0.000	....	....	....
7.0	....	....	....	....
8.0	....	....	....	....

Eq. 11 becomes

$$L = x_2 - x_1 = \frac{y_0}{S_0}(\Pi_2 - \Pi_1) \dots\dots\dots\dots\dots\dots (13)$$

in which L is the length of the surface curve. By definition, L is measured parallel to the bottom of the channel. For a channel with mild slope—that is, a slope less than critical—the distance measured in this manner differs but little from the horizontal distance. It is, therefore, convenient and customary to consider L as a horizontal distance in this case. For a channel with a steep slope, however, it is possible that there may be an appreciable difference between the slope distance and the horizontal distance.

In the form of Eq. 13, the equation of nonuniform flow will be found most convenient.

THE HYDRAULIC EXPONENTS m AND n

Inasmuch as the hydraulic exponent n (see Eq. 5*a*) has been thoroughly explained elsewhere,[7] it is needless to discuss it in detail. The exponent m proposed herein is very similar to n (see Eq. 5*b*) so that for any two values of y and the corresponding values of K and M the values of n and m can be solved for directly by the relationships:

$$n = 2\,\frac{\log K_1/K_2}{\log y_1/y_2} \qquad (14a)$$

and

$$m = 2\,\frac{\log M_1/M_2}{\log y_1/y_2} \qquad (14b)$$

For a rectangular channel,

$$\left(\frac{y_1}{y_2}\right)^m = \left(\frac{M_1}{M_2}\right)^2 = \left(\sqrt{\frac{A^3{}_1/(b_w)_1}{A^3{}_2/(b_w)_2}}\right)^2 = \frac{A^3{}_1\,(b_w)_2}{A^3{}_2\,(b_w)_1} = \frac{(b\,y_1)^3\,(b_w)_2}{(b\,y_2)^3\,(b_w)_1} \qquad (15)$$

However,

$$(b_w)_1 = (b_w)_2 = b \qquad (16)$$

so that

$$(y_1/y_2)^m = (y_1/y_2)^3 \qquad (17)$$

and $m = 3$. It can be shown similarly that for a V-shaped channel $m = 5$, and for a parabolic channel $m = 4$.

In most cases, however, it will be unnecessary to determine these exponents more exactly than to the nearest tenth. For this purpose it is convenient to plot log K against log y, and log M against log y for the channel in question. The slopes of the lines thus obtained will be one half of the values of the exponents.

In the case of m, the relation is based entirely on the shape of the channel. For instance, two channels with the same side slopes, but having different bottom widths, will have the same value of m for values of y between 5 and 8 in the case of a bottom width of 10 and for values of y between 10 and 16 in the case of a bottom width of 20. Using this principle it is possible to plot log M against log y for a channel of unit bottom width, and from the plotting to determine the values of m for any range of depths or bottom widths. Fig. 1 is a graph of this type for channels of unit bottom width with different side slopes.

To illustrate the use of Fig. 1 in determining m, assume that it is desired to determine m for the range of depths from $y_1 = 10$ to $y_2 = 15$ in a channel having a bottom width of 10 and side slopes of 1 on $1\frac{1}{2}$. First divide the depths y_1 and y_2 by the bottom width, giving 1.0 and 1.5, respectively. In Fig. 1, lay a triangle across the points where the 1 on $1\frac{1}{2}$ slope line crosses the 1.0 value and the 1.5 value of y/b (see insert in Fig. 1). With the aid of a straight

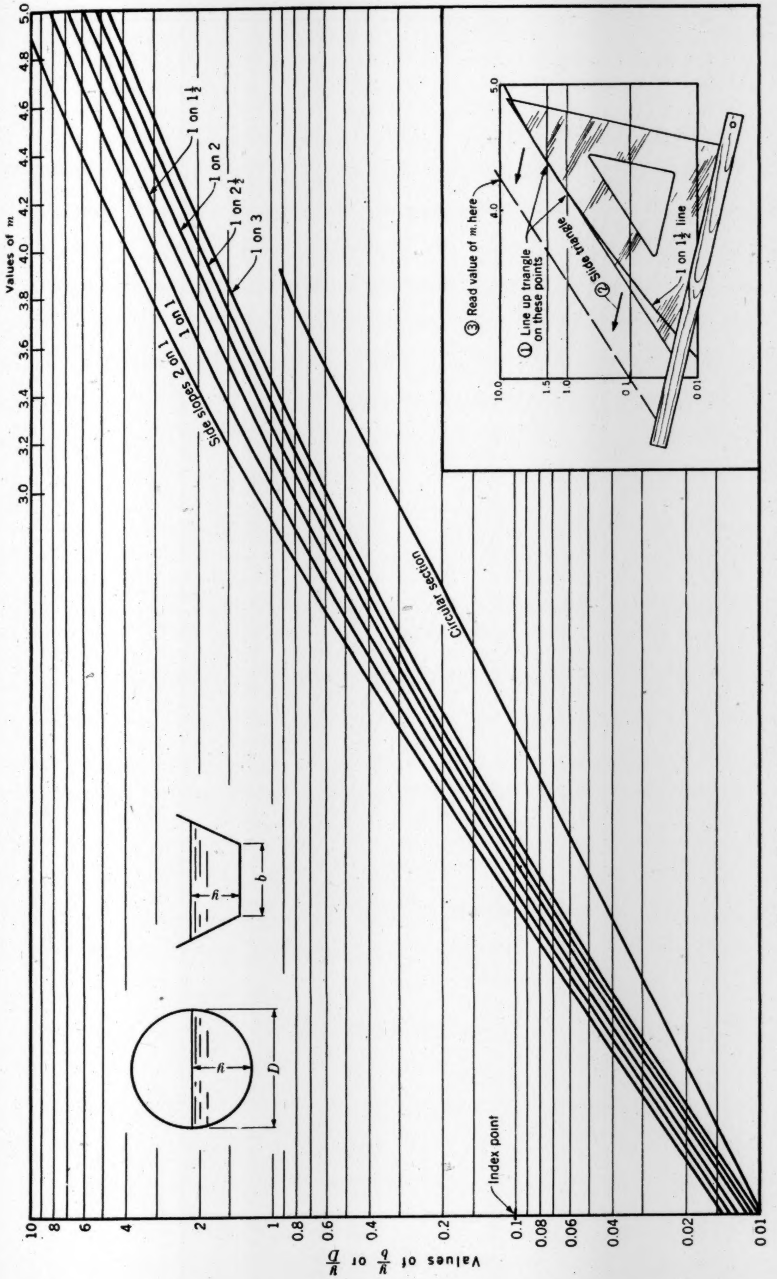


FIG. 1.—CHART FOR DETERMINING VALUES OF THE HYDRAULIC EXPONENT m

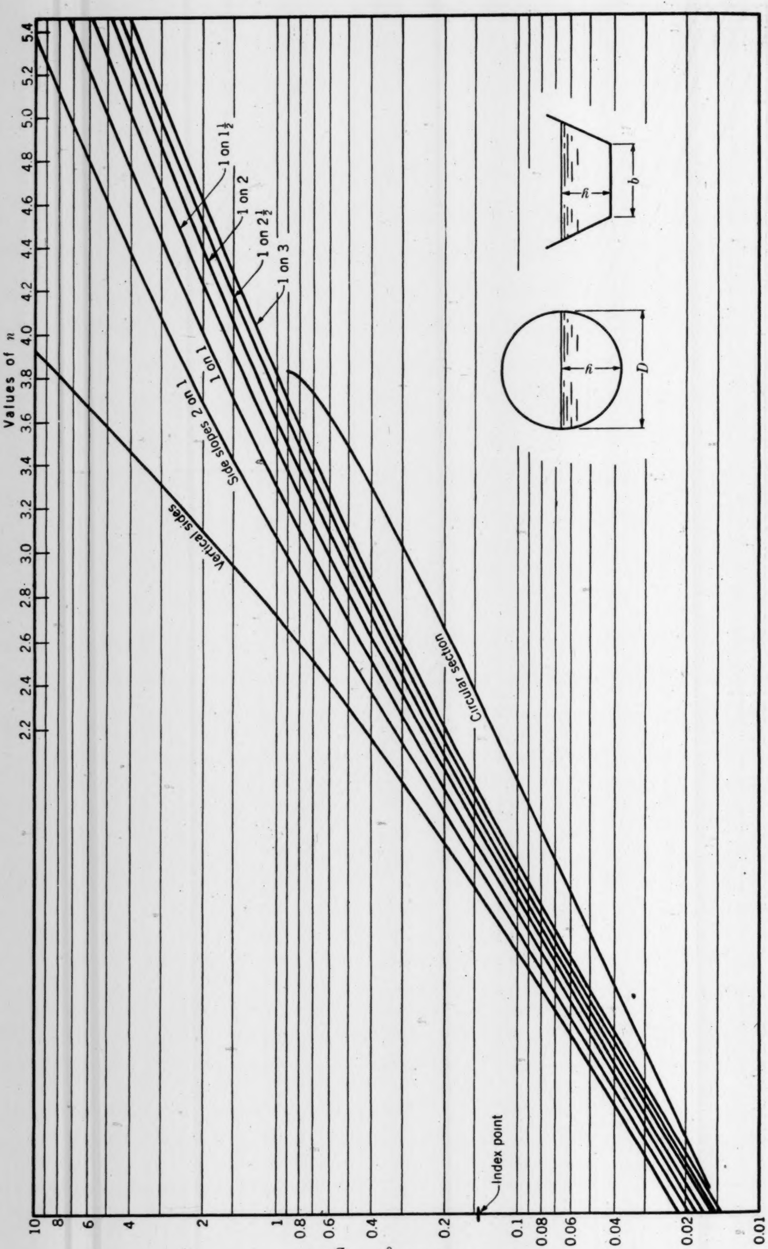


FIG. 2.—CHART FOR DETERMINING VALUES OF THE HYDRAULIC EXPONENT n

edge, slide the triangle to the left until it coincides with the index point at the left of the chart. Read the value of m where the triangle intersects the scale at the top of the chart. In this case, the value of m is slightly less than 4.2. For ordinary computations the nearest 0.1 will be accurate enough.

The value of n can be determined from a chart in a manner similar to the determination of m. Any of the flow formulas which can be reduced to the form:

$$Q = C\,A\,R^a\,S^b \qquad (18)$$

may be used, provided that C contains only factors pertaining to the shape and to the roughness of the channel and that the roughness coefficient is considered constant throughout the cross section. The Kutter formula is not applicable because in that case C contains the slope.

In the Manning formula,

$$K = (1.486/n)\,A\,R^{2/3} \qquad (19)$$

so that

$$\left(\frac{y_1}{y_2}\right)^n = \left(\frac{K_1}{K_2}\right)^2 = \frac{[(1.486/n)\,A_1\,R^{2/3}{}_1]^2}{[(1.486/n)\,A_2\,R^{2/3}{}_2]^2} = \frac{(A_1\,R^{2/3}{}_1)^2}{(A_2\,R^{2/3}{}_2)^2} \qquad (20)$$

in which n, when used with the constant 1.486, is the well-known Manning coefficient of roughness. Thus, the roughness coefficient cancels out and the hydraulic exponent n is a function of the shape of the channel. In Fig. 2 are plotted values of $\log y$ against values of $\log K$ for a unit bottom width and for different side slopes. The chart is used exactly as is Fig. 1.

COMPUTATION PROCEDURE

To illustrate the use of the method in nonuniform flow problems, the following typical examples are shown:

Example 1 (M_1-*Curve*).—In a channel (Fig. 3) with a discharge of 800 cu ft per sec, and with Manning's $n = 0.025$ and $y_2 = 8.0$ ft, it is required to determine the length of the curve. For $Q = 800$ and $S_0 = 0.0004$, $y_0 = 6.73$ ft and $y_c = 2.98$ ft. Because $y_0 > y_c$, and the whole curve lies above the normal depth line, this is an M_1-curve. The values of n and m are obtained by dividing 8.0 ft and 2.98 ft, the values of y_2 and y_c, respectively, by the bottom width of the channel—25 ft. This procedure gives 0.32 and 0.119, respectively; and, from Figs. 1 and 2, $m = 3.4$ and $n = 3.6$, using the 1 on $1\frac{1}{2}$ slope line in each case.

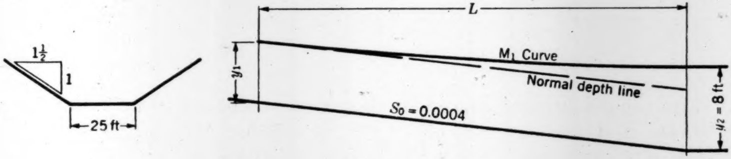


FIG. 3

Theoretically the surface curve would be infinitely long, but it is impractical to extend it past the point where $u = 0.999$. The curve, then, will extend from where $u_2 = 6.73/8.00 = 0.841$ to where $u_1 = 0.999$. From Eq. 13 in which $y_0/S_0 = 16{,}825$ and from Eq. 12 in which $(y_c/y_0)^n = 0.063$, the following values are found:

y	u	Y	$\left(\frac{y_c}{y_0}\right)^m Y$	Z	Π	$\Delta\Pi$
8.00	0.841	0.368	0.023	0.857	0.880	1.493
6.74	0.999	1.769	0.111	−0.724	−0.613	

—from which $L = 25{,}100$ ft. If it is desired to know what the stage would be at a distance of 5,000 ft upstream from y_2 by Eq. 13 it is found that $\Pi_1 = 0.880 - 0.297 = 0.583$.

The value of u which will make Π_1 equal to 0.583 is found by trial and error, as follows:

u	Y_1	$\left(\frac{y_c}{y_0}\right)^m Y_1$	Z_1	Π_1	y_1
0.92	0.554	0.035	0.572	0.607	....
0.93	0.591	0.037	0.525	0.562	....
0.925		(interpolated)		0.583	7.29

Example 2 (M$_2$-*Curve*).—A channel with 1 on 1 side slopes and $S_0 = 0.0003$ ends in an abrupt drop (Fig. 4). It is desired to determine the length of the surface curve if $Q = 800$ cu ft per sec and Manning's $n = 0.030$.

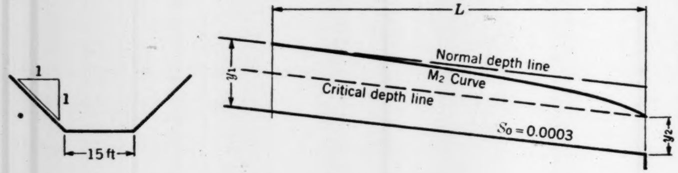


FIG. 4

With the conditions given, $y_0 = 10.8$ ft, $y_c = 4.05$ ft, $y_0/S_0 = 10.8/0.0003 = 36{,}000$, and $(y_c/y_0)^m = (4.05/10.8)^{3.4} = 0.0356$. Also, $y_0/15 = 0.72$ and $y_c/15 = 0.27$; and, from Figs. 1 and 2, $m = 3.4$ and $n = 3.6$. In Eqs. 12 and 13 the values are found to be

y	u	Y	$\left(\frac{y_c}{y_0}\right)^m Y$	Z	Π	$\Delta\Pi$
4.05	2.667	0.259	0.009	−0.002	0.007	1.384
10.79	1.001	2.283	0.081	−1.458	−1.377	

—from which $L = 49{,}800$ ft.

Example 3 (M_3-*Curve*).—A sluice discharges into a rectangular channel laid on a slope of 0.0004; $Q = 300$ cu ft per sec and Manning's $n = 0.015$. The depth at the vena contracta is 0.5 ft and a jump occurs where $y_2 = 0.8$ ft (Fig. 5). It is required to determine the position of the jump.

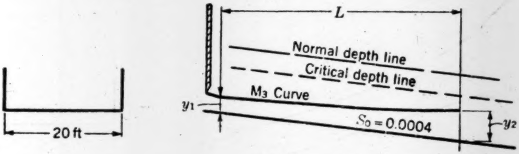


FIG. 5

The position of the jump will be determined by the length of the curve between $y = 0.5$ and $y = 0.8$. For a rectangular channel $m = 3.0$, and by entering Fig. 2 with values of y/b of $0.5/20 = 0.025$ and $0.8/20 = 0.040$, it is found that $n = 3.4$. Then $y_0 = 3.83$ ft, $y_c = 1.92$ ft, $(y_c/y_0)^m = 0.126$, and $y_0/S_0 = 3.83/0.0004 = 9{,}575$, from which the values in Eqs. 12 and 13 are found to be

y	u	Y	$\left(\frac{y_c}{y_0}\right)^m Y$	Z	Π	$\Delta\Pi$
0.50	7.66	0.042	0.005	0	0.005	0.005
0.80	4.79	0.080	0.010	0	0.010	

—from which $L = 48$ ft.

Example 4 (S_1-*Curve*).—A channel with 1 on $\frac{1}{2}$ side slopes and a bottom width of 15 ft carries 1,000 cu ft per sec and empties into a pool so that y_2, the depth at the pool, is 10 ft (Fig. 6). If y_1, the depth after the jump, is 6.2 ft, locate the position of the jump ($S_0 = 0.01$ and Manning's $n = 0.020$).

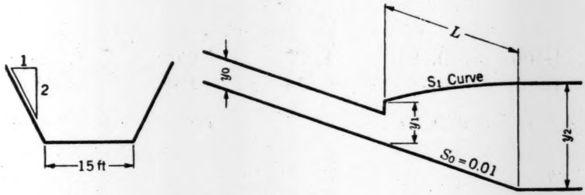


FIG. 6

The first step is to determine y_0 and y_c, which are found to be 3.97 ft and 4.93 ft, respectively. From Figs. 1 and 2, it is found that $m = 3.2$ and $n = 3.2$. Then, $y_0/S_0 = 397$ and $(y_c/y_0)^m = 1.99$; and, in Eqs. 12 and 13, the following values obtain:

y	u	Y	$\left(\frac{y_c}{y_0}\right)^m Y$	Z	Π	$\Delta\Pi$
10.0	0.397	0.061	0.121	2.459	2.580	0.828
6.2	0.640	0.190	0.379	1.373	1.752	

—from which $L = 328$ ft.

Example 5 (S_2-*Curve*).—Fig. 7 shows a typical case of this type of curve, for which it is required to determine the length ($Q = 1{,}000$ cu ft per sec, Manning's $n = 0.013$, and $S_0 = 0.005$).

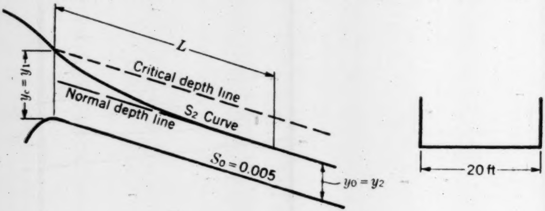

L
Critical depth line
Normal depth line
$y_c = y_1$
S_2 Curve
$S_0 = 0.005$
$y_0 = y_2$
20 ft

FIG. 7

Computations give $y_0 = 3.35$ ft and $y_c = 4.27$ ft. For this range of depths, the hydraulic exponents, n and m, both have a value of 3.0. Therefore, by Eqs. 12 and 13 the following values are found: $y_0/S_0 = 670$, $(y_c/y_0)^m = 2.073$, and

y	u	Y	$\left(\frac{y_c}{y_0}\right)^m Y$	Z	Π	$\Delta\Pi$
4.27	0.785	0.396	0.821	0.877	1.698	1.645
3.353	0.999	2.183	4.525	−1.182	3.343	

—from which $L = 1{,}102$ ft.

Example 6 (S_3-*Curve*).—A sluice discharges into a rectangular channel on a slope of 0.008; $Q = 600$ cu ft per sec, Manning's $n = 0.015$, and the depth at the vena contracta is 1.6 ft. Determine the length of the surface curve (Fig. 8). In Eqs. 12 and 13, $y_0 = 2.25$ ft, $y_c = 3.03$ ft, $y_0/S_0 = 281$, $(y_c/y_0)^m = 2.444$, $m = 3.0$, $n = 3.0$, and the following values are found:

y	u	Y	$\left(\frac{y_c}{y_0}\right)^m Y$	Z	Π	$\Delta\Pi$
1.60	1.406	0.793	1.938	−0.081	1.857	3.168
2.248	1.001	2.788	6.814	−1.789	5.025	

—from which $L = 890$ ft.

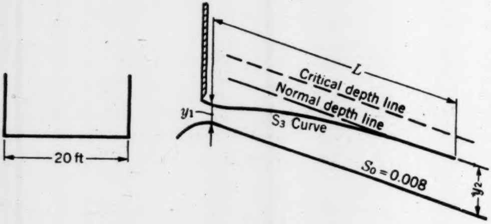
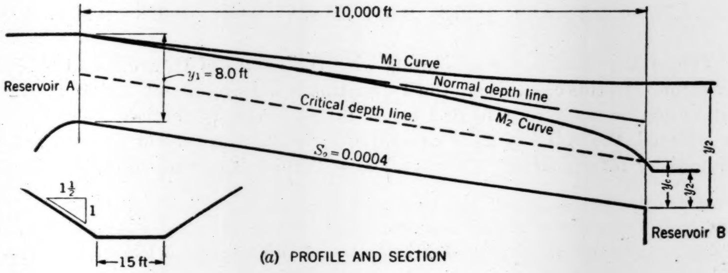

20 ft
L
Critical depth line
Normal depth line
y_1
S_3 Curve
$S_0 = 0.008$
y_2

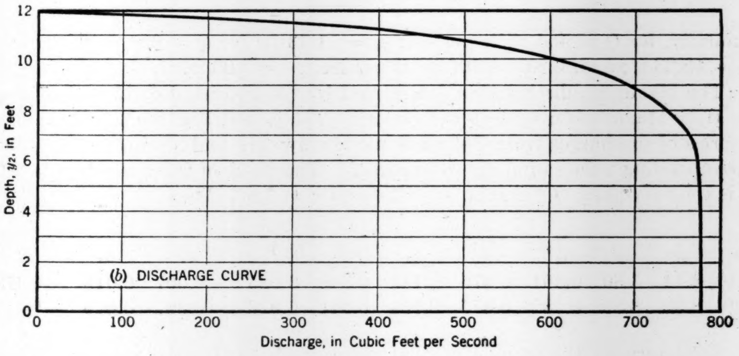
FIG. 8

Example 7 (*Delivery Curve*).—This problem is included to show how easily and quickly the integration method arrives at a solution, whereas the use of a step method would be extremely cumbersome.

A canal 10,000 ft long with 1 on 1½ side slopes (Fig. 9) connects reservoir A with reservoir B, Manning's $n = 0.025$ and $S_0 = 0.0004$. The pool at A is constant, but the pool at B fluctuates. The problem is to prepare a chart that will show the discharge through the canal for any level in pool B. Head lost at the entrance to the canal will be neglected to emphasize the nonuniform flow part of the problem.



(a) PROFILE AND SECTION



(b) DISCHARGE CURVE

FIG. 9

Obviously, when the level at B is at the same elevation as that at A the discharge will be zero. As the pool level at B lowers, the discharge will increase and the surface curve will be of the M_1-type until the pool level is lowered to the point where it coincides with the normal depth line at the lower end of the canal (in other words, to where $y_2 = y_0$). Below this point flow is of the M_2-type and maximum discharge is reached when the pool at B is at or below the level of the critical depth line at the lower end of the canal.

When pool B is at the same level at pool A, $Q = 0$. Also, when $y_2 = y_0$, uniform flow obtains; and, by the Manning formula, $Q = 744$ cu ft per sec. For intermediate points the flow is varied and the value of y_2 corresponding to any selected value of Q is computed by Eqs. 12 and 13. By selecting values of Q, the corresponding values of other quantities for values of $m = 3.4$ and $n = 3.6$ may be tabulated thus:

Q	y_0	$\frac{S_0}{y_0}L$	y_c	$\left(\frac{y_c}{y_0}\right)^m$	u_1	Y_1	$\left(\frac{y_c}{y_0}\right)^m Y_1$	Z_1	Π_1	Π_2
200	3.99	1.003	1.67	0.051	0.499	0.082	0.004	1.938	1.942	2.945
400	5.81	0.688	2.56	0.061	0.726	0.224	0.014	1.182	1.196	1.884
600	7.16	0.559	3.28	0.070	0.895	0.480	0.034	0.676	0.710	1.269
700	7.77	0.515	3.60	0.073	0.971	0.835	0.061	0.238	0.299	0.814

Values of u_2 and y_2 corresponding with the values of Π_2 are found by trial and error. In this case, where the Y-function is not prominent, a close approximation can be made by referring to the tables of the Z-function. Two points are selected, the value of Π_2 is computed for each, and the correct value of u_2 is found by interpolation. Thus, for $Q = 200$ cu ft per sec and $\Pi_2 = 2.945$,

u_2	Y_2	$\left(\frac{y_c}{y_0}\right)^m Y_2$	Z_2	Π_2	y_2
0.32	0.027	0.001	3.105	3.106	....
0.34	0.032	0.002	2.918	2.920	....
0.3385		(interpolated)		2.945	11.79

Similarly, for $Q = 400$ cu ft per sec, $y_2 = 11.33$ ft; for $Q = 600$ cu ft per sec, $y_2 = 10.23$ ft; and, for $Q = 700$ cu ft per sec, $y_2 = 9.00$ ft.

The maximum discharge is determined by transposing Eq. 13 and solving for Π_1. Maximum discharge will occur if $y_2 = y_c$ (see Fig. 9(*a*)). By taking a series of discharges from 744 cu ft per sec upward and setting $y_2 = y_c$, the corresponding values of y_1 are determined. The value of Q for which $y_1 = 8.0$ ft is the maximum Q. Again, $m = 3.4$ and $n = 3.6$. Thus,

Q	y_0	y_c	$\frac{S_0}{y_0}L$	$\left(\frac{y_c}{y_0}\right)^m$	u_2	Y_2	$\left(\frac{y_c}{y_0}\right)^m Y_2$	Z_2	Π_2	Π_1
770	8.14	3.80	0.491	0.075	2.142	0.340	0.025	−0.007	0.018	−0.473
800	8.30	3.89	0.482	0.077	2.134	0.341	0.026	−0.007	0.019	−0.463

To determine the value of y_1 that corresponds to Π_1, trial values of u_1 are selected and trial values of Π_1 are computed. The correct values of u_1 are found by interpolation:

	u_1	Y_1	$\left(\frac{y_c}{y_0}\right)^m Y_1$	Z_1	Π_1	y_1
For $\Pi_1 = -0.473$	1.02	1.450	0.109	−0.643	−0.534	....
	1.03	1.337	0.100	−0.539	−0.439	....
	1.026		(interpolated)		−0.473	7.92
For $\Pi_1 = -0.463$	1.02	1.450	0.112	−0.643	−0.531	....
	1.03	1.337	0.103	−0.539	−0.436	....
	1.027		(interpolated)		−0.463	8.07

Interpolating between these two values of y_1 gives $Q_{\max} = 786$ cu ft per sec and $y_2 = 3.85$ ft. If one point is computed between $Q = 744$ and $Q = 786$—

say, at $Q = 772$—the values in Eqs. 12 and 13 are found to be

Q	y_0	$\frac{S_0}{y_0}L$	y_c	$\left(\frac{y_c}{y_0}\right)^m$	u_1	Y_1	$\left(\frac{y_c}{y_0}\right)^m Y_1$	Z_1	Π_1	Π_2
772	8.15	0.491	3.80	0.075	1.02	1.450	0.109	−0.643	−0.534	−0.043

from which computation by trial and error gives $y_2 = 6.43$ ft. The computed points are plotted and the delivery curve is drawn (Fig. 9(*b*)).

CONCLUSIONS

The average engineer thinks in terms of uniform flow, which for simple problems may give a sufficient answer. In the ordinary canal, however, nonuniform flow is encountered as often as uniform flow, if not more often. Therefore, it is imperative that the engineer learn to think in terms of nonuniform flow. It is important, also, that he have a convenient method of working with nonuniform flow.

An integration method such as the one presented shortens the time required to compute backwater problems. What is probably more important is that such a method makes it possible, successfully and quickly, to attack nonuniform flow problems that are seldom tried by ordinary step methods.

The method presented can be used with the Kutter, the Manning, the Bazin, or any other of the open channel formulas used to compute uniform flow. It can also be applied to natural streams, to channels with horizontal bottom, and to channels of adverse slope, although with the last a special set of tables of the Y-function and the Z-function would have to be computed.

ACKNOWLEDGMENT

The writer wishes to express his appreciation to Mr. Posey of the University of Iowa at Iowa City for his review of this paper and for his words of encouragement.

APPENDIX I. NOTATION

The following letter symbols, defined where they first appear in the text, or by illustration, conform essentially with the American Standard Letter Symbols for Hydraulics (ASA—Z10.2—1942) prepared by a Committee of the American Standards Association, with Society representation:

A = cross-sectional area;
b = width of stream bed; b_w = width of water surface;
C = the Chezy coefficient of flow; also, a constant of integration;
D = diameter of circular section;
e = base of Naperian logarithms;
g = gravitational acceleration;
K = cross-sectional conveyance (defined by Eq. 2*a*); K_0 = cross-sectional conveyance for uniform flow;
L = length of surface curve;

M = a cross-sectional function (defined by Eq. 3*a*); M_c = a cross-sectional function for the critical depth (defined by Eq. 3*b*);
m = an hydraulic exponent (defined by Eq. 5*b*);
n = an hydraulic exponent (defined by Eq. 5*a*); also the Manning coefficient;
p = a substitution factor (defined by Eq. 24);
Q = discharge (rate of flow);
R = hydraulic radius of cross-sectional area;
S = slope:
  S_f = slope of energy grade line (friction slope);
  S_s = slope of water surface;
  S_0 = slope of channel bed (also slope of water surface for uniform flow);
u = a substitution factor (defined by Eq. 7*a*);
V = velocity of flow;
v = a substitution factor (defined by Eq. 7*b*);
x = a substitution factor (defined by Eq. 22); also distance along canal parallel to bottom grade as used in Eqs. 1 to 11;
Y = a substitution factor (defined by Eq. 9*a*);
y = depth of flow:
  y_c = critical depth;
  y_0 = depth for uniform flow;
Z = a substitution factor (defined by Eq. 9*b*); and
Π = a substitution factor (defined by Eq. 12).

APPENDIX II. COMPUTATION OF TABLES OF *Y*-FUNCTIONS AND *Z*-FUNCTIONS

Values of u Less Than 1.0.—Eq. 9*a* can be written as

$$Y = \int (u^{m-2} + u^{n+m-2} + u^{2n+m-2} + u^{3n+m-2} + \cdots)\, du \quad \ldots\ldots(21a)$$

from which

$$Y = \frac{u^{m-1}}{m-1} + \frac{u^{n+m-1}}{n+m-1} + \frac{u^{2n+m-1}}{2n+m-1} + \frac{u^{3n+m-1}}{3n+m-1} + \cdots \quad \ldots(21b)$$

The series converges for u less than 1.0, and for small values of u the convergence is rapid. Thus, only a few terms need be computed in each series. As u becomes large, more terms are required so that at the upper range of values (from u = 0.90 to u = 0.999) it is convenient to transpose the integral. In Eq. 9*a* assume that

$$1 - u^n = x \quad \ldots\ldots\ldots\ldots(22)$$

so that Eq. 9*a* reduces to

$$Y = -\frac{1}{n}\int \frac{(1-x)^p}{x}\, dx \quad \ldots\ldots\ldots\ldots(23)$$

in which

$$p = \frac{m - n - 1}{n} \qquad (24)$$

Expanding and integrating,

$$Y = -\frac{1}{n}\left[C + \log_e x - p\,x + \frac{p\,(p-1)\,x^2}{2 \times 2!} - \frac{p\,(p-1)\,(p-2)\,x^3}{3 \times 3!} + \cdots\right] \qquad (25)$$

in which C is a constant of integration and e is the base of the Naperian logarithms. This expression converges more rapidly as u becomes larger.

For the range between $u = 0.70$ and $u = 0.90$, the tables were computed by differentials, the slope of the curve of Y versus u being

$$\frac{dY}{du} = \frac{u^{m-2}}{1 - u^n} \qquad (26)$$

The difference between the values of Y for u_1 and u_2 is $(u_2 - u_1)$ multiplied by the average slope of the curve between the two points. The interval $u_2 - u_1$ must be small enough to bring this approximation within the limits of accuracy desired.

To compute the Z-table Eq. 9*b* is expanded and integrated to give

$$Z = \frac{1}{u} - \frac{u^{n-1}}{n-1} - \frac{u^{2n-1}}{2\,n-1} - \frac{u^{3n-1}}{3\,n-1} - \cdots \qquad (27a)$$

It will be noted that, if $n = m$, Eq. 27*a* can be written as

$$Z = \frac{1}{u} - Y \qquad (27b)$$

Values of u Greater Than 1.0.—To obtain a converging series for u greater than 1.0, let

$$u = 1/x \qquad (28)$$

Then Eq. 9*a* becomes

$$Y = \int \frac{x^{n-m}\,dx}{1 - x^n} = \int (x^{n-m} + x^{2n-m} + x^{3n-m} + \cdots)\,dx \qquad (29a)$$

or

$$Y = \frac{x^{n-m+1}}{n - m + 1} + \frac{x^{2n-m+1}}{2\,n - m + 1} + \frac{x^{3n-m+1}}{3\,n - m + 1} + \cdots \qquad (29b)$$

Eq. 29*b* was used for values of u from 1.10 to 30.0.

For values of u between 1.001 and 1.10, the number of terms in the series becomes impractical so that it is convenient to transpose the equation. Thus, by letting

$$-x = 1 - u^n \qquad (30)$$

and, by substituting in Eq. 9*a*,

$$Y = -\frac{1}{n}\left[C + \log_e x + p\,x + \frac{p\,(p-1)\,x^2}{2 \times 2!} + \frac{p\,(p-1)\,(p-2)\,x^3}{3 \times 3!} + \cdots\right] \qquad (31)$$

The series represented by Eq. 31 converges more rapidly as u approaches 1.0.

By substituting Eq. 28 in Eq. 9*b*,

$$Z = -\int \frac{x^n\,dx}{1 - x^n} \qquad (32a)$$

which, when expanded and integrated, gives

$$Z = -\frac{x^{n+1}}{n+1} - \frac{x^{2n+1}}{2n+1} - \frac{x^{3n+1}}{3n+1} - \cdots \qquad (32b)$$

All values of the Z-function are negative for values of u greater than 1.0, and it will be noted that, where $n = m$,

$$Z = x - Y \qquad (33)$$

DISCUSSION

Phillip Z. Kirpich,[9] Assoc. M. ASCE.—As stated by the author, Boris A. Bakhmeteff, Hon. M. ASCE, first devised an integration method, applicable to all shapes of channel, for the determination of nonuniform or varied flow profiles. As further expressed by Mr. Von Seggern (see "Introduction"):

> "The Bakhmeteff method, however, evaluates only a part of the integral, thus making it necessary to apply a factor or a series of factors to account for the remainder."

Although these factors are by no means arbitrary, their determination requires further computation and plotting. The author has successfully eliminated much of this labor by:

(*a*) Providing a chart for the exponent n; and

(*b*) Introducing an exponential factor m and developing tables for two new functions, designated Y and Z, in which m and n are parameters.

The same purpose can be achieved by retaining Professor Bakhmeteff's procedure and developing charts for the exponent n and the critical slope σ.[10] Although the factors given in these charts are for rectangular and trapezoidal shapes only, values can be determined for circular, parabolic, or other shapes. From Fig. 10 it is seen[11] that n is a function of y/b only and can be read more directly than in Fig. 2. The critical slope σ is also easily obtained from Fig. 10 except that, in this case, it is necessary to divide by the factor $\left(\frac{1.486}{n}\right)^2 b^{1/3}$, in which n is the Manning coefficient. With the exponent n and σ known, the integration is readily performed using Professor Bakhmeteff's varied flow function tables.[12]

As applied to Example 1 of the paper, first compute $K_M = \frac{1.486}{n} = \frac{1.486}{0.025} = 59.4$; then $K^2_M\, b^{1/3} = 59.4^2 \times 25^{1/3} = 10{,}300$. Values of n and σ, selected from Fig. 10, are as follows:

	y	y/b	n	$\sigma/K^2_M\, b^{1/3}$	σ	β	$1-\beta$
Station 1....	8	0.32	3.7	58	0.0060	0.07	0.93
Station 2....	6.73	0.27	3.65	62	0.0056	0.07	0.93

A quantity β as defined by Professor Bakhmeteff equals $\frac{S_o}{\sigma}$; and $\eta = \frac{y}{y_o}$. Then a quantity $B(\eta)$ is obtained from Professor Bakhmeteff's varied flow function tables using an average value of the exponent n as a parameter, and $\pi = \eta - (1 - \beta)\, B$. With these definitions and with an average value of n

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[9] Hydr. Engr., Knappen Tippetts Abbett Eng. Co., New York, N. Y.

[10] "Dimensionless Constants for Hydraulic Elements of Open-Channel Cross-Sections," by Phillip Z. Kirpich, *Civil Engineering*, October, 1948, p. 47.

[11] *Ibid.*, p. 48, Fig. 4.

[12] "Hydraulics of Open Channels," by Boris A. Bakhmeteff, McGraw-Hill Book Co., Inc., New York, N. Y., 1932, pp. 308–311.

of 3.7 (taken to the nearest tenth), the computation is completed as follows:

	η	$B(\eta)$	$(1-\beta)B$	π	$\Delta\pi$
Station 1.......	1.19	0.303	0.28	0.91	1.46
Station 2.......	1.001	1.668	1.55	−0.55	

—which checks the author's result within 2%. The slight discrepancy is partly caused by the different n-value used—3.7 instead of 3.6.

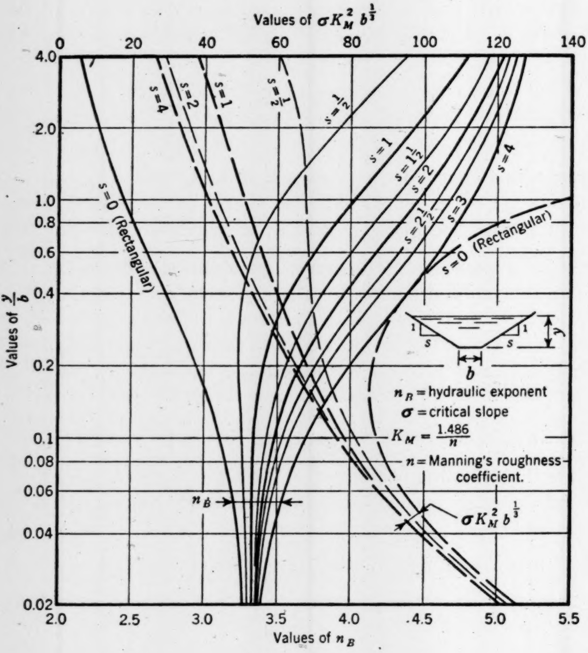

FIG. 10

In cases where alternative procedures are available, the engineer should choose the method with which he is familiar, unless some other method possesses marked advantages. Mr. Von Seggern has, in fact, developed a procedure having the advantage of reducing the labor required in computing varied flow profiles. However, the improvements in Professor Bakhmeteff's original procedure described in this discussion tend to negate this advantage—that is, if the factors in Fig. 10 are used in conjunction with Professor Bakhmeteff's varied flow function tables, the advantage claimed by the author's method is largely lost. At the same time it is unnecessary, using the alternative suggestion presented herewith, to become familiar with two additional functions (designated Y and Z) and to acquire tables of these functions.

Paul V. Hodges,[13] M. ASCE.—The integration of the equation of nonuniform flow by Mr. Von Seggern is a valuable accomplishment toward a more definite knowledge of different flow problems. Eqs. 11 and 13, together with Figs. 1 and 2 and the tables of Y-functions and Z-functions, make possible the easy solution of discharge problems in channels of uniform cross sections with sustaining slopes. The solution of the typical examples adds greatly to the value of the paper and makes it a very useful reference. Because of the curvature of the lines in Figs. 1 and 2, the values of the hydraulic exponents, m and n, tend to increase with an increase in depth y.

In Example 2, the author used the average values of 3.4 for m, and 3.6 for n in computing the total distance of the surface curve. To indicate the relative effect of changes in m and n, the writer recomputed Example 2 by individual reaches, using the values of m and n as determined for each reach, as shown in Table 3.

TABLE 3.—Results of Computations for Example 2, Applying Eq. 13

y	m	n	U	L	ΣL
4.05 6.30	3.3	3.6	2.667 1.715	383	0 383
6.3 8.5	3.4	3.65	1.715 1.27	2,391	383 2,774
8.5 10.38	3.55	3.7	1.27 1.04	12,316	2,774 15,090
10.38 10.59	3.55	3.7	1.04 1.02	5,948	15,090 21,038
10.59 10.79	3.55	3.7	1.02 1.001	27.652	21,038 48,690

The difference in total distance of the surface curve as computed by the two integration methods amounts to only 2.32%.

The computation of the surface curve by the ordinary step method requires relatively small steps to obtain close agreement with the integration method.

TABLE 4.—Comparison of Integration and Step Methods for Example 2

Method	Depths of Flow y (in Feet) for Five Values of L (in Feet)				
	15,000	21,000	31,000	48,690	49,820
Integration Method, with:					
Five reaches	10.38	10.59	10.71	10.79	....
One reach	....	....	....	....	10.79
Step Method; Depths Increased by Steps of:					
0.5 ft	10.44	10.65	10.80	....	....
1 ft	10.54	10.80	....	....	....

In Example 2, step computations for each 0.2 ft or 0.3 ft for the flatter part of the surface curve would be required to find the distances that would agree approximately with the integration method, and consequently such detail computations would be long and would require considerable time.

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[13] Office, Corps of Engrs., Denver, Colo.

A comparison of the results obtained by the integration method and by step methods with 0.5-ft steps and 1-ft steps is given in Table 4.

The application of the integration method to natural streams would be very difficult because of the varying cross section and slopes usually encountered. In their present form, Eqs. 11 and 13, could not be applied to channels with horizontal bottom grade. The integration of the equation for nonuniform flow and the computations of the Y-functions and the Z-functions are important advances in the development of the science of hydraulics.

A. H. Davison,[14] **M. ASCE.**—Agreement with the author's general conclusions presents no difficulty. Practical problems almost never offer a case of strictly uniform flow. The nearest case to a uniform flow problem in the writer's experience was the determination of flow conditions in a power canal that was designed to conform to the capacity output of the hydroelectric plant at the lower end of the canal. Considering peak periods, capacity plant operation occurs for a total of about 2,000 hours per annum which is less than 25% of the total time; and even during the periods of capacity plant operation the stage in the canal does not often remain constant.

A proposed method of working with nonuniform flow should be both convenient and accurate. A method that may be convenient to the specialist who encounters many engineering problems of the same general type may be quite inconvenient and formidable to the engineer in general practice who must achieve a sound engineering solution without spending too much time on a preliminary refresher course in involved technique.

To achieve accuracy, much depends on a correct evaluation of the factor n, the well known coefficient of roughness which appears in the Manning and Kutter formulas.

With respect to artificial channels, the initial evaluation of n can doubtless be a quite accurate assumption based on laboratory experiments; but these experiments were conducted under uniform flow conditions. If actual operating conditions vary both ways from the uniform flow condition as an average, probably the assumed value of n is realistic enough except for effects of frost and weather exposure with the passage of time. If actual operating conditions are otherwise, perhaps some consideration should be given to the probability that the value of n will vary with wetted perimeter.

In cases of natural streams, an assumed value of n is never reliable particularly for high water or flood conditions in pond areas above a dam. In such areas the backwater curve is convex upward, similar to the author's M_2-curve. For this type of curve, a realistic value of n which may be 15% higher than the assumed value means that the length of the backwater curve will decrease from 40% to 60%, depending on circumstances, for the same value of Q. This is an indication that the actual backwater curve will shoot up much more abruptly than expected, facing upstream. An assumed value of n is often too low by 50%. This can have serious results as many power companies have discovered to their embarrassment, when faced by unexpected flood-damage claims involving property that was considered safely located.

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[14] Civ. Engr., Glens Falls, N. Y.

The first step in analyzing nonuniform flow problems is to compute the value or values of n corresponding to the range in discharge, from consideration of high water marks and channel conditions along the reach being studied. Such computed values of n will be much more realistic than any assumptions. The technique and computations applied in analyzing practical nonuniform flow problems should be in such form as to emphasize, rather than obscure, this important factor, which is admittedly difficult to evaluate correctly.

Consider a reach bounded by two channel cross sections (section 1 and section 2, reading downstream) which are separated by the distance L. The equation of nonuniform flow is

$$S_s = S_f - \frac{V^2_1 - V^2_2}{2\,g\,L} \quad \text{(34a)}$$

$$\frac{V^2_1 - V^2_2}{2\,g\,L} = \frac{1}{2\,g\,L}\left(\frac{Q^2}{A^2_1} - \frac{Q^2}{A^2_2}\right) \quad \text{(34b)}$$

and

$$Q = \frac{1.486}{n}\,A\,R^{2/3}\,S^{1/2}{}_f \quad (35)$$

Eq. 35 is the Manning formula, with the factors A and $R^{2/3}$ representing the mean of the corresponding values applicable to the two cross sections of the reach, and the factor n signifying the coefficient of roughness. The expression for the slope of the water surface is

$$S_s = Q^2\left[\left(\frac{n}{1.486\,A\,R^{2/3}}\right)^2 - \frac{1}{2\,g\,L}\left(\frac{1}{A^2_1} - \frac{1}{A^2_2}\right)\right] \quad (36)$$

For uniform flow, the second term in the brackets in Eq. 36 becomes zero, and

$$S_s = \left(\frac{Q\,n}{1.486\,A\,R^{2/3}}\right)^2 \quad (37)$$

For critical discharge,

$$Q = A\sqrt{\frac{A}{b_w}}\,\sqrt{g} \quad (38)$$

For the backwater curve, Eq. 36 may be written

$$\frac{1}{2\,g}\left(\frac{Q^2}{A^2_1} - \frac{Q^2}{A^2_2}\right) = L\left[\left(\frac{Q\,n}{1.486\,A\,R^{2/3}}\right)^2 - S_s\right] \quad (39)$$

substituting $S_s = \dfrac{d_1 - d_2}{L} + S_0$,

$$(d_1 - d_2) + \frac{1}{2\,g}\left(\frac{Q}{A_1} - \frac{Q}{A_2}\right)\left(\frac{Q}{A_1} + \frac{Q}{A_2}\right) = L\left[\left(\frac{Q\,n}{1.486\,A\,R^{2/3}}\right)^2 - S_0\right] \quad (40)$$

In Eq. 40, the factors d_1 and d_2 are the depths corresponding to A_1 and A_2. Except as otherwise stated, other factors in Eqs. 34 to 40 are the same as defined in the paper. Application of these formulas as required, with assistance from tables giving the physical and hydraulic properties of rectangular and trapezoidal sections, produces the solution of a practical problem with slide-rule computations. A certain amount of tabulation is required but the procedure is less tedious than appears at first glance.

No difficulty was experienced in checking closely the coordinates computed by the author for backwater or drop-down curves illustrated by Examples 1, 2, and 7 of the paper. There seems little point in computing the total over-all length of the surface curve in Examples 1 and 2. The author states, "Theoretically the surface curve would be infinitely long * * *." Usually all that is required is to develop the profile of the surface curve within practical limits.

With respect to Example 7, the writer's computation procedure, using Eqs. 34 to 40, established the factors without resort to "trial and error" included in the author's procedure. The slight discrepancy between the resulting rating curves as computed by the two procedures (see Fig. 11) would disappear if the writer had used values of n varying from 0.027 for 200 cu ft per sec, to 0.0256 for 700 cu ft per sec, instead of the constant value of $n = 0.025$ throughout. Since the difference in these n-values is well within the range of allowable difference of opinion with respect to assumptions, the discrepancy in results is unimportant.

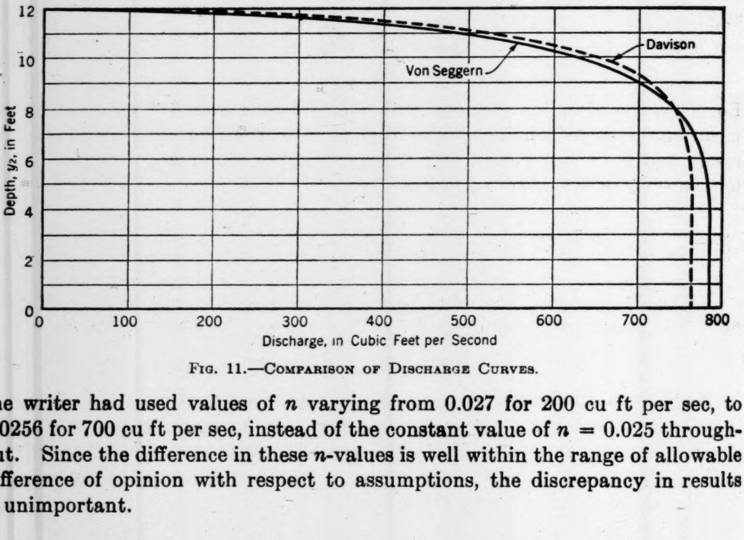

FIG. 11.—COMPARISON OF DISCHARGE CURVES.

STEPONAS KOLUPAILA.[15]—The method concerning the integration of the equation of nonuniform flow in prismatic channels, proposed in the paper, is an interesting extension of the well known method advanced by Boris A. Bakhmeteff, Hon. M. ASCE, in 1912 and 1932. The application of that method, even when facilitated by use of the tables of the varied flow function, is still very complicated, because the expression $(1 - \beta)$ depends on the critical depth. Mr. Bakhmeteff suggests that the variation of $(1 - \beta)$ versus the water depth may be represented graphically and then approximated by successive mean values for the various increments.[16]

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[15] Prof., College of Eng., Notre Dame Univ., Notre Dame, Ind.; formerly on the Faculty of Civ. Eng., Univ. of Vytautas the Great, Kaunas, Lithuania.

[16] "Hydraulics of Open Channels," by Boris A. Bakhmeteff, McGraw-Hill, Book Co., Inc., New York, N. Y., 1932, p. 53.

The method suggested by Mr. Von Seggern is more universal and convenient, as he provides the necessary tables, as well as the diagrams, for determining values of the hydraulic exponents for trapezoidal cross sections of channel. The exponents can be found in these diagrams in terms of the side slope of channel and the ratio of water depth to the bottom width.

The writer would like to suggest that the author search for a possible simple relation between two hydraulic exponents—one for the normal depth and a second for the critical depth. Such a relation would facilitate the procedure of computation.

Mr. Von Seggern's method still requires the determination of the critical depth for each particular discharge. For trapezoidal cross sections, this is rather difficult. The determination of critical depth may be simplified appreciably by the use of diagrams, such as those used by Russian engineers.[17,18]

In eastern Europe the practice is to attempt to simplify the otherwise difficult backwater computation by use of tables, which cover cases of positive slope and negative slope and for the horizontal bottom. In other methods sometimes used, as that of N. N. Pavlovskij,[19] no tables are employed, but each case is solved by integration.

WILLIAM D. MITCHELL,[20] M. ASCE.—An ingenious and worthwhile addition to the procedures for integrating the equation for gradually-varied flow is presented in this paper. In many problems, of which those cited by the author are typical, the procedure is rapid, comparatively simple, and leads to results which are entirely reliable. It also is gratifying to note that, to a considerable extent, the author has conformed to the concepts and notations of Boris A. Bakhmeteff,[2] Hon. M. ASCE. This conformity seems extremely desirable, since Mr. Bakhmeteff's work is generally conceded to be both comprehensive and sound, and may well be accepted both as a point of departure for the discussion of other procedures and as a basis for the correlation of new material. In fact, the principal objective of this discussion will be to point out the basic similarity, and the specific differences, between the two methods.

Before embarking on this principal theme, a thought will be offered concerning the computation of the hydraulic exponent n. At the start of an investigation, there may be available a normal depth-discharge rating for the channel, developed from either computed or observed data. Such a rating, plotted on logarithmic coordinates, is all that is required for a graphical determination of n. Since S_0 is a constant, on logarithmic coordinates the curve for K is parallel to the curve for Q, so that the slopes are identical, and n may be determined from either, using the technique described by Mr. Bakhmeteff.[7] Similarly, $\log Q_1/Q_2$ may be used in Eq. 14a in lieu of $\log K_1/K_2$.

The applicability of the author's method to a rectangular channel has been verified by the writer, using as a criterion of accuracy a series of profiles observed in a 5-ft rectangular channel under carefully controlled laboratory con-

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[17] "Brief Consulting Book of Hydraulics," by N. N. Pavlovskij, Kratkij Gidravličeskij Spravočnik, Techgosizdat, Leningrad, 1940 (in Russian).

[18] "Hydraulics," by Steponas Kolupaila, Kempten, 1947 (in Lithuanian).

[19] "Upon the Nonuniform Flow at Horizontal Bottom of the Channel," by N. N. Pavlovskij, Izvestija Naučno-Melioracionnogo Instituta, 1930, No. 20, Leningrad (in Russian).

[20] Hydr. Engr., U. S. Geological Survey, Champaign, Ill.

ditions.[21] The proposed method yields results that are in very close agreement, not only with observed data, but also with computations by other generally accepted methods. This agreement is to be expected in view of the fact that all reliable methods have much in common. Eq. 1*c*, which may be considered the parent equation from which all the methods derive, might be written

$$dx = \frac{dy}{S_0 - S_f} - \frac{\dfrac{Q^2\, b_w}{g\, A^3}}{S_0 - S_f}\, dy \dots\dots\dots\dots\dots\dots (41)$$

in which the first term on the right represents the effect of the change in depth, whereas the second term on the right represents the effect of the change in kinetic energy. In many cases, this second term has little effect on the final results; in fact, the term may often be entirely neglected in practical engineering problems. However, the entire difference between most of the reliable methods of computing profiles lies in the manner of evaluating this second term. From Eqs. 3*a*, 3*b*, and 5*b*,

$$\frac{Q^2}{g}\frac{b_w}{A^3} = \frac{M^2_c}{M^2} = \left(\frac{y_c}{y}\right)^m \dots\dots\dots\dots\dots\dots (42a)$$

whereas, for example, by Mr. Bakhmeteff's Eqs. (45) and (47),

$$\frac{Q^2}{g}\frac{b_w}{A^3} = \frac{S_0\, K^2{}_0}{g}\frac{g}{\sigma\, K^2} = \beta\,\frac{K^2{}_0}{K^2} \dots\dots\dots\dots\dots (42b)$$

Thus, to the extent to which each assumption is applicable to a given channel, there should be consistency between the results.

By a few simple substitutions, it will become evident that the methods of Messrs. Von Seggern and Bakhmeteff are identical with respect to the evaluation of the first term of Eq. 41. A careful analysis of the Von Seggern development will disclose that

$$Z = \int \frac{dy}{S_0 - S_f} = -\int \frac{du}{u^2\,(1 - u^n)} \dots\dots\dots\dots\dots (43)$$

Bearing in mind that the Von Seggern u is the reciprocal of the Bakhmeteff η, or

$$u = y_0/y = \eta^{-1} \dots\dots\dots\dots\dots\dots\dots (44a)$$

and

$$du = -\,\eta^{-2}\, d\eta \dots\dots\dots\dots\dots\dots\dots (44b)$$

—so that

$$-\frac{du}{u^2\,(1 - u^n)} = \frac{d\eta}{\dfrac{\eta^2}{\eta^2}\left(1 - \dfrac{1}{\eta^n}\right)} = d\eta\,\frac{1}{1 - \left(\dfrac{1}{\eta}\right)^n} = d\eta + \frac{d\eta}{\eta^n - 1} \,.\,. (45)$$

Using the Bakhmeteff notation (Eq. 83),

$$B(\eta) = -\int_0^{\eta} \frac{d\eta}{\eta^n - 1} \dots\dots\dots\dots\dots\dots (46a)$$

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[21] "An Investigation of the Backwater Profile for Steady Flow in Prismatic Channels," by Wallace M. Lansford and William D. Mitchell, *Bulletin No. 381*, Univ. of Illinois Eng. Experiment Station, Urbana, Ill., March, 1949.

Eq. 43 becomes

$$Z = \int \frac{dy}{S_0 - S_f} = \eta - B(\eta) \ldots\ldots\ldots\ldots\ldots\ldots (46b)$$

which is identical to the form developed by Mr. Bakhmeteff, Eq. (90), for the case in which restoration of kinetic energy is neglected.

Now it will be obvious that Table 2, as well as Mr. Bakhmeteff's Table II, may be considered unnecessary, since the required information may be obtained from Mr. Bakhmeteff's Table IA. As an example, let $n = 3.6$, $y = 7.50$, and $y_0 = 3.00$. Then

$$u = y_0/y = 3.00/7.50 = 0.40 \ldots\ldots\ldots\ldots\ldots\ldots (47a)$$

and

$$\eta = y/y_0 = 7.50/3.00 = 2.50 \ldots\ldots\ldots\ldots\ldots\ldots (47b)$$

From Table 2, $Z = 2.464$; and, from Mr. Bakhmeteff's Table II, $Z = \eta - B(\eta) = \phi(\eta) = 2.464$. However, this result may be obtained from Mr. Bakhmeteff's Table IA, which yields $B(\eta) = 0.036$ so that $Z = \eta - B(\eta) = 2.500 - 0.036 = 2.464$ as before. Thus, Mr. Bakhmeteff's Table IA is sufficient to obtain the needed values.

The preceding statement is true, of course, only for the part of the computation that does not involve change in kinetic energy. This additional refinement, involving the evaluation of the second term of Eq. 41, is obtained differently by various investigators, and represents the differences referred to in the third paragraph of this discussion. The concept has been expressed as Eq. 12, which might also have been written as

$$\Pi = Z + \left(\frac{y_c}{y_0}\right)^m Y = \eta - B(\eta) + \left(\frac{y_c}{y_0}\right)^m Y \ldots\ldots\ldots (48a)$$

whereas Mr. Bakhmeteff's Eq. (85) might have been written in the corresponding form:

$$\Pi = \eta - (1 - \beta)\, B(\eta) = \eta - B(\eta) + \beta\, B(\eta) \ldots\ldots\ldots (48b)$$

As mentioned previously, the author's method has been found both useful and reliable for a rectangular channel. Doubtless the application may be extended to many other types of cross sections. However, the writer has been concerned with a flood plain type of cross section—that is, a cross section which, for small values of y, is comparatively narrow but which, as y increases, suddenly expands to a very great width. Several difficulties arise. Among the more serious is the fact that the rapid increase in width results in a decrease in the value of M, so that the values of m range far outside the values shown in Table 1. In fact, in extreme cases, these values of m approach negative infinity. In such cases, it appears that the usefulness of the method is confined to computing those profiles for which the normal depth is substantially greater than that required to submerge the flood plain.

Of two methods which produce equally satisfactory results, one or the other may be the more desirable from the practical viewpoint. If only the total length of reach between two widely different values of y is desired, a method that completely integrates the varied flow equation may result in a great saving of labor. On the other hand, the length of reach between two widely different values of y cannot be taken as an absolute criterion for the comparison of two methods of computing the entire profile. Two profiles which are quite dissimilar may, by chance, intersect at two or more points. For this reason, as well as others, it may be desired accurately to define the shape of the profile for the entire reach, and the reach may need to be subdivided. Under these circumstances computations by the Bakhmeteff method may be made highly satisfactory by varying β from one to another of the subdivisions. As the desired number of points is increased, the step method, with its simplicity, flexibility, and widespread acceptance, becomes more and more attractive; but, for profiles in a channel of simple geometric cross section, for which only a few points are required, the author's method should be welcomed by all hydraulicians.

FRANCIS F. ESCOFFIER.[22]—A method for calculating backwater, which is at once precise and simple to apply, is presented in this paper. The basic assumption of the Bakhmeteff solution, that the quantity $\beta = S_0/\sigma$ (in which S_0 is the normal slope and σ is the critical slope) remains constant, is satisfactory for channels with moderate slopes. However, in steep channels where the value of β is large, the variations of that quantity can be a significant factor in determining the form of the backwater profile. The author's method permits the evaluation of that factor.

One inference that can be drawn from the use of the method is that the conventional backwater profiles shown in most textbooks are in need of modification (except for the special case $n = m$) in that they fail to show the change in character of the profiles that occur at the depth y_r for which the normal discharge is equal to the critical discharge. This depth is given by the formula:

$$y^{n-m}_r = y^n_0 \, y^{-m}_c \qquad (49)$$

The profile shown in Fig. 12 was plotted with the help of Tables 1 and 2 for $n = 3.6$ and $m = 3.4$. The value of $(y_c/y_0)^m$ is 0.9364. It should be noted that the profile dips below the level of the pool into which the water is entering. The point of reversal in slope is in reality a point of transition from an M_1-like profile to a S_1-like profile.

In the following tabulation six such possible transition points are listed (the foregoing illustration falling in case 1):

Case 1. $n > m$ $y_r > y_0 > y_c$ M_1-like to S_1-like
Case 2. $n > m$ $y_r = y_0 = y_c$ M_3-like to S_1-like
Case 3. $n > m$ $y_r < y_0 < y_c$ M_3-like to S_3-like
Case 4. $n < m$ $y_r > y_c > y_0$ S_1-like to M_1-like
Case 5. $n < m$ $y_r = y_c = y_0$ S_3-like to M_1-like
Case 6. $n < m$ $y_r < y_c < y_0$ S_3-like to M_1-like

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[22] Hydr. Engr., Corps of Engrs., Mobile, Ala.

By substituting y_r into Eq. 6,

$$\frac{dy}{dx} = S_0 \frac{1 - \left(\dfrac{y_c}{y_0}\right)^{nm/(n-m)}}{1 - \left(\dfrac{y_c}{y_0}\right)^{nm/(n-m)}} \qquad (50)$$

from which it can be inferred that $dy/dx = S_0$, provided $y_0 \neq y_c$. Thus, in cases 1, 3, 4, and 6, the slope of the profiles at depth y_r is zero. However, in cases 2 and 5, $y_0 = y_c$, and Eq. 50 assumes the indeterminate form 0/0. To determine the slope of the profile, the numerator and the denominator are replaced on the right-hand side of Eq. 6 by their derivatives. The result is

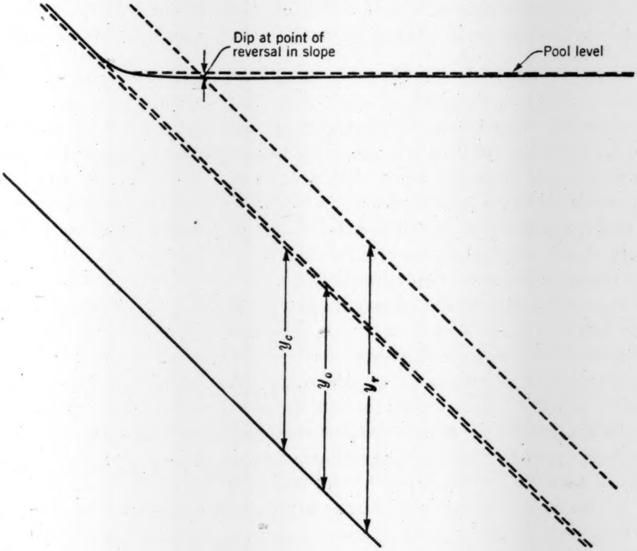

FIG. 12

$$\frac{dy}{dx} = S_0 \frac{n}{m} \qquad (51a)$$

and the slope of the profile is therefore

$$S = S_0 \frac{m - n}{m} \qquad (51b)$$

In cases 2 and 5 the slope of the profile is not zero as is usually shown on backwater profiles but has a finite value which is minus in case 2 and plus in case 5.

GUY C. THATCHER.[23]—The integration method discussed in this paper provides what appears to be an adequate method of analysis for use with uniform channels. However, it is the opinion of the writer that the application of the method must be limited to uniform channels, and that natural channels cannot be classed as uniform in all cases. Standard methods of analysis assume that channel characteristics change uniformly from one section to the next. Thus, Mr. Von Seggern's hydraulic exponents are themselves functions of X, which means that his differential equations do not have the variables separated, and are therefore not proper for use with nonuniform channels.

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[23] Engr., U. S. Bureau of Reclamation, Billings, Mont.

Time has not permitted the writer to investigate the possibility of completing the separation of the variables in such a case, but the fact that the differential equations contain exponential functions of the left-hand variable on the right-hand side indicates that the mathematics may be extremely involved

CHESLEY J. POSEY,[24] M. ASCE.—Together with the complete tables, this paper provides a convenient method for solving problems of nonuniform flow in channels of such cross-sectional shape that the actual variation of area and hydraulic radius with depth must be taken into account. The usual assumption is made that the friction loss varies as indicated by the Manning formula, with the rate of loss assumed equal to that in uniform flow at the same depth. Variations of friction loss and of velocity head with changing depth are quite accurately approximated in the integration, and laborious "step" computations are avoided entirely.

One stumbling block that has caused many engineers to loose interest in complete integration methods has been the necessity of taking "time out" to determine the "hydraulic exponents"—the exponents in the relationships used to approximate the variation of K^2 and M^2. Although these exponents are easily defined, their computation is often time consuming. Nagaho Mononobe's attempt at simplification[4] was unsuccessful, as it was based on the unreliable approximation of the slope of the straight line best representing a section of a curve by the mean of the slopes of the curve at each end of the section. The author's method of simplifying the determination of m and n, as shown in Figs. 1 and 2, is at once easy and theoretically sound, and has the advantage of immediately revealing how large the maximum error of the approximation may be.

The exponent m is useful not only in setting up the equation of nonuniform flow, but also in determining the critical depth. Assume, for example, that one is solving a problem of nonuniform flow in a trapezoidal channel with a bottom width of 5 ft and side slopes of 1 vertical to 1½ horizontal. The exponent m has been determined to be 3.65 for the range of depths to be expected. It is required to find the critical depth for a discharge of 200 cu ft per sec. By making use of the values of area and water surface width for a convenient value of the depth within the range, the value of the constant C_1 in the following equation is found to be 30.

$$\frac{A^3}{b_w} = C_1\, y^{3.65} \qquad (52)$$

For flow at the critical depth,

$$\frac{Q^2}{g} = \frac{A^3}{b_w} \qquad (53)$$

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[24] Prof. of Hydr. and Structural Eng. and Acting Head, Dept. of Civ. Eng., State Univ. of Iowa, Iowa City, Iowa.

Therefore, $\frac{\overline{200}^2}{32.2} = 30\,(y_c)^{3.65}$, from which the value of the critical depth is 2.77 ft.

Similarly, for the channel in Example 1, the constant may be evaluated by using the value $m = 3.4$ obtained by the author and values of the area and water surface for an intermediate depth of 4 ft:

$$\frac{A^3}{b_w} = \frac{\overline{124}^3}{37} = C\,y^{3.4} = 465 \times 4^{3.4} \ldots\ldots\ldots\ldots\ldots (54)$$

Hence, the critical depth when $Q = 800$ may be found by solving for y_c in the equation:

$$\frac{Q^2}{g} = \frac{\overline{800}^2}{32.2} = \frac{A^3}{b_w} = 465\,(y_c)^{3.4} \ldots\ldots\ldots\ldots\ldots (55)$$

This yields $y_c = 3.0$, which agrees closely with the author's value of 2.98.

In obtaining the values of the exponents m and n for Example 1, the author uses a range of depths from 2.98 to 8.00. It would seem that the actual range of depths of the backwater curve, from 6.73 to 8.00, would be more appropriate, although, of course, the difference in results would be inappreciable in this instance. The actual range of depths is apparently used in Example 3.

In evaluating the over-all merit of the author's method, a comparison with the step methods in which the significant variables are rationally accounted for is inevitable.[25] In a channel of uniform cross section the step method does not involve trial-and-error solutions. However, if accuracy is to be preserved, the steps must be short, so that the labor of computation in many problems is still greatly in excess of that required by the author's method. The step method, on the other hand, has the advantage that no tables are required. Thus, it, would seem that the step method will continue to be the favorite of the occasional user but that the author's method should prove valuable to those who have many backwater curves to compute.

A question of greater importance than the comparative convenience of various methods of computing backwater curves, all based on practically the same assumptions, is that of their accuracy in representing the physical phenomena. Following the example set by Mr. Mononobe (who made tests of most of the backwater curves in flumes of different cross sections) Hsu-Hua Hu made tests of M_1-curves, M_2-curves, H_2-curves, and S_1-curves in a rectangular flume,[26] and more recently Wallace M. Lansford and William D. Mitchell, Members, ASCE, have made extensive tests of M_1-curves in flumes of rectangular and "overbank flow" type cross sections.[21] Although the sum total of experimental evidence is inconclusive, it does seem evident that the "precise" methods of computing backwater curves do not always yield results that agree well with experiment. Messrs. Lansford and Mitchell apparently found it necessary to determine the actual rating curve for the cross section and to use this information in computing their theoretical curves. This was equivalent to using a variable value of n, experimentally determined over the range of depths, a procedure not ordinarily available to the backwater curve computer. Until more systematic experimental evidence is available, it would seem that a principal use of the "precise" backwater curve methods— the author's and the step method—would be in facilitating the study of experimental evidence.

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[25] Discussion by J. C. Stevens of "Back-Water and Drop-Down Curves," by Nagaho Mononobe, *Transactions*, ASCE, Vol. 103, 1938, p. 990.

[26] "An Experimental Study of Backwater Curves," by Hsu-Hua Hu, thesis presented to the Graduate College of the State University of Iowa at Iowa City, Iowa, in 1947.

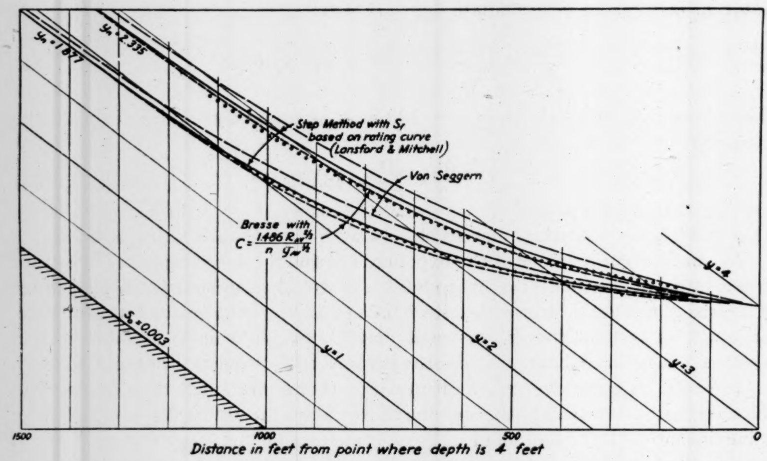

FIG. 13.—COMPARISON OF BACKWATER CURVES

For practical purposes, the simple equation of J. A. C. Bresse, for which tables are available, is sufficiently accurate. In using the Bresse equation the selection of constants is important. The value of the normal depth should be determined by the Manning formula, or better yet, by direct measurement. The value of C should be that which will make the approximation for the friction slope used in the Bresse formula most nearly correct according to the Manning formula, or

$$C = \frac{1.486}{n} \frac{R^{2/3}}{y^{1/2}} \quad \text{.....................} \quad (56)$$

Values of C should be so computed for the depths at each end of the backwater curve. Ordinarily such values of C will differ but little. An intermediate value is then selected for use in the Bresse formula.

Fig. 13 shows a comparision of curves computed by various methods with data selected from the experiments[21] by Messrs. Lansford and Mitchell. The upper set of curves approximates data shown as plotted points for a discharge of 39.98 cu ft per sec in a rectangular channel 5 ft wide, having a bottom slope of 0.003. The sides and bottom of the channel were covered with chain link, 2-in. diamond mesh, No. 9 wire. The normal depth was given as 2.335 ft, corresponding to a Manning's n of 0.027. The solid curve is that obtained by Messrs. Lansford and Mitchell using the step method with friction evaluated by the experimental rating curve with the assumption that the friction slope at a given depth varies with the square of the velocity. The dashed line below the solid curve shows the results obtained by using Bresse's equation with a value of $C = 39$, intermediate between 36.6 and 40.8. The dashed curve above the solid curve shows the results obtained by the author's method, with $m = 3.0$ and the hydraulic exponent n determined from Fig. 2 as 2.4. The agreement might be improved if the value of n were to be based on the experimentally determined rating curve.

As shown in the lower set of curves, similar results are obtained for a discharge of 29.91 with normal depth of 1.877 in the same channel at the same slope.

The author is to be commended for his scholarly paper. His succinct conclusions deserve thoughtful reading.

M. E. VON SEGGERN,[27] ASSOC. M. ASCE.—Some important information regarding nonuniform flow in general has been brought out in the discussion of this paper, and the pros and cons of the writer's integration method have been argued. In this closure the various differences will be elaborated upon.

Mr. Posey raises an important point when he questions the underlying theory behind all present methods of computing backwater curves. The mere fact that, for many years, engineers and hydraulicians have accepted certain assumptions in regard to nonuniform flow is by no means proof that these assumptions are correct. More research on this subject is in order.

Mr. Kolupaila's suggestion of searching for a possible simple relation between the two hydraulic exponents has considerable merit. When Manning's formula is used in the computation of n, it does seem possible that, because both n and m can be reduced to functions of the cross section, a relation between these two exponents could be found. However, the writer has been unable to discover such a relation in a simple form.

Although the reader may have received the impression that Manning's formula is essential to the writer's method, this is not true. Kutter's formula, the Bazin formula, or some other open channel formula may be used; but, if any of these formulas is employed, then Fig. 2 is no longer pertinent, even though the principle behind it still applies.

Mr. Kirpich has devised a convenient chart (Fig. 10) for determining the value of the hydraulic exponent, n, for any given depth, y. The integration of the equation of nonuniform flow as performed by Boris A. Bakhmeteff, Hon. M. ASCE, and by the writer is based on the assumption that log K plotted against log y is a straight line and can be expressed mathematically by the equation:

$$\left(\frac{K_1}{K_2}\right)^2 = \left(\frac{y_1}{y_2}\right)^n \quad \text{.......................(57)}$$

which is an approximation only.

In a practical application it is necessary to obtain the value of the slope, $n/2$, of the straight line which most nearly approximates the curved line of the actual condition. Using Fig. 2, this average is obtained directly; and the degree of the approximation is apparent, as has been aptly stated by Mr. Posey.

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Using Fig. 10, the value of n for the extremes of depth would be averaged arithmetically. The choice of methods is a matter of individual preference.

Mr. Posey has questioned the reason for using the range of depths from 2.98 to 8.00 in Example 1 for determining the values of m and n. This question brings up a point that may have been overlooked by most readers. The writer's equation of nonuniform flow as expressed in Eq. 6 assumes a straight line plot of log y against log M over the range of depths from y_c to y. It is necessary, therefore, to use this range in computing the value of m. For the value of n it may be sufficient to use the shorter range from y_0 to y, but in all the examples the writer used the same range of depths for determining n as for determining m.

The writer is in complete agreement with Mr. Mitchell's discussion. It was, in fact, while attempting to simplify and shorten the work involved in computing the kinetic energy factor of Mr. Bakhmeteff's equation that the writer developed his method. Mr. Mitchell has not emphasized the one important difference between the Bakhmeteff method and the writer's method—the fact that the writer's method integrates both members of the equation whereas the Bakhmeteff method integrates only that part of the equation representing the effect of change in depth.

Mr. Mitchell suggests (and the writer agrees) that the effect of change in kinetic energy can often be neglected in practical engineering problems. According to Mr. Bakhmeteff this term can be safely disregarded only in the M_1-type of profile.[28] With the writer's method, however, there is no need to neglect it in any case, except, perhaps, for natural streams, as the entire computation is relatively short and simple.

Mr. Posey suggests that the equation of J. A. C. Bresse is sufficiently accurate for computing ordinary backwater profiles. The Bresse equation is based on a special case in which the hydraulic radius equals the depth of the water and would apply only to an infinitely wide rectangular channel, although it would be reasonably close for rectangular channels where the width is great compared to depth. However, if the hydraulic radius is assumed equal to the depth, then in the writer's method it can be shown that $n = 3.33$ (using Manning's formula) and $m = 3.0$. If a set of tables is abstracted from the writer's original set for these values of n and m, the necessity of computing the hydraulic exponents for each problem is eliminated, and the result is a method that should give exactly the same answers as the Bresse method—and one that may be even simpler to apply. Thus, some of the disadvantages of the Bresse equation are eliminated, and some of the advantages are retained.

Messrs. Hodges and Thatcher expressed doubt as to the applicability of this method to natural streams. The writer does not claim that the method applies to all cases, but it does to a large proportion of them. In natural streams the M_1-type profile is probably the most common. In this profile the effect of the change in kinetic energy is small and may be neglected. Thus, in Eq. 12 the Y-term can be eliminated so that $\Pi = Z$. Then Eq. 13 becomes

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[28] "Hydraulics of Open Channels," by Boris A. Bakhmeteff, McGraw-Hill Book Co., Inc., New York, N. Y., 1932, p. 98.

$$L = x_2 - x_1 = \frac{y_0}{S_0}(Z_2 - Z_1) \qquad (58)$$

In this form (Eq. 58) the critical depth does not need to be computed or the exponent, m, determined. Only the tables for the Z-function need to be consulted.

As has been stated by Messrs. Hodges and Thatcher, a natural channel has innumerable variations in both cross section and slope. The roughness coefficient also varies considerably from place to place. There are, in fact, so many variations that, except in rare instances, a "precise" computation is out of the question regardless of the method of computation employed. A rating curve at any point along the stream sums up and includes all these variables in the vicinity of that point; but even the rating curve cannot be considered 100% accurate.

If it is acknowledged that a "precise" computation is impossible, then the problem becomes one of deciding how much accuracy is required, and how much trouble and expense are warranted in obtaining data and making computations. For an integration method it will be necessary to know the average river bed slope, the normal depth, y_0, for any given discharge, and the value of the hydraulic exponent, n. If a rating curve or curves are available, log Q can be plotted against log y to obtain an approximately straight line the slope of which is $n/2$. Mr. Bakhmeteff shows one method of making this plot,[29] but the writer prefers the method shown by Horace W. King,[30] M. ASCE. The King logarithmic plot will give n and y_0. If the water spreads thinly over the flood plain, other complications arise, as discussed by Mr. Mitchell.

For backwater problems where data are meager and a preliminary solution is required for the early stages of planning studies, the integration method as presented herein or as given by Mr. Bakhmeteff (the two being nearly identical for natural streams) is particularly well adapted. A reasonably accurate profile can be obtained with a minimum of field data and a minimum of computation.

Even where more detailed studies of the backwater profile are required, the integration method will give satisfactory results, considering the difficulties in estimating realistic values of the roughness coefficient and in selecting the proper location for cross sections. Aggradation of the stream bed by silt deposition upstream from the control structure which will produce the backwater profile may seriously affect that profile, and cannot be predicted in advance with any accuracy. So-called "precise" profile computations on a natural stream appear to the writer to be illusions in most cases; and, except where absolutely necessary, time and money spent on obtaining great volumes of detailed data and in making elaborate computations are unwarranted.

The step method of computation is not outmoded by any of the integration methods. As suggested by some of the discussers, many engineers who compute backwater profiles only occasionally will prefer one of the many step methods that have been published. In some cases it is not practical to apply an integration method.

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[29] "Hydraulics of Open Channels," by Boris A. Bakhmeteff, McGraw-Hill Book Co., Inc., New York, N. Y., 1932, p. 221.

[30] "Handbook of Hydraulics," by Horace W. King, McGraw-Hill Book Co., Inc., New York, N. Y., 1939.

For backwater profiles in uniform, prismatic channels, however, an integration method is simple, quick, and accurate. Some of the step methods published in textbooks and technical papers are no less complicated than is the integration method presented by the writer. Mr. Davison's method is a case in point. He begins with a form of the equation of nonuniform flow (Eq. 34*a*), which is the mathematical equivalent of Eqs. 1. By algebraic substitutions and computations he develops Eq. 40. It is doubtful whether most engineers who only occasionally compute backwater curves would find Eq. 40 any less formidable than Eqs. 12 and 13. The use of the writer's method does not necessarily require a complete understanding of the derivation of all the equations.

Where a complete water surface profile is not desired, but only one or two points are needed, the integration method is especially advantageous.

Mr. Hodges raises the question of applying this method to horizontal bottom grades. A few simple modifications in the equations are necessary for this extension, and the entire process is related to critical flow because y_0 approaches infinity as the bottom grade approaches zero. For a horizontal bottom grade, $S_0 = 0$, and Eq. 2*b* becomes: $S_s = -\frac{dy}{dx}$. For Eq. 7*a*, substitute

$$v = \frac{y_c}{y} \qquad \text{(59)}$$

and let $Q^2 = K^2_c\,\sigma$, in which σ is the critical slope or the slope necessary to make a given Q flow at critical depth.

After substituting in the equation of nonuniform flow and integrating,

$$\Pi = \frac{v^{-n-1}}{-n-1} - \frac{v^{m-n-1}}{m-n-1} \qquad \text{(60a)}$$

and

$$L = x_2 - x_1 = \frac{y_c}{\sigma}(\Pi_2 - \Pi_1) \qquad \text{(60b)}$$

Eqs. 60 correspond to Eqs. 12 and 13 for sustaining slopes. No tables are necessary. The application to a practical problem is the same as for sustaining slopes. Mr. Kirpich's charts for computing critical slope and critical depth are especially useful in these computations.

Mr. Escoffier offers an interesting development which results from a manipulation of the formulas presented in the paper. The deviation of the profile from a horizontal line in the instances he cites is small. The exponents, n and m, represent the slopes of lines which, in reality, are only approximately straight. For this reason, and because of Mr. Posey's pertinent query as to the soundness of the theory underlying the equation of nonuniform flow, the writer hesitates to comment on Mr. Escoffier's conclusion.

The writer is pleased at the response to this paper and wishes to thank all the discussers for their interest. It appears that the discussion provoked has been worthwhile.

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NOMOGRAPHIC ANALYSIS OF RECTANGULAR SECTIONS OF REINFORCED CONCRETE

BY W. P. LINTON,[1] M. ASCE

WITH DISCUSSION BY MESSRS. MAURICE BARRON, M. HIRSCHTHAL, WINFIELD A. MCCRACKEN, KING BURGHARDT, ROBERT V. HAUER, AND W. P. LINTON.

SYNOPSIS

Several methods are in current use for the design of reinforcement in rectangular sections of reinforced concrete subjected to bending and direct stress. However, most of them require the use of numerous diagrams or tables; involve long and laborious computations; and are useful mainly in analyzing designs previously made, rather than in making an original design. In designing it is customary to assume the amount of reinforcement and then to check the assumption. If the assumed reinforcement is inadequate or uneconomical, a new assumption must be made and the process repeated until a satisfactory design is obtained. This process is laborious and time consuming. Many diagrams or tables are usually required, because, on account of the number of variables, d'/d is made constant and one diagram is drawn for each value of d'/d. A few values of d'/d are usually selected but in practice it is frequently found that the actual value of d'/d does not conform with any one of the diagrams. Then the diagram made for the nearest value of d'/d must be used, at the sacrifice of considerable accuracy, or interpolation must be made between two diagrams and that process is inconvenient. The value of n also is assumed to be constant; therefore, if a different value of n is used, another set of diagrams is required so that, for a complete and satisfactory solution with reasonable accuracy, many diagrams are required.

Also an infinite number of combinations of p and p' will satisfy the equations, but usually no provision is made for determining the most economical combination.

The method presented herein obviates these objections, and is believed to be more complete and satisfactory and more easily applied then any other known

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NOTE.—Published in January, 1949, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

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to the writer. Only three nomograms which contain six variables need be used. The nomograms were made primarily for bending and direct stress, but they are equally useful for simple bending because simple bending is merely a special case of the more general one.

The nomograms and examples of their use are explained to make the method of procedure clear. Then the derivation of the formulas and the method of construction of the nomograms are given.

The accepted theory of reinforced concrete is adhered to throughout without modification and the symbols used are standard and well known.

EXPLANATION OF THE NOMOGRAMS AND EXAMPLES OF THEIR USE

Six variables are involved in the three nomograms, Figs. 1, 2, and 3—namely, $p' n$, $p n$, d'/d, k, **B**, and **C** and their significance is made clear in Fig. 4, with the formulas that follow:

$$M = N e \qquad \text{(1a)}$$

$$\mathsf{B} = \frac{N e'}{f_c b d^2} \qquad \text{(1b)}$$

and

$$\mathsf{C} = \frac{N}{f_c b d}\left[\frac{e'}{d} - \left(1 - \frac{d'}{d}\right)\right] \qquad \text{(1c)}$$

For simple bending

$$\mathsf{B} = \mathsf{C} = \frac{M}{f_c b d^2} \qquad \text{(2)}$$

Fig. 1 is a nomogram that contains the variables **B**, **C**, d'/d, and k and gives the value of k which will make the sum of $(p n + p' n)$ a minimum when **B**, **C**, and d'/d are known. Fig. 2, similarly, is a nomogram that contains the variables **B**, $p' n$, d'/d, and k and gives the value of any one of the four when the other three are known. Fig. 3, likewise, is a nomogram that contains the variables **C**, $p n$, d'/d, and k and gives the value of any one of the four when the other three are known. The equations on which Figs. 2 and 3 are based are given in Appendix I and a brief review of the construction of nomograms is presented in Appendix II.

In designing reinforced concrete sections, the allowable unit stresses f_c and f_s, the ratio of the modulus of elasticity of steel to that of concrete n, and the embedment of the steel d' are given in the specifications or are known. The dimensions b and d are fixed by other considerations; and, therefore, the problem in design is to determine the most economical or satisfactory ratios of tensile and compressive reinforcement to fulfil the given requirements.

In checking a design of the reinforcement which has previously been made, $p' n$, $p n$, d'/d and the dimensions b and d are known and the problem is to find f_c and f_s. Since f_c is unknown, the variables **B** and **C** are also unknown.

Because the procedures in the two cases (that is, designing and checking) are different, examples of both are given which will make both procedures clear.

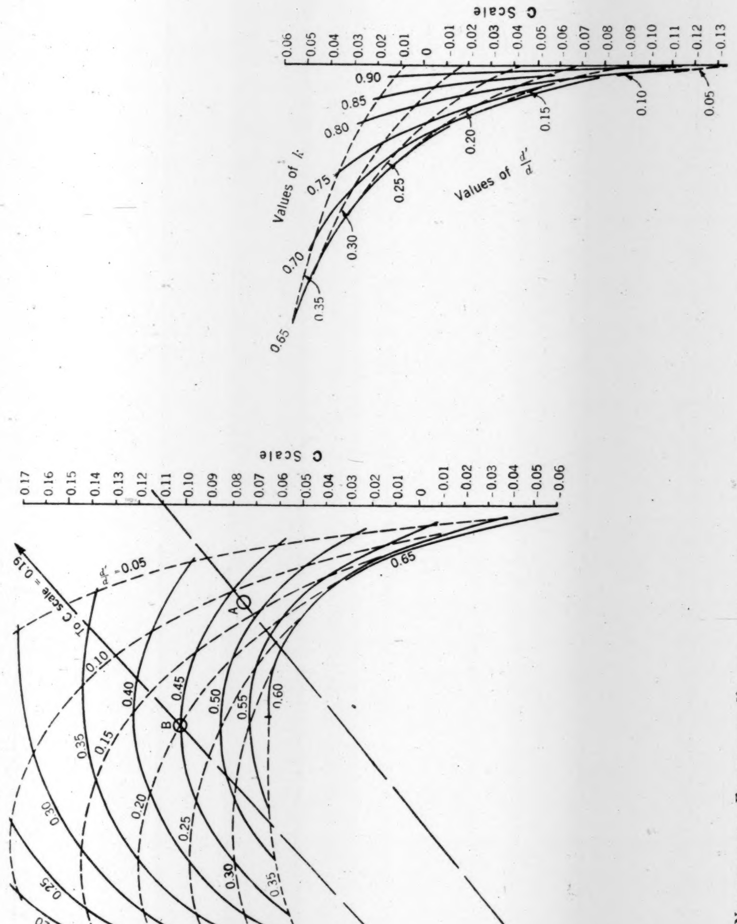

Fig. 1.—Nomogram for Finding the Value of k That Will Produce Minimum Values of $p n + p' n$ for Given Values of B, C, and d'/d

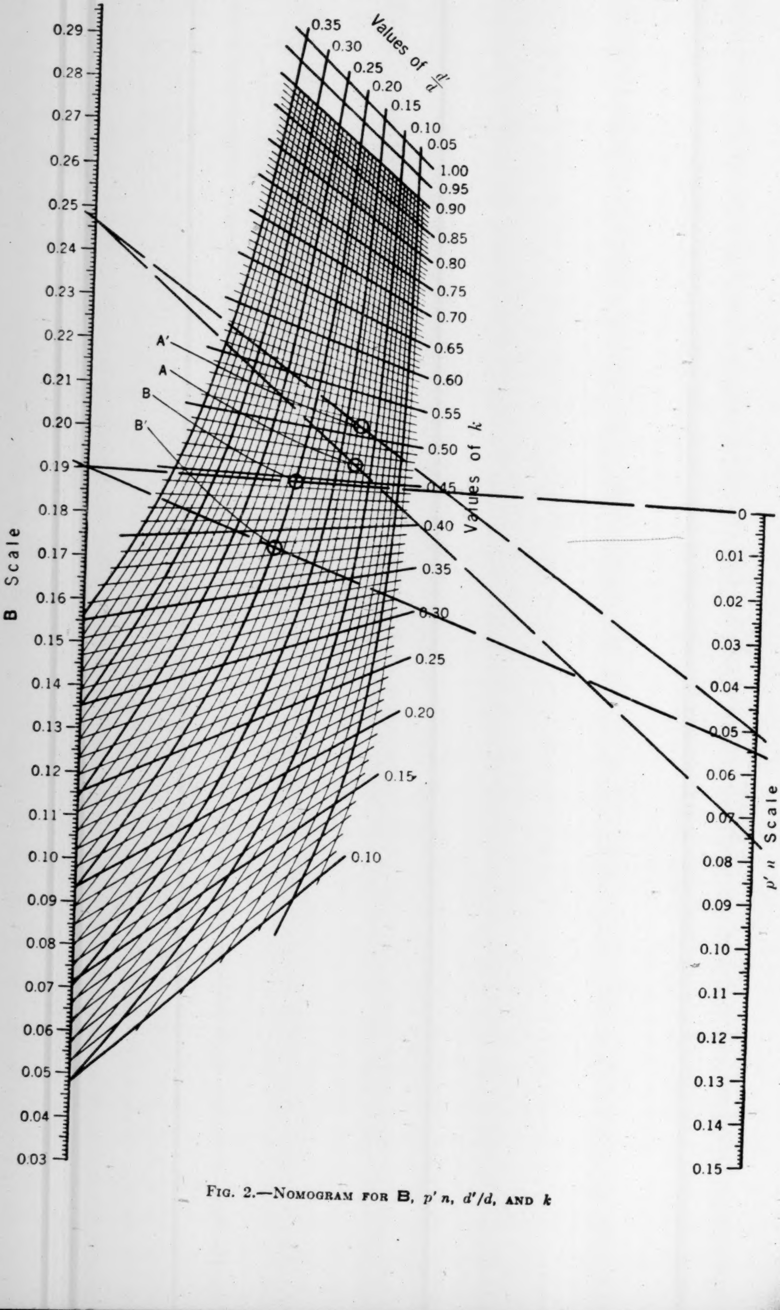

FIG. 2.—NOMOGRAM FOR B, $p'n$, d'/d, AND k

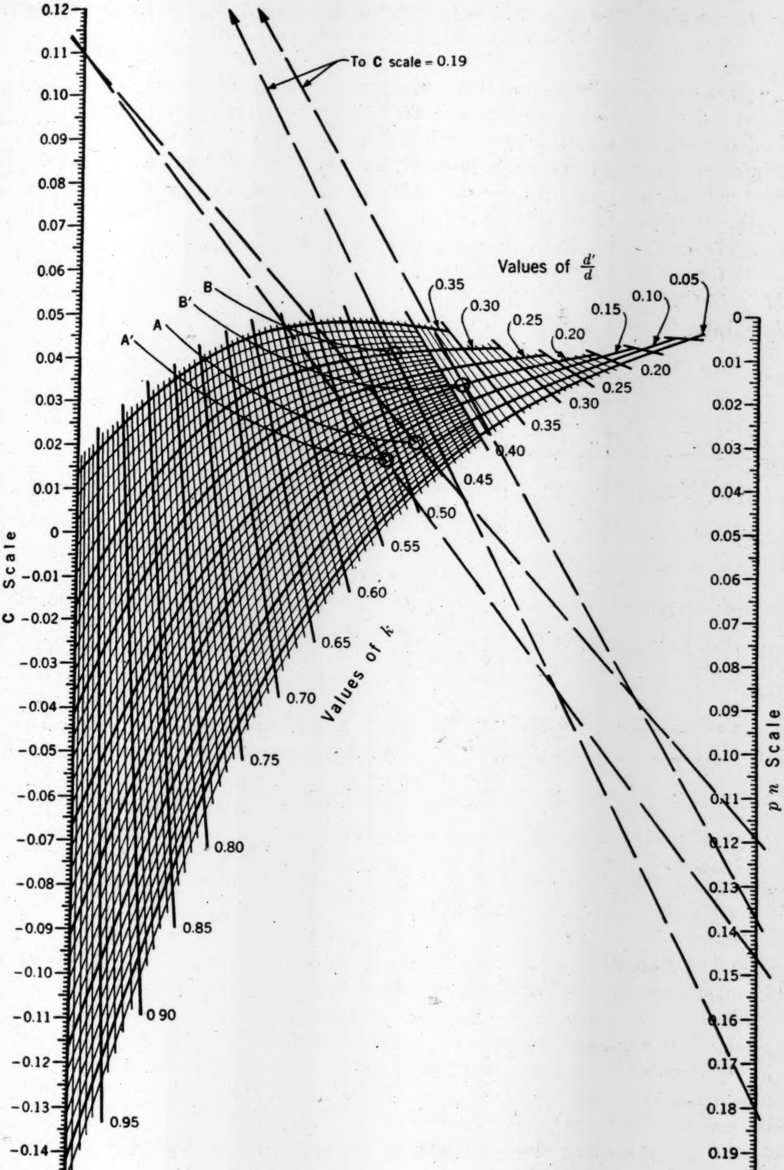

FIG. 3.—NOMOGRAM FOR C, $p\,n$, d'/d, AND k

In designing, the procedure is to find the values of **B**, **C**, and d'/d from the given data and then to determine the values of k, $p'\,n$, and $p\,n$ from the nomograms.

Example 1. Design of the Reinforcement for a Case of Bending and Direct Stress.—It is required to find the longitudinal reinforcement for an eccentrically loaded column or a section of an arch ring, 12 in. wide and 18 in. deep with the reinforcement 2 in. from the surface of the concrete. The column (or the arch ring) is to carry a load or a thrust of 30,000 lb and a bending moment of 550,000 in.-lb. The maximum allowable unit stresses are 1,000 lb per sq in. in the concrete and 20,000 lb per sq in. in the steel, and n is equal to 10.

Referring to Fig. 4, let $b = 12$ in.; $d = 16$ in.; $d' = 2$ in.; $d'/d = 0.125$; $M = 550{,}000$ in.-lb; $N = 30{,}000$ lb; $e = M/N = 18.33$ in.; $e' = 25.33$ in.; $f_c = 1{,}000$ lb per sq in.; $n = 10$; $e'/d = 1.583$; **B** (by Eq. 1b) = 0.247; and **C** (by Eq. 1c) = 0.111.

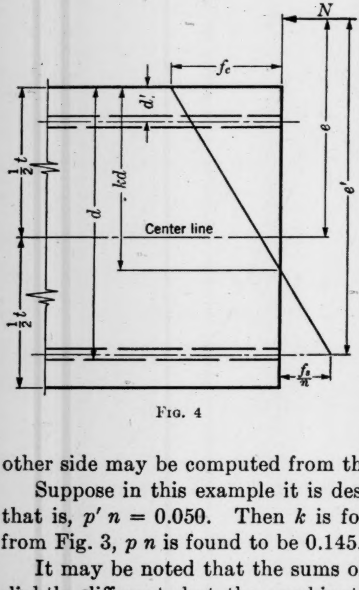

FIG. 4

In Fig. 1 lay a straight edge passing through the points **B** = 0.247 and **C** = 0.111 on the **B**-scale and the **C**-scale, respectively. Where the straight edge cuts the curve for $d'/d = 0.125$ (interpolated), the value of k is found to be 0.470 (point A, Fig. 1). With this value of k, $p'\,n$ is found, from Fig. 2, to be 0.075; and, from Fig. 3, $p\,n$ is found to be 0.118—which are the required values of $p'\,n$ and $p\,n$.

On the other hand, it may be that it is desirable to run the bars through from one section to another to make the number of splices as small as practicable. In that case the given percentage of reinforcement may be used on one side and the percentage required for the other side may be computed from the nomograms.

Suppose in this example it is desired to use 0.5% of compressive steel—that is, $p'\,n = 0.050$. Then k is found to be 0.512 (point A′, Fig. 2); and, from Fig. 3, $p\,n$ is found to be 0.145.

It may be noted that the sums of ($p'\,n$ and $p\,n$) in the two cases are only slightly different, but the combination obtained by using Fig. 1 is smaller. In most cases the difference will be greater than in this example.

Example 2. Simple Bending Design with Both Tensile and Compressive Steel.—It is required to design the reinforcement for a beam 10 in. wide and 18 in. deep with the steel 3 in. from the faces of the concrete and subjected to a bending moment of 363,000 in.-lb. The allowable unit stresses are $f_c = 850$ lb per sq in. and $f_s = 16{,}000$ lb per sq in.

In simple bending the same nomograms are used except that **B** = **C** as in Eq. 2. The given data in this example are $b = 10$ in.; $d' = 3$ in.; $d = 15$ in.;

$d'/d = 0.20$; and $f_c = 850$ lb per sq in.; and, by Eq. 2, $\mathbf{B} = \mathbf{C} = \dfrac{363{,}000}{850 \times 10 \times 15^2}$ $= 0.190$.

The value of k which will give the minimum value of $(p\,n + p'\,n)$ is found to be 0.447 (point B, Fig. 1). From Fig. 2, with $d'/d = 0.20$ and $k = 0.447$, $p'\,n$ is found to be zero; and, from Fig. 3, $p\,n$ is found to be 0.180, or $p = 0.018$.

Of course, as is known, $p'\,n$ should be zero in simple bending when that value of k is used which gives the minimum value of $(p\,n + p'\,n)$.

However, 0.018 is a large percentage of tensile steel and may be considered undesirable. If so it may be reduced by using a small amount of compressive steel. In practice some of this steel is usually placed near the top of a beam for temperature or for other purposes and is frequently neglected in computing the tensile reinforcement because its effect on the tensile steel is thought to be negligible. With the nomograms in this paper it is as easy to give the compression steel its proper consideration as it is to neglect it; therefore, let it be assumed that there is 0.55% of compressive steel, or $p'\,n = 0.055$. Then k is found to be 0.385 (point B′, Fig. 2); and, from Fig. 3, $p\,n$ is found to be 0.137, or $p = 0.0137$. As should be expected, this total percentage of steel (0.0137 + 0.0055) = 0.0192 is greater than 0.0180, the percentage of tensile steel alone.

EXAMPLES OF ANALYSIS

In the analysis of designs previously made, Figs. 2 and 3 are used but the procedure is different. In Fig. 2, $p'\,n$ and d'/d are known but k is unknown and $\mathbf{B}$ is unknown because f_c is unknown. In Fig. 3, $p\,n$ and d'/d are known but k is unknown and $\mathbf{C}$ is unknown because f_c is unknown.

However, the quantity,

$$\frac{\mathbf{C}}{\mathbf{B}} = \frac{\dfrac{e'}{d} - \left(1 - \dfrac{d'}{d}\right)}{\dfrac{e'}{d}} \qquad \text{(3)}$$

can readily be found from the given data and is used in Figs. 2 and 3 to determine k.

Example 3. Checking the Stresses in a Section Subjected to Bending and Direct Stress.—It is required to check the stresses in the steel and the concrete in a member 12 in. wide and 24 in. deep subjected to a bending moment of 600,000 in.-lb and a direct thrust of 75,000 lb. The reinforcement consists of four $\frac{7}{8}$-in.-round bars in tension and two $\frac{7}{8}$-in.-round bars in compression. The embedment of the steel is 3 in. and n is equal to 10. From the given data, $b = 12$ in.; $d = 21$ in.; $d' = 3$ in.; $d'/d = 0.143$; $p'\,n = 0.0475$; $p\,n = 0.095$; $e = M/N = 8$ in.; $e' = 17$ in.; $e'/d = 0.81$; and $\mathbf{C}/\mathbf{B}$ (Eq. 3) $= -0.058$.

The procedure is to assume a tentative value of k and to find tentative values of $\mathbf{C}$ and $\mathbf{B}$ and of $\mathbf{C}/\mathbf{B}$ from Figs. 3 and 2. The tentative value of $\mathbf{C}/\mathbf{B}$ found in the first trial will probably be far from the correct value (-0.058 in this example), but the correct value can soon be reached by a few trials which can be made to approach the correct value.

The ratio $\mathbf{C}/\mathbf{B}$ is always less than unity and it is observed from nomograms $\mathbf{B}$ and $\mathbf{C}$ that, when $\mathbf{C}/\mathbf{B}$ is a small percentage of unity or is negative, k will be rather large. In this example, $\mathbf{C}/\mathbf{B}$ is negative; therefore, with $d'/d = 0.143$, $p\,n = 0.095$, and $p'\,n = 0.0475$, and knowing that $\mathbf{C}/\mathbf{B}$ should be -0.058, try $k = 0.80$. From Figs. 3 and 2, $k = 0.80$; $\mathbf{C} = -0.027$; $\mathbf{B} = 0.328$; and $\mathbf{C}/\mathbf{B} = -0.082$. The value of $\mathbf{C}/\mathbf{B}$ is too low; therefore, k should be smaller. Try $k = 0.75$ and find $k = 0.75$, $\mathbf{C} = -0.010$, $\mathbf{B} = 0.316$, and $\mathbf{C}/\mathbf{B} = -0.032$. This value of $\mathbf{C}/\mathbf{B}$ is too high; therefore, the correct value lies between the two. Try $k = 0.77$ and find:

Variable	$k = 0.770$	$k = 0.775$
$\mathbf{C}$	-0.017	-0.019
$\mathbf{B}$	0.320	0.321
$\mathbf{C}/\mathbf{B}$	-0.053	-0.059

These results are as close as the accuracy of the data justifies; therefore, with $k = 0.775$ and $\mathbf{B} = 0.321$, f_c and f_s are found from the formulas—

$$f_c = \frac{N\,e'}{\mathbf{B}\,b\,d^2} \qquad (4a)$$

and

$$f_s = n\,f_c\,\frac{1-k}{k} \qquad (4b)$$

to be, respectively, $f_c = \dfrac{75{,}000 \times 17}{0.321 \times 12 \times 21^2} = 750$ and $f_s = 10 \times 750\,\dfrac{0.23}{0.77} = 2{,}240$.

Example 4. Checking the Stresses in a Member with Both Tensile and Compressive Reinforcement Subjected to Simple Bending.—Given a member 12 in. wide and 15 in. deep subjected to a bending moment of 300,000 in.-lb, the reinforcement consists of two 1-in.-square bars in tension and one 1-in.-square bar in compression; the embedment of the steel is 3 in.; find f_c and f_s.

The given data are $b = 12$ in.; $d = 12$ in.; $d' = 3$ in.; $d'/d = 0.25$; $p = 0.0139$; and $p' = 0.007$. In simple bending (Eq. 2), $\mathbf{B} = \mathbf{C} = \dfrac{M}{f_c\,b\,d^2}$; $\mathbf{C}/\mathbf{B} = 1$; and

$$f_c = \frac{M}{\mathbf{B}\,b\,d^2} = \frac{173.6}{\mathbf{B}} \qquad (5)$$

As in Example 3, this problem must be solved by assuming tentative values of k until the value is found which makes $\mathbf{B} = \mathbf{C}$ for the given values of $p'\,n$, $p\,n$, and d'/d. The variable $\mathbf{B}$ is always positive; therefore, when $\mathbf{C} = \mathbf{B}$, $\mathbf{C}$ is also positive and, when $\mathbf{B}$ and $\mathbf{C}$ are equal, the value of k is rather low. Try $k = 0.3$; from Fig. 3 it is noted that $\mathbf{C}$ would be unreasonably large. Therefore, try $k = 0.40$; from Fig. 3 find $\mathbf{C} = 0.182$ ($\mathbf{C}$-scale produced); and from Fig. 2 find $\mathbf{B} = 0.194$. Then $\mathbf{C}/\mathbf{B} = 0.94$.

Try $k = 0.38$ and find $\mathbf{C} = 0.194$; $\mathbf{B} = 0.184$; and $\mathbf{C}/\mathbf{B} = 1.05$. It is apparent that k must be about halfway between 0.38 and 0.40 or 0.39; thus:

$k = 0.39$; $\mathsf{C} = 0.189$; $\mathsf{B} = 0.189$; $\mathsf{C/B} = 1.00$; f_c (Eq. 5) $= \dfrac{173.6}{0.189} = 918$; and, by Eq. 4*b*, $f_s = 9{,}180\,\dfrac{0.61}{0.39} = 14{,}350$.

It is usually not necessary to compute f'_s because its value is always less than $n\,f_c$.

From these examples it is apparent that these nomograms can be used for any problem in either bending and direct stress or in simple bending in rectangular sections of reinforced concrete, and they are believed to have considerable merit in comparison with other methods in current use.

APPENDIX I. DERIVATION OF BASIC FORMULAS

In the derivation of the two fundamental formulas on which Figs. 2 and 3 depend, the accepted theory of reinforced concrete is adhered to without modification and the nomenclature is standard and well known.[2] From the condition for static equilibrium, $\Sigma M = 0$, taking moments about the tensile steel:

$$N\,e' = \tfrac{1}{2} f_c\,k\,b\,d^2 \left(1 - \tfrac{1}{3} k\right) + f_c\,p'\,n\,b\,d^2\,\frac{k - \dfrac{d'}{d}}{k}\left(1 - \frac{d'}{d}\right) \quad\ldots\ldots(6)$$

which may be written

$$\tfrac{1}{2} k \left(1 - \tfrac{1}{3} k\right) + p'\,n\,\frac{k - \dfrac{d'}{d}}{k}\left(1 - \frac{d'}{d}\right) = \frac{N\,e'}{f_c\,b\,d^2} \quad\ldots\ldots\ldots\ldots(7)$$

From the condition that $\Sigma H = 0$,

$$N = \tfrac{1}{2} f_c\,k\,b\,d + p'\,n\,f_c\,b\,d\,\frac{k - \dfrac{d'}{d}}{k} - p\,n\,f_s\,b\,d\,\frac{1 - k}{k} \quad\ldots\ldots\ldots(8)$$

which may be written

$$\tfrac{1}{2} k + p'\,n\,\frac{k - \dfrac{d'}{d}}{k} - p\,n\,\frac{1 - k}{k} = \frac{N}{f_c\,b\,d} \quad\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

Multiply Eq. 9 by $1 - \dfrac{d'}{d}$,

$$\tfrac{1}{2} k \left(1 - \frac{d'}{d}\right) + p'\,n\,\frac{k - \dfrac{d'}{d}}{k}\left(1 - \frac{d'}{d}\right) - p\,n\,\frac{1 - k}{k}\left(1 - \frac{d'}{d}\right) = \frac{N}{f_c\,b\,d}\left(1 - \frac{d'}{d}\right) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

Subtract Eq. 10 from Eq. 7,

$$-\tfrac{1}{6} k^2 + \tfrac{1}{2} k\,\frac{d'}{d} + p\,n\,\frac{1 - k}{k}\left(1 - \frac{d'}{d}\right) = \frac{N\,e'}{f_c\,b\,d^2} - \frac{N}{f_c\,b\,d}\left(1 - \frac{d'}{d}\right) \quad\ldots(11)$$

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[2] "Bending and Direct Stress in Reinforced Concrete," by W. P. Linton and Thomas P. Revelise. *Public Roads*, October–November–December, 1943, p. 268, Eqs. 7 and 8.

From Eq. 7,

$$p' = \frac{\dfrac{N\,e'}{f_c\,b\,d^2} - \tfrac{1}{2}\,k\,(1 - \tfrac{1}{3}\,k)}{n\,\dfrac{k - \dfrac{d'}{d}}{k}\left(1 - \dfrac{d'}{d}\right)} \qquad (12)$$

From Eq. 11,

$$p = \frac{\dfrac{N}{f_c\,b\,d}\left[\dfrac{e'}{d} - \left(1 - \dfrac{d'}{d}\right)\right] + \tfrac{1}{2}\,k\left(\tfrac{1}{3}\,k - \dfrac{d'}{d}\right)}{n\,\dfrac{1 - k}{k}\left(1 - \dfrac{d'}{d}\right)} \qquad (13)$$

From Eq. 12, substituting **B** from Eq. 1*b*,

$$p' = \frac{k}{n\left(k - \dfrac{d'}{d}\right)\left(1 - \dfrac{d'}{d}\right)}\left[\mathbf{B} - \tfrac{1}{2}\,k\,(1 - \tfrac{1}{3}\,k)\right] \qquad (14a)$$

and, from Eq. 13, substituting **C** from Eq. 1*c*,

$$p = \frac{k}{n\,(1 - k)\left(1 - \dfrac{d'}{d}\right)}\left[\mathbf{C} + \tfrac{1}{2}\,k\left(\tfrac{1}{3}\,k - \dfrac{d'}{d}\right)\right] \qquad (14b)$$

APPENDIX II. CONSTRUCTION OF NOMOGRAMS

A nomogram can be constructed for any formula containing four variables u, v, r, and s which can be written in the form:[3]

$$U\,F_1(r,s) + V\,F_2(r,s) = 1 \qquad (15)$$

in which U is a function of u alone and V is a function of v alone and $F_1(r,s)$ and $F_2(r,s)$ are two functions of r and s.

Eqs. 14 may be written as follows:

$$\mathbf{B}\,\frac{k}{\tfrac{1}{2}\,k^2\,(1 - \tfrac{1}{3}\,k)} + (-\,p'\,n)\,\frac{\left(k - \dfrac{d'}{d}\right)\left(1 - \dfrac{d'}{d}\right)}{\tfrac{1}{2}\,k^2\,(1 - \tfrac{1}{3}\,k)} = 1 \qquad (16a)$$

and

$$\mathbf{C}\left[-\,\frac{k}{\tfrac{1}{2}\,k^2\left(\tfrac{1}{3}\,k - \dfrac{d'}{d}\right)}\right] + (-\,p\,n)\,\frac{-\,(1 - k)\left(1 - \dfrac{d'}{d}\right)}{\tfrac{1}{2}\,k^2\left(\tfrac{1}{3}\,k - \dfrac{d'}{d}\right)} = 1 \qquad (16b)$$

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[3] "Handbook of Mathematics for Engineers," by Edward V. Huntington, McGraw-Hill Book Co., Inc., New York, N. Y., 1st Ed., 1918.

Eqs. 16*a* and 16*b* are both in the same form as Eq. 15 and from them the two nomograms for **B** and **C** (Figs. 2 and 3) were constructed. In Eq. 16*a*, k and $\frac{d'}{d}$ correspond with r and s in Eq. 15 and **B** corresponds with U and $(-\,p'\,n)$ corresponds with V. Therefore, $U = \mathbf{B}$ and $V = -\,p'\,n$, and

$$F_1(r,s) = \frac{k}{\frac{1}{2}\,k^2\,(1 - \frac{1}{3}\,k)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (17a)$$

and

$$F_2(r,s) = \frac{\left(k - \frac{d'}{d}\right)\left(1 - \frac{d'}{d}\right)}{\frac{1}{2}\,k^2\,(1 - \frac{1}{3}\,k)} \quad \ldots\ldots\ldots\ldots\ldots (17b)$$

To construct the nomogram for **B** (Fig. 2) from Eq. 16*a* draw a pair of x,y axes, and through the point $x = 1$ draw a third axis parallel to the y-axis which may be called the y'-axis. The x-axis and the y-axis need not be at right angles, but the x-axis may be sloped up at any angle desired as shown in Fig. 5.

The y-axis will ultimately become the **B**-scale and y'-axis will become the $(p'\,n)$-scale. Therefore, choose a unit for y such that the **B**-scale and the $(p'\,n)$-scale will be of the desired length.

Fix the point $(x = 0,\ y = 0)$ on the y-scale and $(x = 1,\ y = 0)$ on the y'-scale and connect those points by a straight line which is the x-axis. (The x-axis is not shown in the finished nomogram.) In this case, since **B** is always positive and $(-\,p'\,n)$ is always negative, the height of the nomogram would have to be as high as the sum of the extreme values of **B** and $p'\,n$ if the x-axis were made horizontal. Therefore, it is desirable to slope the x-axis up so that the lowest point on the y'-axis (the maximum value of $p'\,n$) will be about on the same level with the lowest point of the y-axis.

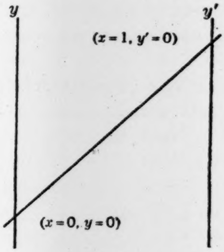

FIG. 5

Next a network of curves is drawn by plotting the points:

$$x = \frac{F_2(r,s)}{F_1(r,s) + F_2(r,s)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (18a)$$

and

$$y = \frac{1}{F_1(r,s) + F_2(r,s)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (18b)$$

This can always be done by giving one of the variables (say, r) successive constant values (0, 0.1, 0.2, $\cdots$, 0.9, 1.0) in the equations, and plotting the curves for each of the constant values of r. Each curve thus plotted is marked with the appropriate value of r and each point plotted for each r-curve is marked with the appropriate value of s. Then the corresponding values of s are connected by smooth curves and each curve is marked with the appropriate value of s.

However, in many cases, it is possible to eliminate either r or s from Eqs. 18*a* and 18*b* and to obtain one equation containing only x, y, and r or x, y, and s, from which one family of curves can be plotted directly.

In this case by substituting the values of $F_1(r,s)$ and $F_2(r,s)$ from Eqs. 17a and 17b in Eqs. 18a and 18b, there results

$$x = \frac{\left(k - \frac{d'}{d}\right)\left(1 - \frac{d'}{d}\right)}{k + \left(k - \frac{d'}{d}\right)\left(1 - \frac{d'}{d}\right)} \quad \text{(19a)}$$

and

$$y = \frac{\frac{1}{2} k^2 \left(1 - \frac{1}{3} k\right)}{k + \left(k - \frac{d'}{d}\right)\left(1 - \frac{d'}{d}\right)} \quad \text{(19b)}$$

The ratio d'/d may be eliminated from Eqs. 19, giving

$$y = \tfrac{1}{2} k \left(1 - \tfrac{1}{3} k\right)(1 - x) \quad \text{(19c)}$$

from which the k-curves may be easily plotted. It is apparent from Eq. 19c that the k-curves are all straight lines passing through the point $(x = 1, y = 0)$; and, when $x = 0$, $y = \frac{1}{2} k \left(1 - \frac{1}{3} k\right)$.

The (d'/d)-curves can then be plotted by giving d'/d successive constant values in Eq. 19b.

Next the **B**-scale is constructed by plotting the values of **B** along the y-axis and the $(p' n)$-scale is constructed by plotting the values of $p' n$ along the $-y'$-axis. The x-axis is then erased and the nomogram **B** (Fig. 2) for the variables **B**, $p' n$, k, and d'/d in Eq. 16a is completed. When any three of the four variables are known, the fourth can be found immediately.

The nomogram **C** (Fig. 3) for the four variables **C**, $p n$, k, and d'/d in Eq. 16b is constructed similarly. In this case,

$$F_1 = \frac{-k}{\frac{1}{2} k^2 \left(\frac{1}{3} k - \frac{d'}{d}\right)} \quad \text{(20a)}$$

$$F_2 = \frac{-(1 - k)\left(1 - \frac{d'}{d}\right)}{\frac{1}{2} k^2 \left(\frac{1}{3} k - \frac{d'}{d}\right)} \quad \text{(20b)}$$

$$x = \frac{(1 - k)\left(1 - \frac{d'}{d}\right)}{k + (1 - k)\left(1 - \frac{d'}{d}\right)} \quad \text{(20c)}$$

and

$$y = -\frac{\frac{1}{2} k^2 \left(\frac{1}{3} k - \frac{d'}{d}\right)}{k + (1 - k)\left(1 - \frac{d'}{d}\right)} \quad \text{(20d)}$$

Eliminating d'/d from Eqs. 20*c* and 20*d*,

$$y = \tfrac{1}{2} k \left(1 - \tfrac{1}{3} k\right) - \tfrac{1}{2} k \left[\left(1 - \tfrac{1}{3} k\right) + \frac{k}{1 - k} \right] x \text{.........} (21)$$

The k-curves plotted from Eq. 21 are straight lines. The (d'/d)-curves are plotted from Eq. 20*d*.

The two nomograms for **B** and **C** can be used without the nomogram in Fig. 1, but it will be noted that Fig. 1 is very useful in finding the most economical values of k.

In designing the reinforcement only two of the four variables in each nomogram are known—that is, in Fig. 2, **B** and d'/d are known whereas k and $p'\,n$ are unknown; and, in Fig. 3, **C** and d'/d are known whereas k and $p\,n$ are unknown.

In some cases the value of $p\,n$ or the value of $p'\,n$ may be fixed by other considerations, such as by the use of the same bars as those in another section. In that case the steel in one face is fixed; then the value of k can be found from the known percentage of steel and that value of k used in finding the unknown percentage.

If neither $p'\,n$ nor $p\,n$ is known, either one of them may be assumed and the other may be found after determining k from the assumed percentage; but this procedure will require several assumptions and trials before the most economical or satisfactory combination is obtained.

To obviate the assumptions and trials, Fig. 1 was devised to give the value of k that will make the sum of $p'\,n$ and $p\,n$ a minimum. To do this, Eqs. 14 were added together, giving

$$\left(1 - \frac{d'}{d}\right)(p\,n + p'\,n)$$

$$= \frac{\frac{2}{3}\left(1 - \frac{d'}{d}\right)k^3 - \left\{\mathbf{B} - \mathbf{C} + \frac{1}{2}\left[1 - \left(\frac{d'}{d}\right)^2\right]\right\}k^2 + \left(\mathbf{B} - \mathbf{C}\frac{d'}{d}\right)k}{-\left[k^2 - \left(1 + \frac{d'}{d}\right)k + \frac{d'}{d}\right]} \text{.} (22)$$

Eq. 22 was differentiated with respect to k and the derivative placed equal to zero, from which the equation—

$$-\frac{d'}{d}(1 - k)^2\,\mathbf{B} + \left(k - \frac{d'}{d}\right)^2\mathbf{C} = \frac{2}{3}\left(1 - \frac{d'}{d}\right)k\left\{k^3 - 2\left(1 + \frac{d'}{d}\right)k^2\right.$$

$$\left. + \frac{3}{4}\left[1 + 6\frac{d'}{d} + \left(\frac{d'}{d}\right)^2\right]k - \frac{3}{2}\frac{d'}{d}\left(1 + \frac{d'}{d}\right)\right\} \text{..........} (23)$$

—was obtained and was written

$$(-\,\mathbf{B})\,\frac{\frac{d'}{d}(1 - k)^2}{\delta} + \mathbf{C}\,\frac{\left(k - \frac{d'}{d}\right)^2}{\delta} = 1 \text{..........} (24a)$$

in which, to simplify typography,

$$\delta = \frac{2\,k}{3}\left(1 - \frac{d'}{d}\right)\left\{k^3 - 2\,k^2\left(1 + \frac{d'}{d}\right)\right.$$
$$+ \frac{3\,k}{4}\left[1 + 6\,\frac{d'}{d} + \left(\frac{d'}{d}\right)^2\right]$$
$$\left. - \frac{3\,d'}{2\,d}\left(1 + \frac{d'}{d}\right)\right\} \ldots\ldots\ldots\ldots\ldots\ldots \text{(24}b\text{)}$$

which is the same form as Eq. 15. Eqs. 24 also represent the form desired for the construction of Fig. 1 which was drawn in a manner similar to that used for Figs. 2 and 3, although in this case it is impractical to eliminate either k or d'/d from the equations for x and y corresponding to Eqs. 20*c* and 20*d*.

DISCUSSION

Maurice Barron,[4] M. ASCE.—One of the important features of the nomograms in this paper is the fact that they contain only nondimensional quantities which permit their use, without adjustment, in either the metric or the English system of units. Because of this nondimensional property, the nomograms are universal in that they can be used for any specification, any combination of stresses, any arbitrary concrete strength, and any embedment of reinforcement.

For several years, the writer has used nomograms[5] by Henri Dumontier virtually identical to these for the review of arch sections; but for the design of the sections other methods are usually preferred which do not require tables, charts, or diagrams.

Fig. 1 is of particular interest, in that it shows the minimum amount of steel reinforcement (tension steel plus compression steel) required for the design of a concrete section subjected to combined bending and direct stress. This nomogram, as well as the others, is based on the straight-line theory, and currently the plastic theory[6] indicates that a stronger structure results when part of the reinforcement intended as compression steel is used as additional tension reinforcement—because of the redistribution of stress resulting from plastic yield.

An interesting and enlightening problem is presented as follows:

The cross section is 12 in. by 24 in., subjected to a moment of 35.6 kip-ft and a direct thrust of 45 kips; depth $d = 21$ in. The specifications permit: $f_s = 18{,}000$ lb per sq in.; $f_c = 1{,}000$ lb per sq in.; and $n = 10$. By Eqs. 1, $B = 0.1573$ and $C = 0.0002$. Using Fig. 1, with these values and $k = 0.357$, the solution is not apparent. From a direct design method the following solution results—$e = \frac{M}{N} = \frac{35.6 \times 12}{45} = 9.5$ in.; $a = \frac{1}{2}t - 3 = 9.0$ in.; $e' = e + a = 18.5$ in.; and $M_s = N\,e' = 45 \times 18.5 = 832$ kip-in. For a balanced section with the full allowable stresses—$k = 0.357$; $j = 0.881$; $K = 157.3$; $M_B = K\,b\,d^2 = 832$ kip-in.; and $A_s = \frac{M_s}{f_s j d} - \frac{N}{f_s} = \frac{832}{18 \times 0.881 \times 21} - \frac{45}{18} = 2.50 - 2.50 = 0$ sq in.

Therefore, this section, subjected to the designated stresses, requires no tension reinforcement and no compression reinforcement. It should be noted that the thrust is considerably outside the middle third of the section and that, for an uncracked section design, tension greater than 200 lb per sq in. is indicated. The paradox becomes more complicated if the embedment in the foregoing problem is 2 in. instead of 3 in. With an increased depth d the required tension steel changes from zero to 0.2 sq in. If the embedment is increased to 4 in., no tension steel is required, but compression reinforcement is necessary. This appears to be another defect in the straight-line theory, and similar results will be obtained frequently for sections that have values of **C** equal to or approaching zero. In Eq. 1*c*, the quantity in brackets approaches zero when e'/d approaches the value of $1 - \frac{d'}{d}$.

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[4] Chf., Structural Design Div., Clarke, Rapuano & Holleran, New York, N. Y.

[5] "La Technique des Travaux," by Henri Dumontier, Société des Pieux Franki, Liége, Belgium, February, 1928, pp. 135–146.

[6] "Plastic Theory of Reinforced Concrete Design," by Charles S. Whitney, *Transactions*, ASCE, Vol. 107, 1942, p. 251.

Since the compression bar in Fig. 4 replaces an equal area of concrete, p' should be multiplied by $(n - 1)$ instead of by n in the derivation. This point is usually ignored for practical simplification in the derivation of the equations and construction of the nomograms. It is a simple procedure to use the nomograms as shown and to apply a correction to the area of compression steel. Mr. Dumontier introduced two nomograms similar to Fig. 1 which the writer considers more convenient in arrangement and scale layout. In these nomograms the paradox demonstrated by the foregoing example falls outside the range and the designer cannot be misled. The unusual case requiring special consideration is brought forcefully to the attention of the designer.

The author is to be commended for a very clear presentation and for making these nomograms readily available to the profession.

M. Hirschthal,[7] M. ASCE.—Mr. Linton has certainly performed a valuable service for the designers of reinforced concrete structures who have the problems of designing for bending and direct stress, or who use steel as compressive reinforcement. The labor expended by the author on the preparation of his nomographic charts is undoubtedly enormous. He has saved each individual using them a corresponding amount of time.

It is all the more regrettable, therefore, that he did not take into consideration the fact that both the Joint Committee on Standard Specifications for Concrete and Reinforced Concrete[8] and the American Concrete Institute in the latest design specifications[9] have given consideration to the fact that the modular ratio in the compression side varies from that of the tension side of a concrete member reinforced for compression as well as tension. The regulation adopted by both these bodies is based on the assumption that the value of n (the modular ratio) for compressive reinforcement is double the value used for the tensile reinforcement. This assumption was adopted because of the low stresses to which compressive reinforcement was subjected under the old assumption of a uniform n on both sides of the neutral axis of the section, and was the result of test observations[10] made on large size columns at the University of Illinois, at Urbana.

Under the old provision, compressive reinforcement in a 3,000-lb concrete, the modular ratio (in tension) of which is assumed as 10, would be stressed to about 9,000 lb per sq in. For a beam such as Fig. 4, in which d is 16 and d' is 2 in., the stress in the compressive steel will be $\frac{5.5}{7.5} \times 1{,}000 \times 10$—which is somewhat more than 7,000 lb per sq in., or somewhat less than one half of the stress capacity of the steel. However, if the value of n for the compression side is taken as double that in tension, the compressive reinforcement will be stressed to about 16,000 lb per sq in., taking advantage of the full capacity of the steel in compression.

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[7] Concrete Engr., D.L. & W.R.R., Hoboken, N. J.

[8] "Recommended Practice and Standard Specifications for Concrete and Reinforced Concrete," *Proceedings*, ASCE, June, 1940, Pt. 2, p. 45.

[9] *Journal*, A. C. I., June, 1945, p. 582.

[10] *Bulletin No. 40*, Univ. of Illinois, Urbana, Ill., June 5, 1934, p. 86.

The writer appreciates the fact that the basic formulas will have to be modified both for the location of the neutral axis and for the resulting percentages of steel reinforcement p and p' in Eqs. 14. However, since **B** and **C** are not affected, the variation is not quite as serious as it otherwise would be. It is to be hoped that the use of $2\,n$ as a modular ratio for reinforcement in compression will be generally adopted since it will result in a marked saving in the reinforcement required in compression.

Such a percentage of saving is certainly worthwhile even at the expense of considerable design calculation labor. This saving will be about 50% as the position of the neutral axis is but slightly affected by this change in the modular ratio.

WINFIELD A. McCRACKEN,[11] Assoc. M. ASCE.—Simplification of Eqs. 1*b* and 1*c* and the addition of one nomogram for determining **B** and **C** make the set of nomograms presented in this paper an excellent design tool.

The moments used in Eqs. 1*b* and 1*c* are taken about the positive and negative reinforcement, respectively. The equations can be simplified by restatement:

$$\mathbf{B} = \frac{N}{f_c\,b\,d^2}\left(e + \frac{d - d'}{2}\right) \quad\text{(25a)}$$

and

$$\mathbf{C} = \frac{N}{f_c\,b\,d^2}\left(e - \frac{d - d'}{2}\right) \quad\text{(25b)}$$

KING BURGHARDT,[12] M. ASCE.—The nomograms presented in Figs. 1, 2, and 3 are considered a decided improvement over the numerous diagrams usually included in treatments of this topic. If conservation of space is important, only Figs. 2 and 3 are necessary, as the economical balance of tension and compression steel may be determined by a "cut and try" method similar to that proposed for checking stresses. However, the inclusion of two additional nomograms would furnish direct solutions for checking stresses or for designing with equal steel in the tension and compression face.

The utility of Fig. 1 is somewhat diminished by the recent tendency to consider that initial compression in reinforcing steel is introduced by shrinkage of the concrete. The 1949 edition of "Standard Specifications for Highway Bridges"[13] of the American Association of State Highway Officials provides for a doubling of the stress and the resisting moment of the compression steel over

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[11] Cons. Engr., Houston, Tex.

[12] Senior Highway Bridge Engr., U. S. Public Roads Administration, Denver, Colo.

[13] "Standard Specifications for Highway Bridges," A.A.S.H.O., Washington, D. C., 1949.

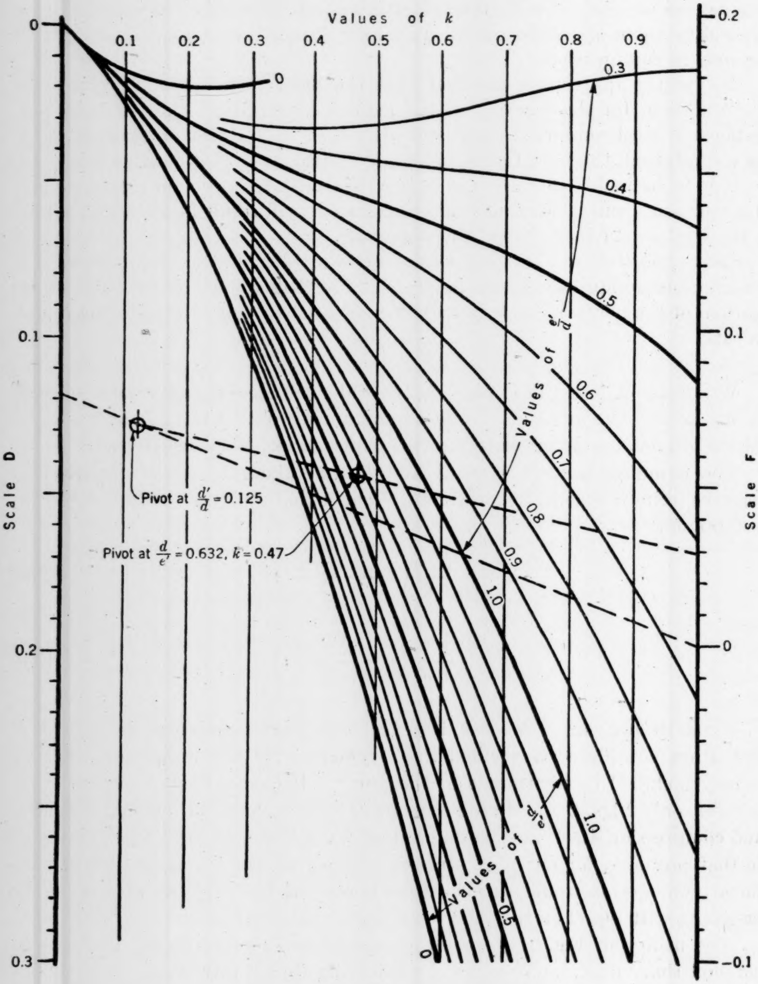

FIG. 6.—INVESTIGATION TO FIND k

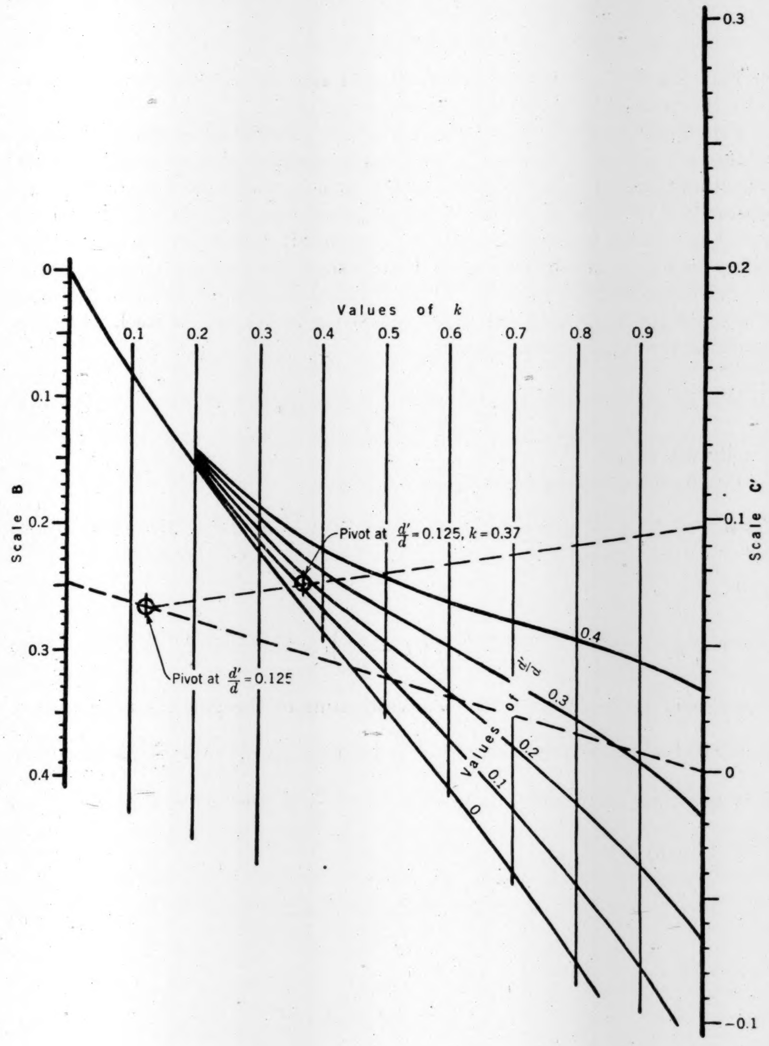

FIG. 7.—DESIGN TO FIND k WHEN $n\,p' = 2\,n\,p$

that calculated by former methods. To comply with this provision, let

$$n\,p'\,b\,d = 2\,n\,A'_s \qquad (26)$$

for Figs. 2 and 6. The minimum total steel area for this condition would require the revision of Eq. 24 and the redrawing of Fig. 1.

Figs. 2 and 3 present quick solutions of Eqs. 12 and 13, respectively. Some values of k and d'/d produce rather vague readings of $n\,p'$ or $n\,p$, merely indicating that the particular value of $n\,p'$ or $n\,p$ contributes very little to the resistance of the section. It might be mentioned that, if the value of k obtained from Fig. 1 produces a negative value of $n\,p'$ in Fig. 2, k should be recomputed for $n\,p' = 0$ (Fig. 2); and this new value of k should be used to find $n\,p$ in Fig. 3. When k is less than the value for balanced design (usually about 0.33), the value of $n\,p$ obtained in Fig. 3 will cause overstress in the tensile steel. For this condition the approximate formula,

$$A_s = \frac{N}{f_s}\left(\frac{e'}{0.9\,d} - 1.0\right) \qquad (27)$$

usually will suffice.

Fig. 6 represents the expressions:

$$\frac{e'\,n\,p}{d}(1 - k) - n\,p'\left(\frac{e' - d + d'}{d}\right)\left(k - \frac{d'}{d}\right) = \frac{k^2\,(3\,e' - 3\,d + k\,d)}{6\,d} \qquad (28a)$$

and

$$n\,p\,(1 - k) - n\,p'\left(\frac{e' - d + d'}{e'}\right)\left(k - \frac{d'}{d}\right) = \frac{k^2\,(3\,e' - 3\,d + k\,d)}{6\,e'} \qquad (28b)$$

thus providing a solution for k for investigating or checking a known section. As the factor $\frac{e'}{d}$ may vary from $\frac{1}{2}\left(1 - \frac{d'}{d}\right)$ to infinity, the value of $\frac{e'}{d}$ is used when it is less than unity and the inverse factor $\frac{d}{e'}$ is used when $e' > d$. When $e' < d$, then, $D = \frac{e'}{d}$ and

$$F = \frac{e' - d + d'}{d}\,\frac{d - d'}{d} \qquad (29a)$$

and, when $e' > d$, then $D = 1.0$, and

$$F = \frac{e' - d + d'}{e'}\,\frac{d - d'}{d} \qquad (29b)$$

To investigate a section, compute d'/d, $D\,n\,p$, and $F\,n\,p'$. In Fig. 6, place a straightedge between $D\,n\,p$ on scale **D** and 0 on scale **F**. At the point where the straightedge intersects $k = \frac{d'}{d}$, pivot the straightedge to $F\,n\,p$ on scale **F**.

At e'/d, or d/e', Fig. 6, read k. Use this value of k in Fig. 2 to find **B** and then compute f_c and f_s. Referring to Example 1, $\frac{d'}{d} = 0.125$; $\frac{d}{e'} = \frac{16.00}{25.33} = 0.632$; $D\,n\,p = 1.0 \times 0.118$; and $F\,n\,p' = \frac{25.33 - 16 + 2}{25.33}\,\frac{16 - 2}{16}\;0.075 = 0.029$. Join 0.118 on scale **D** with 0 on scale **F**. Pivot at $k = \frac{d'}{d} = 0.125$ to 0.029 on scale **F**, and at $d/e' = 0.632$ read $k = 0.47$.

There are occasions when it is desirable to use equal steel in the tension and compression faces. Fig. 7 is a nomogram for the equation:

$$(1 - k)\left[\mathbf{B} - \frac{k\,(3 - k)}{6}\right] = \left(k - \frac{d'}{d}\right)\left[2\,\mathbf{C} - \frac{2\,k\,(3\,d' - k\,d)}{6\,d}\right] \ldots\ldots (30)$$

by which to compute the values of k for this condition. Fig. 7 is drawn for the aforementioned new provision that $n\,p'\,b\,d = 2\,n\,A'_s$. Compute **B** and **C** from Eqs. 1*b* and 1*c*, respectively. Compute $\frac{d'}{d}$ and $\mathbf{C}' = \mathbf{C}\,\frac{d - d'}{d}$. Join **B** on scale **B** with 0 on scale **C′**. At $k = d'/d$, pivot to **C′** on scale **C′**; and, at d'/d in Fig. 7, read k. With k known, compute $n\,p'$ from Fig. 2 or $n\,p$ from Fig. 3. As a check, $n\,p' = 2\,n\,p$. Referring again to Example 1, $d'/d = 0.125$, $\mathbf{B} = 0.247$, and $\mathbf{C}' = 0.111\,\frac{16 - 2}{16} = 0.097$. Join 0.247, scale **B**, with 0 on scale **C′**. Pivot at $k = \frac{d'}{d} = 0.125$ to 0.097, scale **C′**. At $d'/d = 0.125$, read $k = 0.37$. From Fig. 2, $n\,p' = 0.147$; and, from Fig. 3, $n\,p = 0.074$. It is possible to plot a nomogram so that the intersection of a line joining a **B**-value on scale **B** with a **C**-value on scale **C** will show k at d'/d on the chart, but in certain positions there is confusion between the k-lines and the $\frac{d'}{d}$-lines of the chart, making it difficult to obtain a satisfactory reading. For this reason, the somewhat longer method illustrated was preferred.

Figs. 6 and 7 are reproduced at a comparatively small scale. Drawn to a larger size with additional subdivisions of the scales, quite satisfactory readings may be obtained from either nomogram.

In conclusion, the writer believes that the author has made a presentation valuable to all designers who are interested in obtaining economical compression members of reinforced concrete.

ROBERT V. HAUER.[14]—The nomograms presented by the author for the design and the analysis of rectangular reinforced concrete sections subjected to combined direct stress and bending have the interesting feature of being based upon the quantities **B** and **C**, which are the two moments of the axial force about the center lines of the tensile and of the compressive reinforcements, divided by $b\,d^2\,f_c$.

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[14] Structural Engr., Albert Kahn, Inc., Detroit, Mich.

The advantage of this particular choice is readily explained by the facts that the tensile reinforcement is independent of **B** and that the compressive reinforcing is independent of **C**. This makes it possible to split the representation of the functional relations between the six quantities **B**, **C**, $n\,p$, $n\,p'$, k, and d'/d into two separate parts, as the author has done in nomograms 2 and 3.

The same simplification was utilized before by E. Moersch[15] in his textbook on reinforced concrete. Although Mr. Moersch uses different charts for different ratios of d'/d, his representation, in the writer's opinion, is superior in several other respects. The two parts showing the tensile and the compressive reinforcements, respectively, are placed opposite each other in such a manner that not only can each of these reinforcements be read at a glance but so also can their sum. In addition, this arrangement eliminates the need for a special chart for determining the value of k at which the total reinforcement becomes a minimum, and also solves the problem of finding the reinforcement in the case of a predetermined ratio. Such a case (for instance, if symmetrical reinforcement is required) could be solved with the author's nomograms only by a somewhat cumbersome cut-and-try process.

It may be of interest to mention that nomograms 2 and 3 can be used not only if N is a compressive force, but also if N is a tensile force, provided that **B** and **C** denote the moments of the axial force about the center lines of the tensile and compressive reinforcements, respectively, divided by $b\,d^2\,f_c$. For a tensile force N, the variable **B** becomes smaller than **C**, because tensile and compressive reinforcements are interchanged.

On the other hand, it is regrettable that the author did not extend the nomograms far enough to cover the range of small eccentricities, which, at least in building design, is the case most often encountered. Another shortcoming is that the value of f_c must be known, in order to use the nomograms in design; but most building codes now make the allowable concrete stress a function not only of the relative eccentricity (which is known from the data), but also of the percentage of reinforcement—that is, of the very quantities at which the design computation is aimed.

In conclusion, the writer wants to question some remarks made in the "Synopsis"—namely, that methods in current use for the design of reinforcement are useful mainly in analyzing designs previously made, and that in designing it is customary to assume the amount of reinforcement and then to check the assumption. There are many more formulas, tables, and charts available for design than for analysis. The obvious reason is that the design problem is not only much more important practically, but also, because of its very nature, much simpler than the problem of analysis. If the design is based on predetermined stresses, the location of the neutral axis follows from a very simple formula, whereas in the case of analysis k must be computed from a third-degree equation.

W. P. LINTON,[16] M. ASCE.—The reception of this paper, both in the published discussions and in personal correspondence, was gratifying to the writer. The purpose of the paper was to reduce the labor involved in the design and checking of reinforced concrete sections subject to bending and direct stress, while adhering to the accepted theory of reinforced concrete design.

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[15] "Der Eisenbetonbau," by E. Moersch, Konrad Wittwer, Stuttgart, Germany, 5th Ed., 1920–1922.

[16] Senior Highway Bridge Engr. (*Retired*), U. S. Public Roads Administration, Philadelphia, Pa.

Mr. Barron presents an example that tends to show a paradox in the straight-line theory. The writer considered the straight-line theory to be the accepted theory and the nomograms were made accordingly. Mr. Barron also states (truthfully) that:

> "Since the compression bar in Fig. 4 replaces an equal area of concrete, p' should be multiplied by $(n - 1)$ instead of by n in the derivation."

Mr. Hirschthal writes:

> "It is all the more regrettable, therefore, that he did not take into consideration the fact that both the Joint Committee on Standard Specifications for Concrete and Reinforced Concrete[8] and the American Concrete Institute in the latest design specifications[9] have given consideration to the fact that the modular ratio in the compression side varies from that of the tension side of a concrete member reinforced for compression as well as tension. The regulation adopted by both these bodies is based on the assumption that the value of n (the modular ratio) for compressive reinforcement is double the value used for the tensile reinforcement. This assumption was adopted because of the low stresses to which compressive reinforcement was subjected under the old assumption of a uniform n on both sides of the neutral axis of the section, and was the result of test observations[10] made on large size columns at the University of Illinois, at Urbana."

What Mr. Hirschthal states is true, but not so regrettable as one might, at first, think, because it would make no change in Figs. 1 and 2 except in the label of the $(p' n)$-scale. Those nomograms can be used as they are by simply giving any value desired to n. The value of n, to be used in Fig. 2, need not be the same as the value used in Fig. 3. Mr. Hirschthal's suggestion would make a change in Fig. 1. As stated, Figs. 2 and 3 can be used as they are with the understanding that n may have different values in the two nomograms.

Mr. Burghardt proposes two additional nomograms, Figs. 6 and 7. The writer believes that Fig. 6, the nomogram for finding k in a given section, is a very useful and desirable addition. Although the stresses in a given section can be checked by Figs. 2 and 3 alone, as is done in the paper, the direct method, by using Fig. 6, now seems to the writer to be more desirable. Fig. 7, the nomogram for equal steel in tension and compression, will be useful (as stated by Mr. Burghardt) in some cases.

Mr. Hauer calls attention to other charts,[15] which in his opinion are superior in several respects. The writer cannot comment on them because he has not seen them. Mr. Hauer also feels that:

> "* * * it is regrettable that the author did not extend the nomograms far enough to cover the range of small eccentricities, which, at least in building design, is the case most often encountered."

That is the case in which there is no tension on the concrete, and the solution of that problem is not as difficult by other methods, as it is where there is tension. However, it would be desirable to have that case covered by nomograms.

In conclusion, the writer believes that nomograms, in general, are very useful and can, and should, be used much more extensively than they have been. In some cases it is more difficult to construct a nomogram than a diagram, but when constructed, the nomogram is more easily used and more variables can be provided for.

After considering all the discussions, the writer is of the opinion that Figs. 2 and 3 (with the n on the $(p'\,n)$-scale in Fig. 2 changed to n') together with Mr. Burghardt's Figs. 6 and 7, make very useful and convenient tools for the purpose for which they were intended.

The writer wishes to thank all those who have taken an interest in the paper.

AMERICAN SOCIETY OF CIVIL ENGINEERS
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TRANSACTIONS

Paper No. 2393

CITIES CANNOT LIVE WITHOUT TRUCKS

BY HOY STEVENS[1]

WITH DISCUSSION BY MESSRS. HAROLD M. LEWIS, JOHN A. MILLER, L. A. NEES, AND HOY STEVENS.

All goods or commodities purchased by the American people for personal use travel by motor truck at some stage, and usually during many stages, of their journey between the original raw material producer and the ultimate consumer. Any apparent exceptions are goods, materials, or machinery for the use of intermediate industries that either fabricate or transport consumer goods.

As a reminder of the way in which trucks have been intimately woven into the economic pattern, some of the uses of trucks in cities include:

1. Delivery of goods and commodities (*a*) from retailers to ultimate consumers; (*b*) from warehouses to wholesalers, to retail outlets, or to other users; (*c*) from producers and manufacturers to wholesalers or retailers; and (*d*) to and from the collection depots of long distance carriers (such as highway motor truck terminals, railroad terminals, air cargo carrier airports, and steamship docks).

2. Transportation of the service employees of utilities and other service industries that help to make our cities livable and efficient—for example, the Bell Telephone System uses more than 18,000 trucks to maintain its service.

CHICAGO (ILL.) CASE HISTORY

In order to give a more specific idea of the importance of motor trucks to a city, some of the data from a report[2] on a comprehensive study made in 1945 of the flow of truck traffic in and out of Chicago are presented. Data from this study and report, prepared by the Illinois Highway Traffic Advisory Committee to the War Department, indicate the huge volume of goods which must be handled daily over the streets of Chicago in order to meet the needs of the community. It states in part:

> "The role played by highway transportation in war production was only an extension of its service to the pre-war industrial economy, a service which

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NOTE.—Published in December, 1948, *Proceedings*. Positions or titles given are those in effect when the paper or discussion was accepted for publication.

[1] Chf., Equipment and Operations Section, Am. Trucking Assns., Inc., Washington, D. C.

[2] "Chicago Truck Terminal—1945," Illinois Highway Traffic Advisory Committee to the War Dept. (lithograph).

> without a doubt will be further expanded in post-war years. The trucking industry today employs about 4,000,000 people in the United States. Scarcely an item of commerce is produced which is not dependent at some point on motor transportation. Trucks traveling on highways are the sole means of freight delivery to over 54,000 small communities."

Table 1, taken from the Chicago report, shows the tons of freight carried daily into and out of the Chicago area by various types of vehicles and by different types of operators. In all, 32,000 truck units carry 80,774 tons of freight into or out of the area daily. The report states:

> "About ten percent of these (5,000 single unit) carrier-operated trucks have an origin or destination at a transportation terminal. From data of origin-destination, these trucks appear to be engaged in the complicated task of distributing the necessities of life throughout the primary trade areas, and in transporting the goods of the region between numerous warehouses, wholesale stores, retail stores, etc. The over-the-road motor carrier appears to use the single unit trucks largely for the local distribution of goods.
>
> "Of the 6,300 combination units operated daily into and out of Chicago, approximately 1,800 have a definite origin or destination or both in a transportation terminal, and some 1,800 more have an origin or destination or both at a warehouse, which certainly would include characteristics of a freight terminal. * * * These vehicles operate into rail-truck interchange terminals, motor-transport terminals, and, to a minor extent, into air terminals. It is not possible to estimate the tonnage of freight delivered to these various types of terminals from the traffic data alone; but it is estimated that, of the 50,000 tons of freight handled daily into and out of Chicago, at least two fifths, or 20,000 tons, is destined for motor transportation terminals for local delivery, or for interchange with other motor carrier lines.
>
> "Of the 50,000 tons of freight being dispatched daily, about 30,000 tons (estimated by various trucks at 55% to 70%) is handled directly from the shipper's or to the consignee's docks in full truck loads, and without having been first unloaded on a terminal dock for distribution."

TABLE 1.—Daily Truck Operation Into and Out of the Chicago (Ill.) Area

Type of vehicle (1)	Commercial Motor Carriers		Privately Owned Vehicles	
	No. (2)	Tonnage (3)	No. (4)	Tonnage (5)
Trucks	5,000	9,471	18,800	22,753
Tractor-trailer trains	6,300	40,640	1,900	7,910
Total	11,300	50,111	20,700	30,663

This information not only shows the essential daily truck needs of a city but also indicates the enormous traffic problem which such vehicular flow presents.

Fallacies in Thinking About Trucks

Before considering possible remedies for congested street conditions, it would be well to analyze critically some of the fallacies that have crept into popular thinking regarding commercial motor vehicles. Too frequently "sidewalk" thinking about traffic cures overlooks two basic facts: (*a*) That large American communities could not exist without modern truck transportation, and (*b*) that the volume of truck freight is so large and so important that it must be moved continuously and during the regular working hours of business establishments.

One comment frequently heard is "great big trucks." Actually, in so far as frontal width is concerned, none of the regular goods-carrying trucks are larger than other vehicles which are accepted on the highways—namely, vehicles used in the public transportation of passengers. Large straight trucks are no bigger than large buses or trolley coaches, whereas tractor-trailer trains are comparable to streetcars. Admittedly, van-type trucks frequently are higher than other vehicles on the road, generally from 12 ft to 12½ ft, but the height of the vehicle has nothing to do with the area it occupies as it travels through the streets.

The trucking industry prefers to use large units because they are most economical and most efficient. In the last analysis, the customer prefers the most efficient vehicles. However, it may be desirable to define small trucks as to size. It will no doubt be agreed that it will require at least 2 or 3 small van trucks to take the place of 1 large van trailer. Certainly, substituting 2 or 3 small vehicles for 1 large vehicle will not improve traffic situations. The Chicago data estimate that 30,000 tons daily were handled directly from shippers' docks or to consignees' docks in full truck loads without having first been unloaded on the carriers' docks for distribution in small trucks. In thinking of remedial measures, it must not be assumed that the use of 2 or 3 city delivery trucks in place of 1 large efficient van is a remedy for traffic congestion.

SURVEY OF CONGESTION COMPLAINTS

The American Transit Association has made available some data regarding the interference of trucks with the buses, trolley coaches, and streetcars of urban public transportation companies. This information is thought to be particularly valid as an indication of conditions on city streets because the public transportation companies have regularly scheduled operations over the city streets and are thereby more quantitatively able to judge actual interference. Two of the questions asked of the local transportation companies in a questionnaire circulated on June 21, 1946, were:

1. Are your vehicles being delayed and your schedules disrupted by trucks loading and unloading on the streets? and
2. How serious is the situation caused by the trucks; that is, how would you rate vehicle blocks due to trucks among the factors responsible for delays to your vehicles?

A recapitulation of the replies from the various companies shows the answers to the first question to be: Yes, 21; slightly, 14; and no, 14. Answers to the second question were: Minor problem, 29; serious problem, 10; coal trucks, 2; and double parking, 5.

The purpose of presenting these data is merely to emphasize the point that in so far as these regular and mature users of the streets are concerned, trucks are a part of the traffic problem, but definitely are far from the major cause of traffic delays.

CERTAIN UNSATISFACTORY EXPEDIENTS

From New York, N. Y., which has almost reached the ultimate of traffic loads with little possibility of doing anything but keeping even with its problems, come the following comments (from the special inquiry service of the American Transit Association), which are thought to be significant:

> "Various experiments in traffic regulations have been made. As an example, effective in October 1945, the police department prohibited parking along 34th Street between Second and Ninth Avenues from 7:00 A.M. to 7:00 P.M. daily except Sundays. Prior to this prohibition some week-day service on the 34th Street route was considerably behind schedule. Within a few days thereafter, schedules were fairly maintained. However, in the months following the parking restrictions, the volume of moving vehicular traffic along 34th Street has increased to such an extent that the benefits accruing from the parking regulations have largely disappeared. Coach service along 34th Street is again operating behind schedule."

Contained also in these comments are some replies from cities where other expedients had been tried. For example, night deliveries were suggested, but labor organizations oppose them because their members do not want to work nights. Businessmen oppose night deliveries because they would have to maintain a night force and because their establishments would be more vulnerable to robbery and theft when opened up for night delivery.

THE PROBLEM IN REGARD TO TRUCKS

There is no doubt that trucks at rest while being loaded and unloaded are a problem. How can this loading and unloading situation be remedied? In so far as it can be seen it is a matter of off-street loading areas. This idea is not new, but this seems to be the most feasible answer. The common carriers have to load and unload at their shippers' and receivers' facilities. The private truck operator has some choice of his own loading space facilities, especially if he owns the plant or building. Tenants of buildings may have less choice. Fundamentally, the problem of standing trucks for loading or unloading is a matter of building design and all truck operators will welcome any efforts to convince plant and building owners that it is sound business for them to provide ample off-street truck loading areas.

The earliest ordinance requiring off-street loading facilities for industrial or commercial buildings dates back to June, 1931, in Pueblo, Colo.[3] In 1946, zoning ordinances of twenty-one cities required retail stores and wholesale houses to furnish adequate off-street loading facilities. Twenty cities require industrial buildings to furnish off-street loading facilities. Seventeen cities have this requirement for office buildings.

No city has passed a retroactive ordinance which would compel the installation of off-street loading facilities in buildings already in existence at the time the ordinance was passed. However, one city (Greenville, S. C.) has set a period of eighteen years from the date of the ordinance during which the building may continue to exist without conforming to the ordinance. A number of cities now provide that, if a building is substantially altered or repaired, the off-street parking or loading provisions must be complied with.

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[3] "Zoning Applied to Parking," by C. S. McGraw, Jr., and Wilbur S. Smith, *Traffic Quarterly*, The Eno Foundation for Highway Traffiic Control, Inc., Saugatuck, Conn., January, 1947.

Different thinking is needed as to what can or may be done in regard to off-street loading. Why is it necessary to continue to think that alternate routes, overhead routes, and by-passes must be built to permit industrial areas to continue in their motley array of curbstone loadings? Consider what the result would be if overhead roadways were built over commercial streets. Actually, off-street truck loading areas would be provided at the curbstones of the lower street to the extent of each building's front footage. This arrangement still might not provide adequate loading areas for certain buildings and might provide too much for other buildings. In addition, the overhead road structures would depress the value of the buildings to the detriment of their owners, and depress the tax values of the buildings to the detriment of the community.

AMOUNT OF OFF-STREET LOADING AREA

The several ordinances regarding off-street loading facilities are based on three general premises. Some prescribe loading spots based on the gross floor area in the building, others prescribe certain minimum space between an alley and the rear of the building, and others merely prescribe reasonable or adequate facilities so as not to interfere with traffic. Obviously, before any sound development of off-street loading can be had there is a need for formulas by which reasonable or adequate facilities can be estimated.

The New York Regional Plan Association[4] gives a sliding scale for use in computing the required number of truck berths, the number of loading areas being based on the gross floor area for two types of building—office buildings and hotels and buildings for manufacturing, storage, and similar facilities. It may be that loading area formulas based on floor area are types which can produce reasonably consistent results. However, any formula based on floor area overlooks one fundamental—namely, that the trucks move goods and not floor area. Truck loading area requirements should be based on the volume and tonnage of freight to be moved into and out of a building.

The American Trucking Association is engaged in a study of loading platform areas with reference to the tonnage of freight handled. This study, also, is considering the areas required for efficient manipulation of vehicles in front of the loading area. The material is not complete, but some estimated values will indicate the type of coefficients that may be developed. When available the results of this study should be of real assistance in city planning.

For rapid and efficient manipulation of commercial vehicles when they are to be backed against a dock for loading or unloading, the open area between the edge of the loading platform and the outer edge of the manipulation area should be approximately twice the length of the vehicle. This manipulation area may be wholly within the building, or the outer half may be a part of an alley or a street. On heavily traveled streets all the area obviously should be within the property lines. Along the edge of the dock from 10 ft to 12 ft should be allowed per vehicle. These dimensions will permit expeditious backing of trucks into loading stalls.

Of major importance in speeding the loading and unloading of commercial vehicles is the size of the dock area adjoining the trucks. To some extent, the dock area necessarily will be dependent on material handling equipment and arrangements for transporting the freight between the loading dock and the more permanent storage area of the commodities being received or shipped. Sufficient area should be provided on the dock for the truck load accumulation of outgoing shipments or for the truck load receipt of incoming shipments without the necessity of moving the freight to or from the dock while the truck is held at the dock.

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[4] "Traffic and Parking Study," New York Regional Plan Assn., Inc., New York, N. Y., December, 1942, p. 42.

Based on the experience of the trucking industry, it is necessary that the dock area be at least twice the total body floor area of the largest number of trucks that can be docked at one time. This is a logical conclusion when it is realized that freight of average density (from 22 lb per cu ft to 24 lb per cu ft) usually is piled between 5 ft and 6 ft high in the truck, is transported to and from the truck on two-wheeled hand trucks handling a height of stock from 3 ft to 4 ft, and seldom is tiered on the loading dock; hence, allowing for aisles in which the hand trucks can operate, the required dock area is at least twice the floor area of the vehicles at the dock.

With such a dock area it would be possible to accumulate an entire outgoing load for a vehicle—and in a receiving section to have enough area upon which to unload an entire vehicle—opposite each loading spot. Even with dock areas of this size ratio it will still be necessary to transport the freight promptly to and from the dock and building in order to make continuous use of the truck berths. If a loading dock is to be used as a storage area even more space is required. Final formulas will be more specific than this, but this value indicates the basis of present thinking.

In the case of loft buildings, elevator service for the upper floors will be a great factor in determining dock area. Freight placed on an elevator is seldom tiered; hence, calculations of elevator size may be made on an area basis to determine elevator requirements.

CONCLUSIONS

Because cities cannot live without trucks, it is the duty of the communities to plan for or to provide facilities that will reduce the truck factor in traffic congestion for the benefit of all vehicular movement, including trucks. The trucking industry is definitely interested in remedies that will reduce traffic congestion and will improve facilities at shipping and receiving establishments for loading and unloading motor trucks. The business of that industry is moving goods and commodities, and trucks make no money when delayed in traffic or when hindered in loading or unloading.

Individually, as well as an industry, truck operators are becoming more and more interested in problems of traffic as well as in problems of loading and unloading trucks at curbs. The state trucking associations and their members are cooperating in local studies of traffic, zoning, and freight handling facilities in cities where street traffic conditions are unsatisfactory. It is believed that the truck operators have developed sound experience in regard to the handling of general freight. The dock facilities that they have found efficient should give adequate data for shippers and receivers. When the data are complete, it is planned to pass on this accumulated experience in the form of specific recommendations regarding loading docks for use with motor freight vehicles.

DISCUSSION

Harold M. Lewis,[5] M. ASCE.—The timeliness of this paper lies in the fact that it stresses the importance of truck traffic to urban areas, and the importance of so handling such traffic that it may move expeditiously without hampering the other uses of city streets. Any prosperous city must have healthy industrial and business areas, and these require intensive truck services. As the author states, this requisite calls for not only sound street planning, but also the design of buildings and industrial areas for the motor age, with facilities for off-street loading and unloading. Although, to some extent, such facilities can be required under municipal ordinances, adequate provision will in part remain dependent on the architects who design the structures, the engineers and city planners who lay out the street system, and the executives of business and industrial concerns whose cooperation is essential.

Truck Versus Passenger Car Traffic.—There is a wide variation in the percentage of trucks to total vehicles on highways in the United States—not only in regard to the highway, but in regard to the time of day, day of the week, and season of the year. On arterial routes the number of trucks will ordinarily comprise from 5% to 25% of the total vehicles; on some of the bridges in New York City this value reaches 35%. On the interstate crossings operated by the Port of New York Authority trucks average approximately 20% of the total traffic.

In the 1920's, when the modern parkway systems were being developed, stress was laid on the need of express routes for passenger cars. It was assumed that trucks, being slower vehicles, could get along on ordinary streets where they would find plenty of room when some of the passenger cars had been diverted to parkways. The parkway program has encouraged the use of more passenger cars but has not solved the movement of trucks, many of which are now capable of traveling at speeds comparable to those of passenger cars. Looking forward in 1927 to the need for better routes for trucks, the Engineering Division of the Regional Plan of New York (N. Y.) and Its Environs stated that, in addition to the restriction of certain routes to passenger vehicles, it was probable that:

> "* * * there will eventually have to be highways which are intended for trucks only, and which passenger vehicles would either naturally avoid or from which they would be excluded."[6]

Limited access highways available for all types of traffic provide equal facilities for trucks and passenger cars. The writer believes that most of the expressways to be built in the future will be of this type. However, they should be supplemented by parkways through suburban sections of metropolitan areas and along scenic routes. Perhaps, some day, certain highways will be reserved for trucks during the weekdays.

Compulsory Off-Street Loading.—The provision of off-street loading and unloading facilities not only should be required in new buildings exceeding a specified minimum floor area designed or used for such purposes—manufacturing, offices, hotels, etc.—as require service by large numbers of trucks, but also should be made retroactive for buildings of the same types. Such retroactive provisions were recommended in 1942 by the Regional Plan Association, Incorporated, in its study of traffic and parking in New York City.[7] This recommendation was based on observation of the extent to which such facilities had been voluntarily installed, and suggested a five-year period for compliance. An essential part of any such regulation would be provision for a variance in cases where undue hardship would result. The Regional Plan Association recommended the following provision to cover such cases:

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[5] Cons. Engr. and City Planner, New York, N. Y.

[6] "Highway Traffic," Vol. III in "Regional Survey of New York and Its Environs," New York, N. Y., 1927, p. 99.

> "Where, due to an unusual shape of the lot or structural features of the building, the Board of Standards and Appeals shall determine after public notice and hearing that these provisions cannot reasonably be complied with by the owner, that Board may approve such alternative provisions for truck loading and unloading as it deems suitable and adequate to prevent the creation of serious traffic congestion by such loading and unloading operations."

Such retroactive provisions, to be effective within a period of from five years to ten years, have also been suggested by Nathan Cherniack,[8] M. ASCE. On main highways outside central business districts, municipalities, through their police powers, may require substantial setbacks for industrial buildings as well as the provision of adequate loading and unloading spaces which, in most cases, can then be provided in the open spaces on the lot.

The writer agrees with the author that there is need for better designed industrial districts served by streets on one level rather than for elaborate systems of two-level streets. Nevertheless, in some of the older intensively developed districts it may be necessary to provide two levels, as has been done on Wacker Drive and part of Michigan Avenue in Chicago, Ill., and on the west side of Manhattan in New York City, with an elevated highway for passenger vehicles.

Formulas for Off-Street Loading and Unloading Areas.—Any formula for an off-street loading and unloading area incorporated in a municipal ordinance must set the minimum requirement for certain types of buildings. The writer questions the feasibility of the author's suggestion (see "Amount of Off-Street Loading Area") that such requirements "* * * should be based on the volume and tonnage of freight to be moved into and out of a building." These factors will vary with the type of occupancy. Any buildings that will have exceptionally heavy truck service should be provided with more than the minimum legal requirements. If the city supplements its off-street loading requirements with regulations of street use to prevent the abuse of curb side space, this result should be obtained.

In the Regional Plan Association study of 1942,[7] it was shown that in many cases structures within this exceptional classification had voluntarily provided much more than the minimum off-street space which would be required under the regulations proposed. For example, a large department store in New York City had eighty-five off-street loading berths in comparison with the minimum of twenty five which would be required for a building of the same size "used for manufacture, storage, or goods display, for a department store, or for a hospital." Again, the underground truck berths in the Rockefeller Center group of office buildings in New York City had forty loading berths as against the twenty two that would be required for a similar floor area "used for office purposes or for a hotel supplying full facilities for transients."

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[7] "Traffic and Parking Study," New York Regional Plan Assn., Inc., New York, N. Y., December, 1942, pp. 43 and 138.

[8] "A Statement of the Parking Problem," by Nathan Cherniak, *American Planning and Civic Annual*, 1946, p. 129.

The sliding scale used by the Regional Plan Association for computing the number of truck berths, referred to by the author ("Amount of Off-Street Loading Area"), was based on probability tables used by the American Telephone and Telegraph Company in determining the number of trunk lines necessary under different loads to avoid keeping a customer waiting too long to make his connection. It was developed under the assumption that 1 truck out of 4 might have to wait for one eighth of the average loading time.[9]

The minimum sizes of buildings that should come under such compulsory loading space restrictions will vary with the size and character of the community. In New York City the minimum size for buildings under the present regulations is 25,000 sq ft of gross floor area. The Regional Plan of New York and Its Environs recommended that such requirements apply to loft, department store, or office buildings having 8,000 or more square feet, when located in "close suburban or intermediate areas."[10] In several ordinances for small communities a minimum size of building is specified but the amount of off-street space to be required is to be determined by the zoning board of appeals.

Smaller buildings in established centers must continue to be served by curb side space, but cities should forbid the use of sidewalks for the storage and classification of motor truck freight. This regulation would require owners to provide adequate ground floor storage and elevator service to upper floors for such freight.

Supplementary Types of Regulations.—Among the other types of regulations which cities may enforce to relieve street congestion by trucks are:

1. Keep large trucks handling less-than-truck-load shipments out of congested business areas by the provision of union, intercity truck terminals where their freight will be sorted and reloaded on smaller vehicles for delivery. The same procedure in reverse would apply to pickups. Two such terminals in the New York metropolitan area were under construction by the Port of New York Authority in 1949.

2. Limit the hours for pickup or delivery of truck-load lots in congested business areas. This would not necessarily require night service, but might ban any service during the heaviest rush-hour periods.

3. Designate certain streets in central business areas for the routing of trucks desiring to pass through such areas without pickup or delivery.

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[9] "Traffic and Parking Study," New York Regional Plan Assn., Inc., New York, N. Y., December, 1942, p. 43.

[10] "The Building of the City," Vol. II in "Regional Plan of New York and Its Environs," New York, N. Y., 1931, p. 176.

John A. Miller,[11] **M. ASCE.**—In what appears to be a blanket argument against any restrictions on the operation of intercity motor trucks within city limits, the author presents certain statements as basic facts which, to the writer, seem to need supporting evidence. The first such statement (see heading, "Fallacies in Thinking About Trucks") is that: "* * * large American communities could not exist without modern truck transportation." However, the City of New York (N.Y.) had a population of 4,766,883 in 1910 when there was no modern truck transportation. At this same date a considerable number of other American communities could also properly be classified as large. None of them found it impossible to "exist"; nor would large communities today find it impossible to dispense with motor trucks, although they would be considerably inconvenienced, particularly if deprived of the service of small city-delivery trucks.

This distinction between "convenient" and "essential" is extremely important. The expenditure of huge sums of public money may be justified to create improved facilities for something essential to existence when it would not always be justified to create facilities for something that is only a convenience and an advantage.

A second statement (also under the heading, "Fallacies in Thinking About Trucks") is made that: "* * * the volume of truck freight is so large and so important that it must be moved continuously and during the regular working hours of business establishments." Presumably Mr. Stevens means that no restriction should be placed on the hours during which intercity trucks may move within the city. Since the maximum spread of regular working hours of business establishments in any city would not extend beyond the 12-hour period from 7 a.m. to 7 p.m., there remains another 12-hour period from 7 p.m. to 7 a.m. outside regular working hours. When the author states that the volume of heavy truck transportation is so large that it must be operated in city streets during the daytime as well as during the night (when other traffic movement is at a minimum), it is equivalent to declaring that the virtually unrestricted use of roadway facilities for half of the twenty-four hours is insufficient to accommodate all the truck traffic. The writer has never seen any statistics to show that this theory is true, and he suspects that the motor truck operators would be the first to deny its truth if they were asked to pay taxes on that basis. It seems probable, therefore, that the real basis of the author's objection to restrictions on daytime operation is that such restrictions would increase the cost of truck operation rather than that facilities are inadequate.

A third statement (again under the heading, "Fallacies in Thinking About Trucks") is made that: "Certainly, substituting 2 or 3 small vehicles for one large vehicle will not improve traffic situations." To dispute the certainty of that view, the author's own data show that only 37% of the tonnage at Chicago, Ill., was handled in unit, full truck loads. When delivering the remaining 63% of the tonnage, the trucks were running partly empty most of the time. Wherever traffic congestion exists, a partly loaded truck represents a waste of valuable street space. Under such circumstances the use of smaller trucks would be a definite advantage.

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[11] Schenectady, N. Y.

The most serious objection to the use of extremely large trucks is the congestion they cause when making turning movements in congested areas and when stopped at an angle to the curb. In such situations there can be no doubt that 1 large truck does more to create congestion than do 2 or 3 small trucks. This does not necessarily mean that the use of large trucks is never justified, but it does mean that the use of smaller trucks is desirable under many circumstances.

The author seems to conclude that it is the duty of the community to provide such facilities that the truck operator can operate whatever kind of vehicle he wants in whatever way he wants to operate it. As an ordinary citizen and taxpayer in the community, the writer rejects the contention that the community has any such duty.

L. A. Nees,[12] Assoc. M. ASCE.—This thought-provoking paper touches on a feature of city planning and building design to which in great degree, insufficient attention is given, because of either the lack of knowledge of the problem, or the barrier of individual or public opinion. Off-street truck loading takes considerable area and may entail a serious encroachment on the space available for the activity planned for the building. It is the duty of the engineer or architect, in designing a building, to give the client a structure that will serve his purpose efficiently at the lowest possible cost. Except in cases where legal provisions exist or where the loading and unloading of freight are major items in the facility, the client is likely to regard these facilities as a necessary evil, and to view with a jaundiced eye any designs that add considerably to the cost of the structure, or infringe on what he pleases to call his "operating space." This reaction is especially true of smaller industries or businesses where the volume of truck freight to be moved is not large. Therefore, the designer is faced with the problem of convincing his client of the "indirect benefits" involved in adequate loading facilities, especially where they encroach measurably on the available building space. Quite frequently the study and analysis required to demonstrate this fact are beyond the scope of his organization, or his fee, or both.

It appears then that municipal control in the form of codes or regulations is the logical basic approach to the problem. In establishing such codes, a city is faced with two fundamental problems: (*a*) The establishment of requirements for construction in areas as yet undeveloped; and (*b*) the establishment of requirements for construction in existing developed areas. Item (*a*) offers full latitude for the development of straightforward long-range planning. The solution for item (*b*), however, calls for bold and imaginative planning, coupled with an intensive educational campaign to convince the taxpayer that such measures eventually increase the economic health of the community.

The writer suggests that the first steps in the development of a "code" are to zone the city into districts and to study the type of truck freight serving each zone. Such a study should aim at determining the average daily volume of truck loadings and unloadings for a range of truck capacities based on size. It is anticipated that this research would yield a truck size that could be considered average for each zone, together with a clue to the average density of loadings per enterprise, critical street, or zone.

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[12] Headquarters, Air Materiel Command, Wright-Patterson Air Force Base, Dayton, Ohio.

From such data, combined with traffic counts, and growth-of-use forecasts, the planner would be in a position to weigh the necessity of thoroughfare widening in the zone, since this aspect is intimately related to the problem of developing a code for off-street loading facilities.

Based on such studies, a code would contain the following provisions:

1. Establishment of revised property lines as required, for all new construction;
2. Establishment of minimum off-street loading facilities defined by "setback" distance from the street line, width of unloading space in either feet or number of berths, turnout radii to the loading area, etc;
3. Requirement that the construction plans be submitted for approval to an appropriate engineering representative of the city, together with an engineering analysis of anticipated volume and intensity of freight to be handled and types of trucks expected to serve the facility, as a justification of the loading facilities proposed for construction; and
4. A time limit in years at the expiration of which all existing structures must be modified to conform to the code.

It is apparent that the optimum degree of development in any zone is that stage at which the interrelation between structure, truck sizes, and width and type of the traffic artery is such that the total combined annual charges for all phases is a minimum. Achievement of this goal necessitates the introduction into the code of a restriction on the maximum size of truck that may be permitted to deliver freight in any one zone. There is a limit as to how far a city can go in designing its traffic arteries to accommodate larger and larger trucks. It appears that, in planning, a point can and must be reached within which trucking industries can balance their fleets economically to conform to established reasonable zoning restrictions.

Enterprises of like character, and therefore with similar basic freight requirements, have a habit of choosing the same environment. Such development can be observed in many of the larger cities. There is a strong possibility, therefore, that the studies outlined herein will indicate that a great percentage of the freight in any zone is handled within a certain limiting truck size. If such an economic grouping exists, there is no reason why it should be paralyzed in its development by codes that impose restrictions on structures but at the same time permit unrestricted delivery by trucks that are oversized, not only for the type and the volume of freight they deliver, but also for the streets upon which they travel. The writer has observed frequent instances of large tractor trailers delivering small parts of their loads to consignees. It has not been possible to judge the extent of this practice, but its economics for intracity deliveries are seriously open to question except where required by the dimensional character of the freight.

In the future development of a city and its trucking industries, this concept therefore implies a greater development of major "in transit" and "distribution" freight terminals. It implies also that trucking industries will be forced to scrutinize carefully the composition of their truck fleets with a view to determining economic sizes and achieving a proper relation in numbers. It is possible that a thorough study of the problem along the lines indicated for any actual case will demonstrate the validity of a plan leading to uniform development within the zones of a city.

The author's contention that criteria based solely on a general type of enterprise or space values are questionable guides is a fair one. The volume and the character of freight to be moved and the intensity of movement are more reliable indexes.

The statement that the dock area must be at least twice the total body floor area of the largest number of trucks that can be docked at one time is open to question. Although it may be statistically correct in many instances, this requirement may prove too rigorous. For instance, the writer has seen numerous cases of government warehouses which do not meet this criterion where the freight never stays on the dock but is handled rapidly and expeditiously by fork-lift equipment which stocks it in its proper place without rehandling. This procedure is especially applicable where the cargo is all of one type. The dock space requirements are intimately related to the character and the method of storage in the structure. Its design, therefore, must represent an integrated step in the space planning for the entire facility.

The author states in his conclusions that trucking associations are conducting studies on this problem. The writer feels that such studies will be of great value to the engineering profession and looks forward to their publication.

HOY STEVENS[13].—The discussions by Messrs. Lewis, Miller, and Nees indicate a growing appreciation of the problems of commercial transportation in urban communities. Because of the significance of commercial vehicles in industrial cities, it will require cooperative effort of architects, builders, city planners, and truck operators in each community to develop realistic future plans.

Mr. Lewis further stresses the need for formulas by which off-street loading areas and truck berths for commercial buildings can be determined. As he states, a building's use is a predominant factor in the number of truck berths required. More study and data are needed on this subject. Also, in calculating off-street loading areas, consideration must be given to the size of the vehicle that will use these docks. Currently there is a trend toward 16-ft bodies on city freight delivery trucks. Generally these 16-ft bodies are mounted on cab-forward types of motor trucks, but the over-all length is increased slightly.

A quick rule for width of manipulation areas in front of a dock is that the area width from the dock to the limiting boundary shall be not less than twice the over-all length of the vehicle. A greater width makes for easier manipulation and backing into truck berths, and is recommended. The same rule holds true for the longer combinations of tractor semitrailers.

In regard to his recommended supplementary type of regulations, Mr. Lewis' suggestions would not bar tractor semitrailer combinations from making truck load deliveries. With this view the writer is in accord. In general, in congested areas most carriers find it advantageous to make less-than-truck-load "pickups and deliveries" with city trucks, using only road semitrailers for truck-load movements. The economies of these practices are readily apparent in less manual handling of the freight.

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[13] Chf., Equipment and Operations Section, Am. Trucking Assns., Inc., Washington, D. C.

Mr. Miller restates an error that appears frequently in many traffic surveys, when he states that trucks are found partly empty while running in the streets. This is inevitably the case when the amount of loading in the trucks is checked at outlying survey stations. The only time a city delivery truck making deliveries can be full is just when it leaves the carrier's terminal. From then on it is becoming more and more empty as it makes deliveries. This must be the case if the truck makes deliveries. The contrary is true with a pickup truck collecting outbound shipments. In this case the truck is full only between its last pickup stop and the carrier's terminal. This partial empty phenomenon has been recorded in many traffic surveys, and the incorrect analysis of the causes for the phenomenon have resulted in very erroneous conclusions about city truck operation.

Mr. Nees' suggestion regarding a code for off-street loading facilities indicates a reasonable approach to the problem, both as regards new construction and old buildings.

Mr. Nees' objections to the writer's suggestion regarding the size of a loading dock further point out the interrelation between intrabuilding transportation facilities, such as elevators or industrial dock trucks, and dock areas. Since the dock area (even following the broad suggestions in the paper) will be less than the area required for the motor truck manipulation area, there are advantages in dock arrangements (especially in buildings) which will enable the fast loading and unloading of vehicles. The faster that motor vehicles can enter and leave a dock area, including either loading or unloading time, the smaller can be the truck manipulation area and the fewer can be the truck berths required for a given tonnage. For this reason, there are real advantages to a building owner in having his dock and his material handling equipment as efficient as possible for both the loading and unloading of commercial vehicles, and for the moving of the freight within the building.

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DULUTH-SUPERIOR HARBOR

BY HESTON R. COLE,[1] M. ASCE.

WITH DISCUSSION BY MR. HARLAND C. WOODS.

SYNOPSIS

This is a history of the development of the harbor at Duluth, Minn., and Superior, Wis., into the most important bulk traffic port in the world. There is outlined the growth in traffic from the days of the early French explorer and fur trader to the present, with weekly shipments of more than 1,000,000 tons of iron ore.

Construction of the harbor by the Corps of Engineers, Department of the Army, began in the days when there were still many wooden sailing vessels, which were loaded and unloaded by wheelbarrow and plank. Today, the harbor is being maintained and improved to accommodate the fast, deep draft, 600-ft-long, steel vessels with enormous cargoes, which are loaded and unloaded at docks designed to accomplish this work quickly and efficiently by the most modern mechanically operated equipment.

Duluth-Superior Harbor is one of the finest natural harbors and probably the most important bulk traffic port in the world. Herein is sketched briefly its history, the types and tonnages of traffic, and a description of the harbor before and after improvements (see Fig. 1).

The first white men to visit the western end of Lake Superior were French explorers and traders, in 1659 and 1660. They left no written record of visiting Superior and Duluth bays, but there is reliable evidence that they wintered at Chequamegon Bay, 75 miles east of Duluth, and theirs was the first shipment (furs) from Lake Superior. Jesuit cartographers in Paris, France, drew the first map of Lake Superior in 1670–1671 and marked thereon a river flowing into Lake Superior at its western extremity, probably from information furnished by missionaries.

It is not until 1679 that there is written record of a lasting settlement at Head of the Lakes. During this year Daniel Greysolon Sier Du Lhut and his party portaged over Minnesota Point near its base, entered the St. Louis River

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[1] Col., Corps of Engrs., Dept. of the Army, Duluth, Minn.

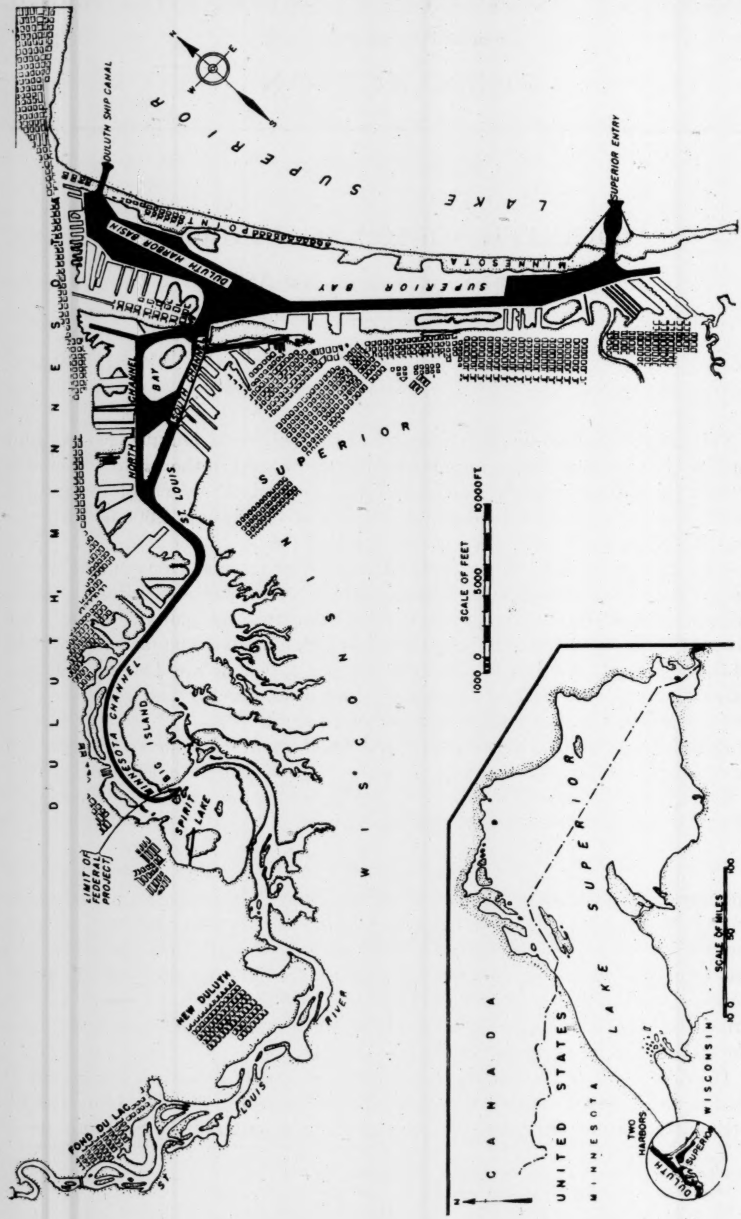
DULUTH, MINNESOTA
DULUTH SHIP CANAL
LAKE SUPERIOR
SUPERIOR ENTRY
MINNESOTA POINT
DULUTH HARBOR BASIN
SUPERIOR BAY
ST. LOUIS BAY
NORTH CHANNEL
SOUTH CHANNEL
SUPERIOR, WISCONSIN
MINNESOTA CHANNEL
BIG ISLAND
SPIRIT LAKE
LIMIT OF FEDERAL PROJECT
NEW DULUTH
FOND DU LAC
ST. LOUIS RIVER
SCALE OF FEET
1000 0 5000 10000FT
N
S
E
W
CANADA
UNITED STATES
MINNESOTA
WISCONSIN
LAKE SUPERIOR
TWO HARBORS
DULUTH
SUPERIOR
SCALE OF MILES
0 50 100

FIG. 1

and established a fur trading post 20 miles west of Duluth, which he named Fond du Lac, or end of lake. In 1856 the struggling townsite on the north side of Superior Bay adopted the name of Duluth in honor of this French explorer. The fur trade continued as the main commerce until 1855, the year the first lock was constructed at Sault Ste. Marie (Mich.).

The modern story of Duluth-Superior Harbor as a port begins in 1852 with the arrival of George Stuntz. Mr. Stuntz had been in charge of a survey party running the Wisconsin-Minnesota boundary line and liked the area so well that he settled on the spit of land later called Minnesota Point. Mr. Stuntz prospected over the area and, in 1865, found iron deposits while looking for gold. In 1873 he persuaded Philadelphia (Pa.) capital to invest in mining operations on the eastern end of the iron range. This was the beginning of the iron ore period. Although this ore was shipped from Two Harbors (Minn.), Duluth became the administrative headquarters for mining operations in northern Minnesota. Discovery of ore at Mountain Iron (Minn.), in November, 1890, by the Merritt brothers, and subsequent development of the Mesabi Range with the ever increasing shipments of ore through Duluth-Superior Harbor, soon made the port one of the most important in the world.

Duluth-Superior Harbor consists of an outer and inner basin. The outer basin, known as Superior Bay, lies between the mainland and the natural sand and gravel breakwater, which is ½ mile wide and about 9 miles long and extends from the Minnesota side to the Wisconsin shore. This spit of land protects the harbor from disturbing wave action in Lake Superior. The inner harbor consists of St. Louis Bay and St. Louis River. Originally, there was only one entrance to the harbor, located at approximately the present position of Superior Entry near the south end of Superior Bay. The natural entrance shifted considerably from year to year, because of wave action and currents, and was often obstructed by shifting sand bars.

Superior Bay (the main harbor) and St. Louis Bay (the inner harbor) were wide expanses of shallow water with depths ranging from 5 ft to 9 ft and containing several small islands. The natural channel, created by the river current, was from 100 ft to 500 ft in width with depths varying from 8 ft to 12 ft. Because of the varying depths of water at the entrance to the harbor, navigation was limited generally to 8 ft or less, and during bad northeast storms this was decreased to 3 ft or 4 ft. These data are taken from a survey of the harbor made in 1861 by then Capt. George B. Meade, Corps of Engineers, who later attained fame as the victor at the Battle of Gettysburg.

The first act performed by the federal government was the building of a lighthouse at the entrance of the harbor in 1857. Parallel timber crib piers were constructed at Superior Entry by the Corps of Engineers from 1867 to 1875.

The present Superior Entry piers and breakwaters were completed in 1914. The piers are 500 ft apart; each has a concrete superstructure, supported by timber piling. The south pier is 1,584 ft long and the north pier, 2,096 ft long. The rubble mound breakwaters converge, forming an arrowhead and providing a stilling basin of approximately 62 acres, which gives safe passage in the stormiest weather to all vessels. Entering from the lake at Superior Entry, the channel depth is 32 ft, decreasing to 25 ft at the Superior Bay end.

A channel between these piers which extends to established docks was dug to a project depth of 12 ft, the work being completed in 1873. This depth was the controlling depth until completion of the 16-ft project depth in 1885. Commerce that year totaled 1,200,000 tons valued at $24,500,000. Arrivals and departures totaled 1,997 vessels, of which approximately 10% were sailing ships. The largest cargo was about 3,000 tons. In 1885 there were 25 docks in the harbor and the elevators had a storage capacity of about 9,000,000 bu of grain. About 600,000 tons of coal were received and 14,000,000 bu of grain shipped.

With the opening of the Canadian and Poe locks at the Soo canals in 1895 and 1896, respectively, and with the advent of larger bulk freighters with steel hulls, the channels in the Duluth-Superior Harbor were deepened to 20 ft, with additional depths provided in the lake approach channels. During the six-year period, from 1896 to 1902, about 21,500,000 cu yd of material was dredged at a cost of about $2,000,000. Contracts were let at the low rates of 7½¢ per cu yd, 8¢ per cu yd, and 10¢ per cu yd, depending on the location. The first shipment of iron ore from Duluth-Superior Harbor was made in October, 1892, and continued shipments soon established the harbor as one of the most important on the Great Lakes. The billionth ton of iron ore was shipped in 1940 and more than 2,000,000,000 tons of total commerce were handled prior to July, 1947.

During the years from 1933 to 1935 the harbor was dredged to provide channel depths of 25 ft to the ore docks and 22 ft to the grain and coal docks. These depths were required to permit loading of the bulk freighters to their maximum usable draft.

In 1884 the bottom of the harbor was somewhat irregular, and vessels experienced a great deal of trouble from striking shoals. In 1893 some of the leading citizens of Superior and Duluth decided to cooperate in their efforts to secure harbor appropriations instead of fighting each other's requests for harbor development. The Joint Duluth-Superior River and Harbor Commission was created. Largely through the efforts of this commission an act was passed by Congress creating a special committee to examine and report on the harbor. In May, 1895, this examination of the harbor was made by a congressional river and harbor committee. While making the examination, the boat conveying the party passed a large freight boat that had accidentally gone aground in the Superior Front Channel. Later in the day, the inspection boat also ran aground and had to be pulled off by a tug. The committee decided the channel should be deepened. The separate harbors of Duluth and Superior were united under the title, "Harbor at Duluth, Minnesota, and Superior, Wisconsin," by the River and Harbor Act of Congress approved on June 3, 1896.

The district engineer at that time also appeared before the river and harbor committee with reliable statistics compiled by his office in 1895. This report showed a total of 10,986 arrivals and departures carrying 6,325,351 tons valued at $95,000,000 for the combined Duluth and Superior harbors.

The story of the dredging of the artificial entrance, or the Duluth Ship Canal, originated as early as 1859. In the charter of the City of Duluth, organized in April, 1870, power was given the council to construct a canal connecting Lake Superior with Superior Bay. In the fall of 1870 the city let a contract for dredging a canal 150 ft wide and 16 ft deep, protected by piers on each side, extending into 18 ft of lake water. The work was pushed with all possible speed, and when winter put a stop to operations, dredging had been completed through about two thirds of the peninsula. On petition of interests in Superior, an injunction was issued by the United States Circuit Court restraining the dredging company and the City of Duluth from constructing the canal, but was later dissolved with the condition that the City of Duluth build a dike across the bay, making two separate harbors of Superior Bay. There followed seven years of litigation, until the United States Supreme Court decided in 1877 against the City of Superior. The dike was constructed but lasted only a short period, being destroyed by storms. The only railroad at that time ran into Duluth and the lack of a dike kept Superior commerce from easy access to the railway dock. Prior to opening the Duluth Canal, most of the Duluth docks were located on the shore of Lake Superior, and the first grain elevator at the Head of the Lakes was located on the lake front. With the opening of the Duluth Canal, development began on the Duluth side of the inner harbor.

The present Duluth Canal piers were completed in 1901. They consist of a concrete superstructure on a rock-filled timber crib substructure supported on piling, and are constructed to withstand the severest storms. The piers are 1,734 ft long and extend about 1,000 ft lakeward, beyond the shore line. The entrance channel has been improved to provide depths decreasing from 32 ft at the lake end to 26 ft at the harbor basin. The piers are 300 ft apart but have been riprapped, thus reducing to 240 ft the width of channel having project depth. The old canal with its timber cribs was 250 ft wide and 1,200 ft long, with channel depths decreasing from 25 ft to 20 ft.

In 1901 a prominent engineer predicted that within twenty-five years the structure would have to be pulled down to provide for larger or better improvements, yet in 1947 the project was adequate for the largest vessels. Moreover, if ocean-going vessels come to the lakes as the result of an improvement of the St. Lawrence Waterway, Duluth-Superior Harbor would require only the deepening of its channels by a few feet to accommodate them.

The vertical lift bridge over the Duluth Ship Canal was authorized by an act of Congress and is owned and operated by the City of Duluth. As originally constructed in 1905, the bridge consisted of two towers and an overhead span. A car suspended from this overhead span traveled on regular trips between the approaches on each side of the canal. The capacity of the car was four vehicles per trip. In 1929 the bridge was modified by raising the fixed span 42 ft, constructing new towers within the towers of the existing bridge, and providing a lift span 393 ft 9 in. in length to handle vehicles and pedestrian traffic. The old towers and fixed span are not essential to the new bridge but are used to carry public utility pipes and cables.

The bridge has several sources of power, the main one being from storage batteries located under the south approach. These storage batteries are recharged by motor generators operated by local commercial power or by a stand-by, gasoline driven charging set. Double sets of trolleys are available for transferring power to the operating room of the lift span; the extra set is considered necessary to assure contact during unfavorable weather conditions, which may cause formation of ice on the trolleys. Four 100-hp motors are available in the operating cage in the lift span, any combination of two being sufficient for raising and lowering the span. A 300-hp gasoline engine, located in the operating house, in an emergency is capable of raising the bridge in 2½ min by merely shifting a gear to connect the engine to the operating mechanism. Normal time of lift is a little less than 1 min.

Vertical clearance when the span is raised is 138 ft; when down, it is 16.5 ft. The concrete counterweights each weigh 450 tons and are built with pockets to contain separate small concrete blocks to be added to adjust variation in weight of the span, particularly after a fall of heavy wet snow or after a sleet storm.

Duluth-Superior Harbor has a frontage of 49 miles behind established harbor lines; the harbor area is 19 sq miles. There are 17 miles of dredged channels with widths ranging from 200 ft to 600 ft and two anchorage areas, one in the Duluth Harbor basin, the other in the Superior Harbor basin, with ample room for anchorage of from 35 to 50 vessels.

The federal government expended a total of $10,915,400.13 in permanent work and maintenance of Duluth-Superior Harbor prior to June 30, 1946. This represents a cost of about $0.005 per ton as applied against the 2,000,000,000 tons of commerce handled by this harbor.

Coal is one item of bulk commodities wherein a large saving to the public results from water shipment over rail shipment. For example, in 1947, the rail charges on coal shipments from the Illinois mines, the nearest supply of coal to the Head of the Lakes, amounted to approximately $4.05 per ton to Duluth, as compared with the rate in 1940 of $0.40 per ton applying to the movement by water of a superior quality of coal from Lake Erie ports to Duluth-Superior Harbor. In 1940 more than 7,000,000 tons of coal were unloaded on docks at the "Twin Ports," resulting in a saving to the public of about $26,000,000 for that year. The total receipts of coal prior to July, 1947, at this harbor exceed 400,000,000 tons.

The major cargoes on the lakes are the bulk commodities of iron ore, coal, grain, and limestone, which, likewise, are the main items of commerce at Duluth-Superior Harbor. About 98% of the cargo tonnage moving through Duluth-Superior Harbor consists of bulk commodities. The majority of the vessels commonly employed on the Great Lakes are a highly specialized type of ship and are known as the bulk freighter. The loading and unloading equipment is also highly specialized to fit the ship design. The lake carriers haul more than twice the normal tonnage borne by the rest of the United States merchant fleet combined. The combined lake fleets of American and Canadian bulk cargo vessels at the close of navigation in 1946 totaled 669 steamers, motor ships, and barges of more than 2,970,000 gross tons. In 1886 about 25% of the vessels were sailboats, and steel ships 300 ft in length were just making their appearance. The cargo vessels in the Great Lakes fleet under United States registry now consist preponderantly of vessels 600 ft or more in length—a far cry from the canoes used in the first commercial traffic recorded in the Duluth-Superior area in 1855.

Duluth-Superior Harbor consistently ranks second only to New York (N. Y.) harbor as the largest port in the United States in point of tonnage, and is probably the most important bulk traffic port in the world—despite the fact that it enjoys a navigation season of only eight months. For the calendar year 1945, the leading ports of the United States in point of tonnage were:[2]

Order	Harbor	Million short tons
1	New York, N. Y.	103.9
2	Duluth-Superior	65.4
3	Philadelphia, Pa.	37.2
4	San Francisco Bay, Calif.	32.2
5	Toledo, Ohio	30.5

In the foregoing comparison the total tonnage shown for Duluth-Superior Harbor originates at, or is destined for, localities outside the vicinity of the port, whereas all but 50,000,000 tons of the commerce credited to New York harbor is classed as local or within port commerce. The following tonnage data of commerce will give some idea of the growth of commerce at Duluth-Superior Harbor:

Year	Million tons
1871	0.093
1890	3
1910	36.5
1930	45.5
1942	74.3
1945	65.4

Shipments of iron ore constitute the largest item of commerce for the harbor; they amounted to about 39,000,000 short tons in 1946, or 71% of the total tonnage of the harbor. Of all the iron ore shipped from Great Lakes ports in 1946, 59% was shipped from the Duluth-Superior docks. During the war, freighters of the Great Lakes carried six sevenths of the ore used in making American steel. There are 308 bulk freighters engaged in the iron ore trade, almost all of which visit this harbor many times during a season. The first ore was shipped from the harbor in 1892. There are seven ore docks at Duluth-Superior with a total length of 15,216 ft and total storage capacity of 819,100 tons. The number of pockets range from 302 to 384, depending on the length of the dock. Thus, it is possible to load four or five large ore vessels at each dock simultaneously. Boats are normally loaded in three or four hours, with an average cargo of 12,000 tons of iron ore.

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[2] "Annual Report of the Chief of Engineers, U. S. Army," U. S. Govt. Printing Office, Washington, D. C., 1945.

Receipts of coal and coke of more than 10,000,000 tons constituted 19% of the total tonnage of the harbor in 1946. A coal dealer's publication in 1935 showed that at Duluth-Superior Harbor is concentrated the world's largest reserve of prepared coals and screenings, ready for immediate shipment to points in Minnesota, Wisconsin, northern Iowa, North Dakota, South Dakota, and northeastern Nebraska. There are now 19 docks receiving coal at Duluth and Superior.

Grain, flaxseed, and screenings receipts and shipments constituted 6% of the tonnage of the harbor in 1946 and amounted to nearly 3,500,000 tons. Of this, 151,000 tons was received from Canadian lake-head ports of Port Arthur and Fort William in Ontario, and the remainder was shipped to lower lake ports. Duluth-Superior shipped 87% of the grain transported in lake movement from American ports on the Great Lakes in 1946. Lake carriers carried one third of the world's grain supply during the World War II. The sixteen elevator systems at Duluth-Superior include thirty elevators with a total storage capacity of nearly 52,000,000 bu.

In addition, vessels at this harbor are engaged in carrying limestone, petroleum products, iron and steel—including scrap iron, automobiles, cement, salt, gravel, sand, manganese ore, and various smaller items. In all, there are 110 terminals at this port.

With the present development of processes to utilize low-grade taconite ore the future of Duluth-Superior Harbor seems certain.

DISCUSSION

Harland C. Woods,[3] **M. ASCE.**—In addition to the inconsistency noted by Colonel Cole in the tonnage data for New York (N. Y.) harbor, there is another tonnage given in this paper that is misleading as quoted. In the list of five important ports of the United States, San Francisco Bay in California is credited with an annual tonnage of 32,200,000 for the year 1945. That amount includes duplications of intraport shipments and is for all the ports in the San Francisco Bay area. Adjustment by one half of the total intraport receipts and shipments reduces it to 26,100,000, and restricting it to San Francisco and the near-by harbors of Oakland and Richmond reduces it to 18,200,000 tons. Tabulating all the ports of the United States having 1945 adjusted commerce of more than 5,000,000 tons, and crediting to each the tonnage through localities closely contiguous to the name harbor, gives the values listed under 1925 in Table 1.

Of the thirty-two harbors listed, eighteen are Great Lakes ports, five are Atlantic Coast ports, five are Gulf Coast ports, and three are Pacific Coast ports. The magnitude of the Great Lakes commerce is further indicated by the data in Table 2 from the annual report of the Chief of Engineers, United States Army, for 1945 commerce.

TABLE 1.—Comparison of Annual Cargo Tonnages

Harbor	1925		1946	
	Million short tons	Descending order	Million short tons	Descending order
New York, N. Y.	78.8	1	82.1	1
Duluth, Minn., and Superior, Wis.	65.4	2	54.2	2
Toledo, Ohio	30.5	3	30.0	6
Buffalo, N. Y.	27.1	4	16.5	14
Philadelphia, Pa.	26.1	5	32.7	4
New Orleans, La.	25.2	6	29.8	7
Houston, Tex.	23.8	7	31.8	5
Los Angeles, Calif.	23.8	8	14.4	18
Baltimore, Md.	23.1	9	34.7	3
Galveston, Tex.	22.5	10	6.8	28
Chicago, Ill.	20.2	11	19.0	11
Two Harbors, Minn.	19.7	12	15.6	16
Cleveland, Ohio	18.6	13	16.5	15
San Francisco Bay, California	18.2	14	17.0	13
Detroit, Mich.	16.0	15	28.0	8
Norfolk, Va.	14.4	16	18.3	12
Conneaut, Ohio	14.0	17	11.9	21
Beaumont, Tex.	13.6	18	23.0	9
Sandusky, Ohio	13.5	19	12.6	19
Ashtabula, Ohio	13.0	20	12.1	20
Indiana Harbor, Ind.	13.0	21	11.2	22
Boston, Mass.	12.8	22	15.0	17
Port Arthur, Tex.	12.2	23	20.1	10
Gary, Ind.	10.5	24	8.6	26
Portland, Ore.	10.5	25	9.3	24
Seattle, Wash.	10.2	26	10.4	23
Calcite, Mich.	9.7	27	....	....
Lorain, Ohio	9.7	28	8.6	25
Erie, Pa.	9.1	29	5.9	31
Milwaukee, Wis.	8.0	30	7.9	27
Escanaba, Mich.	5.5	31	....	....
Ashland, Wis.	5.4	32	....	....
Mobile, Ala.	....	....	6.2	29
Portland, Me.	....	....	5.9	30

TABLE 2.—Relative Annual Cargo Tonnage (Million Short Tons) on Various Coasts

Coast	1925		1946	
	Million short tons	Order	Million short tons	Order
Great Lakes	357.0	1	314.5	2
Atlantic	287.1	2	326.1	1
Gulf	135.6	3	174.7	3
Pacific	123.7	4	104.8	4

In 1945 traffic had not fully recovered from the war dislocation, as will be noted in the 1946 column of Table 1, which illustrates the changes through readjustment of traffic.

Of the thirty-one ports then having an annual tonnage in excess of

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[3] Prin. Engr., Buffalo Dist., Corps of Engrs., Dept. of the Army, Buffalo, N. Y.

5,000,000, fifteen are on the Great Lakes, five are on the Atlantic Coast, six are on the Gulf Coast, and four are on the Pacific Coast (see 1946 data in Table 2). Tonnage at some Great Lakes ports declined largely because of reduced quantities of iron ore and limestone used in war production; that at the Gulf Coast ports increased (some very greatly) through increased shipments of oil and oil products.

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THE RELATION OF URBAN EXPRESSWAYS TO MASS TRANSPORTATION FACILITIES

By Curtis G. Bradfield,[1] M. ASCE

With Discussion by Messrs. Elmer F. Steigelman, Harold M. Lewis, Joseph Barnett, Ralph Wesley Jones, and Curtis G. Bradfield.

Synopsis

Although the provision of urban expressway networks to accommodate all classes of metropolitan vehicular traffic would be a much desired luxury, such construction for other than intercity traffic appears economically unjustifiable. The fact that more than half of the local traffic is unnecessary and could be eliminated by modernizing mass transportation facilities is emphasized. The irreducible part of local traffic could be materially aided in many old cities by correcting inconsistencies in street layouts and establishing reasonable continuity of traffic flow. The pattern for justifiably modernized off-the-surface rapid transit nuclear systems is described and the desirability of extending commuter service into the central business district is recommended.

Introduction

Almost every major city in the United States is being urged to undertake the construction of elaborate urban systems of limited access expressways designed largely for the accommodation of local motorists. Such systems will provide: (*a*) Inner, intermediate, and outer circumferential belt lines; (*b*) radial arterial connections between these loops; and (*c*) continuous flow interchanges at the junction points.

Vast sums of the taxpayers' money will be required to create these facilities. For example, the city and the adjacent county of Los Angeles, Calif., are contemplating an over-all expressway plan estimated to cost $582,000,000. More compact cities can obtain equivalent coverage for somewhat smaller, but still staggering, sums. In addition, these limited access freeways, particularly when depressed, entail the sacrifice of large strips of originally tax-producing private property and thus will add still more to the remaining taxpayers' burdens.

Note.—Published in December, 1948, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

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Nevertheless, although gregarious man likes crowds and busy streets, it is obvious to everyone that traffic conditions in metropolitan areas have become intolerable and some remedy must be found. Apparently there are two basic alternatives. One is to provide the urban expressways and thus to perpetuate the usage of the personal automobile as an uneconomical, illogical substitute for rapid transit facilities; the other, to discourage the unnecessary use of the automobile and to provide, instead, modernized, adequate mass transportation.

The Traffic Problems

In most cities the business district traffic, in reality, presents three problems—first, the intercity traffic following a natural geographic route through the very heart of the "downtown" section which these old highways brought into being; second, the local traffic overcrowding the few available through streets; and, third, mass transportation equipment, such as streetcars, buses, etc., operating almost end to end on the surface of the streets. Each of these phases of the over-all problem has sharply distinctive characteristics. A thorough diagnosis of the malady does not result from merely lumping these problems together and prescribing the urban expressway as the panacea for all these ills.

Instead, each of the traffic types should be analyzed separately. The degree and the extent of actual necessity of each category must be determined and, once they are established, the kind and the extent of relief the city can justify and afford to provide becomes apparent.

Relief for Intercity Traffic

The intercity and, as a subproblem with somewhat similar characteristics, the cross-the-city traffic are types that, obviously, cannot be reduced. In fact, most cities want to encourage such traffic because the ever-increasing volume of heavy commercial vehicles—trucks, trailers, semitrailers, and the like—is as much the lifeblood of trade as any railroad or waterway ever was. Also, that part of intercity traffic consisting of private motorcars should not be shunned, for it is a well-known phenomenon of merchandising that an appreciable amount of trade does result from a stream of travelers passing through a city. Consequently, it would be shortsighted indeed for a community deliberately to discourage the tourist from entering a town.

Nevertheless, although the trucker must make contact with the typical truck-loading exchange terminals usually located on the fringe of the central business district, and the tourist should penetrate a city deep enough to be within reach of hotel, restaurant, and shop, their presence on the old highway lines invariably meandering through the exact center of the business district is certainly a traffic congesting element that is unnecessary and unjustifiable in that particular zone. Therefore, where this type of traffic exists in a volume sufficient to constitute a major traffic problem, the indicated remedy is to remove it from the surface of the city streets.

For intercity and cross-the-city traffic problems, there is little doubt but that the urban expressway is the logical facility. The limited access highway linking city with city has amply demonstrated its worth, and the extension of such construction on through a city, as either a depressed or an elevated expressway, can usually be justified. Such a facility frequently can be routed so as to bend away from the original highway line just sufficiently to pass along the edge of the active central district. When so installed, it will still give access to the business zone and adequately serve the intercity and satellite community traffic terminating and originating in the metropolis, the through traffic (which, although proportionately small numerically, consists largely of highly objectionable heavy trucks, semitrailers, and the like), and the traffic which enters the city from an outlying community for a destination in the city beyond the central business district. In most cities, only a few blocks separate highly valuable property on the edge of the central business district from the deteriorated "fringe" section of low value. If the intercity expressways are located in this fringe, large construction economies will be achieved.

RELIEF FOR LOCAL TRAFFIC

Although the urban expressway is the indicated remedy for the foregoing class of traffic, the projection of this theory to justify similar installations for substantially local traffic flowing to and fro between the edges of a city and the perimeter of the central business district is seemingly unwarranted. In fact, in the second, or local traffic phase of the over-all problem, an entirely different basic condition prevails. In this case, most of the traffic is not unavoidable and it is not at all certain that it should be encouraged.

An origin-and-destination traffic survey conducted in a large eastern city disclosed that, of every hundred people who brought their cars to a parking location in the central business district, seventeen were shopping, thirty were on business trips, and forty three were going to places of employment. For want of a better index, it can be assumed that these proportions are typical of all cities and it is obvious that a large part of such local traffic is probably not genuinely necessary. Nevertheless, this group of cars comprises the local traffic problem for which most of the in-city system of expressways is proposed —people going to work or to a business appointment.

Questions immediately arise. How much of this traffic is really mandatory? Is there not some less expensive method of transporting these people? Is it possible to continue sacrificing more and more of the valuable "downtown" areas for storage spaces for more and more automobiles as cities grow larger and larger? To demonstrate by exaggeration, if it is continually easier and easier for people to bring their cars into the business districts, eventually the central zones will be so filled with automobiles there will be no room left for people—or for business.

However, if complete urban expressway networks are provided, the well-known and most often stated criticisms are:

1. The inner circumferential expressway, in encircling the business district, will effectively fix the ultimate growth limits of that zone and may produce a throttling influence on future expansion. It is a well-known phenomenon of city development that a marked irregularity in the surface plane creates a psychological barrier and a restriction on normal growth. Cases in point are

Park Avenue (New York, N. Y.) before it was covered over, the Canal (Cincinnati, Ohio) before it was filled in, and the elevated trackage along Market Street (Philadelphia, Pa.), all of which were definite walls against urban development.

2. The arterial expressway radial connections between inner and intermediate circumferential belts will be built in the typical, obsolescent intermediate zones between downtown and suburban areas. These areas, usually from 1 mile to 3 miles wide, are in a transitory status and, actually, do not now present too much traffic congestion. However, by building expressways through these sections, it is likely that, in an average city, from 7 min to 12 min may be saved in driving from a suburb to the downtown district. Nevertheless, people have certain specific times for work or for business appointments and, consequently, will invariably use this saving in driving time for time out of the city—that is, they will leave home later when going to work. Therefore, almost everyone will arrive in, or leave from, the downtown area at approximately the same time.

3. As the inner circumferential expressway belt will only encircle, not penetrate, the downtown area, all local cars heading for a destination in the central business section will still be on the surface of the city streets while in the zone of most congestion.

4. To be consistent, if facilities are installed to expedite and encourage local traffic to come into the downtown area, tremendously improved parking or "terminal facilities," as the freeway proponents term them, must be provided to accommodate the cars while they are in the critical space area. Inasmuch as private interests have found little inducement to furnish such "terminals," it is likely that the taxpayer ultimately will also have to underwrite this construction.

Summing up, the net result of a huge investment in in-city expressways will be solely to get cars downtown a few minutes faster, only to add to substantially the same general confusion. Also, once downtown, what is to be done with the cars?

Assuredly, people are wanted downtown, but perhaps their cars are not wanted downtown after all. Therefore, the logical course of investigation is to first determine why people want to bring their cars into the central business district.

In almost every case, people elect to use a relatively expensive type of personal transportation because of the inadequacy of existing mass transportation. This inadequacy results from insufficient coverage, infrequent schedules, discomfort, noise, dirt, and all the other objectionable features so familiar to everyone and, also, from the fact that, since about 1920, the development of mass transportation has not kept pace with the movement volume of the people, because of the increasing use of the automobile for personal transportation.

When there was space in the central business districts to accommodate cars, it mattered little that one person after another discontinued using mass transportation and drove his automobile downtown. This procedure was not discouraged. On the contrary, it was encouraged. Streets were widened, traffic control systems installed, parking garages built, and the taxpayers spent large sums of money to foster the use of the personal automobile. In so doing, the normal growth of mass transportation was stultified and its standards crystallized at the 1920 level.

It is readily apparent to any engineer that, from the standpoint of economics, the only way to move a volume of people from point to point is by mass transportation; that in big cities bulky, cumbersome, personal transportation probably can no longer be used in lieu of mass transportation; and that, whether cities expand horizontally or vertically, with increasing land values, downtown vehicular accommodations cannot continue to be expanded for the convenience of the 43% who merely drive to work and a sizable fraction of the 30% on business trips or the 17% who want to shop.

Therefore, before constructing extensive, expensive, systems of freeways to encourage the use of the private automobile further, it appears that the first obligation is to provide transportation in volume—to reverse the trend of the past and to encourage the development of mass transportation as a primary remedy for the local traffic phase of the over-all traffic problem. Just this result automatically will be accomplished if nothing is done about the local traffic situation. In fact, for all cities it could be stated that the time to do nothing about local traffic occurs when everyone clamors for something to be done—when it is clearly established that the breaking point in tolerance of traffic delays has been reached—for then the situation will begin to rectify itself. One by one, group by group, people will gradually discontinue driving their cars into the downtown area and will resort to the use of mass transportation facilities rather than personally endure the exasperation of operating a car in heavily congested traffic.

Of course, if left to chance, the mistake of 25 years is not going to be undone overnight. Although as more and more people resume using mass transportation, it, in turn, will gradually expand and modernize; such development, unless artificially accelerated, will require many years of slow progress and much more public clamoring to the various public service commissions. However, mass transportation is already recognized as one of the contributing phases of the traffic problem. Therefore, it should be possible to expedite the redevelopment of mass transportation materially at the same time such transportation is being corrected as a traffic problem.

Although improved mass transportation should substantially reduce the surface street congestion, there will always be that irreducible minimum of local traffic arising from cars that have to be brought into the cities. Generally speaking, this problem can be materially alleviated by greater use of one-way streets, synchronized grid traffic light control, elimination of curb parking and truck loading and unloading during business hours, and all the other regulatory measures that are so politically unpopular.

Other steps that are unpleasant to many municipalities, but which are frequently tremendously effective in older cities, are to correct the inconsistencies in street layout in the central zone and to establish a symmetrical traffic artery pattern providing reasonably continuous flow. In many old cities most of the streets go nowhere. Because of a complete lack or an incomprehensiveness of early city planning, almost every street jogs back and forth and terminates at a building, a cemetery, or the water's edge. To reach a destination, traffic must frequently follow a circuitous path involving much unnecessary right and left turning. Each turn delays and slows down traffic flow. Although this condition may be only annoying in a suburb, it becomes of major importance in the congested business districts.

Careful analysis of the street plan of many cities frequently will reveal situations where little-used streets can be cut through existing blocks or buildings so as to join with other little-used streets and thus form a continuous artery or where obsolete indentations of a waterfront can be filled and reclaimed so as to permit the extension and the joining of streets terminating at the water's edge. Such measures often will allow unused streets to become valuable traffic thoroughfares and, as an incidental benefit, will often immeasurably increase land values in deteriorated zones languishing in the backwash of a city's growth.

RELIEF FOR MASS TRANSPORTATION

The obvious and the only genuine solution to the traffic problem occasioned by mass transportation equipment (streetcars, trackless trolleys, buses, etc.) operating almost end to end on the city streets is to remove such equipment from the surface of the streets, at least in the areas where it is a major traffic element. Not only will this measure relieve surface traffic, but it also will greatly expedite the movement of the mass transportation equipment and enhance its value to the patrons. Consequently, no long range plan for traffic correction can be considered adequate unless it will ultimately provide off-the-surface mass transportation—for example, a subway system.

However, the natural outline or geographical shape of most cities in the population brackets from 750,000 to 2,000,000 is such that it would be impossible to build a complete, city-wide conventional subway system that would defray its own cost. Subway construction is extremely expensive, and even to approach a self-liquidating basis each line must serve a tremendous number of patrons. When a city is laid out in a long, narrow rectangle, so that a few lines can serve a large and densely populated area, subways can then at least appear to be a sound financial venture. On the other hand, when a city is fan-shaped, as are most United States cities, with streets radiating in all directions and with none of these streets serving a continuous strip of heavily populated zones, the investment required for subways cannot possibly be economically self-supporting.

Under these conditions, the most that can be justified is a composite system consisting of conventional surface equipment (such as streetcars) in all thinly populated areas; and, in the definitely critical zone, either subsurface tubes for these same streetcars or, as an alternate, a relatively small downtown conventional subway system to which patrons can transfer from the radiating surface lines. The business district subway system is, of course, mandatory when buses or trackless trolleys would be used in lieu of streetcars for surface operation in the outer or relatively thinly populated areas.

Elevated rapid transit systems are also practicable, but past experience with this type of facility has demonstrated many shortcomings. Thus, generally speaking, they can be considered undesirable. Also, because the days of the streetcar appear to be numbered and the bus is becoming more popular, it is probable that the central zone subway system is the logical choice for the average city.

The operation of such a system is obvious. From terminal points located on principal arteries a short distance from the central business district, subway trains run into the central zone where they traverse a common subsurface loop or loops routed strategically under important business streets so as to serve office buildings, shopping districts, commercial areas, theater districts, and the like. Transfer privileges are extended to surface transportation. Passengers from a suburb travel by bus along a principal surface radiating street until the subway terminal at the outer fringe of the central district is reached. At this point the passengers leave the bus, descend to the subway, and continue to their destinations in the downtown area, the buses returning to the suburbs. If the passengers wish to continue diametrically across the city, as a small proportion of them may, they stay on the subway until reaching the opposite side of the central zone and then retransfer to radiating surface buses. Of course, buses and other surface equipment should be almost entirely excluded from the area served by the subways.

Another operating possibility in such a system is the convenience offered those who are forced to use automobiles. Since the subway terminals should be located in the deteriorated zones outside the central district, it should be economically practicable to provide extensive parking lots at such locations. Motorists can then leave their cars at these points and, in a few minutes, reach their destination in the business section via the high-speed subway.

This type of off-the-surface facility lends itself to expansion in keeping with the growth of the city and will in no way hamper the normal expansion of the central district. If modern design criteria in lightweight, high-speed railroad practice are applied to subway equipment, it can be smooth, clean, quiet, and reasonably comfortable. Equitable fares and public pressure, returning to mass transportation after 25 years, can make it so.

At only a fraction of the cost of in-city freeway facilities for local traffic—and that fraction a self-liquidating profit-returning investment instead of a nonrecapturable outlay—mass transportation to the central business district can be immeasurably improved. Under a coordinated attack on the problem, increasing restrictions on parking and operation in the downtown area, coupled with the present exasperating traffic congestion, will open up new sources of mass transportation patronage in the outlying districts. New patronage will justify expanded networks of bus coverage. Increased bus patronage will, in time, justify subway extension. Subway extension, offering still faster transportation to the business section, will open up still more sources of patronage. Thus, the cycle will continue of its own volition.

For some obscure reason, decentralization of cities is viewed with alarm by many city planners and the proponents of expressways argue that one of the principal advantages of this type of facility is that it will reduce the tendency to decentralization. The record does not support such a contention. Timid drivers will not venture onto high-speed vehicular speedways. For example, suburban matrons, obliged to travel by automobile, although they would prefer to shop in the downtown stores will often seek out a community shopping center, rather than brave fast-moving or congested traffic. On the other hand, convenient, comfortable mass transportation will bring them back to the downtown stores. The proof rests in the well-known fact that the cities with the most centralization have the most extensive mass transportation systems; and, on the other hand, cities depending largely on motor cars for transportation are the most decentralized.

Redevelopment of Commuter Service

To reduce still one more traffic type contributing to the "downtown" congestion, another type of mass transportation that can be redeveloped in conjunction with the central zone subway system is the commuter service to outlying communities accessible from existing railroad lines serving the city. Such service has languished in many cities, partly because of the automobile. In most cases, however, the deteriorated service is caused by the inconvenience in getting from railroad stations to office and business districts. As a contact with railroad stations is well-nigh mandatory in any downtown subway system, it would usually be possible to provide subsurface rail interconnections at such points and thus to bring the commuter equipment right into the downtown subway loop. With modern equipment designs, it would be a simple matter to devise lightweight rolling stock with retractable third-rail shoes for conventional subway operation in the city and for diesel-electric operation on the surface rails in the outlying sections. Adequate service of this type, right into the heart of the city, would open up whole new groups of patronage in the one-car subsuburban community class.

Conclusions

From a realistic approach to the functional problems of a city as tempered by an appreciation of the economic condition of such communities, it appears that, although urban expressways seem to be the answer for intercity and cross-the-city vehicular traffic, the logical approach to the purely local traffic problem is first to develop mass transportation to the utmost and then, as a secondary measure, to give consideration to in-city expressways for the convenience of personal, luxury transportation. Expensive as subway construction is, it is still cheaper from an over-all viewpoint because, as the nucleus of a composite mass transportation system, it is concentrated at the hub of the trouble and not spread out along the spokes or the rim of the street pattern. The subway nucleus relieves the traffic precisely where the relief is needed—right in the heart of the business district. It disperses the problem outward instead of compressing it into the dense and already hopelessly overcrowded city core. The use of urban expressways for local traffic appears basically unsound. It will not solve the downtown traffic problem, but will only perpetuate it.

DISCUSSION

Elmer F. Steigelman,[2] **Assoc. M. ASCE.**—Anyone who drives a car will agree with Mr. Bradfield when he states (see "Introduction"), "* * * it is obvious to everyone that traffic conditions in metropolitan areas have become intolerable and some remedy must be found." It might be added that the more intolerable the condition the more drastic the remedy would have to be. The paper presents the problem precisely, and is timely.

Having hotels, stores, theaters, and offices concentrated in a downtown area of a city is an advantage of modern living and decentralization of these facilities is not to be desired. Rather one must accept the fact that there is a centralized downtown area, that people will come to this area in their own cars instead of using public transportation, and that this traffic, together with other traffic, creates an intolerable traffic condition. The writer would like to contribute the following discussion on this condition as it affects the downtown area only:

1. The author suggests that relief for mass transportation could be obtained by removing it from the surface of the street to either subways or elevated structures. However, this procedure is quite expensive and a better one would be to prohibit entirely the use of the downtown area streets by private cars. Traffic in a downtown area is composed of private cars, trucks, and public conveyances (street cars and buses). Of these, trucks are necessary to bring goods to stores during business hours; public conveyances are necessary for people without cars. Thus, the obvious group to eliminate, and one which would relieve the situation considerably, would be the private car.

2. To compensate for the prohibition of private cars and to obviate the necessity of constructing expensive circumferential expressways around the business district, public garages should be erected at the fringe of the district in sufficient number and size to accommodate the cars that would normally park on the streets in the business district. These garages should be placed on the perimeter of the district so that the center of the district would be within walking distance.

If the city would prohibit use of the downtown streets by private cars the incentive for private construction and ownership of the garages would be provided. However, a competent planning organization should determine the location and number of the garages for maximum efficiency. If private financing and construction of the garages could be guaranteed, cities would be relieved of additional strain on their finances. Certainly, the cost of the garages would not add up to the cost of subways and circumferential expressways.

3. If the business district is so large that walking from the perimeter garages would not be convenient, an integrated, circulating, low fare transportation system within the downtown area should be provided. To be attractive to the public these units should be spaced not more than 3 min apart and the fare should be approximately 5¢. There would be no transfer privileges

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[2] Cons. Engr., Berkeley, Calif.

to or from this system. Thus, a person would park his car in a perimeter garage and if his destination were not within walking distance, he would use the circulating transportation system, or a taxi, to reach it.

A public garage system has an advantage over the subway program in that it could be financed privately whereas the subway would probably require bond issues. Time required to secure relief would be shorter for a garage system than for a subway or elevated railway program. By eliminating private cars from the downtown area more safety for pedestrians would be obtained and accidents to vehicles might be reduced.

HAROLD M. LEWIS,[3] M. ASCE.—Although expressways were first developed as approaches to cities through suburban areas, it is now recognized that they should extend into urban areas. Mr. Bradfield accepts this, but states (under the heading, "Relief for Intercity Traffic") that they should be used only for "intercity and cross-the-city traffic" and that "local traffic," which should be reduced in quantity by greater provision and use of mass transportation, can be handled on ordinary city streets with the aid of improved regulation (see "Relief for Local Traffic").

The writer agrees that expressways are justified only in locations where they will be of use to through traffic, but believes that the inner sections of such routes should be so designed that they can also take care of intracity traffic which is moving either to or from the downtown areas or from one side of the city to another. In a small city such intracity traffic may travel only half a mile or more along the expressway, whereas in a large city it will travel on the expressway for at least 2 or 3 miles. Thus, the express route can be used to its maximum efficiency.

Mention should be made, also, of the opportunity for still greater use of expressways through their joint use by private cars and mass transit vehicles in the form of buses operated over the main roadways or of rail facilities located in a part of the right of way reserved for that purpose. Such a treatment of a considerable part of the proposed expressway system in Los Angeles, Calif., was proposed at a "transportation clinic" held in that city in December, 1945;[4] and the Hollywood Parkway was designed to accommodate buses, with special turnouts at which they can pick up and discharge passengers.

The writer agrees that the use of private automobiles in central areas should be discouraged, but would include three additional steps to the provision of "adequate mass transportation" as part of the municipal action necessary to implement such a program. These are: The elimination of the free use of downtown streets for transient parking, the substitution of a limited curb side space for short time parking controlled by parking meters, and supplementary off-street parking facilities. The last of these may be provided in retail centers of small municipalities as free municipal parking lots; in other cases they should be operated on a fee basis. In very large cities some form of municipal subsidy may be required to bring the fees within the range of what the driver is willing to pay.

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[3] Cons. Engr. and City Planner, New York, N. Y.

Although some planners have suggested that the modern city should be crisscrossed by a network of expressways, the writer agrees with the author that such schemes are visionary and impractical and that most of the arterial networks of cities in the United States will continue to be surface level streets. The expressway system within urban areas will not often require more than one loop, or circumferential, route except in large metropolitan areas where both an inner and an outer loop may be desirable. If planned far enough in advance most of the outer route can be of limited access type, constructed at grade with depression or elevation only at connecting thoroughfares.

In many cities the expressway system need be only a single route in the direction of prevailing traffic, or two intersecting routes where traffic is heavy in two directions. In each case it should pass through the city along the edge of the downtown area.[5] In an extreme case an additional link may have to cross the central area. For example, in the New York (N. Y.) region the Mid-Manhattan Expressway shown on the New York City master plan is needed to provide for heavy traffic between the New Jersey and the Long Island, New York, sections of the region.

As cities increase in size, the percentage of people entering downtown areas by private car decreases—primarily because mass transportation has been provided. Experience has shown that when cities continue to grow without comprehensive planning both types of facilities become unsatisfactory because of overcrowding and delays, or both. Good mass transportation has the advantages of cheapness, efficiency of street use in the case of surface transit, and speed in the case of subways. The private automobile has the advantages of comfort and convenience. The solution of the congestion problem in the downtown areas is not simply to ban the private car in such areas and to attempt to shift the entire load to mass transit lines. There must be a balanced transportation system in which both private cars and mass transportation have a place.

The stagnation of street movement in central areas has resulted from three main factors:

(1) City planning that failed to coordinate the design of the street system with control over the bulk of buildings fronting on the streets, and that laid out new arterial routes without giving thought to where the cars they bring in will go or park in the downtown area;

(2) Laxity of municipal authorities in permitting downtown streets to be used as free storage space for long time parkers, either through failure to impose restrictions or through inability or unwillingness to enforce restrictions when imposed; and

(3) Failure of both public and private transit companies to modernize and keep up to date their mass transit facilities.

If large cities will stop providing free parking space for transients and substitute fee parking, economic forces will help to bring about a solution. Such fee parking can be provided through meters for short time parking, where curb

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[4] "City of Los Angeles—Recommended Program for Improvement of Transportation and Traffic Facilities in the Metropolitan Area," by De Leuw, Cather and Co., Harold M. Lewis, and Joe R. Ong, Los Angeles, Calif., December, 1945.

[5] "Express Highway Planning in Metropolitan Areas," by Joseph Barnett, *Transactions*, ASCE, Vol. 112, 1947, pp. 643–644, Figs. 2 and 3.

side space can be set aside for such use, and through off-street parking in lots or garages for long time parking. Most of the workers previously driving into the area and parking all day will then have to shift entirely to mass transit lines or drive only to convenient outlying transfer points. The few who can afford to pay for downtown off-street parking can continue to drive in. This arrangement should leave room on the streets for the salesmen, visitors, executives, and shoppers who will, by either necessity or preference, continue to come by private automobile.

City planners are more often accused of urging decentralization and being willing to let overgrown cities decay than of viewing decentralization with alarm. The decentralization of residences in metropolitan areas has been a steady trend for several decades and when it involves a shift to planned areas it is beneficial. On the other hand, centralization of many types of business activities has distinct advantages, and cities cannot afford to let their established business centers become choked and blighted by inaccessibility. At the same time cities should plan for some recentralization of business in subcenters rather than permit overbuilding in a single center.

In attempting to prove that mass transportation will keep patrons coming to downtown stores because (under the heading, "Relief for Mass Transportation") "* * * cities with the most centralization have the most extensive mass transportation systems* * *" the author may be putting the cart before the horse. Would it not be more logical to say that cities which have developed mass transportation systems have thereby made possible great centralization? This result has certainly been true in New York City, which extended its rapid transit system into unbuilt areas, thereby attracting a vast resident population to those areas and greatly increasing the congestion in the downtown business areas where all the rapid transit lines focused.

The author's proposed cycle of downtown traffic restrictions, increased bus patronage, subway extension, additional outlying bus transportation, and so on, seems to offer hope of a more efficient transportation system, but its effectiveness will depend on how these facilities are planned. If the new transit lines continue to focus on the old downtown center, there is danger of establishing a vicious cycle of new transit lines, bordering residential development, demand for additional parallel transit lines, additional bordering development, and so on—all leading to greater concentration and congestion in the downtown area.

JOSEPH BARNETT,[6] M. ASCE.—Quite properly, this excellent paper emphasizes the need for mass transportation facilities in solving the urban traffic problem but the author's presentation is too strong in implying that advocates of urban expressways contend that they are "* * * the panacea for all these [traffic] ills." Responsible officials recognize that congestion in urban areas will be relieved only by a broad attack on the problem involving several related problems requiring the force and the cooperation of numerous city, state, and federal agencies. The writer has consistently tried to point out and to simplify the broad effort necessary, and has broken it down into the following five-point program:

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[6] Chf., Urban Road Div., Public Roads Administration, Federal Works Agency, Washington, D. C.

(1) The improvement of the arterial route system.

(2) The improvement of the streets that must form distributors for all traffic. Nearly all people and goods transported in motor vehicles must be delivered to their ultimate destinations on local streets.

(3) The improvement of the transit system. By no known plan can all traffic be handled by private vehicles. The transit vehicle, whether it be a railroad car, trolley car, or bus, is a much more efficient user of street space in providing mass transportation than is a private car; and, to an increasing degree, the public must look to the transit vehicle to carry a large part of the passenger load, particularly between home and work during the hours of greatest congestion.

(4) The solution of the terminal problem—adequate provision for parking and for the loading and the unloading of goods. It seems unreasonable to expect a city to use its extremely valuable streets for the storage of vehicles for long periods of time, particularly during rush hours, or for the use of trucks loading and unloading goods. Especially in the case of large commercial and industrial establishments, there seems to be no less justification for requiring off-street loading and unloading areas for trucks than for railroad cars.

(5) The more efficient use of present facilities. In this category the writer would include spot improvements that will make possible more efficient use of streets. Among these might be mentioned widened and improved pavements, markings and signs, more efficient traffic signal control, channelization, and other isolated improvements at points of congestion.

This program will be necessary even though the type of mass transit the author emphasizes (namely, subways) are provided. Subways are economically justified in probably only a few cities. Only five cities in the United States have populations greater than 1,000,000 and not more than fifteen cities have a million people, if the satellite or suburban cities are included. On the other hand, about a thousand cities have populations of more than 10,000. Nearly all these have serious traffic problems. Mass transit for most of these cities consists of gas buses, trolley buses, and trolley cars.

Mass transit is not on the increase as it should be. The impetus given mass transit during World War II has not resulted in permanent gains, and there are signs that mass transit operators are having their old troubles despite the increasing difficulty in moving and parking experienced by drivers of private automobiles in congested cities. Rail operation on its exclusive right of way is excepted, of course. The cause of such difficulties appears to be the vicious cycle in which mass transit operators and the public find themselves. Mass transit operation on city streets is slow and costly because of frequent stops and congestion. As the time of travel on mass transit increases and the fare is raised to meet costs, fewer people ride and more drive, further aggravating congestion.

The arterial routes in most cities are lined with commercial establishments. The routes serve several types of traffic that interfere with one another. Arterial route traffic wants to move fast, without stopping. Neighborhood shopping traffic wants to move in the same manner but only as far as the commercial establishments, where it stops. Buses want to move as rapidly as possible, but they want to stop frequently to pick up passengers. Cross traffic wants to cross without stopping at intersections, and pedestrains want to cross everywhere. Congestion increases as one approaches the center of the city because traffic builds up to higher and higher volumes, width of pavement decreases because of increasing cost of right of way, and intersections become more frequent with higher volume of cross traffic. Street capacity drops sharply just where volume increases and congestion begins. Mass transit vehicles must slow down along with all other vehicles.

The solution being proposed is the construction of separate, free-flowing facilities for arterial route traffic which, in addition to serving such traffic, will release the streets for local traffic including mass transit vehicles. In addition, mass transit of an express nature can then be provided on the free-flowing arterials themselves. The reduced time and cost of mass transit operation on both the new and the old arterials result in a superior service which should encourage their greater use, and the shift back from private vehicles to mass transit would contribute further to lessening congestion. Here are the ingredients for a reversal of the present cycle with benefit for all.

Railed mass transit is being considered on a few expressways and is being provided on one, the West Expressway in Chicago, Ill. The only logical location for it is in a median of sufficient width to enable stations and platforms to be constructed clear of the high-speed through traffic lanes. Climbing on the part of passengers is unavoidable in such cases but there is increasing evidence that, where volume of passengers justifies it, even this effort will be made unnecessary by the use of escalators, which have a long record of reliable service.

RALPH WESLEY JONES,[7] M. ASCE.—The author opposes high-speed urban highways and favors the elimination in the downtown business areas of vehicular traffic (characterized as unnecessary), except mass transport vehicles (subways). Greater traffic volumes in cities do not occur in downtown areas because decentralization and by-pass routes have spread the load over greater areas. Mr. Bradfield claims that most of the older cities have an inconsistent street layout in central zones, which causes traffic congestion, and he recommends razing buildings and filling in harbor indentations to provide continuous routes.

The author's theme in opposing in-city expressways can be crystallized into almost one statement—that the traffic congestion at the central business (downtown) areas of major cities will be corrected if all but mass transport vehicles are ruled off the streets. As further improvement, he suggests that public transport vehicles be placed in subways. Obviously, if public and private vehicles disappear from the streets, there will be no vehicular street congestion; but vehicles do not disappear so readily.

The author states that people are wanted in the central business areas rather than private automobiles—and suggests that if the existing intolerable traffic conditions are deliberately permitted to be aggravated, such unwanted cars will stay away. He proposes that subways be operated in loop arrangement under all downtown business streets to serve offices, shops, theaters, and railroads—with transfer and "park and ride" privileges at terminals of streetcars and buses which would be located along this subway loop at the fringe of the central business area.

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[7] Cons. Engr., Philadelphia, Pa.

Several of the author's premises are open to question. He focuses all attention on traffic congestion at the downtown (central business) area, when traffic surveys clearly show that the greatest concentrations of city vehicular traffic are at points some distance from the center of the city. The author favors intercity expressways, but he opposes them within city limits for local motorists. and tolerates them only for through or cross-the-city traffic. Probably, one need is to elucidate what constitutes a "major city" and what constitutes a "local motorist."

At the 1940 census, there were only nine cities in the United States with populations exceeding 750,000; yet there were thirteen states with populations of less than 750,000. Surely, the author's statement that expressways are unnecessary inside cities, but are needed between cities is debatable. The area of New York, N. Y., covers 650 sq miles, which is half the size of Rhode Island, and one third that of Delaware. Local traffic must include that from metropolitan areas of cities; such areas increase the major city population by 50% or more, and the major city area by as much as 700%. In 1940, there were sixteen metropolitan areas with populations exceeding 750,000. The Philadelphia, Pa., metropolitan area is approximately the size of Rhode Island, so, if interurban expressways are proper for states like Rhode Island, they are more proper at larger cities.

The need for city expressways cannot be disposed of by empirical formulas, but must be determined by individual studies made at each community; nor should the author confine his planning to circumferential expressway belts as the only geometric pattern. Notable examples of satisfactory radial expressways include the East River Expressway and the West Side Highway in Manhattan (New York City) and the Parkway and Fairmount Park drives in Philadelphia. New York's Midtown Tunnel with approaches on Long Island is of incalculable value to this area. Expressways, like high-speed railways and railroads, also tend to enhance the values of certain adjacent benefited realty.

Municipal planners would agree with the author, that the motorist is not entitled to unrestricted parking and driving rights on city streets. A debatable point is that the local motorist should be legislated off the streets, but that the distant tourist should be welcomed for his shopping and hotel dollars. The local motorist is entitled to proper use—but not misuse—of the streets, because he has been assessed for their construction, and he is taxed annually to maintain them. Philadelphia has 33,000,000 sq yd of paved highways and 260,000 registered private automobiles—an average of 127 sq yd per car. What use would the author have the local resident make of such paving that he pays for? If he could not use it, why would he build it, or permit himself to be taxed for it?

It seems as questionable for the author to argue for in-city expressways for the convenience of the out-of-town user (to the exclusion of the local owner of the facility), as for one to suggest developing the United States for the convenience of some foreign nation. In modern highway planning, cities are bypassed for the many who prefer such routes, as well as for the benefit of the city itself; yet, expressways are constructed to connect conveniently the downtown commercial area with the outlying industrial and residential areas. In-city expressways at major cities cannot be justified on the basis of tourists alone. The present trend is for truck terminals to be located near principal through traffic routes, quite some distance from commercial areas in the center of the city.

Quite properly, the writer believes that motorists must not be given unrestricted parking or other traffic privileges; but, the solution to this problem is not too difficult. The author, however, bases his argument on the fact that 43 cars out of 90 cars parked in a central city parking lot were used by persons "going to employment"; such a survey means nothing in the over-all traffic problem which must catalog every vehicle entering or leaving the central area. Traffic problems would practically evaporate if all cars were anchored in parking lots instead of cruising about the streets. By actual statistics, 1,650 trolley cars in a major city made 33,000 trips daily—an average of 20 trips per car. It is probable, too, that some of these 43 drivers find a car essential to their employment, whereas others find public transportation from home extremely unsatisfactory. Origin and destination surveys show that the majority of persons driving to employment do so from necessity—not by choice.

The author's statement that expressways will preclude municipal development beyond the expressway (much as did the Great Wall of China), is not supportable. The fortification wall about Paris, France, could not permanently confine that city; nor could the Market Street Elevated Railway in Philadelphia restrict normal growth on either side, as the author contends. Actually, the reverse is true—the area tributary to this elevated railway developed phenomenally and there was a free interchange of traffic from one side of the "El" to the other, both by private vehicle and by mass transport lines.

The author claims that an expressway would save only from 7 min to 12 min from the suburbs to the downtown district, but that saving would not change the time of arrival at, or departure from, the downtown area. Naturally, the time saved would depend on the conditions prevailing—especially, the distance and the speed possible by both routes; however, with congestion so bad that 15 or 20 min are consumed sometimes, in driving 1 mile during peak traffic on regular streets at Philadelphia, an expressway would save much more than 7 to 12 min for persons living at its 9-mile edge. Expressways are more economical than conventional streets of equal carrying capacities, on the simple basis of possible velocity. The elementary formula for the carrying capacity per hour of a traffic lane is $\frac{\text{velocity (feet per hour)}}{\text{length of vehicle plus clearance between cars}}$. Thus, expressways save highway area. Additional economies of limited access expressways include fewer traffic officers, and less traffic control equipment—plus their added safety.

The points of greatest traffic volume at major cities are not at the downtown business centers—nor have they been for many years. Actually, in a city of 2,000,000 population, the total vehicular traffic at the maximum hour showed the principal downtown crossroads to clock only 40% and 45%, respectively, of the peak hour volume at arterial points 3 miles to the northwest and 4 miles to the north of the downtown point. On the principal highway, vehicular traffic volume increased progressively for 3½ miles from the main business center—then it tapered off continuously. During evenings, every intersecting street feeds additional vehicles into the principal artery of traffic, and also drains some off; in the mornings, the reverse is true.

This same main highway reflects other conditions that oppose the author's arguments against limited access highways. Thus, even in 1930, before today's aggravated conditions, the average vehicular speeds, inbound and outbound, were found to be as follows, for the distances shown from the center of the downtown business area:

Distance from center, in miles	Speed, in miles per hour
0–1	15
1–2	18
2–4	19.5
4–5	20.5
5–6	22
6–8	28

(Maximum traffic volume occurred at a speed of 19.5 miles per hr.) These statistics show that the volume of vehicles, alone, does not control the average speed; rather, a more important influence on speed is the number of intersecting streets where drivers must be "on edge" more constantly.

Although he agrees with the author that expressways will increase the need for parking facilities, the writer feels that such induced increase will be slight and that the large percentage of persons who conveniently can do so will prefer public transportation to driving their own cars; nor do the facts sustain the author's statement that major cities have failed to develop or modernize their public transport facilities in parallel with increasing use of the automobile for personal transportation. Mass transportation by private companies has proved so unsuccessful, financially, that the number of such operators has declined, by merger or otherwise, from 1,200 in 1922, to 478 in 1937. Still, in spite of such decline, total transit passengers gained 50%, from 15,600,000,000 in 1930, to 23,400,000,000 in 1946; in the same period, private automobile registrations, which are some measure of car usage, increased only 28%.

In 1920, the United States had in operation 800 miles of subway and elevated lines; by 1935, this had increased to 1,230 miles. Since the 1920's, the City of Philadelphia has expended approximately $170,000,000 for municipally-owned high-speed transit facilities; New York City has added greatly to its high-speed system. Much additional mileage would have been built had it not been for a long economic depression, followed by a long war that precluded construction. In some major cities, the local governments have stepped in with transit improvements when bankruptcy faced the private corporations.

The author makes a point that expressways do not prevent the healthy tendency to decentralize. Actually, expressways accelerate decentralization.

The fact is that decentralization has been underway, progressively, in many cities, since long before the impact of the motor vehicle. Then, too, as cities expanded, so did business, industry, and homes over larger areas; yet, population statistics of urban places show clearly the absolute need for greater decentralization as follows:

Census year	Population (millions)	Percentage of total
1900	30	39.5
1910	42	45.7
1920	54	51.2
1930	69	56.2
1940	74	56.5

No one should expect to crowd the commercial, industrial, and social activities of such tremendously increased populations into the downtown central city areas as they existed at the turn of the century. Planners dare not attempt to restrict changes in modes of living that improve conditions. Workers, today, travel daily 20 to 25 miles to their jobs when, formerly, the majority of factory employees lived within walking distance of their occupations. Few workers who drive risk the conflict with city traffic, so, wherever possible, they use faster alternate routes.

Statistics show, too, that workers' traffic peaks occur at different hours for cities in different population groups. Industrial employees usually start and quit work as much as one hour earlier than office or store workers. Great numbers of industrial workers seldom see the downtown area, because they live and work in other parts. Decentralization is a proper trend, with community shopping and theater centers in surburban areas; expressways and other highway and transit facilities are made necessary to bind such sections together more effectively. Another traffic peak is found in most cities at about 9 p.m.—the entertainment peak. Surely, this evening vehicular traffic cannot be lured, in any appreciable volume, to the mass transportation lines. This traffic is profitable to general business; it will avoid unreasonably congested routes, but will make practical use of improved facilities, even if they be toll roads.

The author's statement that timid drivers and suburban matrons are afraid of expressways and will not use them is not on sound ground. Records of urban accidents show that 70% more collisions between motor vehicles occur at intersections than at points other than intersections. This is a striking point of argument in favor of limited access highways—with limited intersections. It is further support, and explains the reasons, for the tabulation made in an earlier paragraph, that the slowest urban vehicular speeds are realized nearer the city center although traffic volume here is not maximum. Obviously, the greater number of street intersections and the greater number of vehicles turning onto or off the main traffic artery, as well as pedestrians crossing (as they do near the city business center), slows up the traffic appreciably.

Records of rural areas showed that only one out of 11.6 accidents occured at intersections. Beyond doubt, street intersections are the principal scenes of urban motor accidents, and to eliminate such points of probable accident, by expressways or otherwise, is a step toward safety. The suburban matron on a shopping tour would find other statistics favoring her hours of travel—the incidence of urban traffic accidents occurring in the following order of declining importance:

Severe periods of traffic accidents	Time
1	7 p.m.– 8 p.m.
2	8 p.m.– 9 p.m.
3	5 p.m.– 6 p.m.
4	12 p.m.– 1 a.m.
5	6 p.m.– 7 p.m.
6	9 p.m.–10 p.m.
7	11 p.m.–12 p.m.
8	10 p.m.–11 p.m.
9	4 p.m.– 5 p.m.

On-the-site observation, as well as study of street maps, reveals few larger American cities with the inconsistencies in their central city street layout that the author mentions, and for which he prescribes razing obstructions and filling in harbor indentations to make a more direct route. A rather remarkable rectangular pattern of streets, with certain radials, prevails at New York, Chicago, Ill., Philadelphia, Washington, D. C., Toronto, Ont., Canada, Seattle, Wash., San Francisco, Calif., Portland, Ore., New Orleans, La., Minneapolis, Minn., Saint Paul, Minn., Milwaukee, Wis., Los Angeles, Calif., Kansas City, Mo., Indianapolis, Ind., Detroit, Mich., Denver, Colo., and others, in spite of the fact that many of these face on curving lakes or rivers with the attendant difficulties to uniform layout. Few American cities can be called bad; and, here, it was nature who made certain requirements. For example, cities with difficult layout include Baltimore, Md., Buffalo, N. Y., Cleveland, Ohio, Pittsburgh, Pa., St. Louis, Mo., and (perhaps the worst) Boston, Mass. However, such European cities as Paris, Vienna, Austria, and London, England, where it is difficult to find a rectangular pattern downtown, may really be classed as bad. On the other hand, consider the age of these Eurpoean cities. As for filling in the harbor indentations, as is being done at Rio de Janeiro, Brazil, for a boulevard drive in a park section, this would be impossible at a port city with piers and docks jutting into the harbor. Also, such a filled in route across a harbor would provide a limited access highway, at least for the length of such cutoff, an improvement that the author opposes in favor of city streets with frequent intersections.

If the author has a complete remedy, by simply keeping vehicles out of the city and by putting mass transport underground, he does not need additional corrective measures. Actually, if mass transport were put below ground as he suggests, it could no longer suffer from the motorist; the latter could then blame no one but the motorist for any difficulties, if he had any. Actually, the automobilist would have no difficulty with traffic, if mass transport were off the streets. A winter strike of all operators of subway, elevated, trolley, bus, and taxicab lines at Philadelphia created a utopian condition for the motorist; he wished the strike would never end. This drastic step of taking mass transport off the surface to aid traffic is not recommended; mass transport vehicles are troublesome, but necessary.

The author suggests more one-way streets, more traffic controls, and elimination of curb parking and truck servicing during business hours. Experience shows that most cities have approached the solution of these problems quite well. Philadelphia has 480 miles of one-way streets, nearly 22% of its 2,200 miles of paved streets. Probably, curb parking is the worst offender; and it is the easiest to correct, although the parker has certain rights, especially if he paid taxes for the street on which parking is restricted. Newer buildings are constructed to provide off-street truck handling, but one cannot legislate against the current practice; nor would one sensibly wish to do anything to harrass or destroy industry, the lifeblood of a community. Some relief to this problem is accomplished by pickup trucks, which retail loads between the various buildings and the motor truck terminals, thus reducing the number and size of vehicles on the downtown streets.

The author's statement that subways can be built at lower cost, and on a self-liquidating and profit-returning basis, will find little factual support. Municipal experts and transit operators are agreed that subways cannot be financially profitable. Local expressways could charge tolls similar to the successful practice with superhighways and bridges. Excellent arrangements include planning for joint development and use of a combined right of way for vehicular and rail traffic side by side. An old plan was to use the same bridge for a joint rail and highway crossing, with economy for both users. Some combined usage of limited access facilities by public and private conveyances is just as sane as is the practice of combining them on a city street or state highway.

Mr. Bradfield's statement that cities with densest centralization have the most extensive mass transport systems, and vice versa, is merely an observation of economics; sound railroading depends on revenue per mile, but mass transport cannot be built in sparsely settled sections with few riders per mile. Therefore, one can expect to find ample transit lines in densely populated territory, but one cannot expect to find satisfactory public transport where the area is thinly settled. As citizens tend toward the construction of homes with larger lots of ground away from the city centers, people will find an increasing tendency toward greater use of the private automobile, instead of travel by public utilities entirely, even though the private car may be used only to reach a convenient line of high-speed transit, or for shopping or relaxing at the neighborhood center.

Conclusion.—Cities must be planned today, as they always have been, for the foreseeable needs. Today, the planner can foresee farther than in former times. The earlier American colonial cities were built for the horse traffic of the day—even the horse car lines were not foreseen when many of these cities were planned. The trolley cars of the 1890's were not anticipated in the 1600's or the 1700's, or even in the 1850's, and no provision was made for them. Many downtown streets have never been widened, in spite of the fact that the city's population may have increased fivefold, tenfold, or twentyfold.

The greatness of a city depends upon the ease with which the people and their business can move from one part to another, and from any and all parts to the outer world. Expressways, and high-speed traffic arteries of various types, are as economically sound as is the turbine driven steamship of today in comparison with the sailing vessel of yesteryear. Actually, a given volume of traffic can be carried at lower expense—for construction, maintenance, and operation—on a limited access route than on an ordinary city street. Mass transport, for freight as well as for passengers, must be developed in cities, because merchandise is vital. Although, in any metropolitan area, it may be difficult always to justify each modernization or improvement on its own basis, engineers dare not obstruct progress. Economically, the old community hand pump cost less to operate than does municipal water supply, but people prefer the latter; they must bear the costs if they want the comforts.

In no city of 100,000 or more population can the public be served adequately by mass transport facilities, if the enterprise is to succeed financially. Private automobiles and taxicabs must supplement the streetcar or bus, especially for the off hours. Nevertheless, few cities of 100,000 population would think of expressways, although such cities are entitled to have through streets for faster movement of traffic, as well as by-pass routes. As for subways, if all the mass transport vehicles—that the author claims now move end to end on the surface of the many downtown streets—were to be placed in one or more subway loops, an impossible condition would arise.

Flow of traffic is very similar to flow of water—a common formula for the volume of either being, "flow equals cross section times velocity." Traffic flow, like the flow of water, is greater where the conduit is smooth, with few laterals to create side eddies. Much can be accomplished by observation of the simple principles of flowing water and applying these principles to the handling of traffic.

One fact is clear, when the various recommendations of the author are put together: If all vehicles (private and mass transport) are taken from the streets, expressways will be uneconomic; expressways will also be unnecessary—as, indeed, will be the streets.

Curtis G. Bradfield,[8] M. ASCE.—It is gratifying to the writer to have had such learned discussions of this paper and to find that the little valid adverse comment must indicate rather general assent to the basic premise.

Mr. Steigelman presents an interesting but somewhat extreme projection of the basic theory advanced in the paper when he recommends prohibiting all private passenger cars from the congested areas, and proposes the erection of radial artery perimeter parking facilities as the innermost limits of private vehicular travel. Mr. Steigelman would place these travel termini around the edge of the downtown zone and suggests either walking or surface mass transportation equipment to accomplish circulation within the critical area. At first, this would appear to be a radical step to insure freeing streets from congestion. However, in view of the steady, inexorable increase in motor car registration, who can deny that some day such a drastic move may be necessary?

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[8] Public Works Officer, U. S. Coast Guard Yard, Curtis Bay, Md.

At any rate, although scarcely necessary now or in the foreseeable future (since such problems have a way of solving themselves), the possibility that some kind of prohibition ultimately may be required should be kept in mind by all city planners and devisers of traffic corrective measures. Incidentally, it is a strong argument against a too-liberal provision of urban expressways for local traffic when such facilities are not carefully balanced with adequate mass transportation.

The excellent discussion by Mr. Lewis adds greatly to the paper. The only comments the writer believes justified are the following:

1. The third paragraph, under the heading, "Relief for Intercity Traffic," was intended to convey the thought that the urban expressways needed for intercity traffic would automatically be valuable for much cross-the-city traffic and, similarly, for intracity vehicular movements. In any case, the writer agrees heartily with Mr. Lewis' statement: "* * * the inner sections of such [expressway] routes should be so designed that they can also take care of intracity traffic * * *."

2. As stated in the paper (under the heading, "Relief for Local Traffic"), the writer believes all curb parking should be prohibited during business hours in the congested zone. Meter controlled parking, as suggested by Mr. Lewis, is as much a traffic flow deterrent as unlimited curb parking and there is no justification for such retarding influences.

3. The writer feels it is not putting "the cart before the horse" to state that the cities with the most centralization have the most extensive mass transportation systems. Cities become large for economic reasons, not because of local mass transportation facilities. In such cities, business people tend to group together in the smallest possible cluster of centers because, up to now, no communication device ever used is as satisfactory, in the final analysis, as personal face-to-face contact when consummating, or arriving at, an agreement. Somewhat belatedly, transportation usually follows this grouping to move the workers in and out of these trade or profession centers. The centralization originates first and then transportation somehow blunders along to at least partial adequacy. If, by chance, adequate transportation should fail to evolve, obviously the entire scheme would bog down, decentralization would result, and large investments in downtown properties would be depreciated in an unreasonably short time. This process is now occurring in too many cities.

The only method of maintaining the value (and taxable assessment) of the downtown area is to improve the mass transportation service to that area in keeping with the demands of the greatest number of people desiring to move in and out of the district.

No discussion of urban expressways and urban mass transportation facilities could be considered complete without the comments of Mr. Barnett to provide the reaction of those who lean toward a liberal provision of expressways. Mr. Barnett's statement that there are about fifteen cities with combined city and satellite populations of 1,000,000 or more is, of course, accurate. However, when it is realized that the total population of these metropolises is approximately a fourth of the total population of the entire United States, the urgency of solving the transportation problem properly is readily apparent. Eleven of these fifteen cities do not now have off-the-surface mass transportation of any consequence. If these cities, which are of the utmost importance to the national economy, fail to meet the demand for adequate transportation or incur huge indebtedness for a type of facility which does not halt the too-rapid tendency toward blight in their downtown areas, disastrous municipal financial conditions could easily arise.

As Mr. Barnett so clearly indicates, surface mass transit is in a vicious cycle leading only to ultimate strangulation and, seemingly, is not the answer to the problem. It certainly follows that, if planners make it still easier and more attractive for the motorist to bring his car into the zone of congestion, such a policy can only accelerate this eventual strangulation. The inescapable fact is that the greatest number of people desiring to enter and leave the downtown areas daily would use mass transportation facilities if the service were adequate. These are the people who must be considered, and whether they should be encouraged to use private motor cars by building a network of expressways or whether, instead, they should be discouraged from the use of automobiles by furnishing adequate mass, rapid transit, is the crux of this entire discussion.

It is not the occasional traveling salesman, the service mechanic, or the insurance collector who congests cities when he brings his very necessary car downtown. Instead, it is the intolerably large number of people who merely drive to and from their place of employment and leave their cars idle all day. They are the motorists that must be either accommodated or discouraged.

The writer gathers from the comments of Mr. Jones that the paper was not sufficiently lucid. At least, there was no intention that it should be interpreted as meaning all downtown vehicular traffic is unnecessary as Mr. Jones states. Instead, the paper seemed to the writer to be clear in remarking that much of the downtown vehicular traffic arising from motorists merely going to places of employment or keeping business appointments was not really mandatory and these people could be served by adequate mass transportation.

Again, the paper would be quoted more accurately if Mr. Jones' discussion read "many" instead of "most" when commenting on the number of cities with imperfect grid patterns of street layout.

Still again, the writer is unable to find in the paper any statement which could be commented on, thus: "Surely, the author's statement that expressways are unnecessary inside cities, but are needed between cities is debatable." Instead, the writer tried to state an opinion that expressways were needed in cities for intercity and cross-the-city traffic (and such incidental local traffic as could advantageously use the facilities), but that the immediate need for expressways for purely local shuttle traffic could not be determined accurately or positively until all possible methods of handling this movement volume by improved mass transportation facilities had been thoroughly exploited.

Mr. Jones further states: "* * * nor should the author confine his planning to circumferential expressway belts as the only geometric pattern." No part of this paper recommends any expressway pattern. Incidentally, the more important expressways Mr. Jones describes as "radial" are actually "tangential," as the writer recommends, and are basically for intercity or satellite community traffic.

The paper should not have given the impression that the writer proposed prohibiting the taxpayer from using his downtown streets. It did recommend discouraging the use of these thoroughfares unless the need was actually mandatory—a quite different approach. In this connection, there seems to be no impropriety in a policy of discouraging taxpayers from using various tax-supported facilities unless they have an urgent need for them. Cases in point are old people's homes, welfare clinics, and mental hospitals.

Unfortunately, Mr. Jones assumed the writer's statement "* * * the elevated trackage along Market Street (Philadelphia, Pa.) * * *" as meaning the Market Street Elevated (rapid transit) west of the Schuylkill River. Instead, the elevated trackage (railroad) along Market Street between the Schuylkill River and Broad Street Station was the intended example of an element of interference with consistent city growth. The cited example (a railroad fill) is known in Philadelphia as the "Chinese Wall" and the need for its removal has long been a familiar source of complaint on the part of most Philadelphians.

Other comments the writer feels may be justified when considering Mr. Jones' discussion are:

1. Citing an accident frequency table to contradict the statement that timid drivers avoid high-speed expressways does not appear germane.

2. Mr. Jones' criticism that streets projected over filled in harbor indentations, as recommended by the writer, would be objectionable appears to be hypercritical.

3. Despite Mr. Jones' skepticism, the writer believes it is self-evident that the net cost to the taxpayer of off-the-surface mass transportation facilities confined to the downtown district, even if not entirely self-liquidating, will be measurably less than the tremendous, nonrecapturable outlay required for an extensive network of urban expressways covering every part of a city except the downtown district. Expressways cost about $5,500,000 per mile and subways approximately $7,500,000 per mile. Obviously, the first cost of central zone subways will be immeasurably less than a city-wide network of expressways, and even the most skeptical cannot help but admit that some revenue will be collected to defray part of the rapid transit cost. On the other hand, the expressways will return nothing and will entail a constantly growing unliquidated maintenance cost.

4. The writer agrees with Mr. Jones that "Cities must be planned today * * * for the forseeable needs" but cannot agree that "* * * they always have

been * * *." There is little evidence that past city planning has been any more than an unrelated series of expedient measures arising from waves of indignant pressure on political groups. Furthermore, the writer doubts that the future definitely indicates a return to a modern version of the personal horse and buggy—the personal automobile—and the gradual atrophy of mass transportation. There is little place in modern metropolitan areas for personal luxury transportation—antiquated or modern versions—and the only feasible, sound, procedure for moving large volumes of people from place to place is by mass transportation equipment.

5. Mr. Jones' analogy of a hand pump compared to a modern municipal water supply supports the writer's contention. The hand pump was the personal facility—personal transportation, albeit either horse and buggy or automobile—and the municipal water supply is the mass facility—mass transportation.

6. There seems to be little evidence to support Mr. Jones' contention that subways cannot accommodate all mass transportation equipment. There are relatively few surface transit vehicles operating on Manhattan (New York, N. Y.) streets and the subways, although often overcrowded, successfully bear well-nigh the entire transportation burden.

7. Despite Mr. Jones' statement, the writer does not recommend that: "* * * all vehicles [should be] * * * taken from the streets * * *." He did suggest, however, that unnecessary surface traffic be reduced and that mass transportation be measurably improved. If this is done, it is quite possible that Mr. Jones' remark—"* * * expressways will be uneconomic; expressways will also be unnecessary * * *"—may prove to be true.

If the paper was not entirely clear in all respects, or if erroneous impressions were given, a brief restatement may be justified. The paper was, in essence, a plea for moderation in current city planning—an admonition that broad thinking on the entire subject, not just the motorists' woes, might be indicated before arbitrary steps are taken.

The United States is largely a nation of extremists. Its citizens will endure and tolerate a condition for far too long and then, suddenly, wish to alter the entire situation immediately.

Most cities are already burdened with a heavy tax load brought about, in many instances, by past inadequate expediencies or hastily planned measures which failed to cope with overevaluated or underevaluated problems.

This traffic problem and its effect on the economic well-being of cities in the United States is a matter of great concern. A proper solution of the dilemma must be found. The cost of either an expressway network or a composite system with some expressways and some intensively redeveloped mass transportation facilities will be huge and, if the wrong decision is made, there could be serious economic repercussions.

Some guidance may be found from past experience. There was a time when personal transportation, in the form of a horse and buggy, threatened to drive the lumbering omnibus from the city streets. However, instead of this, the horse car and cable car (the rapid transit of their day) were provided and the seeds of the present growth of modern downtown districts sown. Again there has arisen a form of personal transportation that is strangling the existence of the lumbering buses and street cars. The writer believes it is quite possible that the wiser course would be to concentrate planning efforts on rectifying the shortcomings of mass transportation and redeveloping it to the utmost before saddling the taxpayers with still more debt largely for the benefit of people who desire a luxury type of transportation to their places of employment.

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Paper No. 2396

SOME EFFECTS OF ANAEROBIC DIGESTION ON SEWAGE SLUDGE

By A M Rawn,[1] M. ASCE, and E. J. Candell[2]

With Discussion by Messrs. W. E. Abbott, Robert E. Leaver, A. L. Genter, and A M Rawn and E. J. Candell.

Synopsis

That anaerobic digestion of sewage sludge effects a marked reduction in the biochemical oxygen demand (B.O.D.) and coliform group density thereof, as well as a radical change in sludge characteristics, is well known. The extent of these phenomena, however, does not appear to have been very well explored, and factual data relating thereto have been gleaned from research leading toward other objectives.

This paper relates the results of an investigation aimed at discovering the numerical value of some of the changes occurring in digesting sewage sludge. The work was inspired by experience in the disposal of digested sewage sludge from temporary military establishments during World War II, and by public health regulations in some states to the effect that digested sewage sludge may not be discharged into state waters, fresh or salt.

Results of the investigation indicate that, where sludge banks will not be formed, there is little, if any, reason why well-digested sewage sludge may not be disposed of by dilution along with a primary, clarifier effluent if conditions will permit disposal of the latter with safety.

Introduction

The disposal of digested sewage sludge presents many perplexing problems and, at times, constitutes the major one in sewage treatment. As long as well-digested, wet sewage sludge continues to be viewed as a relatively dangerous substance, its final disposition will be presumed to be accompanied with great risk unless, prior to final disposal, it is rendered completely innocuous by the destruction of all its original, dangerous characteristics.

Note.—Published in November, 1948, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

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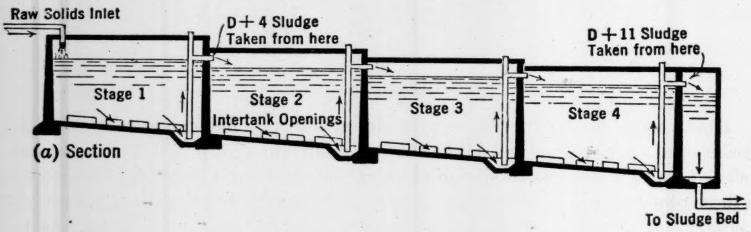
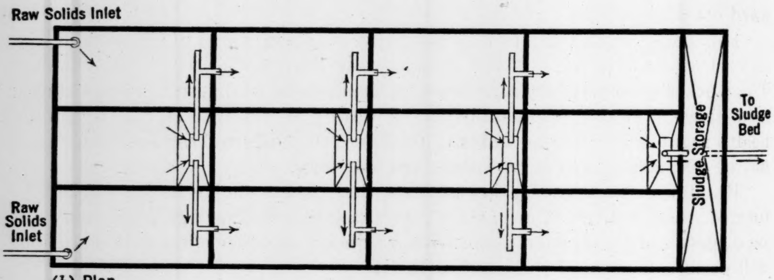
The purpose of this investigation is to evaluate the effect of returning digested sludge to primary effluent prior to disposal of the latter by dilution. Conclusions are drawn by measuring the changes which occur in B.O.D., bacterial population, grease, quantity of solids remaining, and other characteristics, after being digested under varying conditions, from 11 days to 80 days and then computing the effect of recombining digested sludge with primary effluent.

General Description of Experiments

Experimental work for this investigation was conducted at the joint disposal plant of the Los Angeles County Sanitation Districts, in Los Angeles, Calif. Work on the investigation was commenced on February 28, 1946, and still continues in some phases. Tests were made on sludge incubated in bottles and also on sludge taken from the digestion tanks at the disposal plant. In general, the observations of digestion phenomena were made on three series of samples.


Raw Solids Inlet
D + 4 Sludge Taken from here
D + 11 Sludge Taken from here
Stage 1
Stage 2
Intertank Openings
Stage 3
Stage 4
To Sludge Bed

(a) Section


Raw Solids Inlet
Raw Solids Inlet
Sludge Storage
To Sludge Bed

(b) Plan

Fig. 1.—Four-Stage Sewage Sludge Digestion Tanks of the Joint Disposal Plant, County Sanitation Districts of Los Angeles County, California

Before proceeding further with the description of the experiments, attention is directed to the design of sludge digestion tanks at the districts' plant. The tanks are designed in four stages[3] with intertank piping so that sludge discharges by gravity from the bottom of the tanks in each stage into the top of the tanks in the next stage, as shown in Fig. 1. Seed sludge from the last or fourth stage is returned to the first stage. No stirring is provided other than that which results from the flow of sludge through the tanks. The formation of scum mats is actually encouraged in this design. Cleaning of the tanks is necessary at from 4-year to 6-year intervals, but between cleanings the automatic segregation and retention of the lighter solids results in excellent plant performance. At the time of the tests reported herein, the average detention period of the sludge in the digestion tanks was 11 days; and that in tank No. 1, approximately 4 days.

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[3] "Multiple-Stage Sewage Sludge Digestion," by A M Rawn, A. Perry Banta, and Richard Pomeroy, *Transactions*, ASCE, Vol. 104, 1939, p. 93.

A more detailed description of the procedure with each of the three series laboratory tests follows:

Series 1.—The ten specimens comprising this series were each composed of 83% by volume of untreated sewage sludge from primary sedimentation and 17% by volume of digesting sewage sludge from the fourth stage of the sludge digestion tanks.

Series 2.—The five specimens in this series were taken directly from the fourth stage of the digestion tanks and allowed to continue digestion for 60 days. The age of the sludge when it was taken from the tanks was assumed to be 11 days.

Series 3.—The three specimens included in this series were taken from the outlet of the first stage of the sludge digestion tanks and allowed to continue digestion until the age of 60 days. When taken from the tanks, the average age of the sludge, including recirculated seed, was estimated to be 4 days.

In each of the series, sludge digestion was instituted or continued in 2-liter bottles at 85° F under a gas pressure of about 0.33 lb per sq in., thereby simulating conditions obtainable in the districts' digestion tanks.

FIVE-DAY (20° C) B.O.D. OF RAW SEWAGE SOLIDS

As determined by the method given in "Standard Methods for the Examination of Water and Sewage,"[4] 5-day B.O.D. results in values for sludge that appear to be neither true nor reliable. In every case the B.O.D. value obtained indicated oxygen depletion lower than that calculated on the basis of sludge removal from the sewage entering the plant. A typical test, as shown in Table 1, indicates a 5-day B.O.D. for the sample of 12,100 ppm, whereas the determination of removal of B.O.D. from the sewage passing through the treatment plant would indicate that the B.O.D. of the raw sludge should be 20,000 ppm.

TABLE 1.—BIOCHEMICAL OXYGEN DEMAND OF SLUDGE

Sludge[a] (ml) (1)	DEPLETION		B.O.D. (ppm) (4)
	0_2 (ppm) (2)	Available 0_2 (%) (3)	
0.06	3.70	44	15,600
0.10	4.87	58	12,100
0.12	5.19	62	10,900
0.14	5.50	66	9,780
Mean	....	..	12,100

[a] In 250-ml B.O.D. bottle.

The relative strengths of solids contained in sewage have been indicated by W. Rudolfs, M. ASCE, and H. W. Gehm,[5] and their work is accepted as a general pattern to verify the conclusions that district measurements of the B.O.D. of sludge by standard methods are low, but may be corrected to give a fair and consistent value.

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[4] "Standard Methods for the Examination of Water and Sewage," Am. Public Health Assn., New York, N. Y., 8th Ed., 1936.

[5] "Chemical Coagulation of Sewage; IV. B.O.D. of Soluble and Dispersed Materials," by W. Rudolfs and H. W. Gehm, *Sewage Works Journal*, Vol. 8, 1936, p. 537.

C. C. Ruchhoft and W. Allan Moore,[6] as well as H. F. Stephenson,[7] by their experimental work indicate that without constant agitation dissolved oxygen is not continuously available to the oxygen-demanding sludge in a sludge specimen being tested for B.O.D. Because equipment for continuous agitation was not available for this experiment, it was decided to measure the B.O.D. in the usual way, with subsequent application of suitable corrections in accordance with Fig. 2.

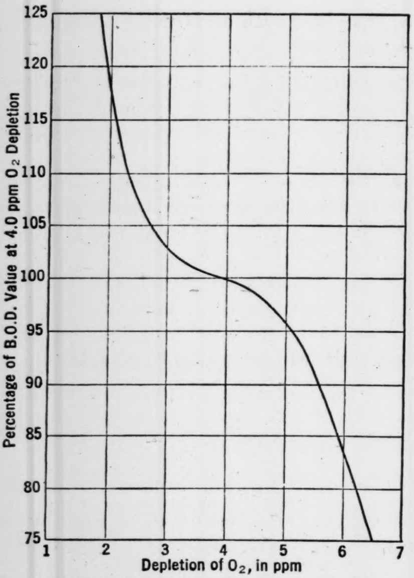

FIG. 2.—RELATIONSHIP OF B.O.D. VALUES OF SLUDGE AT VARIOUS OXYGEN DEPLETIONS TO B.O.D. VALUE AT DEPLETION OF 4.0 PPM

The most practical procedure for the purpose of this research, and one which has been used in correlating results in other district investigations, appears to be to select B.O.D. values corresponding to oxygen depletion of 4 ppm or about 50% of the available oxygen in the incubated mixture. The results so obtained are probably lower than would be found if the most favorable conditions for biochemical oxidation could be maintained; on raw sludge the results appear to be about 75% as great as might be expected from the B.O.D. indicated by analyses of plant influent and effluent. This, however, is not overly important for the purposes to which the tests are put. All conclusions herein are based on percentage reductions of B.O.D. of the sludge, and these percentages are valid as the various B.O.D. results are comparable with one another. It is believed that the arbitrary selection of a depletion of 4 ppm as standard gives results which are comparable.

CHANGES IN B.O.D. DURING DIGESTION

The changes occurring during digestion in 5-day B.O.D. of the three series of sludge samples are shown in Fig. 3. In series 1 (Table 2) the B.O.D. value of the sludge, after dropping to a temporary low on the fourth day, rose by the twentieth day to a value higher than that initially indicated and then declined steadily. In series 2 (Table 3) substantially all that had been accomplished in digestion in series 1 in 80 days was accomplished in the stage digestion tanks in 11 days, and the same was substantially true for series 3 (Table 4), for which the samples were taken after 4 days in the first stage of the four-stage digestion tanks.

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[6] "Determination of Biochemical Oxygen Demand and Dissolved Oxygen of River Mud Suspensions," by C. C. Ruchhoft and W. Allan Moore, *Industrial and Engineering Chemistry*, Analytical Ed., Vol. 12, 1940, p. 711.

[7] "Water Analysis; Effect of Suspended Matter and Daylight on Incubation Tests," by H. F. Stephenson, *Analyst*, Vol. 64, 1939, pp. 344–346.

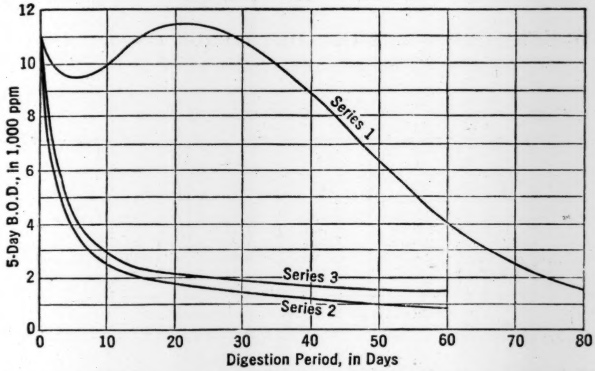

FIG. 3.—CHANGES OCCURRING IN 5-DAY B.O.D. DURING DIGESTION OF SLUDGE SAMPLES

TABLE 2.—OBSERVED 5-DAY BIOCHEMICAL OXYGEN DEMAND FOR SERIES 1

Specimen No.	5-DAY B.O.D. OF SLUDGE AT DIGESTION PERIOD, IN DAYS								
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
1	10,800[a] 0	7,000 10	10,100 20	8,900 30	2,900 40	1,400 50	1,400 60	1,200 72	1,200 80
2	10,700 0	10,100 10	11,900 20	13,600 30	10,400 40	10,000 50	3,900 63	2,100 70	1,400 83
3	12,800 0	9,400 10	13,100 20	10,400 30	11,800 40	11,000 50	8,300 60	3,800 72	1,900 81
4	13,100 0	12,200 10	11,300 20	11,000 30	11,600 40	11,000 50	6,000 60	2,300 70	1,500 79
5	9,300 0	9,000 10	10,000 20	9,700 30	10,100 40	11,400 50	9,300 60	3,200 82	....
6	13,900 0	11,500 10	15,300 20	10,300 30	9,600 40	2,100 60	1,400 73	1,100 80	....
7	10,200 0	10,200 2	10,600 12	10,500 22	9,300 32	5,000 42	2,000 51	1,500 64	1,200 82
8	9,700 0	7,700 8	10,600 18	9,000 28	6,600 38	1,500 62	1,400 70	1,000 80	....
9	10,600 0	7,400 6	12,000 16	11,600 26	10,200 36	11,000 47	9,700 60	6,100 68	2,900 78
10	9,900 0	6,400 4	11,000 14	9,100 24	8,800 34	4,000 43	1,000 56	850 74	....
Mean	11,100 0	....	....	....	....	....	....	....	1,600 ± 80

[a] Upper value is 5-day (20° C) B.O.D. of digesting sludge; lower value, digestion period in days.

This series of experiments was not performed to indicate the efficiency of four-stage digestion or any other type of digestion but to illustrate the very wide difference between series 1, on the one hand, and series 2 and 3 on the other. It should impress the investigator with the time required for adequate digestion in functioning digestion tanks, as compared with the time which might be required in starting from scratch with bottle specimens.

TABLE 3.—Observed 5-Day Biochemical Oxygen Demand for Series 2

Specimen No. (1)	Age of Sludge, in Days						
	0 (2)	11 (3)	17 (4)	24 (5)	30 (6)	45 (7)	60 (8)
1	....	2,040	1,570	1,200	1,160	1,010	980
2	....	2,000	1,460	1,470	1,330	1,160	1,000
3	....	2,660	1,700	1,300	1,330	950	990
4	....	2.640	1,550	1,910	1,000	860	850
Mean	11,100[a]	2,340	1,570	1,220	1,200	995	955

[a] Average B.O.D. of sludge entering digesters.

In series 1 it is clearly indicated that after an initial drop in B.O.D. there is a subsequent rise to a value greater than that initially determined. Although this series of experiments was not conducted to determine the reasons for this occurrence, speculation indicates that, after a short period of digestion, some of the compounds initially resistant to bacterial oxidation may have changed to the extent of rendering them more readily available as bacterial food—for example, by the well-known conversion of certain insoluble material to soluble volatile organic acids. This supposition is verified partly by the continuous and unchecked reduction in organic solids throughout the experiments in series 1 despite the fluctuating B.O.D. values. It is assumed that the conditions in the stage digestion tanks and in other functioning digestion tanks either forestall the rise of B.O.D. noted in series 1 or else mask it. The work of R. S. Ingols and G. M. Ridenour,[8] Assoc. M. ASCE, indicates that B.O.D. changes in raw sewage stored anaerobically varied in an unpredictable manner.

TABLE 4.—Observed 5-Day Biochemical Oxygen Demand for Series 3

Specimen No. (1)	Age of Sludge, in Days					
	0 (2)	4 (3)	7 (4)	11 (5)	30 (6)	60 (7)
1	....	6,300	4,000	3,000	2,200	1,800
2	....	4,700	3,600	2,600	1,400	1,800
3	....	4,600	3,000	2,300	1,200	900
Mean	11,100[a]	5,200	3,530	2,630	1,600	1,500

[a] Average B.O.D. of sludge entering digesters.

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[8] "Effect of Storage of Sewage Upon Its B.O.D.," by R. S. Ingols and G. M. Ridenour, *Sewage Works Engineering and Municipal Sanitation*, Vol. 17, 1946, p. 315.

Examination of the data for series 3 discloses a marked reduction in B.O.D. values in 4 days, and a fairly consistent reduction until the age of 60 days is reached. Comparison of the three series indicates the best reduction for series 2, something less for series 3, and the least effective for series 1.

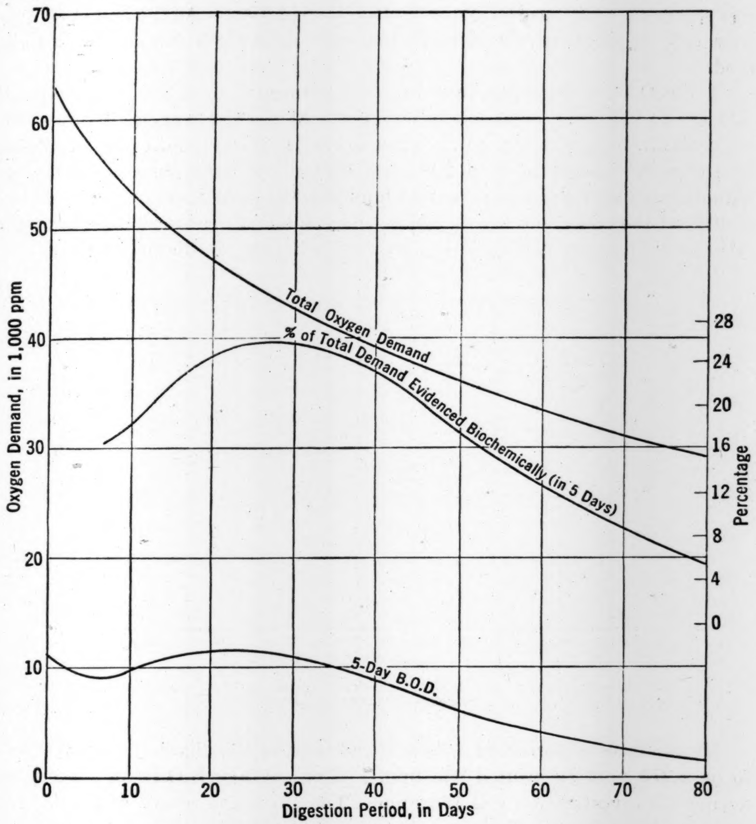

FIG. 4.—RELATIONSHIP OF B.O.D. TO TOTAL OXYGEN DEMAND OF SERIES 1 SLUDGES DURING DIGESTION

As a check on the observed B.O.D. values for series 1, the total oxygen demand of the digesting mass is calculated, using as a basis an estimate, taken from the unpublished notes of Richard Pomeroy, which indicates that each part of the organic matter in the digesting sludge requires 2.1 parts of oxygen for its complete oxidation. Fig. 4 illustrates the conclusion that at all times the apparent 5-day B.O.D. in series 1 was much less than the total oxygen demand of the digesting mass.

The percentage of total oxygen demand exerted biologically in 5 days reached a maximum of 26% between 25 days and 30 days. If 33% is added to the B.O.D. values to allow for error in the method of determination, and, if it is assumed further that this 5-day value is 68% of the "ultimate first-stage B.O.D.," it appears that a little more than one half of the organic matter at this stage is readily susceptible to biological oxidation. At the end of 80 days only 11% of the remaining organic matter is similarly available for bacterial food.

The B.O.D. of the sludge in series 2 was reduced from 11,100 ppm (observed B.O.D.) to 955 ppm, or 91.5% in 60 days. At the Los Angeles districts' disposal plant, sludge is removed from raw sewage in the ratio of 1 part wet sludge (at about 96% moisture) from 208 parts sewage. Estimating a sludge volume reduction (by removal of supernatant liquid and by reduction of volatile matter) of 40% at the end of 60-day digestion, the volume of digested sludge bears the ratio to the primary effluent of 1 part digested sludge to 350 parts effluent.

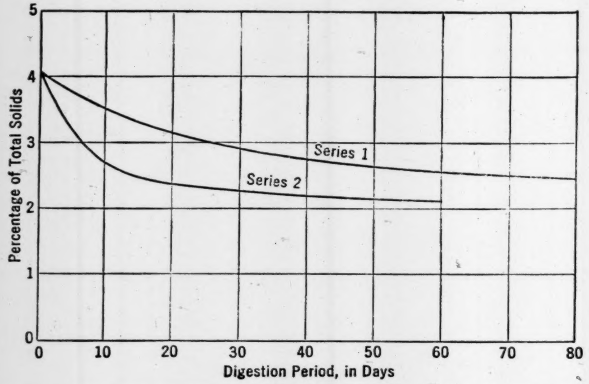

FIG. 5.—COMPARISON OF CHANGES IN TOTAL SOLIDS IN TWO SERIES OF SLUDGE SAMPLES DURING DIGESTION

On the basis of corrected values, the B.O.D. of the digested sludge appears to be 1,275 ppm (measured 955 ppm). The average B.O.D. of the effluent during the investigation was 150 ppm. Therefore, the return of the digested sludge to the primary effluent would increase the B.O.D. of the latter by 2.1%, or from 150 ppm to 153.2 ppm.

SOLIDS REDUCTION DURING DIGESTION

Total solids were determined by weighing and drying a 20-g portion of sludge in the conventional manner. Solids so determined are probably lower than those that are actually present due to the loss of volatile organic matter and ammonium carbonate during the drying process. However, the results are used as so determined. Ash was identified by ignition of the dry solids in an electric furnace at dull red heat for 1 hour. Analytical results for total solids and ash are shown in Tables 5 and 6, and are presented graphically in Fig. 5.

The observations with respect to B.O.D. behavior during the test run are not paralleled in the observations of solids reduction. Series 2 fared better than series 1, and series 3 was the least affected. Comparison of percentage reduction in volatile matter in series 1 and 2 is shown in Fig. 6. Results of series 3 are not shown because of their erratic nature. In part they paralleled series 2 and in part they varied widely. It is thought that the methods of taking samples from intertank connections and of calculating the original

TABLE 5.—SOLIDS DATA FOR SERIES 1

Specimen No.	Ash (%)	TOTAL SOLIDS IN SLUDGE AT DIGESTION PERIOD, IN DAYS (%)								
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
1	0.94	3.60[a] 0	3.02 10	3.02 20	2.84 30	2.47 40	2.25 50	2.19 60	2.16 72	2.16 80
2	1.07	4.26 0	3.47 10	3.49 20	3.33 30	3.26 40	3.10 50	2.86 63	2.62 70	2.62 83
3	1.51	4.85 0	4.51 10	4.09 20	3.98 30	3.76 40	3.60 50	3.42 60	3.14 72	3.19 81
4	1.22	4.80 0	4.18 10	3.96 20	3.63 40	3.48 50	3.17 60	3.02 70	2.90 79	....
5	1.03	4.40 0	3.90 10	3.66 20	3.34 30	3.32 40	3.22 50	3.12 60	3.18 70	2.74 82
6	1.22	4.73 0	4.69 10	3.88 20	3.78 30	3.44 40	3.10 53	2.97 60	2.92 73	2.88 80
7	1.06	3.96 0	3.72 2	3.34 12	3.26 22	3.07 32	2.78 42	2.56 64	2.34 72	2.44 82
8	0.83	3.46 0	3.09 8	2.84 18	2.73 28	2.72 38	2.57 49	2.26 62	2.20 70	2.20 80
9	0.95	3.44 0	3.20 6	3.10 16	3.04 26	3.12 36	3.08 47	2.86 60	2.54 68	2.36 78
10	0.88	3.48 0	3.22 4	2.92 14	2.80 24	2.78 34	2.46 43	2.28 56	2.10 64	....
Mean	1.07	4.10 0	....	....	....	....	....	....	....	2.46± 80

[a] Upper value is percentage of total solids; lower value, digestion period in days.

TABLE 6.—PERCENTAGE OF TOTAL SOLIDS FOR SERIES 2

Specimen No.	Ash (%)	AGE OF SLUDGE, IN DAYS						
		0	11	17	24	30	45	60
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
1	0.94	....	2.56	2.32	2.28	2.24	2.20	2.14
2	0.93	....	2.52	2.32	2.30	2.26	2.07	1.97
3	1.13	....	2.78	2.66	2.56	2.50	2.48	2.32
4	0.97	....	2.76	2.46	2.34	2.33	2.26	2.00
Mean	0.99	4.19[a]	2.66	2.44	2.37	2.33	2.25	2.11

[a] Average solids content of raw sludge for period, corrected for 11% over-all seeding with digested sludge.

solids content are responsible for the nature of the results obtained with series 3.

The total solids content in series 2 was reduced from 4.19 to 2.11, or 49.6%. Assuming that all this would be settleable when returned to the primary effluent, its return in the ratio of 1:350 would raise the average settleable solids of the primary effluent by 1.5 ml per liter.

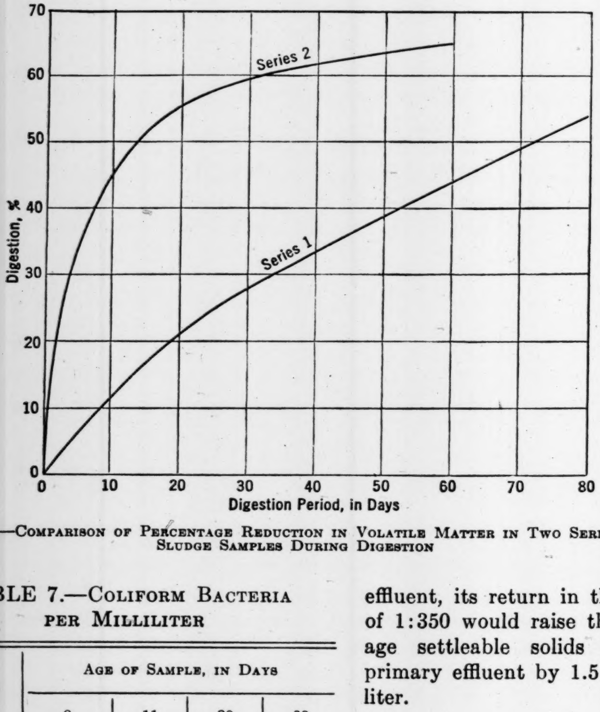

FIG. 6.—COMPARISON OF PERCENTAGE REDUCTION IN VOLATILE MATTER IN TWO SERIES OF SLUDGE SAMPLES DURING DIGESTION

TABLE 7.—COLIFORM BACTERIA PER MILLILITER

Specimen No. (1)	Age of Sample, in Days 0 (2)	11 (3)	30 (4)	60 (5)
1	....	7,000	2,400	1,300
2	....	6,200	2,400	600
3	....	7,000	6,200	2,400
4	....	7,000	600	600
5	....	7,000	2,400	600
Mean	1,504,000[a]	6,800	2,800	1,100

[a] Average coliform group density of raw sludge for test period, corrected for 11% over-all seeding with digested sludge.

REDUCTION IN COLIFORM GROUP DENSITY

The most probable number (M.P.N.) of coliform group bacteria in each test was determined by making lactose broth presumptive and brilliant green bile broth confirmatory tests in accordance with aforementioned procedures.[4] Each M.P.N. was based on paired innoculation dilutions from 10^{-2} to 10^{-7} as indicated in determinations on five samples shown in Table 7.

During the period of this study, primary effluent averaged 51,000 coliform group bacteria per ml. At the age of 60 days the coliform count of digested sludge from series 2 averaged 1,100 per ml, whereas at 11 days the average was 6,800 per ml. It is evident then that the return of 11-day sludge or 60-day sludge from the stage digesters would actually result in a dilution of coliform bacteria and that primary effluent will probably suffer no degradation in this characteristic by the addition of digested sludge.

REDUCTION IN GREASE CONTENT DURING DIGESTION

Raw solids from the untreated sewage averaged 16,000 ppm (wet basis) grease during the investigation. Grease determinations on raw and digested sludge were obtained by acidification with sulfuric acid, drying, and extraction with ethyl ether.

Grease determinations on sludge samples taken from the outlet of the fourth stage of the districts' sludge digestion tanks are presented in Table 8, Col. 2. The grease determinations were inaugurated after the experimental work was well along and, consequently, are not all made on original samples reported in series 1, 2, and 3. However, the tests made on 30-day sludge and 60-day sludge, shown in Table 8, are felt to be sufficiently typical to indicate the numerical value of grease reduction.

TABLE 8.—GREASE REDUCTION DATA, AS TOTAL GREASE IN PARTS PER MILLION (WET BASIS)

Specimen No.	AGE OF SAMPLE, IN DAYS		
	11	30	60
(1)	(2)	(3)	(4)
1	....	....	2,210
2	....	....	2,480
3	....	....	2,840
4	3,930	2,770	2,650
5	3,450	2,580	2,410
Mean	3,690	2,680	2,500

Primary clarifier effluent averaged 34 ppm of grease by acidification and direct hexane extraction during the period of this investigation. Table 8 indicates that sludge from stage digestion at 60 days has a grease content of about 2,500 ppm. If the digested sludge at 60 days is returned to the primary effluent in the ratio of 1:350, the grease content of the latter will be increased from 34 ppm to 41 ppm, or approximately 21%. Similarly, if sludge from the stage digestion tanks with an age of 11 days is returned to primary clarifier effluent, the grease content will be increased from 34 ppm to about 44.5 ppm.

EFFECT OF DIGESTION ON CHLORINE DEMAND

For the purpose of this investigation, the chlorine demand is defined as the parts per million of chlorine added to diluted sludge which are required to produce an apparent residual chlorine content of 0.1 ppm after a 15-min contact period. Dilution of the sludge was required in order to observe color changes. Chlorine demand of raw sludge during the investigations varied from 950 ppm to 1,000 ppm. The 60-day-old sludge had a demand of about 600 ppm, whereas primary effluent had a demand of 36 ppm. If the 60-day-old digested sludge from series 2 were returned to the effluent in the ratio of 1 part of the former to 350 parts of the latter, the chlorine demand of the effluent would be raised to 37.7 ppm, or about 4.7%.

DISCUSSION

Each case involving the disposal of digested sludge with effluent from a primary subsidence treatment process must be considered on its own merits and conclusion 5, stated hereinafter, should be interpreted with special emphasis on the statement, "wherever sludge banks will not be formed." This qualification removes the conclusion from a generalization, but indicates that, under certain conditions, there are situations in which the disposal of digested sludge with primary effluent should be seriously considered as having no public health or nuisance implications and as effecting major economies in sewage treatment and disposal.

One of the major reasons for primary treatment of sewage is to prevent the formation of deposits of raw sewage solids commonly termed "sludge banks." Such deposits will putrefy and cause foul odors if they become exposed to the air; they may release hydrogen sulfide to overlying bodies of water; and they may cause an oxygen demand on the receiving stream greater than is indicated by the B.O.D. of the raw sewage, because the solids may be retained in the vicinity of the point of discharge until a relatively large fraction of the ultimate B.O.D. is satisfied.

These objections largely disappear as the sludge is taken out, digested, and then returned to the effluent. However, the authors would be loathe to conclude that disposal of the digested sludge with primary effluent to a stream, to a lake, or to tidewater is justified where sludge banks of even well-digested sludge will accumulate—hence, the qualification in conclusion 5.

Bacterial pollution of the receiving waters by raw sludge is also a matter of concern. Chlorination of raw sewage is not likely to be 100% effective because the bacteria are sheltered by solids and sludge particles. If, however, the sewage is settled, the effluent may be effectively chlorinated, and if the sludge is digested the destruction of intestinal bacteria is so great that with digestion periods of from 30 days to 60 days under most conditions (and shorter periods in controlled tanks), it appears that the sludge can be returned to the effluent with no more health hazard than is inherent in the disposal of primary effluent, even when the latter is chlorinated.

In the disposal of sewage into the ocean an important reason for primary treatment is to eliminate floating material. Skimming alone will not accomplish this satisfactorily, because some solids carried down by the sludge into the sludge banks may later break loose and come to the surface. In the digestion process the sludge is disintegrated so that substantially all the floatable matter can be separated. If such sludge (minus scum-forming solids retained in the digestion tanks) is discharged with the effluent, no nuisance from floating solids should result.

CONCLUSIONS

1. Under digestion conditions such as those existing at the Los Angeles County Sanitation Districts' joint disposal plant, addition of sewage solids that have been digested for 60 days (the first 11 days of which have been spent

in the four-stage digestion tanks) to primary clarifier effluent (the latter by itself having an average 5-day (20° C) B.O.D. of 150 ppm) will, on the average, change the characteristics of the primary clarifier effluent in a manner similar to the following:

(*a*) Increase the 5-day (20° C) B.O.D. by 3.2 ppm, or 2.1%;
(*b*) Increase the settleable solids by 1.5 ml per liter;
(*c*) Slightly decrease the coliform bacterial population;
(*d*) Increase the grease content from 34 ppm to 41 ppm; and
(*e*) Increase the chlorine demand from 36 ppm to 37.4 ppm, or 4.7%.

2. Substitution of 11-day-old sludge from the districts' four-stage digestion process for 60-day-old sludge will not so materially alter the foregoing conditions as to have any appreciable effect on the characteristics of the mixed effluent and digested sludge.

3. The increase in grease content is probably not of such an order as to influence the appearance of slick at an outfall site.

4. In general, the influence of adding 11-day-old or 60-day-old sludge to primary clarifier effluent is probably less than daily variations in the character of the effluent itself.

5. Wherever sludge banks will not be formed, there appears to be no valid reason why 11-day-old, or 60-day-old, digested sewage sludge may not be disposed of by dilution along with primary clarifier effluent if conditions permit disposal of the latter with safety.

ACKNOWLEDGMENTS

Acknowledgment is made to Richard Pomeroy, consulting chemist, and to H. W. Jorgensen, Assoc. M. ASCE, designing engineer, both of the Los Angeles County Sanitation Districts, for their help in this research work and in the preparation of this paper.

DISCUSSION

W. E. Abbott.[9]—The reduction in volatile solids through digestion for 30 days is of the order of 50% (Tables 5 and 6). Therefore, the proposed discharge of digested sludge will decrease the volatile suspended matter reduction resulting from primary sedimentation by one half—that is, if primary sedimentation removes 70% of the suspended matter, the effluent with the digested sludge will contain 100% minus 35%, or 65% of the initial volatile suspended matter instead of 30%. In other words, volatile suspended matter of the settled sewage is more than doubled through the addition of the digested sludge. The proposed addition of sludge in effect cuts the efficiency of the sedimentation tanks by rather more than half. Surely this is the truest measure of performance of the suggested sequence of operations.

The writer is reluctant to apply the 5-day B.O.D. (biochemical oxygen demand) test to sludges of any type; least of all to digested sludge with countless numbers of anaerobic organisms. The B.O.D. results in the paper appear to be a commentary on the futility of measuring dissolved oxygen changes in diluted sludges rather than evidence that the oxygen balance of a stream will benefit.

Robert E. Leaver,[10] **Assoc. M. ASCE.**—Several problems involving the disposal of sludge by the method advocated in this paper are of interest to the Washington State Department of Health and the Pollution Control Commission of the State of Washington. One example is at Portland, Ore., across the Columbia River from Washington. The aforementioned official agencies question the desirability of going to the expense of removing sludge and then, to accomplish an apparent saving, allowing sludge, digested for varying lengths of time, to be returned to the effluent. Digested sludge properly utilized is a valuable natural resource and it seems inexcusable to waste it to the degradation of streams. It is becoming common practice to utilize sludge in the liquid state, thereby reducing the need for sludge beds. In the State of Washington, the use of liquid sludge under pressure to fertilize beneath the surface of the ground in orchards, hop fields, etc., is gaining much favor. It is granted that considerable effort will be required in certain localities to develop a market for sludge. However, where this effort has been made the economic returns have justified it.

The paper by Messrs. Rawn and Candell opens many questions, one of which is the significance of the 2.1% increase in the 5-day (20° C) B.O.D. (biochemical oxygen demand) in the case of a stream or river below the treatment plant. The organic matter producing B.O.D. in the primary effluent, in general, will remain in suspension and will be further digested by natural aerobic conditions whereas, for the most part, the partly digested sludge will settle to the bottom of the stream and go through benthal digestion, which often requires years.[11] This sludge can accumulate to the extent that dissolved oxygen could be affected adversely. It would appear from this characteristic that the B.O.D. "picture" on sludge must be interpreted from an entirely different standpoint than is normally considered reasonable in connection with the 5-day (20° C) B.O.D.

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[9] Chemist, Nottingham Sewage Disposal Dept., Nottingham, England.

[10] Senior Public Health Engr., Washington State Dept. of Health, Seattle, Wash.

As Messrs. Rawn and Candell state, their method of running B.O.D. analyses on sludge "is neither true nor reliable." When dealing with sludge, the high initial demand and lag combine to obscure the normal monomolecular reaction which characterizes a B.O.D. test.

Even in the rapidly moving rivers of the State of Washington, sludge beds can, and do, form where the settleable solids are not yet removed by treatment. The slowing down of velocity along the banks and bends will cause deposits which are most accessible and can cause the greatest nuisance if they are composed of sewage sludge. In fact, the writer does not know of any streams in the State of Washington where sludge beds will not form.

The effectiveness of the chlorination of digested sludge requires additional study since it is well known that solids give bacteria protection against chlorination and that chlorine residuals are not the complete answer.

The increase of settleable solids in the primary effluent by 1.5 ml per liter by the return of digested sludge does not appear too important to the casual reader, but as a result at least 50.4% of the total solids that were removed to the digester are again returned. Certainly this amount is not insignificant.

The practical difficulties in preventing some of the scum from being carried out with the digested sludge and the 21% to 31% theoretical grease content increase make the evaluation of expected conditions, in a particular case, difficult to say the least.

This paper certainly has stimulated considerable thought on the subject of anaerobic digestion and has shown sanitary engineers in the State of Washington that additional studies are desirable. The Civil Engineering Department of the University of Washington at Seattle has agreed to conduct certain research to try to answer some of the questions that have been advanced in this discussion.

A. L. Genter,[12] M. ASCE.—Sludge digestion undoubtedly diminishes the biochemical oxygen demand (B.O.D.) of sewage sludge. Messrs. Rawn and Candell question the reliability of B.O.D. determinations in measuring this reduction and the writer questions the reliability of the experimental method adopted in the paper to determine the B.O.D. reduction and to evaluate the effects of digesting sludge and returning it to primary effluent for common disposal by dilution in state waters.

In the series 1 laboratory tests the B.O.D. of the seeded mix dropped initially, subsequently rose to values definitely higher than those of the unseeded fresh sludge, and then continued to diminish gradually to some asymptotic low value. Under the experimental conditions used this sequence seems logical. During the active stage of anaerobic multiplication the B.O.D. should increase if the multiplying organisms have a B.O.D. of their own. Their continued multiplication is gradually arrested by the depletion of available food supply in the flasks and possibly by the simultaneous accumulation of decomposition products inimical to such growth in the sludge liquor itself.

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[11] "The Natural Purification of River Muds and Pollutional Sediments," by Gordon M. Fair, Edward W. Moore, and Harold A. Thomas, Jr., *Sewage Works Journal*, March, 1941, p. 270; July, 1941, p. 756; and November, 1941, p. 1209.

[12] Cons. Engr., Baltimore, Md.

The authors remark that this initial drop and subsequent rise and decline in B.O.D. values are not manifest in digestion tanks and state (under the heading, "Changes in B.O.D. During Digestion") that:

> "It is assumed that the conditions in the stage digestion tanks and in other functioning digestion tanks either forestall the rise of B.O.D. noted in series 1 or else mask it."

If this statement be true, and it evidently is, then such small bottle tests do not really simulate "* * * the conditions obtainable in the districts' digestion tanks * * *" as the authors previously claim, under the heading, "General Description of Experiments."

The writer does not place much reliance on conclusions drawn from digesting sludges in 2-liter flasks. In relatively large digestion tanks operating in a treatment plant the biodynamic or biotic fostering activity is renewed daily, whereas in a relatively minute bottle testing system this is far from the case, even though the temperatures and detention times may be identical in both systems. In the actual plant system fresh sludge is added daily to the digesters so that its volume displaces somewhere from the digestion units the equivalent volume of digesting and digested materials in liquid condition. This procedure is not followed with tests conducted in 2-liter flasks where biochemical and chemical equilibriums become different as digestion progresses. The late W. L. Malcolm,[13] M. ASCE, minimized the effects of this possible experimental error in the digestion experiments conducted at Cornell University, in Ithaca, N. Y., in 1936.

However, whether or not bottle tests give a fairly relative indication of B.O.D. reduction through anaerobic digestion and whether or not more reliable methods can be developed for determining the B.O.D. of dilute and concentrated sludges, a closer examination of this method of disposing of sewage sludge reveals the fact that it levies a tax on primary clarifiers that operates in inverse proportion to their efficiency, especially when a plant is collecting sludges heavier and of higher B.O.D. than the sludge listed by the authors. Thus, practically, the better a primary sedimentation system functions in removing suspended sewage solids, the greater becomes the tax rate on the efficiency of the system.

This condition becomes quite evident in plain sedimentation treatment plants handling domestic sewage of about 300-ppm B.O.D. with sludge pumped to digestion containing as high as 9% solids. Here the combination of digested material with the plant effluent of 165-ppm B.O.D. could readily increase the latter by 7% for 30-day digestion and by almost 6% for 60-day digestion. A lowered clarifier efficiency may decrease this percentage but only at the cost of increasing the B.O.D. of the large fraction of plant effluent.

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[13] "Studies in the Digestion of Ground Garbage," by W. L. Malcolm, *Sewage Works Journal*, May, 1937, p. 389.

The authors in accounting for the removal of digester supernatant do not account for its disposal. The writer assumes that it is added back to the plant influent because it is dirty. This return is then accounted for in the 150-ppm B.O.D. of the primary effluent. In this case most of the digested solids returned to plain sedimentation return again to digestion to accumulate for 30 days and 60 days in that space to become a tax on its efficiency.

In any case one should not lose sight of the fact that digester liquor has a B.O.D. that can be double that of the plant influent and the B.O.D. of the wet digested sludge is almost a straight-line function of the amount of digested solids present plus the B.O.D. of the liquor present. Furthermore, even though 30-day digestion of a raw sludge of 9% solids content (as at Winnipeg, Man., Canada) may and does drop the sludge B.O.D. to 10% of its original value (in line with data presented in Table 3), this drop is from a far greater initial B.O.D. level and can leave the liquid digester contents of 5.6% solids with a total B.O.D. of 5,600 ppm, in place of the 1,200 ppm recorded for the digested sludge having 2.33% solids in Tables 3 and 6. Increasing the ratio of sewage treated to liquid digester contents from 208 to 1 in the Rawn-Candell data to 490 to 1 in the case taken from the records of the Greater Winnipeg Sanitary District does not help as much as might be assumed. The penalty actually increases.

Thus, regardless of just how trustworthy the B.O.D. determinations may become, the writer believes that this highly telescoped method of sewage and sludge disposal can serve only to postpone the day of real reckoning. At best its penalty on plant efficiency lightens the taxpayers' burden by operating in some cases with just enough questionable adequacy under the guise of sewage treatment to satisfy the sanitary requirements of those residing above the point of discharge into state waters.

In reviewing various methods of sludge disposal the writer also believes that Mr. Rawn[14] presented a more convincing solution for this problem in 1941, than in the alternative solution now under discussion. In the former he gave excellent experimental evidence of increasing soil fertility assets by adding digested sludge to the soil where it is definitely needed. In the present contribution he and Mr. Candell offer a method of subtracting these fertility values from the soil and adding them to streams where their value is at least questionable.

A M Rawn,[15] M. ASCE, and E. J. Candell.[16]—Several contributors have taken the time to analyze this paper carefully, and have advanced conclusive thought and criticism regarding the experimental work. In response to these and to the many written and verbal comments made directly to the writers it is desired to state that the paper was not designed to advance a plea for a special method of sewage sludge disposal but rather to suggest a tool heretofore apparently not investigated and one which might find logical application in certain situations.

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[14] "Concerning Those Plus Fertility Values of Sewage Sludges," by A. M. Rawn, *Water Works & Sewerage*, April, 1941, p. 186.

[15] Chf. Engr. and Gen Mgr., Los Angeles County Sanitation Dists., Los Angeles, Calif.

[16] Denver, Colo.

Mr. Abbott directs attention to the fact that by returning digested sludge to primary effluent for disposal the volatile suspended solids, as indicated by the writers' computations, may be increased from a total of 30% to a total of 65%. His position is well taken, but he fails to indicate that the 35% increase in volatile solids would result from the return to the primary effluent of those volatile solids in the digested sludge which are slow to digest and which, in general, are nonputrescible and have a low 5-day biochemical oxygen demand (B.O.D.). The volatile solids returned with the digested sludge are similar in composition to leaf mold, peat, or some such matter.

Mr. Abbott's reluctance to apply the 5-day B.O.D. test to sewage sludges is not unusual, although it is believed he would be on firmer ground if he were less reluctant to apply the test to digested sludge than to crude sludge. Results of the test as applied to sludges are erratic at times; but, for what the results may be worth, they are likely to measure, indeed, overmeasure, the oxygen demand placed upon a flowing stream, a lake, or salt water disposal site by digested sludge which will not strand. Mr. Abbott appears to be assuming that the remaining relatively stable volatile matter in the digested sludge will form sludge banks. The writers' admonition as stated in the paper, is not to return the digested sludge to primary effluent for disposal if there is danger of serious stranding.

Mr. Leaver questions the desirability of removing crude sludge from the sewage and then returning it in digested form for disposal. His criticism appears to stem from the fact that digested sludge is a valuable natural resource and should not be wasted to the degradation of streams. In the latter regard, Mr. Leaver directs attention to the accumulation of crude sludge in eddies in the rapidly moving stream waters of the State of Washington. Unfortunately, he tends to apply his reasoning to the Columbia River below Portland, Ore., into which river the crude sewage from cities on tributary streams and on the banks of the Columbia has been emptied for years without significant effect on the waters of that great stream. The effective reduction in the readily putrescible content of a large proportion of the suspended solids in the crude sewage of Portland can surely have only a beneficial effect on existing conditions; and, as applied to the Portland situation, the writers believe that the return of digested sludge to the primary effluent and its discharge into the Columbia River will escape detection by even the best of laboratory techniques.

Again the writers state that, if the solids will strand, the practice of returning digested sludge to primary effluent may not be satisfactory and probably should be practiced only experimentally and with caution until its merit is demonstrated. It is gratifying to note that the civil engineering department of the University of Washington, at Seattle, plans to extend the writers' research.

The writers agree with Mr. Leaver that digested sewage sludge is usually an excellent soil builder and that nothing should be allowed to curtail the activities of those who desire to reclaim and utilize this material for soil building —provided, of course, that the cost of such reclamation is borne by those who profit from the venture. Surely, nothing in the writers' discussion can be construed to indicate that they (who are among the most enthusiastic proponents of the return of digested sludge to the soil for agricultural betterment) desire to waste this material when it can be utilized properly and economically.

Mr. Genter questions the validity of the conclusions drawn from laboratory experiments on digested sludge. However, his position is not well taken with respect to laboratory digestion that is designed to illustrate the changes in sludge if it is allowed to continue digestion without further accretion of crude sludge. This condition is approached in the later stages of controlled stage digestion as currently practiced, and reasonably well demonstrated by the latter part of the curve of controlled laboratory digestion. Mr. Genter states that: "In the actual plant system fresh sludge is added daily to the digesters * * *." This statement is true only of the primary stage; it is not applicable to the later or ripening stages of modern digestion.

Mr. Genter's comments on the concentration of solids in the crude sludge do not apply to the conditions described in the paper. The solids content of the crude sewage is the controlling factor and not the concentration of the solids in the crude sludge. It appears logical that the greater the strength and solids content of the original sewage the less the primary effluent therefrom will be affected percentage-wise by the return of digested sewage sludge which, having originally contained nearly 40% of the volatile suspended solids in the sewage, contains less than half that amount after some 30 days of digestion and only 10% of its originally measured 5-day B.O.D.

Mr. Genter's statement—"* * * the writer believes that this highly telescoped method of sewage and sludge disposal can serve only to postpone the day of real reckoning"—is not deemed appropriate by the authors. They feel that the recommendations contained in the paper will result in the final disposal of digested sludge as completely and adequately as if it were incinerated. Hundreds of thousands of tons of similar organic material from farms, forests, and meadows are disposed of annually in the same manner. Again, in response to Mr. Genter's criticism regarding the wasting of this valuable fertilizer, Mr. Rawn desires to state that the sewerage authority under his direction has produced and sold more digested sludge in the form of fertilizer than any authority west of Chicago, Ill. While still maintaining that the return of this organic material to the soil is of great value, he insists that there are locations where it cannot even be given away and where, if it is processed as a soil nutrient, the operation should be at the expense of those who will profit by the operation. Unless the manufacture of fertilizer is a necessary step in the operation of sludge disposal, at such locations the cost should not be borne by the public.

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MINERAL LAND SURVEYS

By Arthur J. McNair,[1] Assoc. M. ASCE

With Discussion by Messrs. Charles P. Eager, Robert Lenon, John V. Meldrum, Arthur D. Kidder, and Arthur J. McNair.

Synopsis

For the first time in 40 years, there is presented herein a discussion of the history, methods, and instructions for making mineral land surveys in the United States. Public land surveys, although relatively simpler, have been rather thoroughly treated in modern technical literature. Nevertheless, few people realize the importance of mineral land surveys from the standpoint of the number of such tracts of ground or from the standpoint of the entirely different and rather complicated procedure of surveying or resurveying mineral lands. Not only the methods of making a mineral location are presented, but also the importance of having a competent surveyor or engineer make such a survey is emphasized. The changes in regulations and their effect on the survey, together with the different methods of surveying, are rather completely discussed to assist an ordinary land surveyor in resurveys of mineral land wherever tracts of this type occur in public or private ownerships.

Introduction

The survey of mineral bearing lands in the United States for the purpose of segregating them from the public domain so that an individual may obtain a title or "patent" differs from the survey of other public lands in several respects. First, public land surveys are made by the federal government through the General Land Office, United States Department of the Interior, following a regular procedure, whereas mineral land may be originally described by anyone and the official patent survey is made by mineral land surveyors employed by the claimant and working under conditions which vary from place to place. Second, because of the greater value, or, at least, the greater potential value of the land, the precision required for a mineral land survey is higher than that required for ordinary public land surveys. As most mineral land is located in

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Note.—Published in November, 1948, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

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mountainous areas, the difficulty of maintaining the requisite precision is greatly increased. Third, the problem of obtaining, placing, and maintaining adequate survey monuments is frequently difficult, and the restoration of "lost" corners requires considerable patience and judgment. Finally, mineral land surveys are complicated by the conflicts between claims, by the ties to and discrepancies in the public land surveys, and by legal procedures and court decisions, which must be properly interpreted and followed.

This paper deals with the methods employed in making location and patent surveys to segregate mineral lands from the public domain so that they may be patented and also with the important historical developments that have affected surveying procedures. During the days of the westward expansion in the United States, most of the area in the midwest and in the Great Plains was absorbed by the familiar procedure of homesteading, so that the lands were privately owned before minerals were discovered. With the discovery of gold in California and the subsequent mineral discoveries in other Pacific and Rocky Mountain states, the type of land development and the purposes and problems of acquisition changed. All minerals, including even building stone, were included under the original "mining acts" of Congress. The states of Kansas, Missouri, Wisconsin, Michigan, and Minnesota, as well as Texas, were excepted from these general mining acts. Various acts of Congress have restricted the lands which may be patented. For instance, the Leasing Act of 1920 permits only the leasing of land from the United States government for the purposes of prospecting for and extracting coal, phosphates, oil and gas, oil shale, sodium, sulfur, and potash. Also, since 1935 patents for homesteads and for stock raising homesteads have been essentially stopped, so that lands in the public domain which contain metalliferous minerals are about the only lands which a person can freely prospect and develop by individual enterprise with assurance of being able to obtain title to the land so developed. There are in Alaska and the states of Arizona, California, Colorado, Idaho, Montana, Nevada, New Mexico, Oregon, South Dakota, Utah, Washington, and Wyoming more than 90,000 official mineral land surveys, comprising nearly a quarter of a million separately surveyed, monumented, and patented tracts of land, many of which, because of conflicts, may consist of from one to a dozen or more small fractions of land in the original patent. Of course, as is the case with any land, many of these original tracts have been further subdivided. Inasmuch as any civil engineer, land surveyor, mining engineer, or land appraiser is likely to have occasion to resurvey or to subdivide these tracts in these states, it is important for him to have some grasp of the principles of mineral land surveys as presented herein.

TYPES OF MINERAL CLAIMS

There are three recognized types of mineral land claims, classified according to the method of occurrence of the mineral deposits or according to the proposed use of the land: Lode claim, placer claim, and mill site.

Lode Claim.—Of these three the most important is the lode claim, which is laid out to cover mineral deposits that are "in place" in solid rock formations. A lode claim may be a maximum of 1,500 ft long and 300 ft on either side of the center line, a total of 600 ft wide, or 20.661 acres, as established by United States statute in 1872, although the size may be limited to something less than this maximum by state or local rulings. It is preferable, where there are no conflicts with other claims, to survey the lode claim so that it will be rectangular (Fig. 1(*a*)). On many occasions, however, prior claims may have left irregular fractions of land. These can best be covered by a 600-ft by 1,500-ft quadrilateral which is not a rectangle (Fig. 1(*b*)) or by a six-sided or an eight-sided claim (Fig. 1(*c*)). Similarly, when the mineral deposit does not follow a straight line on the surface, to keep all the vein within the limits, the claim must be bent so that tracts of six, eight, or even more sides result (Fig. 1(*c*)). The maximum width or the perpendicular distance between sides is the 600-ft dimension, and the length along the lode line is 1,500 ft; hence the actual lengths of sides or ends may differ from these values, as shown in Figs. 1(*b*) and 1(*c*). It is imperative that the surveyor lay out the lode claim so that the end lines are parallel.

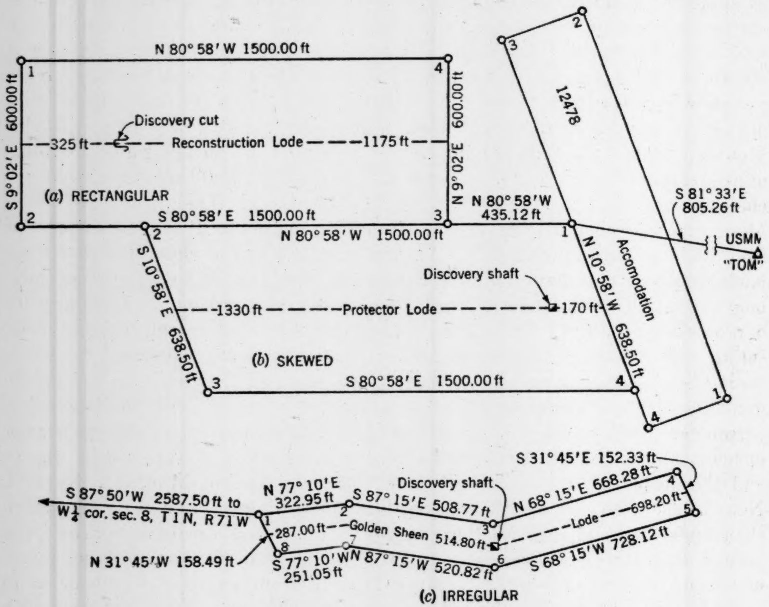
N 80° 58′ W 1500.00 ft
S 9° 02′ E 600.00 ft
N 9° 02′ E 600.00 ft
Discovery cut
325 ft
Reconstruction Lode
1175 ft
(a) RECTANGULAR
S 80° 58′ E 1500.00 ft
N 80° 58′ W 1500.00 ft
N 80° 58′ W 435.12 ft
12478
Accomodation
S 81° 33′ E 805.26 ft
USM
"TOM"
S 10° 58′ E 638.50 ft
N 10° 58′ W 638.50 ft
Discovery shaft
1330 ft
Protector Lode
170 ft
(b) SKEWED
S 80° 58′ E 1500.00 ft
S 31° 45′ E 152.33 ft
N 77° 10′ E 322.95 ft
S 87° 15′ E 508.77 ft
N 68° 15′ E 668.28 ft
S 87° 50′ W 2587.50 ft to W¼ cor. sec. 8, T 1 N, R 71 W
287.00 ft
Golden Sheen 514.80 ft
Lode
698.20 ft
N 31° 45′ W 158.49 ft
S 77° 10′ W 251.05 ft
N 87° 15′ W 520.82 ft
S 68° 15′ W 728.12 ft
(c) IRREGULAR
1
2
3
4
5
6
7
8

FIG. 1.—TYPES OF LODE CLAIMS

Placer Claim.—The next most important type of mineral land claim is the placer claim, which covers mineral deposits that are not in place—for example, deposits which may be mined by washing, screening, or otherwise separating loose unconsolidated soils, sands, or gravels. Entries for placer claims, as for lode claims, are made on the basis of a discovery of mineral. However, there are two important differences in the filing of the claim—first, the discovery must be not just a discovery of mineral, as with the lode claim, but it must be a discovery of commercial deposits of placer material such as gold, silver, platinum, sand, or gravel; and, second, the maximum amount of land which may be taken is 20 acres for one individual, increasing by 20 acres for each copartner to a maximum of 160 acres where eight or more individuals are in partnership on the claim.

There are two types of placer claims, each of which is restricted by the foregoing principles. The first type is taken up by legal subdivision of surveyed public lands. To locate this type, the surveyor must be familiar with the rules for the survey and the subdivision of United States public lands. Most civil engineers and surveyors are familiar with the procedures of rerunning and subdividing a section, and there are several good reference books on the subject.[2,3] Consequently, no further description will be given. Legally, tracts as small as 10 acres may be described by the rectangular system. To take up the desired area, a series of adjacent 10-acre tracts is described. The second type of placer claim is called the gulch placer. This type is used preferably in unsurveyed parts of the public domain where the land has not been sectionized, but is used particularly where the valuable placer deposits are confined to a comparatively narrow strip of land, such as the bed of a well-defined gulch. Claims of this type may be quite long and narrow, more than fifty sides and an over-all length of 2 miles or 3 miles being not uncommon for a 160-acre gulch placer. Maintaining the required precision on these long surveys through the timber and the undergrowth and along the steep slopes which many gulches possess is a problem which taxes the skill of the engineer.

Mill Site.—The last type of mineral land claim is the mill site, which is limited to a maximum area of 5 acres of supposedly nonmineral bearing land and may be taken up in conjunction with a lode claim or as the site of a mill or reduction works. The mill site, which should be reasonably regular in shape, is used as an adjunct to the mining, storing, milling, or treatment of ores from a lode claim.

It should be understood that one claimant, be he an individual, a group of individuals, or an incorporated body, may locate as many claims as he may deem desirable as long as he fulfils all the requirements as though each claim were an entity in itself.

LOCATION SURVEYS FOR LODE CLAIMS

In the early days of mining in the west, the prospector who made a discovery of mineral would post a notice by his discovery hole and proceed to lay out his claim 300 ft by 1,500 ft in size, or whatever the limit was in his particular district. He might look at the sun for directions and pace the distances, putting in his corner stakes about where he thought they belonged. Some prospectors used a compass for obtaining bearing; they might or might not allow for magnetic declination. To locate the position of the claim, bearings were usually taken on one or more mountain peaks or on other prominent natural objects

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[2] "Manual of Instructions for the Survey of the Public Lands of the United States," Bureau of Land Management, U. S. Dept. of the Interior, U. S. Govt. Printing Office, Washington, D. C., 1947.

[3] "Public Land Surveys, History, Instructions, Methods," by Lowell O. Stewart, Collegiate Press, Inc., Ames, Iowa, 1935.

from one or more corners of the claim. Occasionally the lands would be surveyed lands where the section corners could be found; if so, the prospector would tie to a section corner. Probably not 10% of these locations made without the services of a qualified surveyor are truly defensible in case of legal action, and since 1920 the majority of location surveys has been made and the descriptions have been written by engineers.

Legal Requirements for Lode Locations.—The surveyor who undertakes to make a location survey must have in mind the laws and their interpretation by courts so that he will make a legal location. Prior to 1866, the length and the width of lode claims were regulated entirely by local rules. In 1866, the first act of Congress to recognize mining claims as such permitted a maximum length of 3,000 ft. The width was not specified except by local legislation. The Act of Congress of May 10, 1872, the regulation which still prevails in 1948, provided that lode claims could be a maximum of 1,500 ft long and not more than 300 ft nor less than 25 ft on each side of the lode line. These maximum dimensions were frequently restricted by state or local statutes, although since 1930 there has been no place in the United States or Alaska where the dimensions of a lode claim are restricted to less than the federal limits of 300 ft on each side of the vein, lode, ledge, or lead unless limited by conflicts with prior claims.

Discovery Hole.—The surveyor must know that a legal discovery hole should have "such a showing of mineral as would warrant a prudent miner in prosecuting the work with the full expectation of finding pay ore." This discovery must be in solid formation and not in alluvium deposits. It consists of an exposure of a vein, lode, ledge, or lead of mineral to a depth of at least 10 ft below the surface—(hence the term "10-ft hole") and may be a shaft, tunnel, open cut, trench, or even a stripping underground in some mine workings. In this last case, a survey must be run through the mine workings and out onto the surface to a point directly above the point of discovery underground, where directions for reaching the true point of discovery should be posted. At the discovery point, the judgment of the surveyor, and particularly his experience with the minerals and their occurrence in that particular district, may enable him to counsel his client wisely on the advisability of continuing the work.

Apex Rights.—A mineral deposit may occur either as a fissure vein which has an apex or as an outcrop near or on the surface, and extends an indefinite depth into the crust of the earth, usually in igneous rock formations. A mineral deposit may also occur as a "blanket" deposit along seams in, or as replacements of, sedimentary rocks, and may be thought of as replacements or fills of ancient caverns as distinguished from the narrow crack-like fissure veins.

The mineral surveyor, through his familiarity with ore deposits in the area, should recognize the type of the particular deposit he is locating for his client. If he is locating a claim on a "blanket" deposit, vertical planes passing through the side lines and end lines of the claim define the claim. However, if the claim is being located on a fissure vein, the 1,500-ft length should run in the direction of the vein because this entitles the owner to extralateral rights. These extralateral rights allow the owner of the apex of a fissure vein to follow down the dip of the vein as far as he may desire as long as he remains within the end lines, or the end lines extended, of his claim, no matter how far this procedure may carry him beyond the vertical projection of his side lines and under other property. If the end lines of the claim are not essentially parallel, the apex owner loses his rights to mine beyond the vertical projection of his side line and may expend considerable money in developing a mine for his neighbors Needless to state, considerable litigation develops when slight variations from the standard case arise, and most of this litigation is avoidable if the surveyor knows the interpretations of the courts. An excellent treatment of various apex situations, together with a summary of various court decisions on the subject is given elsewhere.[4]

Laying Out the Claim.—If a well-defined vein can be followed on the surface for 1,500 ft, the problem of laying out the claim is simple. The surveyor selects the lengths he desires either way from the discovery point, with a maximum total of 1,500 ft, and runs out the end lines not more than 300 ft on either side normal to the center. In the event that the vein bends so that it would not be contained within the claim, then angle points are put in the side lines, and a claim with six or more sides results (Fig. 1(*c*)). To protect the apex rights of the claimant, it is essential that the end lines of the claim be parallel. If the vein and the occurrence of ore deposits cannot be definitely ascertained at the discovery hole, the surveyor lays out the claim to cover the most desirable land as the owner may decide.

Conflicts with Other Claims.—The lode location is rather simply made if there are no adjacent conflicting claims. However, this lack of conflict is frequently not the case. The surveyor must be aware of the position of any claims in the immediate vicinity which the government has previously patented to prior claimants. It is readily understood that whatever area is patented by the government to an individual or individuals becomes inviolately his or theirs. This fact, however, does not preclude a second claimant from locating a claim which may extend into or even completely across this prior claim as long as the discovery point for the new claim is on open ground. A patent can be obtained to the land within the boundaries of the new claim which is not already patented. A third claim may conflict with both of these prior claims and so on, until, what with the many directions and amounts of overlap of prior claims, the remaining fractions of unpatented land are small and widely scattered. Claims have been known to have conflicts with more than fifty other prior lodes.

Occasionally a patented lode will exclude some part of the area within its limits. In this case, a study of a district map showing the officially surveyed claims will not reveal that the area apparently covered is really not patented at all and hence is open to location. This occasionally occurs when the prior claim is abandoned for some reason or other and reverts to public domain.

Amended Locations.—Occasionally the surveyor may have to amend a location originally made when there were restrictions on the dimensions. This amendment may be necessary to reduce the total number of claims required to cover the ground or to change the location for the purpose of taking in more ground up to the present legal limit. The choice as to which claims would be abandoned would depend on the showing of mineral in the various discovery holes. In the past, not infrequently, locations lay in two districts whose local regulations differed. In this case, each part of the claim was required to conform to the local regulations for the district in which it lay. In case the claims are not patented, the surveyor may have to extend the width of the claims prior to the patenting process. For the case of isolated claims mostly surrounded by previously patented claims, the problem of how to amend the location to take in the most useful ground under the new limiting conditions must be solved for each individual case. A location may be rotated about the discovery point to cover more open ground, if this seems desirable.

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[4] "Mining Rights on the Public Domain," by E. D. DeSoto and A. P. Morrison, Bender-Moss Co., San Francisco, Calif., 16th Ed., 1936.

Equipment and Methods.—Considering the foregoing principles as to location, size, position, and conflicts of lode claims, and knowing something about the geology and mode of occurrence of ores in the district and about the economics of engineering with respect to operation of a mine and its appurtenances, the surveyor is ready to make the actual location survey. The survey party usually is composed of a transitman and a chainman. If it is anticipated that there will be considerable clearing to be done in running the lines, one or two axmen may be added. The following items should be included in the equipment of the survey party:

Light mountain-type 1-min transit with complete vertical circle and an adjustable tripod;
Steel tape (500 ft) graduated at 5-ft intervals, the end 5 ft graduated to 0.1 ft or 0.01 ft;
Reel for 500-ft tape;
Steel tape (100 ft) graduated to 0.01 ft;
Pocket-size reel for 100-ft tape;
Self-winding, 5-ft steel pocket tape graduated to 0.01 ft on one side and to inches on the other side;
Detachable handle for end of 500-ft tape;
Two plumb bobs;
Level rod (12 ft) collapsible to 4 ft;
Axes;
Machete, or brush hook;
Finishing nails (8 penny);
Stiff white cards about 2 in. square;
Stakes (about 3 ft long and 1 in. by 2 in. in cross section);
Transit-type field book;
Standard field tables;
Location notice forms;
Brunton pocket transit; and
Cloth flagging and keel.

The transit should be light enough to carry around in the mountains. An adjustable tripod is a necessity for surveys in the mountains, where rocks, trees, and steep slopes complicate the setups. A complete vertical circle is far preferable to a vertical arc, and the transit should be equipped with stadia hairs. Since a 4½-in.-diameter horizontal circle can be read about as accurately with a magnifying glass as can a larger circle, the smaller transit is preferable.

A method of traversing called the bearing traverse method is a favorite with many mineral surveyors. This method, which was developed by a group of mining engineers and mineral surveyors prior to 1900, is well adapted to land surveys in general and to mineral land surveys in particular. It yields more accurate results than any other method of traversing for the same amount of of time and labor expended. Since the bearing traverse method is not explained in any current text on surveying, it will be described briefly. A transit with two opposite verniers and graduated in quadrants is used. When a survey is started, the transit is oriented at the first station on true north as determined from the compass reading by correcting for magnetic declination, and bearings of lines are read directly from the A vernier. When a sight is taken to the next station, the bearing of the line is read directly from the A vernier and recorded. The transit is moved ahead with the upper motion clamped and the lower motion loose so as to retain this bearing on the upper plate. At the second station, the transit is oriented by backsighting with the telescope normal, and the lower motion is then clamped. Then the upper motion may be loosened, and the A vernier will give directly the bearing of any object sighted. The process is repeated at any other stations.

True bearings are ordinarily determined from a solar observation at some point, but occasionally by turning from a line of known bearing. Solar observations are much preferred over Polaris observations or other star observations because of the inconvenience entailed in being in the field at night and because of the extra lighting equipment which would be necessary. The solar observation usually consists of six observations on the sun with horizontal and vertical angles being read exactly the same as to any other shot in the traverse. The temperature is estimated for approximate correction to vertical angle. Ordinarily, the latitude of the place of observation is taken from a map of the state or of the district, the value being estimated to the nearest minute. Usually observations are taken every day at the beginning of an extended survey, and frequently in both the morning and the afternoon. If morning and afternoon observations do not agree, the assumed latitude may be in error. The six observations may be taken with the sun tangent in quadrants and the average of the observations calculated, but many engineers prefer to make the observations on the center of the sun and to calculate each shot individually. It is advantageous to take and to calculate observations on the center of the sun because the instrumentman can detect erroneous readings as they are taken, the computer can find instrumental errors due to variations in observations taken with the telescope normal or inverted, and the computer can discard "wild" readings instead of averaging all values. The assumed bearings, as determined by the original compass orientation, are adjusted to true bearings very easily, the correction ordinarily being less than a degree unless local attraction as a result of ore bodies is excessive.

There are two principal advantages of the bearing traverse method. First the bearings of all lines are known immediately as the field work progresses—

without making and checking any calculations. This knowledge is particularly convenient when lines and ties of other claims are being retraced and when legal descriptions involving bearings are being written. Second, the precision obtained in turning the angles is as high as that obtained by doubling the angles, using any other method. This precision is due, of course, to the fact that any fractional part of an angle too small to be read on the transit (say, less than 30 sec) remains on the plates and is added to or subtracted from the next angle as it would be when an angle is doubled. Thus, there is no cumulative angular error. The method presumes that any eccentricity in the graduations of the horizontal plate of the transit is less than the error of setting or reading the vernier. To assure that the vernier setting has not been disturbed when the transit was moved forward, the vernier reading should be checked after the backsight is made. This check can be made more rapidly than the vernier could be set to zero, as would be done in running a deflection traverse. If the telescope is inverted in backsighting to orient, the bearing traverse method is essentially the same as that employed in the customary azimuth traverse method.[5] As in other methods of traversing, the magnetic compass bearing is read and recorded as a rough check on the transit angle. It is wise not to have the magnetic declination set off on the annular ring of the compass, as the compass and the transit bearings then will not be the same, and the observer is not likely to make two mistakes in readings which would still check.

The long tape, usually 500 ft long but occasionally longer or shorter, saves many an instrument setup and is much preferable to the usual 100-ft tape used in the flatlands. Of course, as in all other types of land surveys, the dimensions are given in horizontal measure. This theory is the only equitable and tenable principle in mining claims, where the surface measure usually is much longer than the horizontal. As an extreme example, the writer has worked on claims where it was necessary to measure nearly 2,500 ft to obtain the allowable 1,500-ft length of a claim—an average vertical angle of about 54°. Under these conditions it is entirely impractical, both in regard to time consumed and in accuracy attained, to use a short tape and to determine horizontal distances by "breaking tape." Vertical angles are read at the transit to the nail on the stake, and the tape is used to measure from the horizontal axis of the transit to the nail on the stake. The slope distances thus obtained are recorded and usually "reduced" as the survey progresses using cosine or secant functions from field tables so that the desired distance of 600 ft or 1,500 ft may be laid out. The chainman, as usual, carries the zero end of the tape. The instrumentman marks the distance to the horizontal axis of the transit with his fingers or with a pair of parallel pliers as the tape is pulled taut and then uses the 5-ft tape to measure the fractional distance from the 5-ft markers on the long tape.

Much of the measurement will be over rough ground and through timber where the tape will be at least partly supported. When a considerable part of the tape is swinging free, the instrumentman frequently corrects for the sag by applying more tension. Spring balances are practically never used and the correction for sag is not applied. The instrumentman can tell by the "feel" of the tape when he should take the measurement. If the precision required is not high, shots and measurements may be recorded only to the nearest 0.1 ft. However, it is usually necessary and more satisfactory to use the finishing nails to mark the exact points on the stakes and to measure distances to the nearest 0.01 ft. The 100-ft tape, graduated to 0.01 ft, is convenient and a little more precise for short shots. For a rough location survey where the terrain is difficult, a stadia shot might be used for determining a distance. This condition might be encountered where a ravine with considerable undergrowth in it is to be crossed, because of the difficulty of getting the tape across the ravine and clear of the growth. Inasmuch as stadia will not give the precision required for a patent survey, however, regulations of the Land Office prohibit its use.

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[5] "Surveying—Theory and Practice," by R. E. Davis and F. S. Foote, McGraw-Hill Book Co., Inc., New York, N. Y., 3d Ed., 1940, pp. 330–332.

The most convenient-sized stake is about 1 in. by 2 in. in cross section and about 3 ft long. This length stake will stand up high enough that grass and low brush will not have to be cleared, and will enable longer sights to be made. The 8-penny finishing nails, which mark the exact point on the top of the stake, will stick up about 1½ in. so that a white paper or card can be placed on it. This white paper shows up very clearly in any natural terrain, high lighting the nail and making it readily visible at ordinary distances. This nail is more accurate for sighting than a range pole, and is very much simpler and more accurate as a point for measuring.

Precision.—The precision required on the average location survey is a variable quantity depending on several conditions, the most important of which probably is the value or potential value of the land, which may be very difficult to determine at the time of the location survey. Some lode claims have been located where the entire value of the land probably would not pay the price of the survey. On the other hand, some locations have paid the owners at the rate of $1,000,000 per acre. The courts have been rather liberal in construing the intent of a locator. However, the position of the corner posts as actually established on the ground is the most conclusive proof of the intended position of a claim. Consequently, in any case where it is likely that there will be conflicts with other located ("unsurveyed") or patented ("surveyed") claims, the only safe procedure is to decide exactly where the new location should be and then place it there, maintaining a precision of, say, 1 in 1,000. This precision is not sufficient for a patent survey (as is explained later), but will do for most location surveys. In case it is probable that the owners will want to patent the claim rather soon, the surveyor is justified in making a more careful survey and in setting semipermanent points so that the patent survey will be mostly a matter of checking the original survey and setting permanent monuments. True bearings are usually determined by direct observations on the sun and would be correct to less than 5 min of angle; however, on some location surveys the magnetic declination is simply set off from the compass, in which case the bearings may be in error 30 min or more. Compass bearings are read as a check on the transit angles.

Monuments.—The monuments for a location survey ordinarily consist of posts about 4 ft long and at least 3 in. in diameter blazed on the side facing in toward the claim and driven solidly into the ground. The blazed face is marked with keel to show the corner number and the name of the claim. When available—and this is more usual than not—a few stones are piled around the corner to help support the post; to help prevent live stock from scratching themselves on the post; and to mark the spot where the post was set, even though it may rot, fall over, and roll (or be carried) away. If timber is not available, a stone might be set and chiseled with the markings; but, since this is a rather slow procedure, posts are usually used for the locations. In case it would be impossible to set a post so that it would stay, such as in a road, a dump, a sliding talus slope, or a snowslide area, a witness corner should be set along one of the claim lines and plainly marked. In an attempt to be sure that the corners are of at least a semipermanent nature, some states have established laws requiring the posts to be of certain dimensions, a common requirement being a 4-in. square or 4-in.-diameter post at least 4 ft long, set 1 ft into the ground. An alternative of a stone mound 4 ft in diameter and 2 ft high is allowed in some states. All states, as well as the federal government, require that all the corners be marked by substantial posts. Some states (namely, Colorado and Wyoming) require that posts called "side centers" also be set at the center of the side lines (that is, 750 ft from either end for a 1,500-ft claim); one state, Oregon, requires that posts called "end centers" be set at the center of the end lines; and Arizona, North Dakota, and South Dakota require both side centers and end centers to be set in addition to the four corner posts. In most of these states the requirements of such a statute have been ruled by the courts to be directory and not mandatory and a violation does not invalidate the location.

Ties.—Some definite point or points, preferably the corners of the location, must be tied to a natural object or permanent monument so that the position of the claim is definitely fixed. On surveyed public lands where the section corners can be found, the tie might consist of the bearing and the length (course and distance) of the line from a corner of the location to the section corner or the quarter corner. The corner of the location which is thus tied to the section corner would probably be designated corner No. 1. In case there is a corner of some previously patented claim near by, a tie made to this patent corner would be entirely satisfactory. In some areas where there are no section corners or where the section corners cannot be found, United States mineral surveyors have been empowered to establish permanent monuments called United States mineral monuments (USMM) or United States locating monuments (USLM). These locating monuments or mineral monuments are usually a point such as a post, a large stone, or a chisel mark on a rock in place set on some prominent natural feature such as a knoll, a mountain peak, or a bench on a hillside. If the surveyor cannot readily find some recognized monument of one of these three types to which he can tie, he may, for the purposes of a location, tie to prominent natural features. One or more corners of the claim should be tied at least by course to some prominent or readily recognized mountain peak, to some road or stream junction, or to a well-known shaft, mouth of a tunnel, or permanent building. One such tie by course alone is quite poor. Consequently, accepted practice would be to tie to at least two such natural features by courses from one corner of the claim, or to tie to one feature from at least two corners of the claim. This kind of tie, by giving a definite "fix" to the location of the claim, would thereby identify the tract to be described from any other tract.

As previously described, if the locator feels that it would be advantageous for him to take his ground in some shape other than a rectangle, this shape is entirely permissible. If there are more than four corners—as for a six-sided claim—corner monuments must be set at each angle of the claim.

LOCATION SURVEYS FOR PLACER CLAIMS

If the mineral deposit is not in a vein such as quarrystone, oil shale, and the like, or if it is a loose deposit such as placer gravel containing gold, silver, tin, or commercial sand and gravel, it cannot be located as a lode claim. This type of deposit is open, however, to location as a placer claim under the mining laws [6] as amended by the Leasing Act [7] and by various Land Office regulations between 1920 and 1932. No stipulation is made by the federal government as to the dimensions of a discovery hole; but some states, such as Washington and Nevada, require a certain nominal expenditure in developing the claim. Courts have further ruled that, for location of a placer claim, the discovery must be an actual discovery of mineral values which can be recovered commercially.

Comparison of Gulch and Legal Subdivision Placers.—If the valuable placer deposits lie in a narrow, well-defined gulch, it is permissible to survey the 20 or more acres, depending on the number of joint locators, to locate only the valuable deposits. This so-called "gulch placer" results in taking long, narrow, irregular tracts of land, which leave very irregular and generally rather barren land in the public domain. Therefore, it is preferable—in fact, on surveyed public lands it is almost mandatory—to survey the placer claim in accordance with the generally recognized subdivision of a section, the smallest tract of which is a quarter of a quarter of a quarter of a section (containing, nominally, 10 acres). On unsurveyed (unsubdivided) public lands it is preferable to lay out the location with the cardinal directions and in a series of squares which are 10 acres in size so that these tracts could be fitted into a future extension of the public land surveys. Since about 1930, tracts as small as 2½ acres have been subdivided out of the public domain for other purposes, such as governmental withdrawals, and there is no reason to believe that tracts of this size would not be acceptable for placer claims. However, there has not been any test case up to 1948.

If there are any known veins or lodes entering into or crossing a placer, these must be excepted from the area which is located and which may be patented. If it can be proved that a vein disappears and does not extend beyond the end line of the lode claim, it is permissible to locate a placer adjacent to this line. If one or more lode claims extend entirely across a placer location, the area will be divided into separate tracts, although it is necessary to have only one discovery for the one placer location, even though this location may be in several parts. If the mineral values are confined to a relatively narrow gulch, the same valuable mineral land may be more efficiently covered by one gulch placer of less than maximum size than by two legal subdivision placers of approximately 160 acres each. With larger-scale placer operations as compared to the hand operations of the past, taking up extra land of little or no mineral value in order to fill out 10-acre lots is less important. If $2\frac{1}{2}$-acre tracts are to be permitted, the problem will be insignificant. Rules and regulations of the Land Office, as well as the decisions of various courts, differ as to whether or not the boundaries for the location of a legal subdivision placer claim need to be staked, and the surveyor must familiarize himself with local regulations on this point.

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[6] Act of Congress, May 10, 1872, Section 10; Revised Statutes of the United States, Sections 2329, 2330, and 2331; 30 United States Code, Sections 35 and 36.

[7] "An Act to Promote the Mining of Coal, Phosphate, Oil, Oil Shale, Gas and Sodium on the Public Domain," Act of Congress, February, 25, 1920.

Methods of Survey.—When the placer claim lies entirely in one section which is well monumented, the survey problems are comparatively simple. However, the placer may lie in several sections. In the usual mountainous areas frequently the section corners not only are difficult to find in the first place, but are usually not placed very accurately or consistently if, indeed, at all. These difficulties necessitate rather extensive resurveys to relocate the section corners and to subdivide the sections into the desired tracts. A survey covering the several square miles involved of necessity must be a rather careful job to maintain sufficient precision. Otherwise, the methods of surveying a placer claim are no different from those for a lode claim. Corner posts would be of the same type as that described for lode claims, with the markings indicating the corner number and name of the placer. In the case of a legal subdivision placer, the tracts are described by their position in the section. If the section corners are in place, no other ties are necessary for the identification of the tract. In case some corner of the placer claim does not coincide with a section corner or USMM, ties would be run on the ground, as is done with lode claims.

Location Surveys for Mill and Tunnel Sites

The mill site, which is the third standard type of mineral land claim, is probably the simplest to survey. It is limited to a maximum area of 5 acres; the shape is not restricted, but it should form a reasonably compact or regular tract. Of course, more than one mill site can be taken up by one locator provided that a need for and use of the land can be shown. As previously stated, a 5-acre mill site may be taken up with a lode claim. The need for this type of mill site has nearly disappeared with the advent of present transportation and a few large ore treatment plants. A mill site may also be located and patented as a site for an actual ore reduction plant. The ground which a mill site covers should contain no known mineral deposit, which usually precludes the location of a mill site on the end line of a lode claim. The problem in surveying consists in laying out the tract and setting a monument at each corner of it, as is done in the survey of a lode claim location. It is permissible to have the tract composed of two or more near-by parcels of land.

There is one other type of location on mineral land, on which, however, the government will not issue a patent. This is a tunnel site. The tunnel site may claim 3,000 ft in advance of the face of and along the line of a tunnel, together with 1,500 ft on either side of the tunnel. The filing of this location protects an investor who is developing an area with a deep cross-cut tunnel and grants him the exclusive prospecting rights in the 3,000-ft-square area. If any new lodes are discovered as the tunnel progresses, they would then be located in the same manner as an ordinary lode claim with 600-ft by 1,500-ft dimensions. Lodes known to exist prior to a tunnel location do not restrict location at the time they are cut by the tunnel, but are open to location in the usual manner. If desirable, a square "dump site" 125 ft on either side of and 250 ft below the mouth of the tunnel may also be staked out, if this area is on nonmineral land. Staking a dump site consists of monumenting the four corners; staking a tunnel site consists of monumenting the four corners and also the end of the tunnel center line 3,000 ft from the mouth, or from the face, as the case may be.

References or ties to some permanent monument or natural object are the same for mill site or tunnel site locations as they are for lode locations. A mill site located in conjunction with a lode claim should give a tie by course and distance between a corner of the lode and a corner of the mill site.

Completing and Maintaining the Location

The surveyor, after staking out the lode or placer claim, should post a notice of discovery and location at the point of discovery and should advise his client as to the validity of the discovery hole. The surveyor should fill in a location certificate form for his client to file in the office of the county clerk. This certificate should include the name or names of the locators, the date of location, and the dimensions of the claim, including such a description as will positively identify the intended tract of ground. The steps necessary for completion of a mill site or a tunnel site location are essentially the same, with the obvious exception that a discovery does not have to be made.

After initiating a mineral land claim, the locator has certain rights to the land he embraces as to the methods of mining or development. He may retain these rights for himself, or he may lease or sell them to another. However, in order that the location may not become void and the land thereby "jumpable" —that is, open to location by another—the locator or his heirs, assigns, or legal representative must do certain development work (usually called assessment work) which amounts to $100 worth of labor or improvements on each claim during each year. For a contiguous group of claims all this assessment work may be done in one place, as long as it can be shown definitely that such work is a benefit to each of the individual claims.

Title to the land claimed by a location remains vested in the federal government until such time as a patent is issued. Consequently, the locator pays no taxes on any locations. Assessment work is not required on a mill site or a tunnel site. There have been a few depression years and war years during which Congress suspended the requirement for annual assessment work on lode and placer claims.

The matter of assessment work may seem to have little to do with surveying, but in actual practice the surveyor must be able to judge whether ample assessment work has been performed to enable his client to establish and to maintain a clear title to the claim. The United States mineral surveyor must be conversant with the costs of performing various types of mining work, inasmuch as he must swear under oath that at least $500 worth of mining improvements have been performed toward the development of each claim on which a patent is sought.

PATENT SURVEYS

As soon as a locator has determined that his claim or claims have sufficient value to warrant his obtaining a fee simple from the government, he applies for a patent. At this point, the surveyor reenters the picture to survey the land completely and carefully so that the government may adequately note the position of (segregate) and issue title (patent) to the tract of land. This time the survey is made under official order by a duly qualified and appointed United States mineral surveyor.

Mineral Surveyors.—Mineral surveyors are commissioned on application after proof of technical qualification, general reputation, and fitness for the position. The appointment is made by the United States office cadastral engineer through the public survey office for the particular land district or state to which the application is made. The mineral land surveyor, on receiving his commission, must post an official bond in the sum of $5,000 before he may make official mineral surveys.

In 1948 there were 118 men commissioned as mineral surveyors in the United States and Alaska. These men follow the regulations and are under the supervision of the Public Survey Offices, although they are not under civil service nor do they receive any compensation whatever from the federal government. Prior to 1925, the surveyors were called United States deputy mineral surveyors and were under the office of the surveyor general for each of the so-called public land states. The appointment to the position of surveyor general for a state was political, so that the term of office was usually short, and the position was frequently held by some one who might know very little about mineral surveys. Prior to the issuance of the first manual of instructions [8] the survey procedure was often different in different states, and even different in one state from time to time. These differences in procedure would be occasioned when the commissioner of the General Land Office issued special instructions by letter to a surveyor general or to a deputy mineral surveyor to cover some special condition, which might be treated differently if a similar case arose elsewhere. Occasionally much confusion ensued. In 1895 the manual of instructions was issued by the commissioner of the General Land Office as an average of the practices in various states. With each subsequent reissue of the manual, variations in survey practice have decreased until at present, with only slight exceptions in Alaska, all surveys conform to the standards of the 1947 manual.[2] Although 118 mineral surveyors in the United States and Alaska may seem a small number of men to carry on the mineral survey field work, especially when it is realized that more than 300 surveyors held commissions as deputy mineral surveyors for Colorado alone in 1900 as compared to 22 mineral surveyors in 1948, still there are only 40 commissioned surveyors of all the public lands of the United States. In other words, there are at present more than twice as many mineral land surveyors as there are public land surveyors, indicating the importance of mineral land survey work.

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[8] "Manual of Instructions for the Survey of the Mineral Lands of the United States," Gen. Land Office, U. S. Dept. of the Interior, U. S. Govt. Printing Office, Washington, D. C., 1895.

Patent Application.—The procedure for obtaining a patent to mineral land is much different from and more complicated than the process of obtaining a patent to any other of the public lands. This situation derives principally from the fact that for agricultural or similar lands entry is made on a quarter section which has already been surveyed, monumented, and platted according to Land Office methods, whereas for mineral lands a survey of the particular tract must be made, monuments must be set, and a plat must be made so that conflicts can be shown and computed before the local land office can issue a patent.

The owner or owners of a location or locations should first employ a United States mineral surveyor of his or their own choosing to make the patent survey. Before the claimant applies for an order for an official survey, the surveyor, if he did not make the original location survey himself, should make a careful survey to ascertain that the stakes on the ground are where they belong, that the claim is of the proper size, that the desired ground is covered, and that the location certificate properly describes the claim. In a great many cases it is necessary to amend the location by filing an amended location certificate to accomplish these objectives. The official survey is bound by the location certificate, only slight variations being permitted, so that the major changes usually required by poor certificates can be made only with great difficulty, if at all. The survey made for the purpose of ascertaining the correctness of the location, and for obtaining the data for the amended location certificate, is conducted with the same care as the patent survey itself. Indeed, it is a necessary part of a patent survey, even though it is made before the order for official survey is issued. If an irremediable mistake between the location as staked and the location as described in the certificate is discovered after the official survey order is issued, it will be necessary to amend the location certificate, cancel the old survey order and number, and obtain a new order and survey number. The mineral surveyor will have to certify that $500 worth of labor and improvements have been made for each claim for which patent is applied.

The claimant, who must be a citizen of the United States, or a person who has declared his intention to become a citizen, or a corporation chartered in the United States, applies for an order for survey. The application must be accompanied by certified copies of the location certificates or of the amended location certificates, as the case may be, obtained from the county recorder. The public survey office estimates the cost of office work on the survey; and, when the claimant has deposited this amount, the order for mineral survey is sent to the deputy selected by the claimant, together with copies of the certified copies of the location certificates.

Making the Survey.—The mineral surveyor proceeds with reasonable diligence to make the official survey, progress depending somewhat on the season of the year and on similar factors.

Ties to Monuments.—In beginning the patent survey, whether it be the relocation survey or the true patent survey, the surveyor should determine first whether the claim is on surveyed or on unsurveyed land. If the survey is on surveyed land in good standing, he must find the nearest section corner to the claim, provided one exists within 2 miles. In the mountains this frequently is difficult. The section corners in most of the mineral districts are stones or, occasionally, posts. If there have been other mineral patent surveys in the area, the field notes of these claims will show what monument was tied to and may assist the surveyor in his search. In the event that a public survey corner cannot be located within 2 miles of the claim being surveyed, it is possible that a USMM or a USLM has been established previously in the vicinity. A monument of this type may also be used to tie a mineral claim to if it is within 2 miles from the claim. If the mineral surveyor cannot find a section corner, a USLM, or a USMM within 2 miles of the claim, he is empowered to establish a USMM according to the procedure set forth in the manual of instructions.[9] In times past, the mineral monuments or locating monuments have been designated by letters in the greek alphabet, by Roman numerals, by the names of the mining district, or by the names or the survey numbers of the claim being surveyed when the monument was established. There are (1948) more than 700 monuments of this type in Colorado alone. The connecting tie from corner No. 1 of the claim to the section corner or USMM is supposed to be run directly on the ground, or the connecting traverse is supposed to be described in the field notes which the surveyor submits to the public survey office. Nevertheless, common practice is to traverse or to triangulate the tie, and then to calculate the direct line and report its bearing and length. The tie may be calculated through a tie from some near-by claim if that tie has been verified.

Ties to Conflicting Claims.—The surveyor should next find the corners of any conflicting claims. There are two prerequisites in the search for these corners: First, a map showing the district with the patented claims platted on it, available from the public survey office of the state as a "segregated sheet" or published by a mineral surveyor in the district; and, second, an abstract of the field notes of the patented claims likely to be in conflict with or near the claim to be surveyed, obtained by consulting the original field notes on file in the public survey office or the notes published by some individuals.[10] In some of the larger districts, index books have been published giving the township and sections in which each patented claim is located.[11] Such an index book may also help to locate any conflicting or near-by claim.

Recovery of Corners—Starting at the discovery hole the surveyor proceeds to locate any patent corners which are on lines of existing patented claims conflicting with the proposed claim. If there are no conflicts, the problem is simple. The claim is laid out in conformity with the location stakes and the location certificate; monuments are set and marked at the corners of the claim; a tie is run to a section corner or other public land monument; and, under favorable circumstances, the survey may be completed in a day or two. However, in the old established districts, with many conflicts being the usual situation, the surveyor may spend a week or more just looking for and recovering old corners.

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[9] "Manual of Instructions for the Survey of the Mineral Lands of the United States," Bureau of Land Management, U. S. Dept. of the Interior, U. S. Govt. Printing Office, Washington, D. C., 1947, pp. 451–452.

[10] "Abstract of Survey Field Notes in Cripple Creek District, Colo.," by Fred J. McNair, Denver, Colo., 1897.

[11] "Cain's Index of Mining Claims in Lake County, Colo.," by William P. Cain, Leadville Pub. & Printing Co., Leadville, Colo., 1906.

As has been the case with all land surveys from time immemorial, the corners of early mining claims have been poorly monumented according to modern standards. In addition to the usual ravages of the elements, inasmuch as most mining districts are located in mountainous areas, the monuments generally fall on slopes where they are subject to the general subsidence of the land form, the destructive forces of snowslides, damage from rolling rocks, burial by dumps from mines, and other destructive factors not common to the average land survey monuments. The mortality among these patent corners is high. Because there are practically never any fence lines along the claim boundaries, and because the timber which may have existed and been blazed at the time of the survey has usually been cut for use in the mines or has been swept by forest fire either accidentally or purposely, the problem of retracing the boundaries and recovering the monuments is a difficult one indeed.

Most of the early corners were wood posts. They were required to be a minimum of 4 in. square and 4 ft long, squared at each end and set at least 1 ft deep in a dug hole (not pointed and driven into the ground, as was occasionally done). The proper markings were carved into the wood, some deputies exhibiting great artistry with a knife. At the time these monuments seemed very satisfactory. The posts stood up above the surrounding land and, particularly above timber line or on open land, were prominent and easy to find. With the usual mound of rocks around it, the post was reasonably stable and would last for quite a few years. If the deputy used care in selecting the post and chose a cedar, or "pitch" pine, or other rot-resistant wood rather than an aspen or cottonwood, and if the post was hewn rather than sawed, its lifetime might be doubled or tripled. Probably not more than 10% of the original post corners lasts more than 30 years without being repaired or replaced.

Some of the better mineral surveyors in the early days, recognizing the importance of monumenting the corners in a more permanent manner, would set a stone corner if they could do so without too much additional labor. These stone corners are, for the most part, still extant. Many of the earlier stone corners, because they were too small or were poorly set, have subsequently been destroyed. Some of the stones were of poor quality; and, if the surveyor finds that the chisel markings on these stones are entirely obliterated, he may have some reasonable doubt as to the authenticity of the monument. For a good monument a hard, weather-resistant rock at least 24 in. long and of at least 6-in. minimum thickness in transverse section (say, as large as one man can handle or roll into place) should be selected, set with two thirds of its mass underground, and firmly braced in place. In addition, it should have an earth backfill thoroughly tamped around it and should be surrounded by a pile of rocks. Such a monument will withstand all but the most extreme of natural destructive agencies.

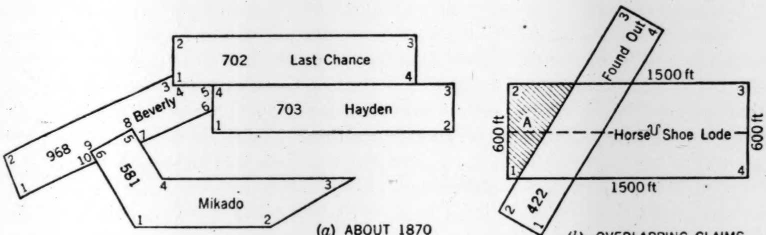
It is not especially unusual for a corner in the mountains to fall on a large boulder or on solid rock. In this case, the chisel markings, if made properly, will last almost indefinitely. Occasionally, a tree may be at the corner point; if so, it will be blazed and properly scribed, with a nail placed at the corner point. A suitable tree, adequately marked, makes a good permanent monument. In rare cases, cut stone, heavy mine rail, light railroad rail, or concrete posts have been used at the corners. Since 1930 a very few iron pipes with flaring base, concrete core, and brass or aluminum cap for marking with steel dies, similar to the type adopted for public land survey monuments, have been used. These have been highly satisfactory except in the rare cases where minerals in the soil or water may attack the iron pipe. Therefore, despite the additional expense for installation, pipes will probably be used to a large extent in the future. However, all these last few materials are usually relatively expensive and difficult to transport to the site. It is difficult to impress the average owner with the necessity of establishing monuments that will be as permanent as possible, although more expensive.

From the abstract of the original field notes the surveyor ascertains that the corner he is looking for consists of a certain material. If there are several conflicting claims, there may be several references from one survey to another which may help to show just where a corner may have been previously found, and in what condition. Of course, the corners may have been reset or replaced and may possibly be of some other material than the original, although the latter case is unusual. As he makes his search, the surveyor should have a full description of any local bearings such as bearing trees or bearing rocks. This information is particularly important for post corners, since in most cases these local bearings may be the only evidence left as to the position of the corner.

Historical Considerations.—After one corner has been identified to start from, it is frequently possible to set the transit over this point and to rerun the old lines and ties described in the abstract of field notes, thus recovering other necessary corners. However, historical considerations as described hereinafter frequently complicate the seemingly simple operation of retracing old lines.

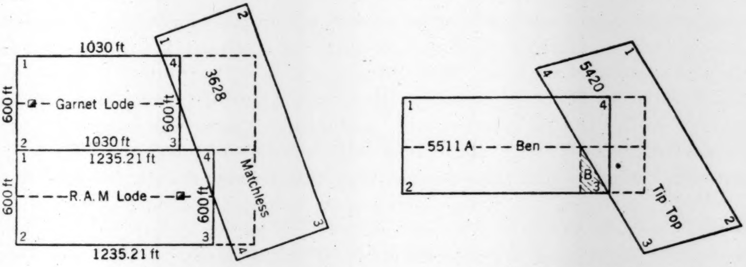
As previously described, the variations in individual instructions caused much confusion in the very early patent surveys, as each surveyor general issued his own instructions for the conduct of each survey. In addition, in each mining district in a state, each claim had its own mineral survey number, beginning with No. 37 (presumably on the hypothesis that there were already thirty-six sections or tracts in a township). Between 1879 and 1890 all these mining districts were coordinated in each state, and since then the numbers in each state have been assigned consecutively. Colorado, with the greatest number, has 20,709 survey orders or a total of 27,940 including the subtotals from the seven separate districts. Montana, with the second highest number, has 10,802 survey orders. Until about 1883, in general, only single claims were patented; after that time two or more contiguous claims could be patented as a group and with only one survey order number.

The very earliest patented claims were cut off where they conflicted with another claim as in the case with the Beverly lode (Fig. 2(*a*))—resulting in an odd-shaped tract. This practice was soon discontinued, because any error in the survey or in the setting of the monuments might leave a small fraction of unpatented ground between the claims. For the next few years, one lode was prohibited from lapping across another prior lode as is done by the Horseshoe lode location to acquire area A (Fig. 2(*b*)). Between the years 1891 and 1900 a ruling required that a claim such as the Garnet lode (Fig. 2(*c*)), which conflicted with a prior lode, must be cut off at the point where its lode line intersected the prior lode. This practice caused many small unpatentable fractions of land to be created. Fig. 2(*d*) shows a special case of this same ruling where, by agreement of the two owners, the Ben lode could be extended to obtain a patent to additional tract B. Since 1900, the claims have been permitted to extend into one another. Of course, if no vacant ground is covered, there is no advantage in extending a claim the full 1,500 ft in length or the full 600 ft in width, since a conflict with other claims adds to the cost of the survey. In all cases where circumstances permit it, the most ground is covered with the least conflict by laying out the claim so that it has a skewed end line, as in Fig. 1(*b*).


702 Last Chance
703 Hayden
968
Beverly
581
Mikado
1
2
3
4
5
6
7
8
9
10
Found Out
1500 ft
600 ft
A
Horse Shoe Lode
422

(*a*) ABOUT 1870

(*b*) OVERLAPPING CLAIMS PRIOR TO 1891


1030 ft
600 ft
Garnet Lode
3628
1235.21 ft
R. A. M Lode
Matchless
1
2
3
4
5420
5511 A
Ben
B
Tip Top

(*c*) 1891 TO 1900

(*d*) 1891 TO 1900 SPECIAL CASE

FIG. 2.—CUTOFF OF CLAIMS UNDER VARIOUS LAND OFFICE REGULATIONS

Originally, claim corners were numbered either clockwise or counterclockwise. In some groups of claims, a common corner might be numbered—for each claim. In Colorado during the period from 1886 to 1892, the corners were numbered consecutively through the claims in the group. This system of numbering is convenient because a person finding corner No. 18 knows immediately where he is with respect to the entire group, instead of knowing only that if he has found a corner No. 2 of some one of perhaps a dozen claims in a group. From 1892 to 1925 the corners were simply numbered in rotation from 1 to 4 (or 6, or 8, depending on the number of sides) around each individual claim, which led to much confusion. Since 1925, the corner nearest the section corner, or USMM, must be No. 1 and the mineral surveyor has been required to put the initial or enough of the name of the claim on each corner to permit its identification.

The main cause of confusion and difficulty in retracing old surveys is the so-called "section-tie" ruling which was in effect from June, 1899, to August, 1904. This Land Office regulation required that existing claims were to be tied to at the place where the approved field notes described them, regardless of where any existing corners might place the claims. At least on paper, this ruling would appear to simplify the surveying and to insure the most economical coverage of the ground. However, this regulation would be comparable to a ruling which would require that henceforth all section corners are to be considered as located where the original subdivision survey notes place them, rather than where they actually may be found on the ground. Such a ruling is contrary to all court decisions, since the position of existing monuments on the ground is considered the best evidence of the true and desired location of a tract of land. The largest differences between the described position and the monumented position of a claim occurred when there was an error in the tie to the section corner or USLM, hence the name "section-tie" ruling. These errors varied from negligible amounts up to several thousand feet, the latter usually occurring where a section corner was not properly identified. Of course, differences in the claim were also caused by errors or mistakes in the survey of the boundaries themselves. The status of all claims found to have erroneous descriptions, or to be in conflict with claims which did have, was very uncertain during the period that the "section-tie" ruling was in effect. Consequently, many of the old claims with imperfect descriptions were "repatented" even though the monuments of both surveys fell at the same place. Similarly, many of the claims which were patented during this period and which were adjacent to or conflicted with claims having erroneous descriptions had to be resurveyed and "repatented" after the repeal of the Land Office ruling, because during this period, if the claims were to occupy a certain vacant parcel of ground, the erroneous ties and lines of the adjacent surveys had to be called correct and reconsideration of claims proceeded on that basis. In searching for old corners and in making the present survey and description, the mineral surveyor must keep all the foregoing facts and approximate times in mind.

In the earliest days of mineral land surveys many of the deputies used the the old link chain for making measurements. As these chains were used, and as the links wore, the measurements gradually became longer and some of the claims were thereby made too long. During the early 1890's, the chain was superseded by steel tapes, and thereafter errors from this source disappeared. On the other hand, many of the early claims are too short because the deputy surveyor had been making public land subdivision surveys and was familiar with "breaking chain." He would badly misjudge the mountain slopes and it is not unusual for the engineer on a resurvey to find a claim described as 1,500 ft long from as much as 15 ft to 30 ft short.

Recently it has been determined that general earth movements after many of the claims were surveyed (in some cases as much as 75 years ago) have caused movements of entire claims or of one corner of a claim with respect to the other corners. This movement is, of course, most noticeable in the steeper mountains, where the shape of the slope may seem intact; and, indeed, the individual corners may appear to be very firmly set in their original positions. Nevertheless, careful surveys made at intervals of 10 years or more plainly show gradual movement and settling of an entire mountainside. In one particular case where a rather valuable claim in Colorado was surveyed for patent in 1876, side-center monuments were set on a claim, one of the side-line points falling on a ledge of solid rock, and therefore being marked by an *X* chiseled on the ledge. The claim lies along the face of a steep mountain and on a slope composed mostly of slide rock. All the corners have apparently remained "in place" continuously. In 1900, this claim was resurveyed by another surveyor during the process of surveying an adjacent claim, and it was discovered that the side line departed from a straight line by about 3 ft at the end—that is, there was an angle of about 0° 05′ in the claim at the side center as shown in Fig. 3. By 1928, the movement had become so large that several apparent fractions amounting to as much as 5 ft or 6 ft had appeared, the movement being such as to cause the apparent apex of the vein to cross the end line rather than the side line; and these fractions were then patented to insure complete coverage of the vein. In this case the title to more than $1,000,000 worth of ore was affected. In 1938, the surveyor who had made the survey of 1900, and others at various intervals since that time, had occasion to make a rather extensive survey of all the claims in the area. By rerunning the lines as they existed in 1900, he was able to show that the fractions between the original claims were probably nonexistent in the surveys of 1876.

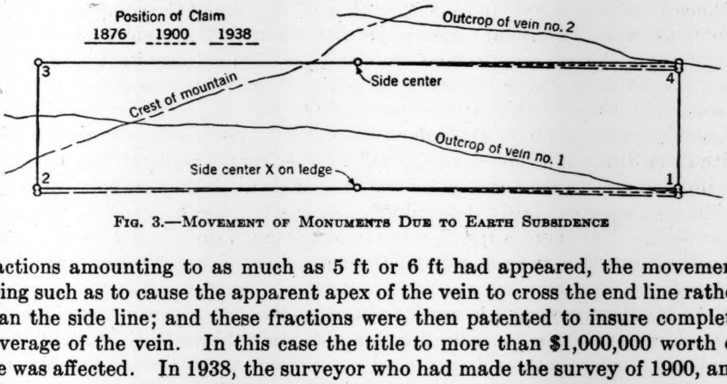


FIG. 3.—MOVEMENT OF MONUMENTS DUE TO EARTH SUBSIDENCE

Another instance of this earth movement is a claim which has one end line on top of a mountain ridge and which extends 1,500 ft down a slope consisting of slide rock inclined about 35°. One of the monuments on the lower end line is a large stone set in a niche on a point of solid rock. To all appearances this stone is as stable and permanent as possible. This claim, as determined by several independent measurements, was 1,500 ft long when it was patented in 1901. By 1942 the monuments had moved apart, the lower point of solid rock apparently settling downhill, so that the length of the claim between monuments was 1,504 ft. The examples just cited are believed to be unusual in the

United States because of the time interval between unquestionable observations and the magnitude of the movement. Some studies have been made of the movement of rock glaciers (great masses of broken rock which "flow" in a manner similar to the familiar ice glacier). The report of these studies,[12] some of which are as yet unpublished, may throw some light on the apparent errors in some old mineral surveys which, although made by reputable surveyors, show unaccountable errors.

Earth movements are also caused by known active faults or by settling due to mining operations. All instances of movements of the monuments pose very interesting and complex problems of title in which engineers and land surveyors are vitally concerned, particularly in court cases.

Precision.—On all mineral land surveys made since 1872, the required precision, measured as an error of closure around the claim, is 1 in 2,000. This error "must not be such as to make the claim exceed the statutory limit"—that is, if one side line is 1,500 ft long, the other must be between 1,499.25 ft and 1,500.00 ft long. However, even in difficult terrain, the mineral surveyor can and usually does, maintain a precision of about 1 in 5,000. The angular variation allowed is 2 min.

Prior to 1872, the magnetic compass was used for determining the bearings of lines on some mineral land surveys. Since 1872, use of the compass has been prohibited and direct solar observation with a transit has been required. For many years Polaris observations were not permissible, despite the fact that they can be more precise than a solar observation. At present (1948) either a solar or a Polaris observation is acceptable, but, as previously stated, most mineral surveyors prefer the solar observation. Ordinarily six observations with three direct and three reversed will produce results which will agree within 2 min of angle. True bearings for the lines of many of the first surveys in any given area were carelessly determined, perhaps because compass values alone were read. However, as the land has become more valuable and other surveys have been made with greater care, the determinations have been quite satisfactory.

As in all surveying field work, each surveyor has his own technique and procedure—some use the deflection method, some the azimuth method; some double the angles, others do not; and some orient by the method of plate reversals, some by plunge reversals. Probably the method used by more mineral surveyors than any other is the bearing traverse method previously mentioned. In all cases where the ground is at all irregular or steep a long tape (usually 500 ft long) is used. Vertical angles are read, and slope measurements are made and reduced to give true horizontal distances. Stadia measurements are not permitted on patent surveys. Occasionally the terrain lends itself to a triangulation network, particularly if several claims are included in one survey order so that a large area is covered. In this case the lines between the triangulation points might span some of the rough or heavily timbered terrain and only short traverse lines would have to be run to the discovery holes or to the position for corners. Conditions would vary depending on the extent of the triangulation network, but rarely would the angles be repeated more than three times with a 1-min transit. The base line for such a system would probably be double chained, and a spring balance and temperature corrections might be used to obtain greater accuracy for an extensive system. Since aerial photographs are available for most of the mineralized area in the United States and Alaska, mineral surveyors will become more familiar with their use. Particularly where several locations are embraced in one survey, the aerial photo gives an over-all view of the area and is an invaluable reconnaissance aid to the surveyor. Essentially, it gives him a map of the area before he starts into the field so that he can select, for his traverse lines and triangulation stations, positions which will avoid heavy clearing and difficult field work.

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[12] "Rock Glaciers in the Colorado Front Range," by Ronald L. Ives, *Bulletin*, Geological Soc. of America, Vol. 51, July–December, 1940, p. 1271.

Monuments for the corner points would be composed of rock in place, iron pipes, set stones, or posts, in that order of preference, as indicated previously. At each corner one or more bearing trees or bearing rocks should be located with respect to the corner wherever this is possible. Such local bearings preferably should be within 50 ft of the corner. If the position of the corner falls in a road, stream, loose bank, or other place where a permanent monument could not be established or maintained, a witness corner would be set on one of the lines of the survey and as near the true corner as possible. Of course, complete descriptions of the corner monuments, bearing objects, or witness corners would be included in the official field notes to insure identification of the true corner at any future time.

Field Notes.—The surveyor is required to run a tie between the nearest public survey monument and the corner of each of the claims he is patenting which is nearest to the monument. In cases where the claim being surveyed conflicts or lies near previously surveyed claims, the surveyor in his official field notes reports how he identified the corners which he found on these old claims and what condition they were in or how he may have reset or re-marked them. He is also required by the Land Office to run a connecting line from some corner of the claim he is surveying to a corner of each conflicting surveyed claim, and also to a corner of any conflicting unsurveyed locations which will exclude area from his claim. In addition, more recently, connections have been required to any near-by claims even though they may not conflict. Also, for the claims which do conflict, the surveyor's field notes must report the distance along the line of the present survey to the point of intersection with the line of the old survey and the course and the distance from that point along the line of the surveyed claim to one or the other of the corners. If any corner of these conflicting claims falls within the claim being surveyed, the course and the distance from the point of intersection of the lines to this particular corner should be given. Of course, to give the bearing and the length along a line, monuments at the ends of the line must be located. In the event that all the corners required for fixing the positions of conflicting lines cannot be recovered, the mineral surveyor must exercise considerable judgment based on past experience and on the evidence at hand as to the probable location of the conflicting claim. The record position of the claim may be adopted as a basis for defining the conflict. In any case, the field notes must give a complete description of the evidence at the time of the survey and the reasons for the adopted values.

Joint Surveys.—In addition to reporting the condition of the corners which are found, the mineral surveyor must report in his official field notes any differences which he finds in the approved field notes of the conflicting claims. In earlier days whenever a deputy reported another claim in error, the deputy who made the conflicting survey was required to resurvey the disputed area and to file amended notes concerning the errors reported. In case the deputy who surveyed the first claim denied that there was an error, a joint survey was ordered by the surveyor general to be made by the two deputy surveyors jointly. The deputy found in error paid all expenses and $10 per day to the surveyor found to be substantially correct. This practice generally produced good results because no reputable surveyor wants to be shown that he has made a mistake. However, the system became impractical and was abandoned about 1925, because so many of the earlier surveyors had died that joint surveys were usually no longer possible. At the present time (1948), errors are reported and notations to that effect are made on the original official plat by the public survey office.

Placer and Mill Site Surveys.—The procedures for the survey of placer or mill site claims are essentially the same as have been described for lode claims. If a lode and a mill site are both included in the same survey, the lode claim survey number is followed by the letter A and the mill site number is followed by the letter B. A tie must be run from corner No. 1 of the mill site to some corner of the lode and also to a public survey or a mineral monument. If a placer claim is separated by known lodes into two or more tracts, each of these sections should have a separate series of corner numbers. To avoid confusion some surveyors call the corners in one tract Nos. 1, 2, 3, and so forth, and those in another tract A, B, C, and so forth. When a placer conforms exactly to the public land system, no survey is necessary to segregate the area, and a patent may be issued directly. The placer then has a mineral entry number similar to the familiar homestead entry numbers. If there is any doubt as to the exact positions of the public survey monuments, as there is in many mountainous regions, the applicant will be wise to apply for an official mineral survey, to be sure his corners cover the land he wants. It is entirely possible that one claimant might have contiguous lodes, together with mill sites and placers, all to be surveyed under one order of survey.

Improvements.—At some time during the progress of the survey the mineral surveyor must locate all mining and other improvements. On lode claims these improvements preferably will be located by course and distance from some corner of the claim, although they may be tied to some point on the lode line—usually the center line—of the claim. A total of $500 of improvements since the date of discovery must be shown to obtain a patent, and the mineral surveyor must certify to this value. The only improvements whose value applies toward the necessary $500 are mining expenditures made by the claimant since the date of discovery. The discovery hole is the first such improvement. Except for the discovery hole, the improvements may be outside the claim if such improvement actually benefits the claim—for example, a drainage tunnel for deep development of the claim. Where more than one claim is being patented in a group, all the development work may be done in one place, with each claim being credited with an equal share of the total value of the improvement if all these claims are in a common ownership. Other improvements such as roads, buildings, and other structures ordinarily not classified as mining expenditures, as well as any topographic features such as streams, roads, or cliffs, should be noted since all these may help to identify the position of the claim. On a mill site taken up in conjunction with a lode, no expenditures are required beyond the $500 required on the lode. On a mill site taken up as a site for a reduction plant, the structures should be located.

Since September, 1882, a descriptive report has been required[13] on all placer claims, including type of soil, type of timber, size and position of streams, improvements by the claimant, and many other items. The field work of locating the various improvements and topographic features is the same as for any ordinary mine survey or topographic survey and need not be described in detail.

Occasionally a claimant may attempt to obtain a patent when he has not performed sufficient work on his claim or claims. In this case the mineral surveyor must remember his code of ethics and that he is under oath; he must refuse to make his official statement on the value of the improvements until the work has been done. Honest differences between the actual cost and the apparent value of some improvements occasionally arise. Each of these cases must be considered by the mineral surveyor on its own merits.

The surveyor prepares a plat of the survey on a scale of 200 ft to the inch, writes up the field notes, makes his area calculations by the method of double meridian distances, and sends copies of all these to the public survey office for the state. The form for each of these is specified, the details being given in the manual of instructions.[14] If everything is in order and the notes and calculations are correct, the cadastral engineer approves the survey. If a mistake is found, the notes are returned for correction. Occasionally, through some irregularity, an amended survey of the area may be ordered by the Director of the Bureau of Land Management. Special instructions as to the object and methods of the amended survey are issued to the mineral surveyor. Every place the survey number of this amended survey appears, whether on the monuments in the field or in the notes, the abbreviation "Am." is used after the number.

The services of the mineral surveyor are supposed, by law, to be completed at this point. However, there are many legal forms to be filled in and land descriptions to be written. If no other claimant files an adverse claim to the area embraced in the survey within 60 days, the patent is then issued to the claimant on his payment of $5 for each acre or fractional part of an acre for lode claim or mill site areas, or of $2.50 for each acre or part of an acre for placer acreage.

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[13] "49 Land Decisions," General Land Office, U. S. Dept. of the Interior, Washington, D. C., April 11, 1922, Vol. 58, Paragraph 167.

[14] "Manual of Instructions for the Survey of the Mineral Lands of the United States," Bureau of Land Management, U. S. Dept. of the Interior, U. S. Govt. Printing Office, Washington, D. C., 1947, pp. 453–458.

CONCLUSION

Nowhere (as of 1948) in literature is there any complete presentation of the history, methods, and instructions for the survey of tracts of mineral land—although this land is the only type of public domain patentable in the United States and although all land is rising in value. Because a land surveyor, civil engineer, or mining engineer may have to resurvey or subdivide these tracts of mineral land, which are numerous, particularly throughout the western part of the United States, this paper has presented the general features of mineral land surveying. The day of the small prospector is about gone; and, as the individual claims are consolidated into larger groups under one management, it will be increasingly important to survey and patent the small, irregular, unpatented fractions of land left between the original claims. This important work will necessitate a thorough knowledge of the historical conditions at the times of those earlier surveys, as herein presented. In new mineralized districts, yet to be found, the claims no doubt will be laid out in more regular form so as to avoid all the conflicts and exclusions of the older system. Moreover, the importance of monumenting the tracts of land as permanently as possible cannot be overemphasized, since the engineer retracing old boundaries finds the greatest difficulty and spends the most time with this phase of the work. This paper, in presenting some of the techniques of the past, should help the engineer to save time on resurveys involving tracts of mineral land.

ACKNOWLEDGMENTS

The writer wishes to express his gratitude for early training and experience gained in mineral land surveying under the careful direction of Fred J. McNair, United States mineral surveyor for Colorado for 50 years, as well as for the use of Mr. McNair's records and maps, which have assisted greatly in the preparation of this paper. The writer also is indebted to Russell K. Allen, administrative cadastral engineer, and to Leo Peterson, mineral surveyor, both in the Public Survey Office, Denver, Colo., for their time and suggestions as to various phases of the subject discussed. In addition, the writer wishes to express his appreciation to Warren Raeder, M. ASCE, head of the Department of Civil Engineering of the University of Colorado at Boulder, for his interest in and assistance with the preparation of this paper.

DISCUSSION

Charles P. Eager.[15]—The presentation of this comprehensive paper is opportune in view of the current activity in mineral land surveying. The scope of the engineering technique (which, of course, is of paramount importance in a paper of this nature) apparently covers the subject. Only where the author touches upon matters pertaining to United States mining laws and to the Bureau of Land Management practice do a few questionable points arise.

An adequate discovery of mineral within the boundaries of any mining claim, whether it be lode or placer, is prerequisite to valid location. Therefore, the specification under the heading, "Types of Mineral Claims: Placer Claim," that

> "* * * the discovery [upon a placer] must be not just a discovery of mineral, as with the lode claim, but it must be a discovery of commercial deposits of placer material * * *"

appears to be, at least, an unauthentic differentiation—based, no doubt, on a singular court decision.

The statement in the same section of the paper, as to "* * * an over-all length of 2 miles or 3 miles being not uncommon for a 160-acre gulch placer" probably applies to the earlier official mineral surveys. It is a fact that the rule enunciated in the mining regulations fixing a limitation on the length of a placer claim will not be applied where the mineral deposits are confined within a narrow strip of land in the bed and on the banks of a small stream in a canyon flanked by abrupt walls or rocky slopes on each side, containing no mineral, agricultural, or timber value.[16] However, the matter of whether a placer location conforms sufficiently to requirements is a question of fact for determination by the Bureau of Land Management in the light of the showing made in each particular case, and the locator should bear in mind that it is not the current policy of the federal government to sanction entries or locations that cut the public domain into long narrow strips or grossly irregular and fantastically shaped tracts.[17]

The Act of Congress of May 10, 1872, the regulation[6] which, as Mr. McNair states under the heading, "Location Surveys for Lode Claims," still prevails in 1948, apparently did not specify that a lode location—other than a lode within a placer[18]—could not be less than 25 ft on each side of the lode line. Section 2320 of the Revised Statutes states: "* * * nor shall any claim be limited by any mining regulation to less than twenty-five feet on each side of the middle of the vein at the surface."

The smallest legal subdivision of the public surveys provided for by the mining laws is a tract of 10 acres, normally in square form; and it is very improbable that, in the location and entry of placer mining claims, rectangular

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[15] Cadastral Engr.-Mineral, Bureau of Land Management, Region II, U. S. Dept. of the Interior, Glendale, Calif.

[16] "Land Decisions," U. S. Govt. Printing Office, Washington, D. C., Vol. 53, 1933, p. 431.

[17] *Ibid.*, Vol. 37, 1909, p. 250.

[18] Revised Statutes of the United States, Section 2333, U. S. Govt. Printing Office, Washington, D. C., 2d Ed., 1878.

"tracts as small as 2½ acres" (as stated under the heading, "Location Surveys for Placer Claims") would be recognized and treated as legal subdivisions.[19]

Rules and regulations of the Bureau of Land Management very definitely do not require that the boundaries of a placer located by legal subdivisions be staked because, to stake a placer in this manner, a survey obviously would be required.[20]

In preparing a location certificate for a placer that is to be officially surveyed for patent, it also is very important to include the name of the claim.[21] Under the mining laws of certain states (which are supplemental to those of the United States), this feature is mandatory (Public Resources Code of California, Section 2301).[22] Also, the possessory right to a mining claim is taxable in some states —as, for example, in California. (Section 3617 of the California Political Code defines "right to the possession of land" as real estate (see People versus Scherer, 30 California 645).)

In perusing the section on "Patent Surveys," especially the opening and closing paragraphs, and the statement following the subtitle, "Field Notes," an uninformed reader quite probably would infer that an application for an official mineral survey is tantamount to a patent application. However, these are two separate and distinct procedures. The application for a survey is merely a preliminary action, addressed to the Public Survey Office, for the purpose of obtaining an official description of the claim. After the approval of the survey by the Public Survey Office, there is no limited period (if ever) during which the mineral claimant may make the actual application to the District Land Office for a patent to the property.

Mineral land is not "the only type of public domain patentable in the United States," as stated under the heading, "Conclusion." Disposals are currently being made through the Bureau of Land Management in practically all classifications permitted under the public land laws.

Robert Lenon.[23]—In commenting upon Mr. McNair's excellent paper, it might be well to note that, although the writer has held a commission as mineral surveyor since 1932 except during World War II, his experience has been limited to Arizona and consequently this discussion will be based on such local experience.

Under the heading, "Location Surveys for Lode Claims," Mr. McNair mentions that "* * * since 1920 the majority of location surveys has been made and the descriptions have been written by engineers." Such is not the case in Arizona; the writer, for example, has made very few such surveys in the course of almost 20 years of practice, and it is quite unusual to find location notices on record with bearings any more specific than "westerly, southeasterly, etc."

Mr. McNair mentions the necessity for "bending" claims to keep all the vein within limits ("Types of Mineral Claims: Lode Claim"). A fairly common case is that in which a bend in the outcrop is caused solely by the topography, the vein itself being as straight as possible. For example, consider the case of a rather flat-lying vein striking across a flat, and up a hill. Where the slopes of the hill and of the vein intersect (on the outcrop), a definite deflection will be apparent on the horizontal projection, increasing as the hill steepens or as the dip flattens.

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[19] "Land Decisions," U. S. Govt. Printing Office, Washington, D. C., Vol. 34, 1906, p. 260.

[20] *Circular No. 1278*, Bureau of Land Management, U. S. Govt. Printing Office, Washington, D. C.

[21] "Code of Federal Regulations," U. S. Govt. Printing Office, Washington, D. C., 1940, Pt. 185.14, title 43.

[22] *Circular No. 127*, California Div. of Mines, State Printing Office, Sacramento, Calif., 1944.

[23] Cons. Mining Engr., Mineral Surveyor, General Land Office, Patagonia, Ariz.

It is not the present policy of the Land Office to require side-center monuments to be set, in Arizona, or to be mentioned in the field notes, although end-center monuments (or their counterparts at each end of the lode line when the lode does not cut the end line exactly at its center) must be set.

Although he has not seen the law in writing, it is the writer's understanding that, in Arizona, a mill site may not be preempted or patented contiguous to a lode claim.

Although the length of the lode line may not exceed 1,500 ft, the side lines may be longer—in fact, up to a length of almost 1,530 ft. The latter is possible in the case of a keystone-shaped claim with a 1,500-ft lode line, with one end line 600 ft long and the other, say, 10 ft long. However, triangular claims are not allowable, no doubt because of the ensuing extralateral rights which require differentiation between side lines and end lines not possible with a figure of this shape.

In the case of lode claims, no limit is placed, by Arizona laws, on the number of claims an individual may hold, although each mining district may enact such limitations. With placer claims, however, each locator may hold but one 20-acre claim, whereas an association may hold 20 acres for each individual within the organization. Although the laws on this matter are rather vague and incomplete, they have been interpreted to mean that an eight-man group, for example, can hold 160 acres of placer ground either as eight 20-acre claims or as one 160-acre claim.

Mr. McNair mentions the inclusion of a level rod in his survey outfit, but does not state how it might be useful during the actual prosecution of the survey aside from its obvious use as a stadia rod for a rough check of taped or triangulated distances. The writer's party carries a rod quite frequently as an adjunct to the work of claim surveying for patents. It is useful for "spotting" less important workings, buildings, and roads. In the frequent cases where a wide canyon is spanned by a long tape it is quite handy to note the position of the center of the gulch by stadia with sufficient accuracy for the purpose, and such improvements as roads and pipe lines may be readily located by traversing by stadia from convenient points (not necessarily corners) along any of the traverse lines being taped. As in other such work, it is well to tie in both ends of such a traverse rather than to leave it with no position check on the far end.

Mr. McNair does not mention that mineral surveyors are precluded, by Land Office ruling, from acting, directly or indirectly, as attorneys for the claimants seeking patents. Many claimants believe that the mineral surveyor is the person to aid them in filling out their requests for survey orders and other subsequent required paper work. Although, in some respects, he is the logical person to consult in these matters he is not normally fitted by training (or temperament!) for this phase of the procedure.

JOHN V. MELDRUM.[24]—This excellent paper on mineral surveys is, as the author states, the first technical discussion presented in many years that goes beyond the bounds of the "Manual of Instructions for the Survey of the Public Lands of the United States."[2] Although in 1948 there were 118 commissioned mineral surveyors, it must be realized that many of these men are in the higher age brackets. There is room in the field for younger men; hence, the paper is timely. Furthermore, since changes in the mining laws are being considered, the paper offers an excellent basis for appraising the present status of patented mineral lands.

The original paper was being prepared during the reorganization of the old General Land Office, which was in effect a merger of that office with the Grazing Service under the new Bureau of Land Management. Therefore, the following may serve to clarify several conflicting statements.

Mineral surveys in the western United States, together with all public land surveys, were made originally under the direction of a surveyor general's office for each state or territory, this office being a division of the General Land Office. In 1925 the several offices of surveyors general were abolished and in their stead public survey offices were established under the supervision of an office cadastral engineer. At that time the supervisor of surveys (a position established in 1910 for coordinating cadastral engineering work) assumed direct responsibility for the functioning of the public survey offices and reported directly to the commissioner of the General Land Office. As a result of President Harry S. Truman's Reorganization Plan No. 3, effected July 16, 1946, the position of supervisor of surveys was abolished along with the General Land Office. The United States including Alaska was divided into regions, each headed by a regional administrator to whom, in accordance with the general theme of decentralization, has been delegated much of the authority and many of the duties formerly centered at Washington, D. C. The Bureau of Land Management, which, in addition to surveying, has assumed all the former functions of the General Land Office and the Grazing Service, is headed by a director at Washington. The public survey offices have remained essentially the same, and the office cadastral engineer accepts applications for, and approves, all mineral surveys. All records of mineral surveys are on file in these offices. Although appointments of mineral surveyors are made by the regional administrator, applications are directed to the proper public survey office.

The technical information contained in the paper is exceptionally accurate, and any criticism offered could only be in a picayune manner. However, the author states (under the heading, "Introduction") that "* * * since 1935 patents for homesteads and for stock raising homesteads have been essentially stopped * * *" and again he writes in the "Conclusion": "* * * this land [mineral land] is the only type of public domain patentable in the United States * * *." It is true that mining claims may be located anywhere on the public domain except in obvious reservations such as national parks and monuments and areas withdrawn from entry for a specific purpose, such as power site withdrawals.

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[24] Mineral Engr., Public Survey Office, Bureau of Land Management, Denver, Colo.

It seems advisable to point out that the homestead laws are still in effect and that the decline in homesteading activity may be attributed mainly to the lack of desirable land. In accordance with the strong policy of conservation being so vigorously prosecuted at this time (1949) in order to assure a continued high standard of living for the United States, and because of the many demands for other uses of the land, patents are not granted promiscuously to any applicant without close scrutiny of his qualifications under the public land laws. This fact, coupled with the character of available land, explains the lessening number of patents issued. However, applications under the various land laws are welcomed and isolated tracts of public domain that cannot be administered effectively by the Bureau of Land Management are offered at public auction.

ARTHUR D. KIDDER,[25] M. ASCE.—A substantial contribution to the subject of the mineral land surveys within the western states of the former public domain has been made by Mr. McNair. The paper supplies basic information that is needed by both the claimant and his surveyor where new discoveries of mineral are made on the vacant public land and, also, for the identification of the boundaries of the older mineral claims wherever that may be required. There has been much less activity in this field since the issue of the "Manual of Instructions for the Survey of the Mineral Lands of the United States," in 1909.[26] That manual contained the rules of survey and entry then in effect, together with citations of the United States statutes.

Concurrently with the period from 1895 to 1909, a most valuable article on the subject of the problems and field procedure incident to the making of mineral surveys was prepared[27] by Robert S. Stockton and Edward P. Arthur, Jr. Appendix B[28] of the textbook in which this chapter appeared was a reprint, in full, of the official "Mineral Manual"[8] of 1895. Chapter XI[27] and Appendix B[28] of the textbook (which was prepared at a time of great activity in mining), together with the elucidation of the subject by Mr. McNair nearly 50 years later, are veritable "gold mines" for tracing the history and practice in the execution of mineral surveys. Literature in this category is rare, and difficult to find at hand just when needed.

The administrative work incident to the mineral surveys in the offices of the surveyors general, and after 1925 in the public survey offices, General Land Office, United States Department of the Interior, decreased very materially between the years 1909 and 1930. In 1930 the technical rules, regulations, and practice then in effect for the making of the mineral patent surveys were published as Chapter X in the general "Manual of Instructions for the Survey of the Public Lands of the United States," in 1930.[29] No historical data regarding mineral surveys were supplied in the 1930 manual. At that time the greater part of the regulations concerning the making of a mineral location on the vacant public lands was issued in the form of departmental circulars, each limited to a particular scope of the subject. The circulars were brought up to date from time to time keeping pace with new legislation or departmental or executive orders, and the decisions in mineral contest cases. The general manual of 1947[2] brought the technical practice up to date. At this time the deletions from the earlier manuals were those parts on nontechnical subjects, which are better treated in circular form as required to conform with current legislation.

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[25] Kidder and Thoma, Terre Haute, Ind., Commr. of the Supreme Court of the United States, designated for the marking of the boundary between the states of Colorado and New Mexico; formerly Cadastral Engr., Gen. Land Office and Bureau of Land Management, U. S. Dept. of the Interior.

[26] "Manual of Instructions for the Survey of the Mineral Lands of the United States," Gen. Land Office, U. S. Dept of the Interior, U. S. Govt. Printing Office, Washington, D. C., 1909.

[27] "The Theory and Practice of Surveying," by J. B. Johnson, John Wiley & Sons, Inc., New York, N. Y., 16th Ed., 1902, Chapter XI.

[28] *Ibid.*, Appendix B.

[29] "Manual of Instructions for the Survey of the Public Lands of the United States," Gen. Land Office, U. S. Dept. of the Interior, U. S. Govt. Printing Office, Washington, D. C., 1930, Chapter X.

Basically, the technical procedure for the making of the mineral patent survey as expressed in the general manual of 1947 is little different from the practice of 1895 to 1909. However, the paper is indeed a most important treatment for those who now for the first time are brought face to face with these questions.

It is well to bear in mind that contest cases between mineral claimants concerning the right of entry are subject to adjudication by the director, Bureau of Land Management, under the supervision of, and the right of appeal to, the Secretary of the Interior. After a patent has been issued, any possible controversy concerning the position of the boundaries, and other questions relating to the title to the property, come under the state law and the jurisdiction of the courts of that particular mining district or county, which possess the sole authority to adjudicate such cases. In this respect mineral suits are similar to other boundary and title litigation, except for the rules that give the right to follow the lateral inclination or dip of the lode underground beyond the vertical projection of the side lines.

A point of much importance in understanding the practice on mineral surveys is that the early discoveries of free gold in California, Colorado, South Dakota, and other western states were made during the frontier days, in some cases even preceding territorial status for those districts. Each new discovery brought a rush of prospectors. Primitive rules of law and order were set up by the miners themselves, including their own plan for staking out a mineral claim, and posting a public notice. Later, to perfect the title, legislation was enacted by Congress to set up a uniform legal right of entry under which patent by the United States might issue to the rightful claimant. Thus, in effect, the miners themselves initiated the first rules. These were then clarified and codified into the general mining laws of 1872, as stated.

The present general laws relating to the public lands, and the regulations thereunder, are available in the "Code of Federal Regulations, Title 43—Public Lands, Interior," the most recent of which are printed in the "Cumulative Supplement."[30]

The federal code includes many subjects such as the legal right of mining or development of the natural resources of oil and gas, potash, phosphate, coal, and other minerals to be found on the vacant public lands, and also the procedure, if authorized by law, for the exploitation of deposits that may be discovered within the various reserved public lands. Another important consideration is also suggested: A large number of patents that have issued in recent years for entries under the general land laws contain a clause that reserves the mineral right in the United States, the entryman thus gaining title to the surface right only. The prospector, geologist, mining engineer, mineral surveyor, or others concerned, when confronted with these questions, should consult the nearest public survey or regional office of the Bureau of Land Management or request the information from the Washington (D. C.) office.

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[30] "Code of Federal Regulations, Title 43—Public Lands, Interior," Cumulative Supplement: Chapter I, Subchapter L, and Pt. 185, Supt. of Documents, U. S. Govt. Printing Office, Washington, D. C.

The research of record data should include an examination of the field notes and a plat of the township survey, if the area has been subdivided, as these will show the location of the mineral surveys that were in existence at the time of the subdivisional survey. Also, the meaning and importance of the General Land Office "supplemental plats" within these townships should be noted. The supplemental plats were prepared from time to time as needed to accommodate one or more nonmineral entries on the public land that adjoined the patented mineral lands. In these cases new lottings were designated in order to give a suitable description for the nonmineral fractional lots. No new field work was done for the preparation of the supplemental plat, the latter being based on the field note record of the mineral patent survey when platted through the direction and the length of the lines connecting to the subdivisional survey.

In an active field, new supplemental plats were frequently required as new mineral discoveries were made, but only as some nonmineral applicant initiated an entry on the adjoining land. To a lesser extent, and only in recent years, "mineral segregation" surveys have been made under the special administrative instructions of the Department of the Interior where needed in order to accommodate a nonmineral entryman on lands adjacent to a known valid mineral claim where the latter had not proceeded to mineral patent.

Looking ahead, it is almost certain that new discoveries will be made, some of them on the surface, and that others will be found much deeper in the ground within the boundaries of the older mineral patents, which are possibly the parent source of the early surface discoveries.

Arthur J. McNair,[31] **Assoc. M. ASCE.**—In this paper the writer presented material on the history and methods of making mineral land surveys because he felt that the subject had not received any attention from the engineering profession since the turn of the twentieth century. The comments emphasize the timeliness of the subject because the old-timers familiar with the procedures peculiar to this type of surveying are rapidly diminishing in numbers. Furthermore, the number of both patented and unpatented mining claims has been increasing continually during these years while at the same time the corner monuments fixing their positions on the ground have become increasingly difficult to find.

This situation, as stated by Mr. Meldrum, creates a fertile field for the development of young engineers. As large sections of the old mining districts are consolidated for unified operation by larger companies, the need for resurvey of the old claims, together with the location and patenting of new claims to cover the vacant fractions, is immediately felt. It is of more than passing interest to note that titles to claims in an old established mining district are usually in such a condition that at least two to three years of work is required by legal counsel before a mining company can consolidate ownerships and begin operations.

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[31] Associate Prof. of Civ. Eng. and Head of Surveying Dept., Cornell Univ., Ithaca, N. Y.

The discussions indicate that there are continual changes in the administration of the lands in the public domain and that these have produced some changes in the procedure of making mineral land surveys. A substantial contribution to this paper has been made by the discussions which describe the functions of the Bureau of Land Management, United States Department of the Interior, which now administers the lands of the public domain.

Because the discovery of gold—and other mineral values—was frequently the reason for the early settlement of many of the western states, and because each mining district grew up autonomously, it is not surprising that today there are local variations in mining laws and regulations. The hope of the writer that these variations would be presented by those discussing the paper, thereby making it more applicable to the entire United States, has been at least partly fulfilled. Since California probably does more placer mining than any other state, the discussions presenting the variations in placer surveys in that state as compared to others are important. It is interesting to note that the possessory right to unpatented claims in the form of locations is taxable in California as this sets up different economic risks for a locator than are established where locations are not taxed.

The requirement in Arizona that "* * * end-center monuments (or their counterparts at each end of the lode line when the lode does not cut the end line exactly at its center) must be set * * *" differs from the procedure in other states. The writer cannot understand the necessity for end centers. Certainly they serve no useful purpose, and one cannot help but surmise what the mineral surveyor might designate as the lode line for a claim located in a region of disseminated copper porphyries where there are no precise boundaries, such as in Arizona, or in a region of sedimentary or so-called blanket deposits such as occur in so many places in the western states.

With the introduction of European-make transits into the United States, a considerable part of the writer's discussion concerning the precision of mineral land surveys will be modified. Some of these instruments are not only much lighter and easier to carry but are more accurate and much easier to read than the conventional American transits described in the paper. In no phase of surveying are portability, accuracy, and ease of reading more essential than in mineral surveying, and the next few years will doubtless see an improvement in procedures corresponding to the improvement in instruments.

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PHOTOGRAMMETRIC SPACE RESECTION

By Carl Chow,[1] Jun. ASCE

Synopsis

The principal object of this paper is to present a direct mathematical method of finding the elevation of the exposure station and consequently the tilt and the swing, if desired, of an aerial photograph. Since three points in space determine a plane (Fig. 1), the three vertical distances z_a, z_b, and z_c (Fig. 2), from a horizontal plane through the exposure station to the images, a, b, and c on the photograph of the three ground control points, A, B, and C, determine the position of the photograph in space. Therefore, the swing of the photograph is not necessarily required provided the values of z_a, z_b, and z_c have been specified. Furthermore, in the solution of the space resection problem only relative quantities are required expressive of the three ground points.

The problem of photogrammetric space resection has been solved in various ways.[2,3,4,5,6] The present paper is added to the list for several reasons. The method developed:

(1) Demonstrates an adaptation of vector analysis which is novel and interesting;
(2) Offers a more direct procedure than its predecessors, permitting the exact altitude of the exposure station in an aerial photograph;
(3) Provides a procedure of adapting photogrammetric work to areas where the customary type of geodetic control is impossible or infeasible;
(4) Yields all data necessary for the rectification of tilted photographs;
(5) Avoids the necessity of observing the swing;

Note.—Published in February, 1949, *Proceedings*. Position and title are those in effect when the paper was received for publication.

[1] Research Fellow, Army Map Service, Dept. of the Army, Washington, D. C.

[2] "Analytical Computations in Aerial Photogrammetry," by Earl Church, *Photogrammetric Engineering*, October, November, December, 1941, p. 212.

[3] "Manual of Photogrammetry," Am. Soc. of Photogrammetry, Pitman Pub. Corp., New York, N. Y., 1944.

[4] "Revised Geometry of the Aerial Photograph," by Earl Church, *Bulletin No. 15* in a series on aerial photogrammetry, Syracuse Univ., Syracuse, N. Y., June, 1945.

[5] "Applied Photogrammetry," by Ralph O. Anderson, Edwards Bros., Inc., Ann Arbor, Mich., 4th Ed., 1946, p. 391.

[6] "Space Resection Problems in Photogrammetry," by P. H. Underwood, *Transactions*, ASCE, Vol. 112, 1947, p. 921.

(6) Involves only simple computations and produces exceptionally accurate results; and

(7) Is a basic theory in the writer's plan to project his investigations of aerial photogrammetry into geodetic triangulation.

SECTION 1. DERIVATION OF FUNDAMENTAL FORMULAS

In Fig. 3, L indicates the position of the emergent nodal point of the lens of the aerial camera, and the length of Lo is equal to f, the focal length of the camera lens. Point L is referred to as the exposure station, and is used as the origin for the coordinate system. The coordinate axes, x and y, are the lines through L parallel to the geometric axes of the photograph, and the z-axis coincides with the lens axis.

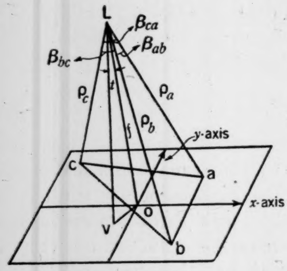
L
βca
βab
βbc
ρc
t
ρa
ρb
f
y-axis
c
a
x-axis
o
v
b

FIG. 1.—PLATE PYRAMID

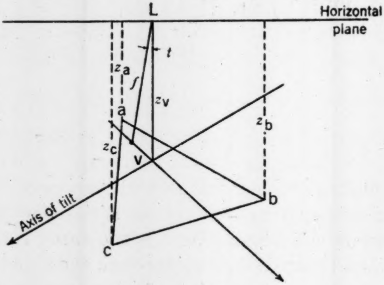
L
Horizontal plane
t
za
f
zv
a
zb
zc
v
b
Axis of tilt
c

FIG. 2.—VERTICAL RELATIONS

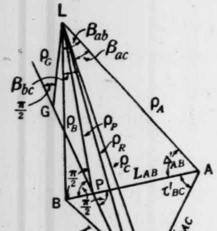
L
βab
βac
ρG
βbc
π/2
G
ρA
ρB
ρP
ρR
ρC
LAB
τAB
A
π/2
P
τ'BC
B
π/2
LBC
LAC
C
R

FIG. 3.—GROUND PYRAMID

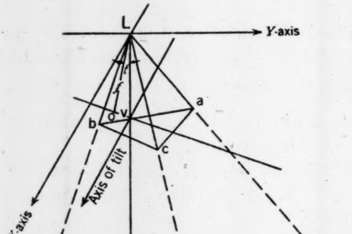
L
Y-axis
t
f
a
b
o
v
c
Axis of tilt
X-axis
A
B
Z-axis
C

FIG. 4.—SPACE RELATIONS AND THE PLUMB-LINE COORDINATE SYSTEM

If the plane coordinates of the images a, b, and c, measured on the photograph, are x_{pa},y_{pa}; x_{pb},y_{pb}; and x_{pc},y_{pc}, the space coordinates of the three images are x_{pa},y_{pa},f; x_{pb},y_{pb},f; and x_{pc},y_{pc},f. Then

$$\overline{La} = \rho_a = \sqrt{x^2_{pa} + y^2_{pa} + f^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1a)$$

$$\overline{Lb} = \rho_b = \sqrt{x^2_{pb} + y^2_{pb} + f^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1b)$$

and

$$\overline{\mathrm{Lc}} = \rho_c = \sqrt{x^2_{pc} + y^2_{pc} + f^2} \ldots\ldots\ldots\ldots\ldots\ldots (1c)$$

If these rays are regarded as vectors,

$$\overline{\mathrm{La}} = \bar{\rho}_a, \quad x_{pa}\, i + y_{pa}\, j + f\, k \ldots\ldots\ldots\ldots\ldots\ldots (2a)$$

$$\overline{\mathrm{Lb}} = \bar{\rho}_b, \quad x_{pb}\, i + y_{pb}\, j + f\, k \ldots\ldots\ldots\ldots\ldots\ldots (2b)$$

and

$$\overline{\mathrm{Lc}} = \bar{\rho}_c, \quad x_{pc}\, i + y_{pc}\, j + f\, k \ldots\ldots\ldots\ldots\ldots\ldots (2c)$$

in which i, j, and k are unit vectors referred to the corresponding axes in space. In accordance with the fundamental principles of vector analysis scalar products,

$i.i = 1,\ j.j = 1,\ k.k = 1,\ i.j = j.k = k.i = 0$ give

$$\overline{\mathrm{La}}.\overline{\mathrm{Lb}} = \bar{\rho}_a \cdot \bar{\rho}_b = x_{pa}\, x_{pb} + y_{pa}\, y_{pb} + f^2 = \rho_a\, \rho_b \cos \beta_{ab} \ldots\ldots (3a)$$

$$\overline{\mathrm{Lb}}.\overline{\mathrm{Lc}} = \bar{\rho}_b \cdot \bar{\rho}_c = x_{pb}\, x_{pc} + y_{pb}\, y_{pc} + f^2 = \rho_b\, \rho_c \cos \beta_{bc} \ldots\ldots (3b)$$

and

$$\overline{\mathrm{Lc}}.\overline{\mathrm{La}} = \bar{\rho}_c \cdot \bar{\rho}_a = x_{pc}\, x_{pa} + y_{pc}\, y_{pa} + f^2 = \rho_c\, \rho_a \cos \beta_{ca} \ldots\ldots (3c)$$

in which the angles β are the face angles of the plate pyramid Labc. Therefore,

$$\cos \beta_{ab} = \frac{x_{pa}\, x_{pb} + y_{pa}\, y_{pb} + f^2}{\rho_a\, \rho_b} \ldots\ldots\ldots\ldots\ldots\ldots (4a)$$

$$\cos \beta_{bc} = \frac{x_{pb}\, x_{pc} + y_{pb}\, y_{pc} + f^2}{\rho_b\, \rho_c} \ldots\ldots\ldots\ldots\ldots\ldots (4b)$$

and

$$\cos \beta_{ca} = \frac{x_{pc}\, x_{pa} + y_{pc}\, y_{pa} + f^2}{\rho_c\, \rho_a} \ldots\ldots\ldots\ldots\ldots\ldots (4c)$$

Now if the lengths ab, bc, and ca, Fig. 1, are designated by l_{ab}, l_{bc}, l_{ca}, respectively,

$$l^2_{ab} = (x_{pa} - x_{pb})^2 + (y_{pa} - y_{pb})^2 \ldots\ldots\ldots\ldots\ldots\ldots (5a)$$

$$l^2_{bc} = (x_{pb} - x_{pc})^2 + (y_{pb} - y_{pc})^2 \ldots\ldots\ldots\ldots\ldots\ldots (5b)$$

and

$$l^2_{ca} = (x_{pc} - x_{pa})^2 + (y_{pc} - y_{pa})^2 \ldots\ldots\ldots\ldots\ldots\ldots (5c)$$

Then, if angle cab (Fig. 1) = τ_a, angle abc = τ_b, and angle bca = τ_c, vector operations identical to those leading to Eqs. 4 give

$$\cos \tau_a = \frac{(x_{pb} - x_{pa})\,(x_{pc} - x_{pa}) + (y_{pb} - y_{pa})\,(y_{pc} - y_{pa})}{l_{ab}\, l_{ca}} \ldots\ldots (6a)$$

$$\cos \tau_b = \frac{(x_{pc} - x_{pb})\,(x_{pa} - x_{pb}) + (y_{pc} - y_{pb})\,(y_{pa} - y_{pb})}{l_{cb}\, l_{ab}} \ldots\ldots (6b)$$

and

$$\cos \tau_c = \frac{(x_{pa} - x_{pc})\,(x_{pb} - x_{pc}) + (y_{pa} - y_{pc})\,(y_{pb} - y_{pc})}{l_{ca}\, l_{bc}} \ldots\ldots (6c)$$

Similarly,

$$\cos \mathrm{aob} = \cos \theta_1 = \frac{x_{pa}\, x_{pb} + y_{pa}\, y_{pb}}{r_a\, r_b} \ldots\ldots\ldots\ldots\ldots (7a)$$

$$\cos \mathrm{boc} = \cos \theta_2 = \frac{x_{pb}\, x_{pc} + y_{pb}\, y_{pc}}{r_b\, r_c} \ldots\ldots\ldots\ldots\ldots (7b)$$

and

$$\cos \text{coa} = \cos \theta_3 = \frac{x_{pc}\, x_{pa} + y_{pc}\, y_{pa}}{r_c\, r_a} \tag{7c}$$

and

$$\cos \text{Lab} = \cos \Delta_{ab} = \frac{x_{pa}\,(x_{pb} - x_{pa}) + (y_{pb} - y_{pa})\, y_{pa}}{\rho_a\, l_{ab}} \tag{8a}$$

$$\cos \text{Lba} = \cos \Delta_{ba} = \frac{x_{pb}\,(x_{pa} - x_{pb}) + y_{pb}\,(y_{pa} - y_{pb})}{\rho_b\, l_{ab}} \tag{8b}$$

$$\cos \text{Lbc} = \cos \Delta_{bc} = \frac{x_{pb}\,(x_{pc} - x_{pb}) + y_{pb}\,(y_{pc} - y_{pb})}{\rho_b\, l_{bc}} \tag{8c}$$

$$\cos \text{Lcb} = \cos \Delta_{cb} = \frac{x_{pc}\,(x_{pb} - x_{pc}) + y_{pc}\,(y_{pb} - y_{pc})}{\rho_c\, l_{bc}} \tag{8d}$$

$$\cos \text{Lca} = \cos \Delta_{ca} = \frac{x_{pc}\,(x_{pa} - x_{pc}) + y_{pc}\,(y_{pa} - y_{pc})}{\rho_c\, l_{ca}} \tag{8e}$$

and

$$\cos \text{Lac} = \cos \Delta_{ac} = \frac{x_{pa}\,(x_{pc} - x_{pa}) + y_{pa}\,(y_{pc} - y_{pa})}{\rho_a\, l_{ca}} \tag{8f}$$

In Eqs. 6 and 7, r_a, r_b, and r_c represent the absolute lengths of oa, ob, and oc, respectively.

SECTION 2. DETERMINATION OF THE LENGTHS OF THE EDGES OF THE GROUND PYRAMID

The rays from the exposure station L to the ground points A, B, and C are designated ρ_A, ρ_B, and ρ_C; and the inclined, or natural ground, distances AB, BC, and CA, L_{AB}, L_{BC}, and L_{CA}, respectively. Then, by the law of cosines,

$$L^2_{AB} = \rho^2_A + \rho^2_B - 2\, \rho_A\, \rho_B \cos \beta_{ab} \tag{9a}$$

$$L^2_{BC} = \rho^2_B + \rho^2_C - 2\, \rho_B\, \rho_C \cos \beta_{bc} \tag{9b}$$

and

$$L^2_{CA} = \rho^2_C + \rho^2_A - 2\, \rho_C\, \rho_A \cos \beta_{ca} \tag{9c}$$

If three approximate values (ρ_A), (ρ_B), and (ρ_C) are substituted for ρ_A, ρ_B, and ρ_C, as given by

$$(\rho_A) = \frac{\rho_a\, L'_{AB}}{l_{ab}} \tag{10a}$$

$$(\rho_B) = \frac{\rho_b\, L'_{BC}}{l_{bc}} \tag{10b}$$

and

$$(\rho_C) = \frac{\rho_c\, L'_{CA}}{l_{ca}} \tag{10c}$$

in which L'_{AB}, L'_{BC}, and L'_{CA} are horizontal distances of AB, BC, and CA, respectively, then

$$L^2_{AB} = [(\rho_A) + \Delta\rho_A]^2 + [(\rho_B) + \Delta\rho_B]^2 - 2\,[(\rho_A) + \Delta\rho_A]\,[(\rho_B) + \Delta\rho_B] \cos\beta_{ab} \,.\,.\,(11a)$$

$$L^2_{BC} = [(\rho_B) + \Delta\rho_B]^2 + [(\rho_C) + \Delta\rho_C]^2 - 2\,[(\rho_B) + \Delta\rho_B]\,[(\rho_C) + \Delta\rho_C] \cos\beta_{bc} \,.\,.\,(11b)$$

and

$$L^2_{CA} = [(\rho_C) + \Delta\rho_C]^2 + [(\rho_A) + \Delta\rho_A]^2 - 2\,[(\rho_C) + \Delta\rho_C]\,[(\rho_A) + \Delta\rho_A] \cos\beta_{ca} \,.\,.\,(11c)$$

Let

$$L^2_{AB} - (\rho_A)^2 + (\rho_B)^2 - 2\,(\rho_A)\,(\rho_B) \cos\beta_{ab} = w_{AB} \dots\dots\dots (12a)$$

$$L^2_{BC} - (\rho_B)^2 + (\rho_C)^2 - 2\,(\rho_B)\,(\rho_C) \cos\beta_{bc} = w_{BC} \dots\dots\dots (12b)$$

and

$$L^2_{CA} - (\rho_C)^2 + (\rho_A)^2 - 2\,(\rho_C)\,(\rho_A) \cos\beta_{ca} = w_{CA} \dots\dots\dots (12c)$$

Omitting the second power terms in Eqs. 12,

$$2\,(\rho_A)\,\Delta\rho_A + 2\,(\rho_B)\,\Delta\rho_B - 2\,(\rho_A)\,\Delta\rho_B \cos\beta_{ab} - 2\,(\rho_B)\,\Delta\rho_A \cos\beta_{ab} = w_{AB} \,.\,(13a)$$

$$2\,(\rho_B)\,\Delta\rho_B + 2\,(\rho_C)\,\Delta\rho_C - 2\,(\rho_B)\,\Delta\rho_C \cos\beta_{bc} - 2\,(\rho_C)\,\Delta\rho_B \cos\beta_{bc} = w_{BC} \,.\,(13b)$$

and

$$2\,(\rho_C)\,\Delta\rho_C + 2\,(\rho_A)\,\Delta\rho_A - 2\,(\rho_C)\,\Delta\rho_A \cos\beta_{ca} - 2\,(\rho_A)\,\Delta\rho_C \cos\beta_{ca} = w_{CA} \,.\,(13c)$$

A solution of these three linear equations (Eqs. 13) yields $\Delta\rho_A$, $\Delta\rho_B$, and $\Delta\rho_C$. Then the second approximate values of ρ_A, ρ_B, and ρ_C will be $(\rho_A) + \Delta\rho_A$, $(\rho_B) + \Delta\rho_B$, and $(\rho_C) + \Delta\rho_C$, respectively. Repeated substitutions of these values in Eqs. 13 give accurate values of ρ_A, ρ_B, and ρ_C.

Section 3. Determination of the Exposure Station Altitude

Fig. 3 shows a plane passed through L perpendicular to any side, say, AC, of the ground triangle ABC, cutting AC at the point R, cutting the side BC at point Q, and cutting the side AB at point P. This plane is also perpendicular to the plane ABC containing the three ground control points. By the law of sines,

$$\frac{\sin\Delta'_{AB}}{\rho_B} = \frac{\sin\beta_{ab}}{L_{AB}} \dots\dots\dots\dots (14a)$$

$$\frac{\sin\Delta'_{BA}}{\rho_A} = \frac{\sin\beta_{ab}}{L_{AB}} \dots\dots\dots\dots (14b)$$

$$\frac{\sin\Delta'_{BC}}{\rho_C} = \frac{\sin\beta_{bc}}{L_{BC}} \dots\dots\dots\dots (14c)$$

$$\frac{\sin\Delta'_{CB}}{\rho_B} = \frac{\sin\beta_{bc}}{L_{BC}} \dots\dots\dots\dots (14d)$$

$$\frac{\sin\Delta'_{CA}}{\rho_A} = \frac{\sin\beta_{ca}}{L_{CA}} \dots\dots\dots\dots (14e)$$

and

$$\frac{\sin \Delta'_{AC}}{\rho_C} = \frac{\sin \beta_{ca}}{L_{CA}} \tag{14f}$$

in which the symbol Δ' refers to the face angles at the base of the ground pyramid L–ABC (Fig. 3). Eqs. 14 can each be transposed to express a value of $\sin \Delta'$.

Since the angles β are all measured by the photograph and computed as shown in Section 1 and since the inclined ground distances L_{AB}, L_{BC}, and L_{CA} are already known from the ground surveying, and ρ_A, ρ_B, and ρ_C have been found by the formulas in the preceding paragraph, the angles $\Delta'_{AB}, \cdots, \Delta'_{BC}$ can be computed. Also the angles CAB $= \tau_A$, ABC $= \tau_B$, and BCA $= \tau_C$ in the plane of the triangle ABC can be computed by the three inclined ground distances L_{AB}, L_{BC}, and L_{CA} or by the following formulas, if the coordinates of the three ground points are known:

$$\cos \tau_A = \frac{(X_B - X_A)(X_C - X_A) + (Y_B - Y_A)(Y_C - Y_A) + (H_A - H_B)(H_A - H_C)}{L_{AB} L_{CA}} \tag{15a}$$

$$\cos \tau_B = \frac{(X_C - X_B)(X_A - X_B) + (Y_C - Y_B)(Y_A - Y_B) + (H_B - H_C)(H_B - H_A)}{L_{BC} L_{AB}} \tag{15b}$$

and

$$\cos \tau_C = \frac{(X_A - X_C)(X_B - Y_C) + (Y_A - Y_C)(Y_B - Y_C) + (H_C - H_A)(H_C - H_B)}{L_{CA} L_{BC}} \tag{15c}$$

in which X is the x-ordinate of a ground point; Y is the y-ordinate of a ground point; and H is the elevation of a ground point. By reference to Fig. 3,

$$\overline{AR} = \rho_A \cos \Delta'_{AC} \tag{16a}$$

$$\rho_R = \overline{LR} = \rho_A \sin \Delta'_{AC} \tag{16b}$$

$$\overline{AP} = \overline{AR} \sec \tau_A \tag{16c}$$

$$\overline{PR} = \overline{AR} \tan \tau_A \tag{16d}$$

$$\rho^2_P = \overline{LP^2} = \rho^2_A + \overline{AP^2} - 2\,\rho_A\,\overline{AP} \cos \Delta'_{AB} \tag{16e}$$

With ρ_R, ρ_P, and the length PR all determined, the area A_{pr} of the triangle LPR (Fig. 3) can be computed from the three sides. The usual formula in plane geometry is

$$A_{pr} = \sqrt{\Sigma\,(\Sigma - \rho_R)\,(\Sigma - \rho_P)\,(\Sigma - \overline{PR})} \tag{17a}$$

in which

$$\Sigma = \tfrac{1}{2}\,(\rho_R + \rho_P + \overline{PR}) \tag{17b}$$

Let a line LG (Fig. 3) be drawn from L perpendicular to PR, and let its length be designated ρ_G. Then, $A_{pr} = \frac{1}{2}\,\rho_G \times \overline{PR}$; or

$$\rho_G = \frac{2\,A_{pr}}{\overline{PR}} \tag{18}$$

which is the perpendicular distance from the exposure station L to the plane ABC containing the three ground control points. (It is not the elevation of the exposure station since the plane ABC is not at sea level, and since LG is not a vertical line.)

In addition to the use of Eq. 18, ρ_G also can be obtained as follows: Let V be the volume of the ground pyramid L–ABC, then[7]

$$6\,V = \rho_A\,\rho_B\,\rho_C \begin{vmatrix} 1 & \cos\beta_{ca} & \cos\beta_{bc} \\ \cos\beta_{ca} & 1 & \cos\beta_{ab} \\ \cos\beta_{bc} & \cos\beta_{ab} & 1 \end{vmatrix}^{\frac{1}{2}} = \rho_A\,\rho_B\,\rho_C\,\delta^{\frac{1}{2}} \ldots\ldots \tag{19a}$$

in which

$$\delta = 1 - \cos^2\beta_{ab} - \cos^2\beta_{bc} - \cos^2\beta_{ca} + 2\cos\beta_{ab}\cos\beta_{bc}\cos\beta_{ca} \ldots \tag{19b}$$

Also,

$$6\,V = \rho_G\,L_{AB}\,L_{AC}\sin\tau_A = \rho_G\,L_{AB}\,L_{BC}\sin\tau_B = \rho_G\,L_{BC}\,L_{AC}\sin\tau_C \ldots \tag{20}$$

Therefore,

$$\rho_G = \frac{\rho_A\,\rho_B\,\rho_C\,\delta^{\frac{1}{2}}}{L_{AB}\,L_{AC}\sin\tau_A} = \frac{\rho_A\,\rho_B\,\rho_C\,\delta^{\frac{1}{2}}}{L_{AB}\,L_{BC}\sin\tau_B} = \frac{\rho_A\,\rho_B\,\rho_C\,\delta^{\frac{1}{2}}}{L_{BC}\,L_{AC}\sin\tau_C} \ldots\ldots \tag{21}$$

Let $R_1 = \dfrac{\rho_A}{L_{AB}}$, $R_2 = \dfrac{\rho_B}{L_{BC}}$, and $R_3 = \dfrac{\rho_C}{L_{CA}}$,

$$\rho_G = R_1\,R_3\,\rho_B\,\delta^{\frac{1}{2}}/\sin\tau_A \ldots\ldots \tag{22a}$$

$$\rho_G = R_1\,R_2\,\rho_C\,\delta^{\frac{1}{2}}/\sin\tau_B \ldots\ldots \tag{22b}$$

and

$$\rho_G = R_2\,R_3\,\rho_A\,\delta^{\frac{1}{2}}/\sin\tau_C \ldots\ldots \tag{22c}$$

Next,

$$\overline{\mathrm{AG}} = \sqrt{\rho^2{}_A - \rho^2{}_G} \ldots\ldots \tag{23a}$$

$$\overline{\mathrm{BG}} = \sqrt{\rho^2{}_B - \rho^2{}_G} \ldots\ldots \tag{23b}$$

$$\overline{\mathrm{CG}} = \sqrt{\rho^2{}_C - \rho^2{}_G} \ldots\ldots \tag{23c}$$

$$\overline{\mathrm{PG}} = \sqrt{\rho^2{}_P - \rho^2{}_G} \ldots\ldots \tag{23d}$$

and

$$\overline{\mathrm{RG}} = \sqrt{\rho^2{}_R - \rho^2{}_G} \ldots\ldots \tag{23e}$$

—all of which can be computed.

The elevation of point P (Fig. 3) can be found by extrapolation between H_A and H_B, the given elevations of points A and B, since both AB ($= L_{AB}$) and AP are now known. The elevation of point R can be found by extrapolation between the given elevations of points A and C, since both AC ($= L_{CA}$) and AR are now known. These calculated elevations of points P and R are called H_P and H_R, respectively. Finally the elevation of the point G (Fig. 3) can be found by interpolation between H_P and H_R, since the distances PG and RG are now both known.

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[7] "Applications of the Absolute Differential Calculus," by A. J. McConnell, Blackie and Son, Ltd. London, 1931, pp. 48–50.

The area of the triangle ABC in the inclined plane of the three control points is

$$A = \sqrt{\Sigma\,(\Sigma - L_{AB})\,(\Sigma - L_{BC})\,(\Sigma - L_{CA})} \tag{24a}$$

in which L_{AB}, L_{BC}, and L_{CA} are the inclined distances, and in which

$$\Sigma = \tfrac{1}{2}\,(L_{AB} + L_{BC} + L_{AC}) \tag{24b}$$

Also the area A' of the horizontal projection of the triangle ABC is

$$A' = \sqrt{\Sigma'\,(\Sigma' - L'_{AB})\,(\Sigma' - L'_{BC})\,(\Sigma' - L'_{CA})} \tag{25a}$$

in which L'_{AB}, L'_{BC}, and L'_{CA} are the horizontal distances, and where

$$\Sigma' = \tfrac{1}{2}\,(L'_{AB} + L'_{BC} + L'_{CA}) \tag{25b}$$

Then, if T represents the angle of inclination of the plane ABC to a horizontal plane,

$$\cos T = \frac{A'}{A} \tag{26}$$

This value of $\cos T$ can be found as follows: If the coordinates of the three ground control points are known as X_A,Y_A,H_A; X_B,Y_B,H_B; and X_C,Y_C,H_C, respectively, the equation of a plane which passes through the three ground control points is

$$\begin{vmatrix} X & Y & Z & 1 \\ X_A & Y_A & H_A & 1 \\ X_B & Y_B & H_B & 1 \\ X_C & Y_C & H_C & 1 \end{vmatrix} = 0 \tag{27}$$

Eq. 27 can be written in the form:.

$$Z = f(X,Y) \tag{28}$$

since, then, by differential geometry,

$$\cos T = \frac{1}{\sqrt{1 + \left(\dfrac{\partial f}{\partial x}\right)^2 + \left(\dfrac{\partial f}{\partial y}\right)^2}} \tag{29}$$

The altitude (Z) of the exposure station is then given by

$$Z = H_G + \rho_G \cos T \tag{30}$$

Section 4. Determination of the Tilt of the Photograph

In this section a space coordinate system is used with the origin at the exposure station L, a plumb line through L is designated as the Z-axis; a horizontal line through L parallel to the tilt axis of the photograph, as the X-axis; and a horizontal line through L perpendicular to this line, as the Y-axis—the last line being situated in the principal plane of the photograph (see Fig. 4).

Since the value of Z (the elevation of the exposure station) and values of ρ_a, ρ_b, ρ_c, ρ_A, ρ_B, and ρ_C have already been computed, and since the altitudes H_A, H_B, and H_C of the three ground points are given, the following relations—

$$z_a = \frac{\rho_a}{\rho_A}(Z - H_A) \quad \text{(31a)}$$

$$z_b = \frac{\rho_b}{\rho_B}(Z - H_B) \quad \text{(31b)}$$

and

$$z_c = \frac{\rho_c}{\rho_C}(Z - H_C) \quad \text{(31c)}$$

—can be used to compute z_a, z_b, and z_c, which are the z-coordinates of the images a, b, and c on the tilted photograph, referred to the coordinate system defined by Fig. 4.

The lengths l_{ab}, l_{bc}, and l_{ca} of the sides ab, bc, and ca, respectively, of the image triangle are known from the image coordinates measured on the photograph. The area a of this image triangle in the plane of the photograph can be found from

$$a = \sqrt{\sigma(\sigma - l_{ab})(\sigma - l_{bc})(\sigma - l_{ca})} \quad \text{(32a)}$$

in which

$$\sigma = \tfrac{1}{2}(l_{ab} + l_{bc} + l_{ca}) \quad \text{(32b)}$$

The horizontal projection of the sides ab, bc, and ca (l'_{ab}, l'_{bc}, l'_{ca}) can be found from

$$l'_{ab} = \sqrt{l^2_{ab} - (z_a - z_b)^2} \quad \text{(33a)}$$

$$l'_{bc} = \sqrt{l^2_{bc} - (z_b - z_c)^2} \quad \text{(33b)}$$

and

$$l'_{ca} = \sqrt{l^2_{ca} - (z_c - z_a)^2} \quad \text{(33c)}$$

Then the area a' of the horizontal projection of the image triangle in space is given by

$$a' = \sqrt{\sigma'(\sigma' - l'_{ab})(\sigma' - l'_{bc})(\sigma' - l'_{ca})} \quad \text{(34a)}$$

in which

$$\sigma' = \tfrac{1}{2}(l'_{ab} + l'_{bc} + l'_{ca}) \quad \text{(34b)}$$

Then the tilt of the photograph is given by

$$\cos t = \frac{a'}{a} \quad \text{(35)}$$

SECTION 5. DETERMINATION OF SPACE COORDINATES OF THE GROUND POINTS AND OF THEIR PHOTOGRAPHIC IMAGES, REFERRED TO THE PLUMB-LINE COORDINATE SYSTEM

Since the exposure station L is taken as the origin, its coordinates on the plumb-line coordinate system explained in the preceding section are O,O,O; and the ground points A, B, and C will have coordinates $X_A, Y_A, (Z - H_A)$; $X_B, Y_B, (Z - H_B)$; and $X_C, Y_C, (Z - H_C)$, respectively—H_A, H_B, and H_C being the elevations of these points. The horizontal coordinates are to be determined. By analytical geometry,

$$\rho^2_A = X^2_A + Y^2_A + (Z - H_A)^2 \tag{36a}$$

$$\rho^2_B = X^2_B + Y^2_B + (Z - H_B)^2 \tag{36b}$$

and

$$\rho^2_C = X^2_C + Y^2_C + (Z - H_C)^2 \tag{36c}$$

$$L^2_{AB} = (X_B - X_A)^2 + (Y_B - Y_A)^2 + (H_A - H_B)^2 \tag{37a}$$

$$L^2_{BC} = (X_C - X_B)^2 + (Y_C - Y_B)^2 + (H_B - H_C)^2 \tag{37b}$$

and

$$L^2_{CA} = (X_A - X_C)^2 + (Y_A - Y_C)^2 + (H_C - H_A)^2 \tag{37c}$$

Also, when ρ_A, ρ_B, and ρ_C are regarded as vectors,

$$\bar{\rho}_A, \quad X_A\, i + Y_A\, j + (Z - H_A)\, k \tag{38a}$$

$$\bar{\rho}_B, \quad X_B\, i + Y_B\, j + (Z - H_B)\, k \tag{38b}$$

and

$$\bar{\rho}_C, \quad X_C\, i + Y_C\, j + (Z - H_C)\, k \tag{38c}$$

The scalar products of pairs of these vectors are as follows:

$$\bar{\rho}_A \cdot \bar{\rho}_B = \rho_A\, \rho_B \cos\beta_{ab} = X_A\, X_B + Y_A\, Y_B + (Z - H_A)\,(Z - H_B) \tag{39a}$$

$$\bar{\rho}_B \cdot \bar{\rho}_C = \rho_B\, \rho_C \cos\beta_{bc} = X_B\, X_C + Y_B\, Y_C + (Z - H_B)\,(Z - H_C) \tag{39b}$$

and

$$\bar{\rho}_C \cdot \bar{\rho}_A = \rho_C\, \rho_A \cos\beta_{ca} = X_C\, X_A + Y_C\, Y_A + (Z - H_C)\,(Z - H_A) \tag{39c}$$

which, solved for the face angles β, yield

$$\cos\beta_{ab} = \frac{X_A\, X_B + Y_A\, Y_B + (Z - H_A)\,(Z - H_B)}{\rho_A\, \rho_B} \tag{40a}$$

$$\cos\beta_{bc} = \frac{X_B\, X_C + Y_B\, Y_C + (Z - H_B)\,(Z - H_C)}{\rho_B\, \rho_C} \tag{40b}$$

and

$$\cos\beta_{ca} = \frac{X_C\, X_A + Y_C\, Y_A + (Z - H_C)\,(Z - H_A)}{\rho_C\, \rho_A} \tag{40c}$$

Since Z is known, Eqs. 36 and 40 suffice for the determination of the six coordinates X_A, Y_A, X_B, Y_B, X_C, and Y_C.

Since t and z_a, z_b, and z_c are known in Section 4, the following equations will give y_a, y_b, y_c and x_a, x_b, x_c:

$$y_a = \frac{z_a - f \sec t}{\tan t} \tag{41a}$$

$$y_b = \frac{z_b - f \sec t}{\tan t} \tag{41b}$$

and

$$y_c = \frac{z_c - f \sec t}{\tan t}; \tag{41c}$$

and

$$x_a = \pm (\rho_a^2 - y_a^2 - z_a^2)^{\frac{1}{2}} \tag{42a}$$

$$x_b = \pm (\rho_b^2 - y_b^2 - z_b^2)^{\frac{1}{2}} \tag{42b}$$

$$x_c = \pm (\rho_c^2 - y_c^2 - z_c^2)^{\frac{1}{2}} \tag{42c}$$

The signs of x_a, x_b, and x_c are selected according to the positions of the images of a, b, and c, with respect to the intersection lines of the XZ-plane and the YZ-plane with the plane of the photograph. These intersection lines can be drawn as follows:

By the lengths of l_1, l_2, and l_3, the axis of tilt (intersection line of the XZ-plane) can be determined, thus:

$$\frac{z_v - z_b}{z_a - z_b} = \frac{l_1}{l_{ab}} \tag{43a}$$

$$\frac{z_v - z_c}{z_b - z_c} = \frac{l_2}{l_{bc}} \tag{43b}$$

and

$$\frac{z_v - z_a}{z_c - z_a} = \frac{l_3}{l_{ac}} \tag{43c}$$

in which

$$z_v = \overline{\mathrm{Lv}} = \sqrt{f^2 + f^2 \tan^2 t} = f \sec t \tag{43d}$$

A line drawn through o and perpendicular to the axis of tilt is the intersection line of YZ-plane and its direction is taken from o to v as positive. The direction of the axis of tilt is taken according to the right-hand rule from the x-axis to the y-axis.

Since x_a,y_a,z_a; x_b,y_b,z_b; and x_c,y_c,z_c and their signs are known, then

$$X'_A = \frac{\rho_A}{\rho_a} x'_a; \qquad Y'_A = \frac{\rho_A}{\rho_a} y'_a \tag{44a}$$

$$X'_B = \frac{\rho_B}{\rho_b} x'_b; \qquad Y'_B = \frac{\rho_B}{\rho_b} y'_b \tag{44b}$$

and

$$X'_C = \frac{\rho_C}{\rho_c} x'_c; \qquad Y'_C = \frac{\rho_C}{\rho_c} y'_c \tag{44c}$$

can be calculated. Substituting these values from Eq. 44*a* to Eq. 44*c* into Eqs. 36 and 39 or 40, the latter equations should be satisfied. However, because of the inaccuracy of cos t and the inaccuracy of computing large values from small values, the residuals from the substitutions are unavoidable. The accurate values of X'_A,Y'_A; X'_B,Y'_B; and X'_C,Y'_C will be obtained, as follows:

$$\rho_A^2 - (Z - H_A)^2 = (k_1) \tag{45a}$$

$$\rho_B^2 - (Z - H_B)^2 = (k_2) \tag{45b}$$

and

$$\rho_C^2 - (Z - H_C)^2 = (k_3) \tag{45c}$$

$$\rho_A \rho_B \cos \beta_{ab} - (Z - H_A)(Z - H_B) = (k_4) \tag{46a}$$

$$\rho_B \rho_C \cos \beta_{bc} - (Z - H_B)(Z - H_C) = (k_5) \tag{46b}$$

and

$$\rho_C\,\rho_A \cos\beta_{ca} - (Z - H_C)\,(Z - H_A) = (k_6) \tag{46c}$$

Then

$$(k_1) = (X'_A + \Delta X_A)^2 + (Y'_A + \Delta Y_A)^2 \tag{47a}$$

$$(k_2) = (X'_B + \Delta X_B)^2 + (Y'_B + \Delta Y_B)^2 \tag{47b}$$

$$(k_3) = (X'_C + \Delta X_C)^2 + (Y'_C + \Delta Y_C)^2 \tag{47c}$$

$$(k_4) = (X'_A + \Delta X_A)\,(X'_B + \Delta X_B) + (Y'_A + \Delta Y_A)\,(Y'_B + \Delta Y_B) \tag{47d}$$

$$(k_5) = (X'_B + \Delta X_B)\,(X'_C + \Delta X_C) + (Y'_B + \Delta Y_B)\,(Y'_C + \Delta Y_C) \tag{47e}$$

and

$$(k_6) = (X'_C + \Delta X_C)\,(X'_A + \Delta X_A) + (Y'_C + \Delta Y_C)\,(Y'_A + \Delta Y_A) \tag{47f}$$

Let the second power terms of ΔX and ΔY be omitted and

$$(k_1) - [(X'_A)^2 + (Y'_A)^2] = (l_1) \tag{48a}$$

$$(k_2) - [(X'_B)^2 + (Y'_B)^2] = (l_2) \tag{48b}$$

$$(k_3) - [(X'_C)^2 + (Y'_C)^2] = (l_3) \tag{48c}$$

$$(k_4) - (X'_A\,X'_B + Y'_A\,Y'_B) = (l_4) \tag{48d}$$

$$(k_5) - (X'_B\,X'_C + Y'_B\,Y'_C) = (l_5) \tag{48e}$$

and

$$(k_6) - (X'_C\,X'_A + Y'_C\,Y'_A) = (l_6) \tag{48f}$$

Then Eqs. 47 become

$$(l_1) = 2\,X'_A\,\Delta X_A + 2\,Y'_A\,\Delta Y_A \tag{49a}$$

$$(l_2) = 2\,X'_B\,\Delta X_B + 2\,Y'_B\,\Delta Y_B \tag{49b}$$

$$(l_3) = 2\,X'_C\,\Delta X_C + 2\,Y'_C\,\Delta Y_C \tag{49c}$$

$$(l_4) = X'_A\,\Delta X_B + X'_B\,\Delta X_A + Y'_A\,\Delta Y_B + Y'_B\,\Delta Y_A \tag{49d}$$

$$(l_5) = X'_B\,\Delta X_C + X'_C\,\Delta X_B + Y'_B\,\Delta Y_C + Y'_C\,\Delta Y_B \tag{49e}$$

and

$$(l_6) = X'_C\,\Delta X_A + X'_A\,\Delta X_C + Y'_C\,\Delta Y_A + Y'_A\,\Delta Y_C \tag{49f}$$

A solution of Eqs. 49 gives ΔX_A, ΔY_A, ΔX_B, ΔY_B, ΔX_C, and ΔY_C. Then the second approximate values of X_A, Y_A, X_B, Y_B, X_C, and Y_C will be $X'_A + \Delta X_A$, $Y'_A + \Delta Y_A$, $X'_B + \Delta X_B$, $Y'_B + \Delta Y_B$, $X'_C + \Delta X_C$, and $Y'_C + \Delta Y_C$, respectively. Usually the second approximate values are accurate enough; if not, the same procedure can be repeated until the designer is satisfied.

After X_A, Y_A, X_B, Y_B, X_C, Y_C, and Z have been found, the accurate coordinates of the corresponding images can be computed directly by the following relations:

$$\frac{x_a}{X_A} = \frac{y_a}{Y_A} = \frac{z_a}{Z - H_A} = \frac{\rho_a}{\rho_A} \tag{50a}$$

$$\frac{x_b}{X_B} = \frac{y_b}{Y_B} = \frac{z_b}{Z - H_B} = \frac{\rho_b}{\rho_B} \tag{50b}$$

and

$$\frac{x_c}{X_C} = \frac{y_c}{Y_C} = \frac{z_c}{Z - H_C} = \frac{\rho_c}{\rho_C} \qquad (50c)$$

From Fig. 2,

$$z_a = y_a \tan t + f \sec t \qquad (51a)$$

$$z_b = y_b \tan t + f \sec t \qquad (51b)$$

and

$$z_c = y_c \tan t + f \sec t \qquad (51c)$$

From any two of Eqs. 51,

$$z_a - z_b = (y_a - y_b) \tan t \qquad (52a)$$

$$z_b - z_c = (y_b - y_c) \tan t \qquad (52b)$$

and

$$z_c - z_a = (y_c - y_a) \tan t \qquad (52c)$$

Hence,

$$\tan t = \frac{z_a - z_b}{y_a - y_b} = \frac{z_b - z_c}{y_b - y_c} = \frac{z_c - z_a}{y_c - y_a} \qquad (53)$$

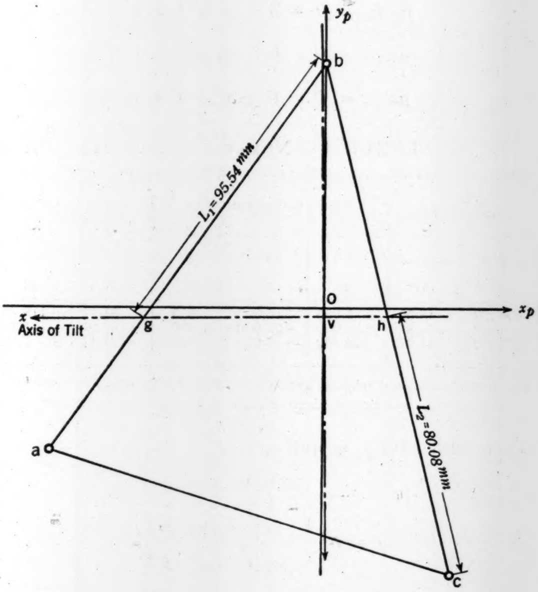

FIG. 5.—GRAPHICAL DETERMINATION OF THE AXIS OF TILT

Theoretically the three values of tan *t* in Eq. 53 should be identical. Actually, exact checks are seldom obtained, especially when the tilts approach zero, because the differences between pairs of *z*-values are so small that the tangents become very sensitive to small errors therein. However, the average of the three values of tilt will be very accurate.

Since t is known—

$$\overline{\mathrm{Ov}} = f \tan t \qquad (54)$$

—gives the distance from the principal point to nadir point on the principal line (see Fig. 5). Now suppose the vectors of the rays from the exposure station to the three image points and the principal point are as follows:

$$\overline{\mathrm{La}} = \bar{\rho}_a, \quad x_a\, i + y_a\, j + z_a\, k \qquad (55a)$$

$$\overline{\mathrm{Lb}} = \bar{\rho}_b, \quad x_b\, i + y_b\, j + z_b\, k \qquad (55b)$$

$$\overline{\mathrm{Lc}} = \bar{\rho}_c, \quad x_c\, i + y_c\, j + z_c\, k \qquad (55c)$$

and

$$\overline{\mathrm{Lo}} = \bar{\rho}_o, \quad 0 + y\, j + z\, k \qquad (55d)$$

in which x, y, and z represent the unknown coordinates of the principal point referred to the current plumb-line coordinate system. Taking scalar products,

$$\bar{\rho}_a \cdot \bar{\rho}_o = f^2 = 0 + y_a\, y + z_a\, z \qquad (56a)$$

$$\bar{\rho}_b \cdot \bar{\rho}_o = f^2 = 0 + y_b\, y + z_b\, z \qquad (56b)$$

and

$$\bar{\rho}_c \cdot \bar{\rho}_o = f^2 = 0 + y_c\, y + z_c\, z \qquad (56c)$$

TABLE 1.—NUMERICAL EXAMPLE; PHYSICAL DATA FOR

Point[a] (see Fig. 3)	PHOTO COORDINATES		GROUND COORDINATES			Δx	Δy	l	ρ
	x	y	X	Y	Z				
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
a_3 a	−81.92	−43.59	+ 5,000	+10,000	1,800	+ 81.92	+118.41	143.985	176.383
c_5 b	0.00	+74.82	+15,000	+25,000	1,000	+ 39.22	−155.89	160.748	167.625
d_1 c	+39.22	−81.07	+20,000	+ 5,000	900	−121.14	+ 37.48	126.806	174.959

[a] In Col. 1, a_3, c_5, and d_1 are the original symbols for the three particular images, made by Earl Church,

If Eqs. 56 are solved for x, y, and z,

$$x = 0 \qquad (57a)$$

$$y = f^2 \begin{vmatrix} 1 & z_a \\ 1 & z_b \end{vmatrix} \Big/ \begin{vmatrix} y_a & z_a \\ y_b & z_b \end{vmatrix} \qquad (57b)$$

and

$$z = f^2 \begin{vmatrix} y_a & 1 \\ y_b & 1 \end{vmatrix} \Big/ \begin{vmatrix} y_a & z_a \\ y_b & z_b \end{vmatrix} \qquad (57c)$$

If it is desirable to utilize the photographic axes to locate the nadir point, the following procedure can be adopted. Let the coordinates of the nadir point be the unknown quantities x_{pv}, and y_{pv}, and let the known quantity $z_{pv} = f$, be referred to the photographic coordinate system; thus (compare Eqs. 55),

$$\bar{\rho}_a, \qquad x_{pa}\, i + y_{pa}\, j + f\, k \tag{58a}$$

$$\bar{\rho}_b, \qquad x_{pb}\, i + y_{pb}\, j + f\, k \tag{58b}$$

$$\bar{\rho}_c, \qquad x_{pc}\, i + y_{pc}\, j + f\, k \tag{58c}$$

and

$$\bar{\rho}_v, \qquad x_{pv}\, i + y_{pv}\, j + f\, k \tag{58d}$$

Taking scalar products (compare Eqs. 56).

$$\bar{\rho}_a \cdot \bar{\rho}_v = \rho_a\, \rho_v \cos V_a = x_{pa}\, x_{pv} + y_{pa}\, y_{pv} + f^2 \tag{59a}$$

$$\bar{\rho}_b \cdot \bar{\rho}_v = \rho_b\, \rho_v \cos V_b = x_{pb}\, x_{pv} + y_{pb}\, y_{pv} + f^2 \tag{59b}$$

and

$$\bar{\rho}_c \cdot \bar{\rho}_v = \rho_c\, \rho_v \cos V_c = x_{pc}\, x_{pv} + y_{pc}\, y_{pv} + f^2 \tag{59c}$$

in which V is the angle between ρ and the Z-axis.

In Eqs. 59, since Z is known, values of V can be computed by

$$\cos V_a = \frac{Z - H_A}{\rho_A} \tag{60a}$$

$$\cos V_b = \frac{Z - H_B}{\rho_B} \tag{60b}$$

THE DETERMINATION OF ELEVATION Z AND ANGLE OF TILT

ΔX (11)	ΔY (12)	ΔH (13)	HORIZONTAL		INCLINED		Point (1)
			$(L')^2$ (14)	L' (15)	L^2 (16)	L (17)	
+10,000	+15,000	−800	325,000,000	L'_{AB} = 18,027.756	325,640,000	L'_{AB} = 18,045.489	a_2 a
+ 5,000	−20,000	−100	425,000,000	L'_{BC} = 20,615.528	425,010,000	L'_{BC} = 20,615.771	c_3 b
−15,000	+ 5,000	+900	250,000,000	L'_{CA} = 15,811.388	250,810,000	L'_{CA} = 15,836.982	d_1 c

Assoc. M. ASCE. In this paper, these three points are regarded as a, b, and c, respectively.

and

$$\cos V_c = \frac{Z - H_C}{\rho_C} \tag{60c}$$

From Eqs. 59 one can solve directly for ρ_v, x_{pv}, and y_{pv}. By the values of x_{pv} and y_{pv}, the distance Ov (Fig. 5), and consequently the tangent of the angle of tilt, can be computed.

Numerical Example.—To demonstrate the theory recommended herein it is proposed to test it against a standard problem that has been solved elsewhere.[8] The focal distance f is given as 150 mm and the ordinates and other physical data are presented in Table 1. With these data, Eqs. 3 can be solved and the results entered in Table 2 as shown; and Eqs. 4 then yield the values of cos β in line 1, Table 3.

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[8] "Manual of Photogrammetry," Am. Soc. of Photogrammetry, Pitman Pub. Corp., New York, N. Y., 1944, p. 558.

As a first approximation, Eqs. 10 are solved to yield: $\rho'_A = 22{,}084$; $\rho'_B = 21{,}497$; and $\rho'_C = 21{,}816$. Substituting these approximate values of ρ in Eqs. 9, successive approximation yields:

Distance	ρ	ρ^2
ρ_A	21,532	463,627,024
ρ_B	21,646	468,549,316
ρ_C	22,305	497,513,025

These values, with the aid of Table 3, then permit the solution of Eqs. 14, as follows:

Angles Δ'	Sine	Cosine
Δ'_{AB}	+0.910 870	+0.413 141
Δ'_{AC}	+0.948 072	+0.318 054

TABLE 2.—COMPUTATION OF FACE ANGLES β, FIG. 1

Eq. No.	First quantity +	Second quantity +	f^2 =	$f(\rho) \cos \beta$
3*a*	0.000	−3,261.404	+22,500.000	+19,238.596
3*b*	0.000	−6,065.657	+22,500.000	+16,434.343
3*c*	−3,212.902	+3,533.841	+22,500.000	+22,820.939

TABLE 3.—NATURAL SINES AND COSINES OF FACE ANGLES β, FOR SOLUTION OF EQS. 14

Line	Function	β_{ab}	β_{bc}	β_{ca}
1	cos	0.650 6956	0.560 3731	0.739 5039
2	cos²	0.423 4048	0.314 0180	0.546 8690
3	sin²	0.576 5952	0.685 9820	0.453 1310
4	sin	0.759 339	0.828 240	0.673 150

TABLE 4.—SOLUTION OF EQS. 15

Line	(X_B-X_A) ×	(X_C-X_A) +	(Y_B-Y_A) ×	(Y_C-Y_A) +	(H_A-H_B) ×	(H_A-H_C) =	$L_{AB} L_{CA} \times \cos \tau_A$
1	15,000	20,000	25,000	5,000	1,800	1,800	....
2	5,000	5,000	10,000	10,000	1,000	900	....
3	+10,000	+15,000	+15,000	−5,000	+800	+900	....
4	+ 150,000,000		− 75,000,000		+ 720,000		....
5	$\Sigma = 75{,}720{,}000$						$L_{AB} L_{CA} \times (+0.264\ 95296)$

Likewise (see Fig. 3), $\rho_P = \rho_A \sin \Delta'_{AB} = 19{,}613$; and $\overline{\mathrm{AR}} = \rho_A \cos \Delta'_{AB} = 8{,}896$. A tabular solution of Eq. 15*a* is demonstrated by Table 4 which reveals that $\cos \tau_A = +\ 0.264\ 95296$, and therefore $\tan \tau_A = 3.639\ 368$. Again, from Fig. 3, $\overline{\mathrm{PR}} = \overline{\mathrm{AR}} \tan \tau_A = 32{,}376$; and $\overline{\mathrm{AP}} = +\ \overline{\mathrm{AR}} \sec \tau_A$

= 33,576. Similar to Eq. 16*c*, then, $\rho^2_R = \rho^2_A + \overline{\text{AR}}^2 - 2\,\rho_A\,\overline{\text{AR}}\cos\Delta'_{AC}$; that is, $\rho^2_R = 1{,}131{,}095{,}158$ and $\rho_R = 33{,}632$. Similar to Eq. 17*b*, $\Sigma = \frac{1}{2}(\rho_P + \rho_R + \overline{\text{PR}}) = 42{,}810$. For substitution in Eq. 17*a*, $\Sigma - \rho_P = 23{,}197$; $\Sigma - \rho_R = 9{,}178$, and $\Sigma - \overline{\text{PR}} = 10{,}434$. With this value of A, Eq. 18 yields $\rho_G = 19{,}050$. Other quantities needed in the solution are

$$\text{(a)}\qquad \frac{H_A - H_C}{L_{CA}} = \frac{900}{15{,}837} = 0.056\,8289$$

$$\text{(b)}\qquad \frac{H_A - H_B}{L_{AB}} = \frac{800}{18{,}046} = 0.044\,3324$$

$$\text{(c)}\qquad \Delta H_R = (a) \times 33{,}576 = 1{,}908$$

$$\text{(d)}\qquad \Delta H_P = (b) \times 8{,}896 = 398$$

$$\text{(e)}\qquad H_R = H_A - (c) = 1{,}800 - 1{,}908 = -\,108$$

$$\text{(f)}\qquad H_P = H_A - (d) = 1{,}800 - 398 = +\,1{,}406$$

$$\text{(g)}\qquad \overline{\text{PG}}\ (\text{Fig. 3}) = (\rho^2_P - \rho^2_G)^{\frac{1}{2}} = 4{,}666$$

$$\text{(h)}\qquad H_R - H_P = -\,(e) - (f) = -\,1{,}514$$

$$\text{(i)}\qquad \frac{H_R - H_P}{\text{PR}} = \frac{(h)}{32{,}376} = -\,0.046\,763\,034$$

$$\text{(j)}\qquad H_G = (f) + (i) \times (g) = 1{,}188$$

The equation of a plane that passes through the three ground control points is

$$X\begin{vmatrix} 0 & 0 & 1 \\ 15{,}000 & -800 & 0 \\ -20{,}000 & -100 & 0 \end{vmatrix} - Y\begin{vmatrix} 0 & 0 & 1 \\ 10{,}000 & -800 & 0 \\ 5{,}000 & -100 & 0 \end{vmatrix} + Z\begin{vmatrix} 0 & 0 & 1 \\ 10{,}000 & 15{,}000 & 0 \\ 5{,}000 & -20{,}000 & 0 \end{vmatrix} = 0$$

In Eq. 30,

$$\frac{\partial Z}{\partial X} = +\,0.063\,6363636; \qquad \frac{\partial Z}{\partial Y} = +\,0.010\,9090909;$$

$$\left(\frac{\partial Z}{\partial Y}\right)^2 = +\,0.000\,1190083; \qquad \text{and} \qquad \left(\frac{\partial Z}{\partial X}\right)^2 = +\,0.004\,0495868.$$

TABLE 5.—COMPUTATION OF HORIZONTAL SIDES l'_{ab}, l'_{bc}, AND l'_{ca}

Line	Symbol	Side ab	Side bc	Side ca
1	$Z - h$	$Z - H_A = 18{,}398$	$Z - H_B = 19{,}198$	$Z - H_C = 19{,}298$
2	z	$z_a = 150.713$	$z_b = 148.668$	$z_c = 151.372$
3	l^2	$l^2_{ab} = 20{,}731.6802$	$l^2_{bc} = 25{,}839.9195$	$l^2_{ca} = 16{,}079.7616$
4	Δz^2	$(z_a - z_b)^2 = 4.1698$	$(z_c - z_b)^2 = 7.3116$	$(z_c - z_a)^2 = 0.4382$
5	$(l')^2$	$(l'_{ab})^2 = 20{,}727.5104$	$(l'_{bc})^2 = 25{,}832.6079$	$(l'_{ca})^2 = 16{,}079.3234$
6	l'	$l'_{ab} = 143.9705$	$l'_{bc} = 160.7252$	$l'_{ca} = 126.8043$

Therefore, $\cos T = 0.997\,9222$; and $\rho_G \cos T = 19{,}010$. Eq. 30 then yields $Z = 19{,}010 + 1{,}188 = 20{,}198$ ft.

Determination of the Tilt and the Intersection Lines of the ZX-Plane and the ZY-Plane with the Photograph.—From data compiled in Table 5, values of σ' in

Eqs. 34 (and corresponding values of σ') are determined as follows:

Symbol	Horizontal projection (symbols primed)	Inclined plane
l_{bc}	160.725	160.748
l_{ca}	126.804	126.806
l_{ab}	143.971	143.985
2σ	431.500	431.539
σ	215.750	215.7695
$\sigma - l_{ab}$	71.779	71.785
$\sigma - l_{bc}$	55.025	55.022
$\sigma - l_{ca}$	88.946	88.965

TABLE 6.—Solution of Eq. 53

Quantity	Point A (or a)	Point B (or b)	Point C (or c)
X	$X_A = +9{,}987$	$X_B = -78$	$X_C = -4{,}946$
Y	$Y_A = +5{,}040$	$Y_B = -9{,}999$	$Y_C = +10{,}032$
z	$z_a = +81.81$	$z_b = -0.60$	$z_c = -38.80$
y	$y_a = \frac{176.383 \times 5{,}040}{21{,}532} = +41.29$	$y_b = \frac{167.625 \times (-9{,}999)}{21{,}640} = -77.45$	$y_c = \frac{174.959 \times 10{,}032}{22{,}305} = +78.69$
Δy	$y_a - y_b = 118.74$	$y_b - y_c = 156.14$	$y_c - y_a = 37.40$
Δz	$z_a - z_b = 2.0423$	$z_b - z_c = 2.704$	$z_c - z_a = 0.6617$
Tan t	0.017 19976	0.017 31779	0.017 69251

TABLE 7.—Solution of Eqs. 59 and 60 to Determine the Nadir

Point	x	y	ρ (photo pyramid)	$Z - H$	ρ (ground pyramid)	$\cos V$	$\rho \cos V$
a	−81.92	−43.59	176.383	18,398	21,532	0.854 4492	150.7103
b	0.00	+74.82	167.625	19,198	21,646	0.886 9075	148.6679
c	+39.22	−81.07	174.959	19,298	22,305	0.865 1872	151.3723

By Eq. 3, $\cos t = \underline{a'/a} = 0.999\,8335$; and $t = 1° 02.5'$. Since $z_v = f \sec t$, $z_v = 150.02498$ and $\overline{\mathrm{Ov}} = f \tan t = 150 \times 0.0181 = 2.715$ mm. By geometry (see Figs. 2 and 5),

$$\frac{z_v - z_b}{z_a - z_b} = \frac{l_1}{l_{ab}} = \frac{1.357}{2.045} = \frac{l_1}{143.985} \qquad (61a)$$

$$\frac{z_c - z_v}{z_c - z_b} = \frac{l_2}{l_{bc}} = \frac{1.347}{2.704} = \frac{l_2}{160.748} \qquad (61b)$$

and

$$\frac{z_c - z_v}{z_c - z_a} = \frac{l_3}{l_{ca}} = \frac{1.347}{0.659} = \frac{l_3}{126.806} \qquad (61c)$$

Eqs. 61 each contain the "ingredients" for determining the values of l, in millimeters, respectively, as follows (see Fig. 5): $l_1 = \overline{\mathrm{bg}} = 95.545$; $l_2 = \overline{\mathrm{ch}}$

$= 80.077$; and $l_3 = 259.192$. Values of $\tan t$, computed by Eq. 53, are given in the last line of Table 6. The average of the three values is 0.017 40302, from which a value of $t = 0° \, 59.8'$ is computed. Computations for the determination of the nadir point with respect to the photograph coordinate system (see Fig. 5) are listed in Table 7. From these data, two equations can be set up for simultaneous solution (by Eqs. 59):

$$-\,0.5436\, x_{pv} - 0.79250\, y_{pv} - 2.0510 = 0 \quad \text{(62a)}$$

and

$$-\,0.25910\, x_{pv} + 1.03884\, y_{pv} + 2.7039 = 0 \quad \text{(62b)}$$

—from which $x_{pv} = +\,0.016$ and $y_{pv} = -\,2.600$. Since $\overline{\text{Ov}} = 2.6000$, Eq. 54 yields, $\tan t = 2.600/150 = 0.0173333$; and $t = 0° \, 59.6'$. Likewise, $\tan s = -\,0.006\,15385$; and $s = 179° \, 39.0'$.

SECTION 6. AIR BASE AND SPACE INTERSECTION

Of two aerial photographs, taken successively, the first can be analyzed by the procedure demonstrated in Section 5. The second photograph, containing a different exposure station (point 2, Fig. 6), can also be analyzed by that procedure if it contains images of three control points that appear in the first photograph. In other words, the elevation of the second exposure station (point L₂, Fig. 6) can be determined, as well as the space coordinates (X'_A, Y'_A, X'_B, Y'_B, X'_C, and Y'_C) of the three ground points (A, B, and C, Fig. 6) referred to the second plumb-line coordinate system.

POINT WITH RESPECT TO THE PHOTOGRAPH COORDINATE SYSTEM

NUMERICAL COEFFICIENTS IN EQS. 59			Point
$f(x)$	$f(y)$	$f(f)$	
$\frac{x_{pa}}{\rho_a \cos V_a} = -\,0.54356$	$\frac{y_{pa}}{\rho_a \cos V_a} = -\,0.28923$	$\frac{f^2}{\rho_a \cos V_a} = +\,149.2930$	a
$\frac{x_{pb}}{\rho_b \cos V_b} = +\,0.00000$	$\frac{y_{pb}}{\rho_b \cos V_b} = +\,0.50327$	$\frac{f^2}{\rho_b \cos V_b} = +\,151.2440$	b
$\frac{x_{pc}}{\rho_c \cos V_c} = +\,0.25910$	$\frac{y_{pc}}{\rho_c \cos V_c} = -\,0.53557$	$\frac{f^2}{\rho_c \cos V_c} = +\,148.6401$	c

Air Base.—In Fig. 6(*a*), the images of points A, B, and C are visible in two photographs taken in sequence. It is possible to compute the space ordinates (X_2 and Y_2) of the second exposure station, with respect to the first plumb-line coordinate system, by utilizing the scalar products of the second set of vectors (ρ'_A, ρ'_B, and ρ'_C). Thus (compare Eqs. 38):

$$\rho'_A\text{:}\quad (X_A - X_2)\, i + (Y_A - Y_2)\, j + (Z_2 - H_A)\, k \quad \text{(63a)}$$

$$\rho'_B\text{:}\quad (X_B - X_2)\, i + (Y_B - Y_2)\, j + (Z_2 - H_B)\, k \quad \text{(63b)}$$

and

$$\rho'_C\text{:}\quad (X_C - X_2)\, i + (Y_C - Y_2)\, j + (Z_2 - H_C)\, k \quad \text{(63c)}$$

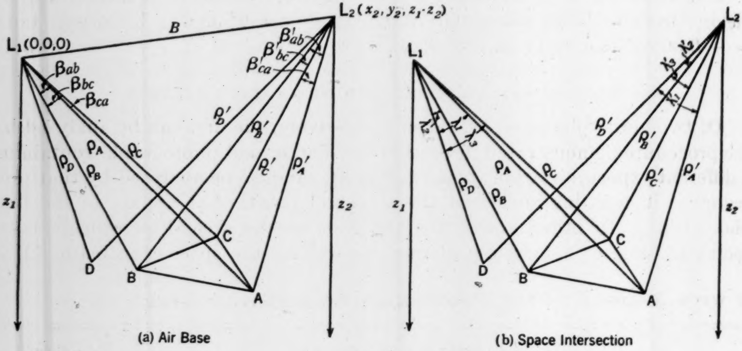
The scalar products of pairs of these vectors are

$$\rho'_A \, \rho'_B \cos \beta'_{ab} = (X_A - X_2)(X_B - X_2) + (Y_A - Y_2)(Y_B - Y_2) + (Z_2 - H_A)(Z_2 - H_B) \quad \text{(64a)}$$

$$\rho'_B \, \rho'_C \cos \beta'_{bc} = (X_B - X_2)(X_C - X_2) + (Y_B - Y_2)(Y_C - Y_2) + (Z_2 - H_B)(Z_2 - H_C) \quad \text{(64b)}$$

and

$$\rho'_C \, \rho'_A \cos \beta'_{ca} = (X_C - X_2)(X_A - X_2) + (Y_C - Y_2)(Y_A - Y_2) + (Z_2 - H_C)(Z_2 - H_A) \quad \text{(64c)}$$



(a) Air Base

(b) Space Intersection

FIG. 6

Therefore,

$$\rho'_A \, \rho'_B \cos \beta'_{ab} - (Z_2 - H_A)(Z_2 - H_B) = (X_A - X_2)(X_B - X_2) + (Y_A - Y_2)(Y_B - Y_2) \quad \text{(65a)}$$

$$\rho'_B \, \rho'_C \cos \beta'_{bc} - (Z_2 - H_B)(Z_2 - H_C) = (X_B - X_2)(X_C - X_2) + (Y_B - Y_2)(Y_C - Y_2) \quad \text{(65b)}$$

and

$$\rho'_C \, \rho'_A \cos \beta'_{ca} - (Z_2 - H_C)(Z_2 - H_A) = (X_C - X_2)(X_A - X_2) + (Y_C - Y_2)(Y_A - Y_2) \quad \text{(65c)}$$

$$6\,V' = \begin{vmatrix} X_2 & Y_2 & Z_1 - Z_2 & 1 \\ X_A & Y_A & Z_1 - H_A & 1 \\ X_B & Y_B & Z_1 - H_B & 1 \\ X_C & Y_C & Z_1 - H_C & 1 \end{vmatrix} = \rho'_A \, \rho'_B \, \rho'_C \begin{vmatrix} 1 & \cos \beta'_{ca} & \cos \beta'_{bc} \\ \cos \beta'_{ca} & 1 & \cos \beta'_{ab} \\ \cos \beta'_{bc} & \cos \beta'_{ab} & 1 \end{vmatrix}^{\frac{1}{2}} \quad \text{(66)}$$

in which V' is the volume of the second ground pyramid; and H_A, H_B, and H_C are the elevations of the ground points A, B, and C, respectively. Eq. 66 is a linear equation with two unknowns of X_2 and Y_2. Since X_A, Y_A, H_A, X_B, Y_B, H_B, X_C, Y_C, H_C, Z_2, ρ'_A, ρ'_B, ρ'_C, $\cos \beta'_{ab}$, $\cos \beta'_{bc}$, and $\cos \beta'_{ca}$ now are known quantities, any one of Eqs. 65 and Eq. 66 can be solved theoretically for X_2 and Y_2. In the solution of these second degree equations, the signs of X_2 and Y_2 can be determined in accordance with the position of the image of the second exposure station on the first photograph. Then $\overline{L_1 L_2} = B: X_2 i + Y_2 j + (Z_1 - Z_2) k$; and therefore the length of the "air base" will be

$$B = \sqrt{X^2_2 + Y^2_2 + (Z_1 - Z_2)^2} \quad (67)$$

The horizontal length of (B) is equal to $\sqrt{X^2_2 + Y^2_2}$; and the direction cosines of B are $\cos \mu_x = \frac{X_2}{B}$, $\cos \mu_y = \frac{Y_2}{B}$, and $\cos \mu_z = \frac{Z_1 - Z_2}{B}$.

Space Intersection.—If there is any new point (D, Fig. 6) common to the two photographs, the coordinates of its image on the first photograph with respect to the first plumb-line coordinate system will be designated as x_d, y_d, z_d; and those of its image on the second photograph with respect to the second plumb-line coordinate system, by x'_d, y'_d, z'_d. The coordinates x_a, y_a, z_a; x_b, y_b, z_b; and x_c, y_c, z_c of the images of the three ground control points A, B, and C in the first photograph, with respect to the first plumb-line coordinate system, are known. Likewise, the coordinates x'_a, y'_a, z'_a; x'_b, y'_b, z'_b; and x'_c, y'_c, z'_c in the second photograph, with respect to the second plumb-line coordinate system, are known. Taking scalar products of the vector ρ for the first set,

$$\rho_a \rho_d \cos \lambda_a = x_a x_d + y_a y_d + z_a z_d \quad (68a)$$

$$\rho_b \rho_d \cos \lambda_b = x_b x_d + y_b y_d + z_b z_d \quad (68b)$$

and

$$\rho_c \rho_d \cos \lambda_c = x_c x_d + y_c y_d + z_c z_d \quad (68c)$$

and, for the second set,

$$\rho'_a \rho'_d \cos \lambda'_a = x'_a x'_d + y'_a y'_d + z'_a z'_d \quad (69a)$$

$$\rho'_b \rho'_d \cos \lambda'_b = x'_b x'_d + y'_b y'_d + z'_b z'_d \quad (69b)$$

and

$$\rho'_c \rho'_d \cos \lambda'_c = x'_c x'_d + y'_c y'_d + z'_c z'_d \quad (69c)$$

In Eqs. 68 and 69, λ_a, λ_b, and λ_c are the apex angles between ρ_d and ρ_a, ρ_b, and ρ_c, respectively; and λ'_a, λ'_b, λ'_c, the apex angles between ρ'_d and ρ'_a, ρ'_b, and ρ'_c, respectively. Solving for x_d, y_d, and z_d in Eqs. 68 yields the coordinates of the image of point D in the first photograph and solving for x'_d, y'_d, and z'_d from Eqs. 69 yields the coordinates of the image of point D in the second photograph. Again, taking scalar products of the rays from the first exposure station L_1 to the ground points,

$$\rho_A \rho_D \cos \lambda_a = X_A X_D + Y_A Y_D + (Z_1 - H_A)(Z_1 - H_D) \quad (70a)$$

$$\rho_B \rho_D \cos \lambda_b = X_B X_D + Y_B Y_D + (Z_1 - H_B)(Z_1 - H_D) \quad (70b)$$

and

$$\rho_C \rho_D \cos \lambda_c = X_C X_D + Y_C Y_D + (Z_1 - H_C)(Z_1 - H_D) \quad (70c)$$

and, from the second exposure station to the ground points,

$$\rho'_A \rho'_D \cos \lambda'_a = X'_A X'_D + Y'_A Y'_D + (Z_2 - H_A)(Z_2 - H_D) \quad (71a)$$

$$\rho'_B \rho'_D \cos \lambda'_b = X'_B X'_D + Y'_B Y'_D + (Z_2 - H_B)(Z_2 - H_D) \quad (71b)$$

and

$$\rho'_C \, \rho'_D \cos \lambda'_c = X'_C \, X'_D + Y'_C \, Y'_D + (Z_2 - H_C)\,(Z_2 - H_D) \quad (71c)$$

By plane trigonometry, from the first set of the relations, as shown in Fig. 6(*b*),

$$\overline{AD}^2 = \rho^2_A + \rho^2_D - 2\,\rho_A\,\rho_D \cos \lambda_a \quad (72a)$$

$$\overline{BD}^2 = \rho^2_B + \rho^2_D - 2\,\rho_B\,\rho_D \cos \lambda_b \quad (72b)$$

and

$$\overline{CD}^2 = \rho^2_C + \rho^2_D - 2\,\rho_C\,\rho_D \cos \lambda_c \quad (72c)$$

and, from the second set,

$$\overline{AD}^2 = (\rho'_A)^2 + (\rho'_D)^2 - 2\,\rho'_A\,\rho'_D \cos \lambda'_a \quad (73a)$$

$$\overline{BD}^2 = (\rho'_B)^2 + (\rho'_D)^2 - 2\,\rho'_B\,\rho'_D \cos \lambda'_b \quad (73b)$$

and

$$\overline{CD}^2 = (\rho'_C)^2 + (\rho'_D)^2 - 2\,\rho'_C\,\rho'_D \cos \lambda'_c \quad (73c)$$

After eliminating $\overline{AD}$, $\overline{BD}$, and $\overline{CD}$ from Eqs. 72 and 73,

$$\rho^2_A + \rho^2_D - 2\,\rho_A\,\rho_D \cos \lambda_a = (\rho'_A)^2 + (\rho'_D)^2 - 2\,\rho'_A\,\rho'_D \cos \lambda'_a \quad (74a)$$

$$\rho^2_B + \rho^2_D - 2\,\rho_B\,\rho_D \cos \lambda_b = (\rho'_B)^2 + (\rho'_D)^2 - 2\,\rho'_B\,\rho'_D \cos \lambda'_b \quad (74b)$$

and

$$\rho^2_C + \rho^2_D - 2\,\rho_C\,\rho_D \cos \lambda_c = (\rho'_C)^2 + (\rho'_D)^2 - 2\,\rho'_C\,\rho'_D \cos \lambda'_c \quad (74c)$$

Eliminating ρ^2_D and $(\rho'_D)^2$ from Eqs. 74,

$$\begin{aligned}\rho^2_A - \rho^2_B + 2\,\rho_D\,(\rho_B \cos \lambda_b - \rho_A \cos \lambda_a) \\ = (\rho'_A)^2 - (\rho'_B)^2 + 2\,\rho'_D\,(\rho'_B \cos \lambda'_b - \rho'_A \cos \lambda'_a)\end{aligned} \quad (75a)$$

and

$$\begin{aligned}\rho^2_A - \rho^2_C + 2\,\rho_D\,(\rho_C \cos \lambda_c - \rho_A \cos \lambda_a) \\ = (\rho'_A)^2 - (\rho'_C)^2 + 2\,\rho'_D\,(\rho'_C \cos \lambda'_c - \rho'_A \cos \lambda'_a)\end{aligned} \quad (75b)$$

Let

$$\tfrac{1}{2}\,\{\rho^2_A - \rho^2_B - [(\rho'_A)^2 - (\rho'_B)^2]\} = w_1 \quad (76a)$$

and

$$\tfrac{1}{2}\,\{\rho^2_A - \rho^2_C - [(\rho'_A)^2 - (\rho'_C)^2]\} = w_2 \quad (76b)$$

Also, let

$$\rho_B \cos \lambda_b - \rho_A \cos \lambda_a = a_1 \quad (77a)$$

$$\rho'_B \cos \lambda'_b - \rho'_A \cos \lambda'_a = b_1 \quad (77b)$$

$$\rho_C \cos \lambda_c - \rho_A \cos \lambda_a = a_2 \quad (77c)$$

and

$$\rho'_C \cos \lambda'_c - \rho'_A \cos \lambda'_a = b_2 \quad (77d)$$

Then

$$a_1\,\rho_D - b_1\,\rho'_D + w_1 = 0 \quad (78a)$$

and

$$a_2\,\rho_D - b_2\,\rho'_D + w_2 = 0 \quad (78b)$$

The solution of Eqs. 78 yields the values of ρ_D and ρ'_D, after which the elevation of points D,H_D, will be given by either of the two equations (compare Eqs. 31):

$$\rho_D = \frac{Z_1 - H_D}{z_d} \rho_d \quad \text{(79a)}$$

and

$$\rho'_D = \frac{Z_2 - H_D}{z'_d} \rho'_d \quad \text{(79b)}$$

Substituting the values of ρ_D, ρ'_D, and H_D from Eqs. 79 back into Eqs. 70 and 71, the solution of any two of Eqs. 70 gives the coordinates of point D with respect to the first plumb-line coordinate system; and the solution of any two of Eqs. 71 gives the coordinates of point D with respect to the second plumb-line coordinate system. The determination of the same point with respect to two different coordinate systems is equivalent to a determination of the angular relationship between the two systems, since their origins are known, one being referred to the other. By the standard formulas of translation and rotation of axes,

$$x_1 = x_o + x_2 \cos\alpha - y_2 \sin\alpha \quad \text{(80a)}$$

and

$$y_1 = y_o + x_2 \sin\alpha + y_2 \cos\alpha \quad \text{(80b)}$$

in which x_o,y_o are the coordinates of the origin of the second system on the first system; x_1,y_1 and x_2,y_2 are the coordinates of a specific point under consideration with reference to the two systems, respectively; and α is the angle between the corresponding axes of the two systems. Since x_o,y_o; x_1,y_1; and x_2,y_2 are known, the values of $\cos\alpha$ and $\sin\alpha$ can be obtained from Eqs. 80, in which x_o,y_o; x_1,y_1; and x_2,y_2 are equivalent to X_2,Y_2; X_D,Y_D; and X'_D,Y'_D, respectively, in Eqs. 63 to 71.

The procedures explained in this section can be applied to any other points that have common images on the two successive photographs.

Again, if there is a new point D, common to the two successive photographs, and this new point has its image coordinates in the first photograph, x_d,y_d,z_d, with respect to the first plumb-line coordinate system, the direction cosines of the line L_1D, from the first exposure station to the ground point D, will then be

$$\cos\alpha_1 = \frac{x_d}{\overline{L_1d}} \quad \text{(81a)}$$

$$\cos\beta_1 = \frac{y_d}{\overline{L_1d}} \quad \text{(81b)}$$

and

$$\cos\gamma_1 = \frac{z_d}{\overline{L_1d}} \quad \text{(81c)}$$

in which

$$\overline{L_1d} = \rho_d = \sqrt{x^2_{pd} + y^2_{pd} + f^2} \quad \text{(82)}$$

All the coordinates of the ground points A, B, and C and the second exposure station with respect to the first photograph plumb-line coordinate system are $X_A,Y_A,(Z_1 - H_A)$; $X_B,Y_B,(Z_1 - H_B)$; $X_C,Y_C,(Z_1 - H_C)$; and X_2,Y_2,Z_2, respectively. Consequently, the formulas—

$$\frac{\rho'_a}{\rho'_A} = \frac{x'_a}{X_A - X_2} = \frac{y'_a}{Y_A - Y_2} = \frac{z'_a}{Z_1 - Z^*_2 - H_A} \quad \text{(83a)}$$

$$\frac{\rho'_b}{\rho'_B} = \frac{x'_b}{X_B - X_2} = \frac{y'_b}{Y_B - Y_2} = \frac{z'_b}{Z_1 - Z^*_2 - H_B} \quad \text{(83b)}$$

and

$$\frac{\rho'_c}{\rho'_C} = \frac{x'_c}{X_C - X_2} = \frac{y'_c}{Y_C - Y_2} = \frac{z'_c}{Z_1 - Z^*_2 - H_C} \quad \text{(83c)}$$

—give the coordinates of images a′, b′, and c′ in the second photograph with respect to a plumb-line coordinate system on which the second exposure station is taken as the origin. In Eqs. 83, Z^*_2 is not the elevation of the second exposure station, as usually designated in the preceding sections. It is really equal to $Z_1 - Z_2$ in the sense of the absolute flight heights. The three rectangular axes are parallel to the axes of the first plumb-line coordinate system. Thus, the vectors of ρ'_a, ρ'_b, ρ'_c, and ρ'_d will be as follows:

$$\rho'_a: \quad x'_a i + y'_a j + z'_a k \quad \text{(84a)}$$

$$\rho'_b: \quad x'_b i + y'_b j + z'_b k \quad \text{(84b)}$$

$$\rho'_c: \quad x'_c i + y'_c j + z'_c k \quad \text{(84c)}$$

and

$$\rho'_d: \quad x'_d i + y'_d j + z'_d k \quad \text{(84d)}$$

Taking scalar products of the vectors in Eqs. 84,

$$\rho'_a \rho'_d \cos \lambda'_a = x'_a x'_d + y'_a y'_d + z'_a z'_d \quad \text{(85a)}$$

$$\rho'_b \rho'_d \cos \lambda'_b = x'_b x'_d + y'_b y'_d + z'_b z'_d \quad \text{(85b)}$$

and

$$\rho'_c \rho'_d \cos \lambda'_c = x'_c x'_d + y'_c y'_d + z'_c z'_d \quad \text{(85c)}$$

in which λ'_a, λ'_b, and λ'_c, the corresponding apex angles with respect to the second photograph, are the known quantities. The coordinates of image d in the second photograph are computed by these three equations for x'_d, y'_d, and z'_d. Accordingly, the following equations give the direction cosines of line L_2D, Fig. 6, a line drawn from the second exposure station to the ground point D:

$$\cos \alpha_2 = \frac{x'_d}{\rho'_d} \quad \text{(86a)}$$

$$\cos \beta_2 = \frac{y'_d}{\rho'_d} \quad \text{(86b)}$$

and

$$\cos \gamma_2 = \frac{z'_d}{\rho'_d} \quad \text{(86c)}$$

By analytical geometry,

$$\frac{X_D}{\cos \alpha_1} = \frac{Y_D}{\cos \beta_1} = \frac{Z_1 - H_D}{\cos \gamma_1} \quad \text{(87a)}$$

and

$$\frac{X_D - X_2}{\cos \alpha_2} = \frac{Y_D - Y_2}{\cos \beta_2} = \frac{Z_1 - H_D - Z^*_2}{\cos \gamma_2} \ldots\ldots\ldots\ldots (87b)$$

Eqs. 87 give the coordinates of point D, with respect to the first plumb-line coordinate system. Since all the coordinates discussed are on the same coordinate system and are then known, it is a simple matter to calculate the geodetic position of one point with respect to the other.

SECTION 7. CONCLUSION

The comparison of the results obtained in this paper with the results obtained by Earl Church, Assoc. M. ASCE, from his space-geometry method[8] lends confidence to a conviction that the procedure adopted in this paper as demonstrated in Section 5 is absolutely correct. For example,

Method	Z, in feet	Tilt	Swing
Church	20,201	0° 59.7′	180° 29.0′
Chow	20,198	$\begin{Bmatrix} 0° 59.8' \\ 0° 59.6' \\ 1° 02.5' \end{Bmatrix}$	179° 39.0′

If only the altitude of the exposure station and the tilt of photograph are to be found, the following simplified procedure is recommended:

(1) Measure the image coordinates with respect to the photographic axes;

(2) From the coordinates of the images find ρ_a, ρ_b, and ρ_c, l_{ab}, l_{bc}, and l_{ca}, and $\cos \beta_{ab}$, $\cos \beta_{bc}$, and $\cos \beta_{ca}$;

(3) By the successive approximation method find ρ_A, ρ_B, and ρ_C;

(4) From ρ_A, ρ_B, ρ_C, and $\cos \tau_B$ (or $\cos \tau_C$ or $\cos \tau_A$) calculate the altitude of the exposure station;

(5) Using the altitude of the exposure station compute $\cos V_a$, $\cos V_b$, and $\cos V_c$, and then find ρ_v, x_{pv}, and y_{pv} from Eqs. 60; and

(6) Compute $\overline{\text{Ov}} = \sqrt{x^2_{pv} + y^2_{pv}}$, from which $\tan t$ is computed by Eq. 54.

This procedure involves only simple computations and affords very accurate results.

Acknowledgments.—Several methods of solving the problem outlined in this paper have appeared in the technical literature, some of which have already been cited herein; as, for example, the "space-geometry method"[2] and the "direction cosine solution"[4] devised by Professor Church. Solutions have also been offered by R. O. Anderson,[5] Assoc. M. ASCE, and P. H. Underwood,[6] M. ASCE. The writer is particularly indebted to Professor Church for his inspiration and instruction while this paper was in the course of preparation, and especially for supplying the data used in Section 5.

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PROPERTY SURVEYS MUST FIT THEIR TITLES

By William C. Wattles[1]

With Discussion by Messrs. Robert F. Borg, S. A. Bauer, Rupert F Carroll, Henry D. Southerland, Jr., and William C. Wattles.

Synopsis

The basic purpose of this paper is to emphasize the fact that land surveying is a specialized profession. The relation between a land title and a property survey is discussed, thus leading into a statement of steps preceding a field survey, a brief description of office and field relocation methods, the legal status and value of physical monuments, and the tracing of land titles from record.

Property boundary location is an art and profession as distinct and different from civil engineering as dentistry is distinct from medicine. The mere fact that the mechanics of surveying, in the use of transit, tape, and computing tables, is common to both is no criterion of identity. The fact that a person is versed in the tactics of surveying as taught in school or college does not warrant the assumption that he is a land surveyor any more than that he is a civil engineer.

Many people hold the idea that the surveying art is a mechanical adjunct to engineering procedure, rather than an informative or determinative function. This is particularly true in the case of property surveys. Often an owner will endeavor to ascertain certain desired relations between title and possession by all the available methods other than survey, even though such survey is the cheapest, quickest, and, frequently, the only method of solution.

That surveying, notably in measurements of land, is a specialized occupation is evidenced historically by early records in Egypt, Rome, and elsewhere. The surveys of the land holdings along the Nile and the special courses of education for surveys in Rome are cases in point. No land location project of modern times is originated or completed without an accurate boundary survey.

It is self-evident that the value of a property survey is correlated with, and dependent on, the ownership or title of the land under consideration.

Land ownership presupposes a valid claim of a proprietor in the physical possession, actual or potential, of the area in question. The claim is further

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Note.—Published in February, 1949, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Glendale, Calif.; formerly Chf. Engr., Title Ins. and Trust Co., Los Angeles, Calif.

supported by a written record, describing the boundaries of the land in such a manner that they may be recognized and, if necessary, be relocated in their original position. Specifically then, the physical location and the record description must be conformable and consistent to provide a correct title in the boundary. The accomplishment of this function is the province of the land surveyors.

The correct definition of boundaries of a parcel of land includes:

1. Examination of title ownership of record;
2. Claimant ownership of the ground area by occupation or other evidence;
3. Monumentation, both physical and as recorded;
4. Deeds, maps, descriptions, and other documentary evidence;
5. Adjoiner agreement or conflict;
6. Surveys of the instant and associated lands;
7. Abstracts of title or title insurance policies; and
8. Any other matters connected with the property in question.

No accurate survey of such land can be made without considering and weighing all the foregoing items. In addition, the surveyor must perform his work, both field and office, to mechanical and mathematical standards commensurate with the best practice. As a chain is no stronger than its weakest link, so is a land location survey no better than the error caused by omission or neglect of any of the factors entering into the determination.

The written record of a land holding is fixed and permanent; the physical location and monumentation are subject to change or destruction from many causes. In most cases, therefore, it becomes the duty of the surveyor to reestablish the record as nearly as possible, and to indicate its variance with the existing occupation. The legal questions of ownership or preference between positions of record, or physical claim, must be decided by the courts; it is the surveyor's province to point out the differences.

The location of a parcel having arbitrarily determined boundaries is comparatively simple, requiring only accurate dimensional treatment and a correctly established and proved tie point referenced to record. Whether the field work precedes or succeeds the description for record, the creating of one from the other is readily accomplished; and with reasonable care, the ground work will agree with the description. Slight variations discovered in relocation are nominally assignable to ordinary natural causes such as earth movement of various kinds or to the common personal errors of observation and measurement of the surveyor. The acceptance and use of monuments in such a case will generally fix them as paramount to the dimensional position of record; it may be presumed that a new description, with new dimensions, but reciting the monuments, will constitute a perpetuation of the original record, modified to represent the present facts.

Relocation surveys, or those made to establish boundaries of recorded existing parcels require the best skill, analytical faculty, and judgment of the land surveyor. He may take nothing for granted, but must verify every monument, mathematical computation, measurement, description, and legal interpretation. The final decision must be based on sound judgment of his analysis and a close adherence to the theory of majority probability. Inasmuch as the surveyor has no judicial authority, he must do his work and submit his findings in the manner best calculated to maintain and support his opinion of the location in question.

The first step in relocation is to gather all information relating to the land to be surveyed—the deed description, title policy, description, tract or subdivision maps of the property and vicinity, descriptions and maps of adjoining properties, street center-line survey notes, and location and ties of existing monuments in the area if such ties are available.

The data thus assembled are to be platted, the existing dimensions computed for agreement and closure, adjoiner boundaries related to the instant property, and a preliminary paper location determined.

The plat so prepared is taken into the field; for reference, a connected survey is made, the surveyor discovering and tying all monuments and other topographic features into a single net. Back in the office, the field work and the record are correlated, monuments and descriptions are interpreted in mutual relationship, and a new plat is prepared for the final showing of the survey.

The second trip to the field (assuming that all necessary and discoverable data have been previously obtained) will set the monuments of the relocation; and these of course will conform with the plat and will bear any authorized markings or structure regulated by law.

The relocation of corners of sections, or of fractional parts of sections, is governed by the method and procedure as defined by the Bureau of Land Management (formerly, the General Land Office), U. S. Department of the Interior.[2] Very little deviation from the rules can be permitted; yet close study of the manual is necessary to arrive at the precise variant applicable to the case in hand.

Argument that monuments govern regardless of description or record is pertinent only if the original survey is precedent to the record. If the ownership is established, as in many cases, by a description drawn from a plat, or by compilation from record sources, then the monuments must be assumed as confirmatory rather than determinative and subject to valid replacement in conformity with interpretation of the documentary positioning. If the contrary view is held, it can be maintained only by support of legal decrees, or through modification by agreement of adjoiners properly recorded.

Monuments discovered in a survey cannot be accepted in themselves, as definitive of the corners they purport to occupy. It is required in every instance to identify them by a conformal record dimensional relation with some other physical object, to prove reasonably (1) that they are in the position of the original corners; (2) that in the event they are found as replacements of originals, they have not been deliberately or accidentally misplaced; and (3) that all persons concerned with their establishment and use, consent, actually or impliedly, to the accuracy of position.

Maps of groups of parcels or lots, when recited as a part of the description, must be considered as a whole in the location of any part. A reference to a record in a document incorporates therein all the information and substance of such record applicable to the interpretation of the terms of such document. Consequently, the location of the corners of a lot in a block must be proportionately related to the corners of all other lots in the block. The surveyor must establish the block corners correctly, either from accepted street center lines, monuments shown on the map or existing physically, or from other sources. Having so fixed the exteriors, he sets his lot corners by the prorate method of measurement. Such a method is not rigid, however, because other control factors, for example, verifiable original lot corner positions, may force the abandonment of the prorate measurement. Here is where the evidence, theory, and judgment of the surveyor must be weighed and properly concluded.

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[2] "Manual of Surveying Instructions for the Survey of the Public Lands of the United States," U. S. Govt. Printing Office, Washington, D.C., 1947.

Regardless of the fact that the basis of land ownership is the possession, visibly, by structural objects or boundary monuments of the land itself, a great percentage of present day holding and transfer of property is maintained through the record only; and the ground location is not identified by monumentation until the owner wishes to avail himself of its actual use. It follows then that, in most cases, the surveyor must predicate his work first on the record, and, second, on the possession.

Whenever the surveyor is called upon to locate by title of record, disregarding possessory claim, his procedure is governed almost entirely by the proper interpretation of the terms of the deed description. Except for specific physical identification of the tie and reference calls of the deed, the field examination reduces to a mere correlation of available information for purposes of analysis. In cases of this kind, the correct placement of a boundary from the record is superior to any occupational claim.

As far as the surveyor is concerned, title is the relation between the record claim of the proprietor and the actual ground position of such claim. It is his business to make that relation exact and correct. If, due to error in the record or holding, such conditions cannot be met under certain circumstances, he must inform his client of the facts, and withhold final location until proper adjustment is made, either by agreement or court action.

The questions of finance, relative value of property to survey, ethical distinctions, or efficiency of mechanical processes involved, are not considered in this paper. They have no bearing on the premise or conclusion that no survey location of land areas is complete unless it corresponds and coordinates with the record title or a proper modification of that title.

SUMMARY

A broad view of the field of property boundary surveys and an analysis of the factors governing correct location of such boundaries indicate the following conditions necessary to successful operation:

(1) Skilled, experienced, and able professional men, trained in field, office, and title practices;

(2) Proper equipment for all necessary work;

(3) Complete information and data of record, together with any other available documentary material affecting the project at issue;

(4) Sufficient preliminary field work to supply corroborative physical information in connection with the project;

(5) Compilation, and sound, intelligent analysis of the title and legal requirements necessary to fix a proper ground location of title boundaries, or to provide a basis of adjustment through the medium of record agreement or court action; and

(6) Physical monumentation of boundaries when finally determined, together with recordation of any necessary maps and documents which will correctly delineate and describe such final boundaries—the physical location and the record title are thereby brought into agreement.

DISCUSSION

ROBERT F. BORG,[3] JUN. ASCE.—The stimulating remarks in this paper are welcomed for their profitable exposition of the several features useful to those engaged in property surveys. It is to be regretted that the paper was kept so brief as to permit only generalizations with regard to some of the problems of descriptions which are not true under all circumstances.

The primary purpose of property surveys is to insure that the owner of a piece of land, and the entire world, know the exact location and extent of his property, chiefly as embodied in deeds. When disputes over descriptions arise, if they cannot be settled by qualified land surveyors, they are resolved by courts of law. Since the law is the final arbitrator in these instances, its rules, whether or not based on logic and reason, must temper the thinking of surveyors.

The author states that if the ownership is established (as in many cases) by a description drawn from a plat, or by compilation from record sources, the monuments must be assumed to be confirmatory and subject to replacement. This categorical statement can be both misleading and incorrect. If the description, as embodied in the deed, refers to monuments not in existence at that time, but which will be erected at a later period, such monuments, once erected, will bind the parties should there be any discrepancy between the deed description and the actual monument locations (Lerned versus Morrell, 2 N.H. 197; Emery versus Fowler, 38 Me. 99).

Naturally, where there are no references to any monuments in the property description in the deed, it is assumed that the taker under the deed did not intend to be bound by any incorrectly located monuments that may have existed (Murdock versus Chapman, 9 Gray (Mass) 156; White versus Loring, 93 U.S. 514). The courts have uniformly upheld this latter contention, and the author's statement is correct under those circumstances.

Again, the author states that, regardless of description or record, monuments govern only if the original survey is precedent to the record. This statement must also be considered true only for certain factual situations. If, in their deed description, the parties do not in any way refer to monuments then in existence (or later to be erected) and if, subsequent to the date of record, they either establish monuments or run a line, holding that line and occupying the land defined by it, for a long time, then the line or monument thus erected will bind the parties where the deed description is ambiguous (Baldwin versus Brown, 16 N.Y. 359; Breaker versus Woolsey, 149 Mich. 86). On the other hand, where the deed description is certain and unambiguous, as the author indicates, any line or monument subsequently established is not binding on the parties (Iverson versus Swan, 169 Mass. 582; Hall versus Eaton, 139 Mass. 217), except by adverse possession (under common law, this would be open and notorious possession for more than 20 years) (Burr versus Smith, 152 Ind. 469; Roberts versus Birks, 223 Ill. 291).

In the last analysis, it is the true intent of the parties that will govern in any particular case. Thus, where, through some misfortune, the survey description made for a deed is at variance with the actual locations of physical features or monuments, the court will weigh all the facts and circumstances in an endeavor to decide the case in a manner that will best express the probable intent of the parties who used the property survey (White versus Loring, 93 U.S. 514, 524).[4]

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[3] Estimator, Psaty & Fuhrman, Inc., Contractors and Engrs., New York, N. Y.

The author is to be congratulated for bringing to the attention of the profession the need for thoughtful and accurate property surveys.

S. A. Bauer,[5] Assoc. M. ASCE.—A scholarly exposition of a basic principle of property surveying is presented by Mr. Wattles. The paper is particularly welcome and important to the practicing surveying profession in that it stresses the true essence of property surveying—title and law, research and logic—and relegates to a proper position of secondary importance the mechanics of the profession, the operation of equipment, and techniques. After many centuries, the profession of property surveying is having a renaissance of spirit. Recent years have seen, in the professional literature of the United States, more professional cooperation among property surveyors, and more original thought applied to the problems of the profession, than has any prior period.

The task of the property surveyor has been defined as the engineering interpretation, definition, and representation of legal documents of land ownership, a task in which the tools of the engineer, his mathematical knowledge, and his capacity for analytical and logical thinking are applied to the problem of reducing to tangible and explicit form the legal records of land titles.

The failure on the part of the lay public to appreciate the true function and importance of the property surveyor has long concerned the profession. Far greater damage to the profession, however, results from the failure of the engineering profession itself to appreciate the true nature of the responsibilities of the property surveyor. The causes of those failures are many. In a large measure, years of apathy within the profession itself form one cause. Another is the almost total lack of property surveying instruction in engineering colleges. It is the present problem of the profession to educate its own members to their responsibilities and importance, and then the engineering profession and the lay public.

Mr. Wattles' main point, the primary importance of the legal title, is recognized, of course, as basic in all accepted current thought on the subject. The "Technical Standards for Property Surveys," [6] compiled and adopted by the American Congress on Surveying and Mapping in 1946, begins with the statement:

> "Each parcel of land whose boundaries are surveyed by a licensed surveyor, should be made conformable with the record title boundaries of such land."

Nevertheless, this statement still comes as a shock to many engineers and some property surveyors, to whom the old rule, "occupation controls," has almost Biblical significance. There are many reasons for the development of the "occupation controls" rule. Very important among these reasons is the fact that, since the beginning of the twentieth century, much of the responsibility of land description has been left to nontechnical "scriveners." These men are responsible for the prevalence of such forms of descriptions as:

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[4] "A Treatise on the Law of Property," by William F. Walsh, Baker, Voorhis and Co., New York, N. Y., 2d Ed., 1937, p. 742.

[5] Cleveland, Ohio.

[6] "Technical Standards for Property Surveys," Am. Cong. on Surveying and Mapping, Washington, D. C., 1946.

> "My house and the land upon which it stands."

or

> "Bounded on the west by the State Road; on the north by the land now or formerly owned or claimed by Mary McNamara; on the east by land now or formerly owned by Agnes Beebe; and on the south by the land of The Harbor Building Company."

Since such descriptions are entirely lacking in specific definition, the surveyor's only recourse is the use of occupation.

An additional factor in the development of the "occupation controls" rule is the fact that the occupation survey is so often the easiest solution, and is welcomed by the poorly qualified surveyor, to whom the law and titles are likely to be something unknown and mysterious. It is also often the solution preferred by the client, whether proper or not, since it conforms to his own assumptions as to his ownership.

The erroneous idea that adverse possession over the statutory period automatically grants title also adds to the misconception. Modern title practice, dealing as it does with the recorded document, rather than the property itself, must of necessity pin its responsibility to the written deed. As a result, the interpretation of the deed by the surveyor becomes an increasingly meticulous process, in which the title becomes the framework on which the surveyor hangs his findings of occupation, possession, adjoiners, etc. Modern high land values, relatively small parcels of land, and extensive improvements of land, particularly in urban areas, tend to make the deed and the precise wording of the description of greater significance than in the past. It is inevitable that, in a growing community, the title as recorded must be recognized as the controlling factor in land ownership.

The property survey, then, becomes an exposition of all pertinent facts relative to the record title (those in agreement as well as those not) rather than a determination of ownership. The surveyor becomes a fact finder, rather than a judge of ownership. His recommendations and opinions, nevertheless, have extrajudicial value in direct ratio to his reputation for thoroughness, knowledge, and judgment.

Mr. Wattles is to be commended on a particularly clear statement of a too little recognized fact. It is to be hoped that his paper will be widely read and digested by both practitioners of property surveying and by those civil engineers who, in the past, have not always viewed the practice of property surveying with sufficient seriousness.

RUPERT F. CARROLL,[7] ASSOC. M. ASCE.—Land surveying is a highly specialized profession. That point is frequently overlooked by men in other branches of engineering who regard surveying as merely the field work necessary to locate structures and provide information from which cost estimates may be prepared. The paper covers this point adequately, and it also lists factors which the engineer must consider in relocating property to conform to its legal and record title. There are special considerations, however, that must be taken into account when applying these factors to survey problems in a given state.

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[7] Office Civ. Engr., West Texas Div., Humble Oil & Refining Co., Midland, Tex.

For example, there are no United States public land surveys in Texas. The patents or record titles to most Texas lands are based on original field notes by surveyors from several nations; surveys were made piecemeal and in some instances without enough data on adjoining or conflicting surveys.

Much has been published concerning the special conditions affecting the position of Texas in the current Tidelands controversy due to the state's retention of its public lands. Texas surveyors know that this same retention created problems which are not generally faced by members of their profession practicing in other states. Most persons are vaguely aware that Texas was an independent nation for 10 years, but they do not realize that a colonization race by at least three other nations brought on the war that made Texas a free republic. Also, because of boundary disputes, this colonization did not cease with the independence of Texas in 1836, but was actually accelerated in certain sections, notably the vast south Texas region between the Nueces River and the Rio Grande.

Texas was originally claimed by the United States as a part of the Louisiana Purchase, but this claim was dropped in 1821. It was also claimed by Spain as a part of Mexico and was still considered as such by Mexico after that nation won its independence from Spain. Mexico opened the territory of Texas to foreign colonization in the period from 1821 to 1835 after the boundaries had been closed for more than 200 years.

American colonists moved into east Texas from Louisiana and Arkansas and pushed southward until they came into contact with the Mexican colonists moving northward from the Rio Grande. Disputes arising from this meeting precipitated a war in which the Texans were forced to fight Mexico without organized aid from the United States government. The Texans, therefore, considered that their fight was for independence from Mexico, and having won it, they established the Republic of Texas. Mexico was forced to recognize this new nation but considered the Nueces River as the proper international boundary, whereas the Texans claimed their sovereignty extended to the Rio Grande. During the 10 years of the Texas Republic, both Texas and Mexico colonized the disputed territory extensively. Prior to 1836 it had been populated by a few Mexicans holding large tracts of land by grants from the Spanish crown and the Mexican government.

Texas was annexed to the United States by a treaty in 1845 which specified, among other things, that the new State of Texas would retain title to all public lands within the state. The United States thus became a party to the boundary dispute which precipitated the Mexican War in 1846. This war ended with the Treaty of Guadalupe Hidalgo in 1848. The treaty established the Rio Grande as the international boundary, but recognized the validity of Spanish and Mexican land titles granted prior to November 13, 1835. Incidentally, this treaty also included the cession to the United States of the territory now occupied by New Mexico, Arizona, Nevada, and California. The Texas Legislature passed the Confirmation Act in 1852 which recognized Spanish and Mexican land titles granted prior to November 13, 1835, but invalidated the titles to lands given by Mexico to its citizens as a reward for military service.

Patents to the valid grants were issued by the State of Texas based on the original Spanish or Mexican field notes which contained a minimum of description, and which in many cases were tied to some of the canceled grants. The recognized titles included several huge tracts, the largest of which was the San Juan de Carricitos grant in Kenedy and Willacy counties, covering approximately 650,000 acres. These grants were usually described in the field notes merely by the four corners with sometimes no course and distance calls between. The field notes were in Spanish and the distance calls, if included, were in cordeles, the cordel being 50 varas or approximately 139 ft. The cordel usually employed was of fiber rope and the length was subject to variation by moisture.

There were several ceremonies connected with the Mexican and Spanish procedure of placing a citizen in possession of his land, but the only factor of surveying importance was the requirement in many instances that the landowner set up permanent markers of stone and mortar at the survey corners. These had to be set within a given time or the landowner would forfeit all right to the land. This feature probably accounts for the respect with which their corners were preserved by the Mexican and Spanish colonists. In these early surveys, great importance was paid to natural objects which were used for corners wherever possible. These grants were almost universally excessive since acreage at that time was of small importance.

The General Land Office of the Republic of Texas was established in 1836, and one of the first acts of the first land commissioner was to gather all the available title papers and records from the various colony headquarters and from the Mexican land offices. Many papers were never collected, but the documents obtained are now preserved in the Spanish archives in the Texas General Land Office.

Colonists who had not received land titles from the Mexican government were given lands by the Republic of Texas depending on the marital status of the immigrant and the time of his settlement in Texas. Lands were given for military service and to the heirs of men who had died in the Texas war for independence.

In the early days of the republic, the General Land Office had no accurate maps and the surveyors of the various land districts were required to furnish maps showing the valid surveys in their territory. Today these maps seem at least inaccurate, but when the low fees for surveys at that time are considered, along with the constant danger from marauding Indians and the distances involved, respect for the early surveyor is increased.

When Texas became a state in 1845, the public domain amounted to about 172,700,000 acres, which the United States government could have acquired by the payment of approximately $13,000,000 in debts accumulated by the Republic of Texas. However, Congress allowed the Texans to retain their land and pay their own debt—a very fortunate decision for the people of Texas.

As stated, Texas used the land to pay colonists and soldiers; after statehood, other land was granted to encourage colonization. Millions of acres were given away for the building of railroads, a state capitol building in Austin, the University of Texas at Austin, and many other public improvements. The remainder was finally set aside for the Public Free School Fund. Several thousand acres of unsold land still are retained in this fund.

Many of the land grants were made with the provision that the grantee would survey the land and submit the field notes to the General Land Office. Most of these surveys made in south and west Texas were intentionally excessive, but the fact was not shown in the record field notes which were plotted on maps in the General Land Office. Some of the surveys were carefully made and permanent markers were established. In other cases, it is reasonably certain that the location field notes were written from the land office maps by a surveyor who was never on the ground. Other surveyors who may have been on the ground left no trace of their footsteps.

As the land was settled, it was resurveyed in many instances and these corners were well marked. Where the original survey could be found, the resurvey was called to fit the same corners and natural objects regardless of excess. Sometimes this excess was reported to the land office by the corrected field notes showing the true dimensions.

Thus, a situation is created wherein a citizen has more acreage in his survey than was granted to him by the state, but his title to the entire survey including excess is clear as long as the original survey can be traced. The courts have held that it is not what a surveyor intended to do, it is what he actually did that will govern. In case of mineral lands or other valuable property, it is generally true that the landowner purchases the excess acreage from the state in order to protect himself should the original corners ever be lost.

The other situation is the so-called "office" survey wherein the original surveyor was not on the ground, or wherein he left no trace of his work. In many cases this survey is built from an older survey, run on the ground and well monumented with excessive dimensions. The office survey calls for adjoinder to the older survey, but the very nature of the calls often shows that the surveyor making them had no knowledge of the survey to which he was building his work. Most of the "office" surveys were resurveyed on the ground with the same excess found in the older survey to which it was called to adjoin. This construction was not contested so long as the land had little value; but court decisions involving the title to mineral lands have held that where a surveyor's work cannot be found, it must be constructed according to the called course and distance from the beginning corner. Even the adjoinder calls were not valid if it could be reasonably proved that the surveyor making such calls did not know where the corners were that he called to adjoin. Under these circumstances, the customary excess cannot be placed in the system of surveys. If it exists on the ground as occupied, it must be considered as vacant or unsurveyed land.

By act of the Texas Legislature in 1939, a landowner may purchase from the state such vacant land found within his enclosure as a "good faith" claimant. If the land is more than 5 miles from a commercial gas or oil well, the State of Texas retains a one-sixteenth royalty interest. If it is less than 5 miles from such a well, the interest retained is one eighth of the oil and gas produced. Other provisions in the law dictate the procedure to be followed in case the vacancy is found and applied for by a so-called "vacancy hunter" before an application to purchase is filed by the "good faith" claimant. These provisions make it very advantageous for a landowner to find and purchase such unsurveyed land before an application is filed by an outsider. Therefore, a surveyor should be very careful in his work of retracing old surveys, and if he cannot definitely prove the position of the title survey, he should advise his client to purchase the doubtful acreage as unsurveyed land, rather than risk a filing by a vacancy hunter.

Surveying of subdivided land in Texas is little different from similar work in other states. However, land holdings in south and west Texas are so large that a property survey usually represents the location of blocks or even of several systems of surveys.

For this reason, a surveyor cannot begin a job of any size without considerable preliminary preparation. He should first consult the General Land Office records, paying particular attention to the following: (1) Patents to the surveys in question; (2) original field notes and all resurvey field notes on the surveys to be located and surrounding surveys; (3) old district and county maps and surveyors' sketches; (4) General Land Office correspondence files; and (5) deed record data from county records or private files.

He should also consult the records of the county surveyor; they sometimes reveal information not available in the General Land Office. If possible, aerial photographs of the area should be obtained as well as good maps by the United States Geological Survey or the United States Coast and Geodetic Survey.

Careful plotting of the patent field notes in chronological order will often reveal discrepancies before the ground work is begun. These should be plotted on tracing paper or cloth on the same scale as the aerial pictures if such are available. By superimposing the field notes on the photographs a good idea of the original locations may be obtained.

In the ground survey, it often pays to know something of the characteristics of the original surveyor. It is useless to look for monuments of some surveyors because it has been found from past experience that either they were not on the ground, or (if they were) they left no visible trace of their work. Some surveyors had definite habits concerning "excess" in their work; others were very accurate and careful. Most early surveyors attempted to run on true bearings, but some followed the magnetic course. The field notes usually show the variation or magnetic declination used in the survey which, for work in south Texas and west Texas, usually was the variation in San Antonio at the time of the survey. The extent of the San Antonio or Bexar Land District was so large that this procedure resulted in bearing errors of as much as two degrees in some parts of west Texas.

To summarize, the surveyor should be careful in his report to record only what he found or did not find on the ground, and leave the decisions on points of fact and law to the courts.

HENRY D. SOUTHERLAND, JR.,[8] **ASSOC. M. ASCE.**—The author is to be commended for his orderly presentation of the subject of real property title surveys to the engineering profession for discussion. The paper presents in a logical manner some of the basic principles of land title surveys, particularly as instructions for the general guidance of surveyors of land for title purposes.

The paper is somewhat provocative, however, in that it does not cover the solution to problems concerning real property surveys which do not fit their titles. The contents of the paper appear to deal with "making property surveys fit their titles" but from its title one would expect to find the reasons why, and conclusive proof that, "property surveys must fit their titles." Frequently the careful instructions of the author have not been followed and there are many title descriptions that do not adequately delineate the boundaries of the land intended to be described. It goes almost without saying that the property surveys "should" fit their titles, but by what sanctions "must" property surveys fit their titles? What is the consequence of a failure to conform with the author's instructions and what steps should one take to remedy the problems arising from inaccurate property descriptions? Reference is made to supplementary sources[9,10,11] as complete answers to these questions hardly could be given in a short paper.

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[8] Birmingham, Ala.

It is to be regretted that the author has made several broad generalizations in the interests of brevity. Copious notes and citations would allow those persons who are interested to delve into the source material for a better understanding of the author's intended meaning and for additional information on the subject. For example, it is not probable that the correct definition of the boundaries of a parcel of land should be so broad as to include "any other matters connected with the property in question." The author probably had reference to easements, encroachments, adverse possession, and any other matters connected with or affecting the legal title or the description of the boundaries, monuments, or area of the parcel of land in question, or of contiguous parcels. Although an accurate survey, by chance, may "be made without considering and weighing all" the items enumerated by the author, one cannot be assured of an accurate survey without considering and weighing these factors to the extent necessary, including "any other matters connected with the property in question" which may affect the accuracy of the survey.

The author refers to the relations between title and possession—that the surveyor must predicate his work first on the record and then on possession, and that the survey may be located by title of record disregarding the possessory claim just as if the possession were always a thing apart, distinct, and different from the title of record. The record title may be the evidence which delineates the property possessed. The color of title of record usually describes the land that goes along with the house occupied or the land used (Hicks versus Coleman, 25 Calif. 122, 85 Am. Dec. 103 (1864)).

The extent to which property surveys must fit their titles varies with the manner in which the result of such a survey is rendered and the purposes for which it is used. A verbal written description is the common method of using the results of a property survey and the penalty for errors therein varies with the purposes for which such title description is to be used and the parties involved. Property descriptions may be used in transfers by conveyance, transfers by will, in mortgages, in taxation, and for other purposes. The particular questions of controversy may involve adverse possession, statute of limitations, tax rolls, notice of sale for defaulted taxes, notice to third parties by recording, or other questions. The effects may vary depending upon the jurisdiction in which the land is located, upon local statutes, and upon the parties involved. For example, misdescription may not be fatal in showing

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[9] "Treatise on the Law of Surveying and Boundaries," by F. E. Clark, Bobbs-Merrill Co., New York, N. Y., 1939.

[10] "Boundary Problems and Development Problems," by D. G. W. Ricketts, Pelican Pub. Co., New Orleans, La., 1939.

[11] "The Surveyor and His Legal Equipment," by A. H. Holt, *Transactions*, ASCE., Vol. 99, 1934, p. 1155.

valid taxation for 10 years and thus title by adverse possession (McCraw versus Lindsey, 209 Ala. 314, 95 So. 898 (1923)), but misdescription in the notice of sale for taxes would be fatal in a purported transfer of title at a tax sale (Craig versus Swader, 225 Ala. 366, 143 So. 553 (1932)). A description of real property for ad valorem tax purposes by reference to lot and block numbers of an unrecorded plat would be insufficient to give the court jurisdiction in rem in a tax foreclosure proceeding (Napier versus Runkel, 9 Wash. 2d 246, 114 P 2d 534, 137 ALR 175, 184 (1941)).

The author is very casual in dealing with the most important problem of the relative value of the property to the cost of making the survey by the simple statement that it is not considered in this paper; yet, the surveyor "must perform commensurate with the best practice." An "accurate boundary survey" readily may exceed the value of the property. One of the first considerations in any engineering survey should be the limits of accuracy and the permissible error of closure, which may be quite different in the survey of valuable city properties from those necessary in property surveys of inexpensive land located in rural areas.

One method by which the detailed work of the engineering profession is presented to the public is through land surveyors in the form of real property title descriptions. It would seem that the profession should make a stringent effort to have such work impeccable in order that the profession may build up esteem with the public through property owners, real estate agents, insurance agents, mortgage firms, taxing officials, courts, lawyers, and others who frequently come in contact with these detailed engineering descriptions which are filed as a matter of public record. Other detailed engineering work normally is handled by a restricted group of engineers, architects, contractors, or officials and does not come into contact with the general public except in the form of a completed project. The extent to which boundary disputes and inaccurate title descriptions still exist is an indication that such problems should receive greater attention. The author's paper should assist in accomplishing this objective.

WILLIAM C. WATTLES.[12]—Readers and discussers of this paper should realize that any rational enlargement of the subject matter, beyond the basic principles and generalities given, would produce a volume far beyond any reasonable scope. An example of this scope is afforded by the valuable work on this subject, presented by the late Ray H. Skelton, M. ASCE, in 1930.[13]

Each survey of land boundaries establishing the location of record title and relating the same to physical conditions presents a particular problem, different in most of its aspects from any other, except as to basic principles. These differences are responsible for many of the variances in legal decrees given in superficially similar cases.

Mr. Borg's claim that certain statements in the paper may be misleading, quoting legal citations to support his discussion, is not proved. The paper, taken as a whole, indicates the uncertainties present in any survey, and cautions against the flat acceptance of any principle by itself, but requires its analytical association with others. Item 8 ("Any other matters connected with the property in question") in the list defining (determining) boundaries, includes examination of any citations covering matters similar to the problem at hand.

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[12] Glendale, Calif.

[13] "The Legal Elements of Boundaries and Adjacent Properties," by Ray H. Skelton, Bobbs-Merrill Co., Indianapolis, Ind., 1930.

The question of reliability or superiority of physical monuments over the description calls of record is a matter of investigation and analysis. If the surveyor is making a location from record, it is a general rule to give secondary value to physical monuments of whatever kind, as against the calls and nonphysical legal monuments recited in the description. If the description calls for physical monuments, or if those existing on the ground can be sufficiently identified in position with the record, then the monuments will have superiority. A subsequent court decision, adverse to this rule, should be construed as a modification because of other factors in the case, and not as a failure of the rule.

Mr. Borg is correct in stating that the intent of the parties will finally govern, but it is the court, not the surveyor, who may assume an intent beyond that specifically expressed in the record.

Mr. Bauer's discussion is an excellent and complete summary of the necessary coordination between survey and title. Also, he points out the methods of attaining such coordination through education of the surveyor and layman to recognize facts as opposed to assumptions.

Mr. Carroll is to be complimented on his discussion of survey difficulties in Texas; it is a valuable addition to survey information. His statement regarding a knowledge of the practice of previous surveyors is very pertinent and applies elsewhere as well as in Texas. Often such knowledge marks the difference between correct and incorrect location.

It is impossible to answer fully Mr. Southerland's implied questions within the scope of this discussion, for he approaches the realm of original error in record titles and the corrective procedure thereof; this is not within the authority of the surveyor nor the province of the title location survey.

It is perfectly true that many descriptions are not "locatable" from their own statements or recitals and that such descriptions, if in a conveyance of any kind, cannot pass a perfect title; therefore, any survey under such conditions cannot fit title, but must be considered in the light of a fact-finding procedure for purposes of adjustment by legal methods.

It is recognized that surveys are made for many purposes other than property boundary relocation. In such cases the matter of relative values and accuracies may enter. A property survey in accord with the title must be as particular and exact as such title can be determined; any departure, such as assumption of possession boundaries and agreed lines not confirmed by record, converts the title location survey to another form.

The purpose of the paper was to present a series of principles of property boundary location founded on the record title as the best and most consistent base of operation. These principles must be associated or modified as need be, in any particular description or case. In the event of an inability to establish a correct technical position, because of an error or an insufficiency in the record, the problem resolves into a pure location for purposes of adjustment by legal methods.

Matters of cost, relative values of land and survey expense, mechanical technique, allowable errors, and the like, have no part in the analysis of property survey conformable with title.

The writer wishes to thank the discussers for their timely, pertinent, and friendly consideration of the paper.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 2400

EFFECT OF DRIVING PILES INTO SOFT CLAY

BY A. E. CUMMINGS,[1] M. ASCE, G. O. KERKHOFF,[2] AND R. B. PECK,[3] ASSOC. M. ASCE

WITH DISCUSSION BY MESSRS. LEONARDO ZEEVAERT; JACOB FELD; JOSEPH C. FEGAN; GREGORY P. TSCHEBOTARIOFF; PHILIP C. RUTLEDGE; A. W. SKEMPTON; JAMES R. SCHUYLER; B. A. KANTEY; D. P. KRYNINE; ROBERT F. LEGGET; STUART B. AVERY, JR., AND STANLEY D. WILSON; GEORGE L. FREEMAN AND JAMES D. PARSONS; L. LEROY CRANDALL; W. S. HOUSEL; AND A. E. CUMMINGS, G. O. KERKHOFF, AND R. B. PECK.

SYNOPSIS

This paper describes the results of field and laboratory tests made to determine the extent of the disturbance produced by driving piles into a soft, glacial, lake clay deposit in Detroit, Mich. The results indicate that the natural structure of the clay was appreciably disturbed close to the piles, but that the amount of disturbance did not approach that of complete remolding in the laboratory. At a distance of about two times the diameter of the pile, the clay within a pile group was disturbed only a negligible amount. The process of consolidation initiated by the disturbance was not of a type that would cause settlement of the ground surface.

INTRODUCTION

Publication in 1932[4] of the results of unconfined and confined compression tests on undisturbed and completely remolded samples of plastic clays from the Laurentian Valley, Canada, and from the subsoil of Boston, Mass., brought to the attention of engineers for the first time the great increase in compressibility which some clays undergo when disturbed. As a consequence of this discovery, it was pointed out that driving piles into soft clay deposits might cause sufficient disturbance to produce appreciable settlement of the surface of the deposits merely under the weight of the soil itself, and that the presence of the piles might prove detrimental rather than beneficial. On the basis of the informa-

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NOTE.—Published in December, 1948, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

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[4] "The Structure of Clay and Its Importance in Foundation Engineering," by A. Casagrande, *Journal*, Boston Soc. of Civ. Engrs., April, 1932.

tion available at that time, it was suggested that the clay immediately surrounding the pile would be completely remolded to a distance of one half of the diameter of the pile, and that to a distance of one and one-half diameters it would be sufficiently disturbed to cause a rather large increase in its compressibility. On the assumption that the presence of the piles had no influence on the settlement of the surrounding clay, other than to produce the disturbance, it was computed that the settlements of certain buildings in Boston might actually be twice as great if they were founded on friction piles than if they were supported directly by the clay.

Although A. Casagrande, M. ASCE, cautioned that, in order to get conclusive data concerning the remolding effect of pile driving, it would be necessary to conduct elaborate field investigations, the tentative conclusions which he drew have been widely accepted. Some engineers even believe that pile driving in soft clays is always detrimental and should be avoided.

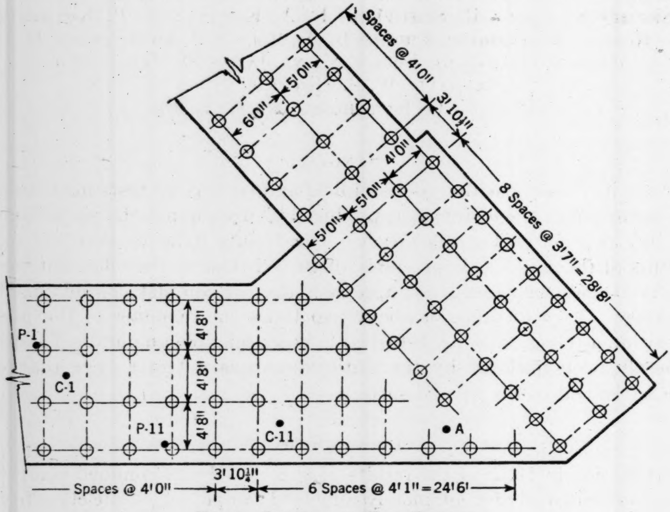

FIG. 1.—LOCATION OF TEST BORINGS

In 1944 an opportunity presented itself to obtain quantitative information concerning the remolding effect of pile driving when the Detroit Industrial Expressway was under construction. At the site of the grade separation between the expressway and the main line of the Michigan Central Railroad, the soil for a depth of about 80 ft is the typical soft blue glacial lake clay deposit found throughout the River Rouge (Mich.) area. It was decided to establish the abutments and center pier of the structure on cast-in-place concrete piles driven through the soft clay to bearing on the hardpan. The diameter of the piles increased from 10 in. at the point to 16 in. at the butt, the footing layout calling for piles at approximately 4½-ft centers. At this spacing, the ground surface between the piles heaved about 2 ft during the driving operations.

Before construction was started, the natural water contents and the stress-strain characteristics of the clay were determined by laboratory tests on soil samples obtained from a number of borings. After the pile driving was completed, additional borings were made to determine the values of these same properties for some samples taken close to the piles and for other samples taken within the pile group at the greatest possible distance from the surrounding piles. Two pairs of borings were made, one pair about 3 weeks after the piles were driven, and the other about 11 months after pile driving. For convenience, the first pair will be referred to as the 1-month borings and the second as the 11-month borings. One boring of each pair was made adjacent to a pile and the other in the center of a group. The location of one of the original borings and of the four additional ones with respect to the piles is shown in Fig. 1. The original boring is designated by the symbol, A. The 1-month and 11-month borings adjacent to a pile are designated P-1 and P-11, respectively. Those in the center of a group of surrounding piles are denoted as C-1 and C-11, respectively. The 1-month borings were made before the pile cap was poured; the 11-month borings were made through sleeves cast in the concrete pile cap.

GENERAL SOIL CONDITIONS

A composite vertical section through the piles and the subsoil is shown in Fig. 2, which is based on the results of six borings made at various locations

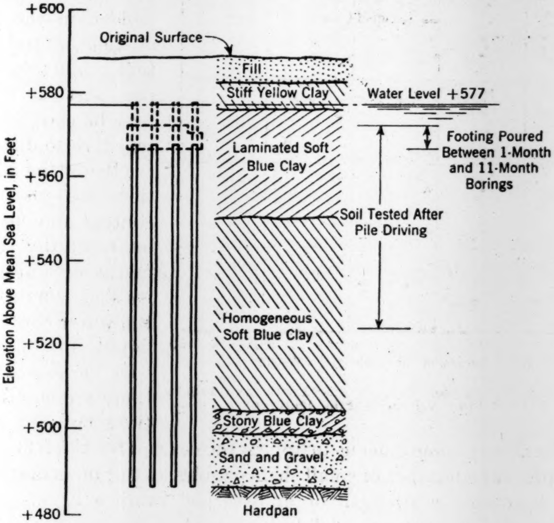
+600
+580
+560
+540
+520
+500
+480
Elevation Above Mean Sea Level, in Feet
Original Surface
Fill
Stiff Yellow Clay
Water Level +577
Laminated Soft Blue Clay
Footing Poured Between 1-Month and 11-Month Borings
Soil Tested After Pile Driving
Homogeneous Soft Blue Clay
Stony Blue Clay
Sand and Gravel
Hardpan

FIG. 2.—COMPOSITE VERTICAL SECTION OF SOIL IN PILE AREA

on the site before any piles were driven. These borings indicated that the soil was essentially uniform in its properties in horizontal directions over the entire area to be occupied by the structure.

The soft clay is divided at an elevation of approximately 550 ft into upper and lower regions with rather different characteristics. This fact is shown in Fig. 3, in which the variation of the natural water content, liquid limit, and plastic limit is plotted as a function of depth. Above the elevation of 550 ft the soft clay has a laminated structure consisting of alternating layers of silty clay and highly plastic clay, commonly only a few hundredths of an inch in thickness. The natural water content of the laminated clay is extremely variable and similar variation occurs in the values of the liquid and plastic limits. In their undisturbed state, the samples had an unconfined compressive strength approximately eight times that of the same samples when remolded in the laboratory at their natural water content. Therefore, the upper part of the deposit may be classified as a clay sensitive to disturbance.

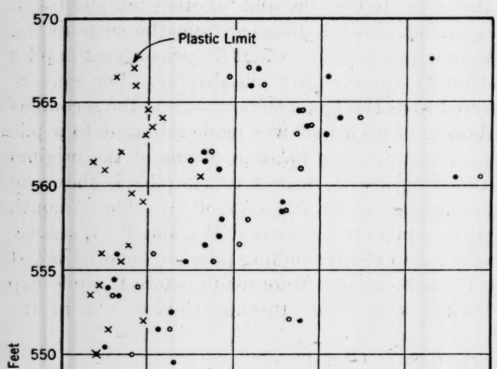

Fig. 3.—Natural Water Content, Plastic Limit, and Liquid Limit at Boring A Before Pile Driving

Below the 550-ft elevation the natural water content and the Atterberg limits of the clay are remarkably uniform. There is a slight tendency for both the water content and the liquid limit to increase with increasing depth, but there are no erratic variations between any of the values. The clay is homogeneous in appearance except for the fact that it contains an appreciable number of small shale pebbles up to $\frac{1}{4}$ in. in diameter. The unconfined compressive strength of undisturbed samples is only about twice that of the material when completely remolded at the same water content. Therefore, the clay in the lower part of the deposit may be classified as considerably less sensitive to disturbance.

The nature of the disturbance due to pile driving in the stratified clays in this region is illustrated in Fig. 4. After a temporary pile trestle had served its purpose, the clay surrounding the piles was removed by excavation. Fig. 4 shows the distortion of the clay surrounding one of the piles at a depth of about 12 ft below the ground surface.

METHOD OF SAMPLING AND TESTING

All the samples were taken in steel tubing having a 2-in. external diameter and a wall thickness of $\frac{1}{16}$ in. The tubes were 2½ ft long and beveled to a sharp cutting edge. Samples were taken continuously through the clay layer using casing whenever necessary. Usually the casing was driven after two tube samples had been recovered. The tubing was forced into the ground under static pressure and was not driven.

Boring A was made from the original ground surface before the piles were driven. The samples were practically undistorted (Fig. 5). On the basis of this evidence, it was considered satisfactory to proceed with the investigation by using 2-in. diameter samples.

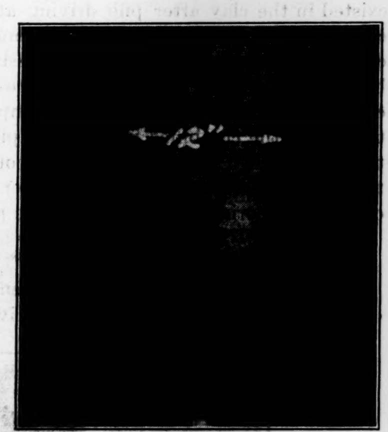

FIG. 4.—CLAY DISTORTION AROUND PILE OF TEMPORARY TRESTLE

Each tube was sealed in the field and shipped to the laboratory, where it was cut into 6-in. sections. The cylinders of soil were then removed and tested, each specimen being subjected first to an unconfined compression test in its intact state. It was then remolded to form another compression specimen and tested once more. A large part of the remolded sample was then used for the determination of the natural water content; the balance was used for determining the Atterberg limits. In the upper, laminated part of the soil deposit, a thin vertical section was cut from the undisturbed sample immediately

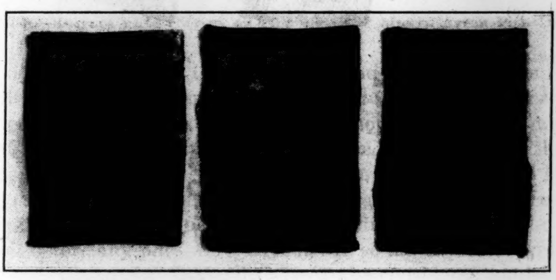

FIG. 5.—SAMPLES FROM BORING A BEFORE PILE DRIVING

after the first compression test. This section was dried slowly in order to disclose the stratification, and later photographed.

Boring P-1 was made as close to a pile as possible. The pile itself was enclosed within a corrugated metal shell consisting of sections 12 ft long. Inasmuch as the bumping of the casing over the corrugations could be detected during the boring operations, it is certain that the samples from this boring were taken within 3 in. of the pile. Because of the high stresses that still existed in the clay after pile driving at the time this boring was made, the clay squeezed up into the casing as much as 1 ft before some of the samples could be obtained. This is an excessive amount of squeezing, and even on boring P-11 it was observed that there was a very noticeable squeezing of the clay into the boring casing. The samples next to the pile were therefore disturbed by the combined influence of pile driving and the flow of the soil into the casing. No such difficulty is encountered in normal boring operations in this area, and none was observed either in the borings made before pile driving or in the C-borings made between the piles after driving.

RESULTS OF TESTS

The thin sections of samples taken adjacent to a pile 1 month after pile driving and 11 months after pile driving are shown in Figs. 6(*a*) and 6(*b*), respectively. Examination shows that the laminated structure of every sample was preserved. Although the layers were distorted, and in the case of the samples from boring P-1 severely distorted, they nevertheless retained their identity. Hence, it may be said that the disturbance by no means amounted to

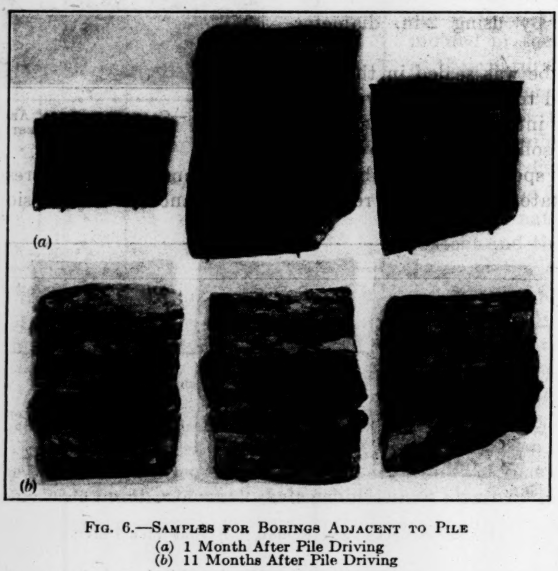

FIG. 6.—SAMPLES FOR BORINGS ADJACENT TO PILE
(*a*) 1 Month After Pile Driving
(*b*) 11 Months After Pile Driving

complete remolding, a process by which all the structure and stratification of a sample are destroyed.

Inasmuch as the disturbance of the soil could not have increased by any conceivable method during the period between 1 month and 11 months after pile driving, it is reasonable to assume that the specimens from boring P-11 indicate most reliably the real distortion of the soil adjacent to the pile. These samples show much less distortion than those from boring P-1. The additional disturbance of the samples from boring P-1 can therefore be attributed exclusively to the squeezing of the clay into the casing during the sampling process.

Close examination of the specimens from boring P-11 suggests that the layers were compressed horizontally and that faults occurred. This would seem inevitable because of the displacement of the clay produced by the insertion of the piles. Nevertheless, the distortion is not excessive.

Samples taken from boring C-1, in the center of a surrounding group of piles, are shown in Fig. 7. The upper inch of the undisturbed sample (Fig. 7(*b*)) is more distorted than the remainder, and the laminations have a shape which indicates that the distortion was due to sampling rather than to the effects of pile driving. Figs. 7(*a*) and 7(*c*) show sections that were cut after the unconfined compression tests were performed. Therefore, they show the shear planes along which the specimens failed in the laboratory. It is apparent that none of the three specimens shown in Fig. 7 has experienced distortions more severe than those produced by the sampling processes in the undisturbed clay deposit before any piles were driven. In so far as distortion is concerned, the soil at the locations of C-1 and C-11 can be considered undisturbed.

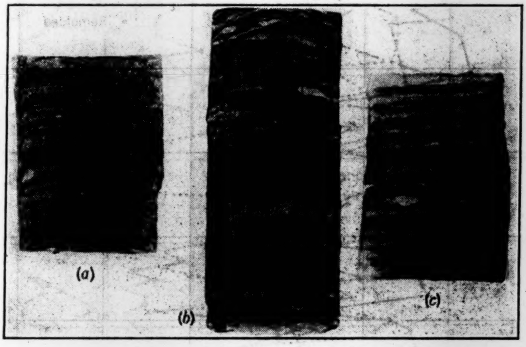

FIG. 7.—SAMPLES FROM CENTER OF PILE GROUP, 1 MONTH AFTER PILE DRIVING

The effect of pile driving on the soil adjacent to the pile and in the center of a pile group is shown quantitatively by the unconfined compressive strength diagram (Fig. 8) of samples from borings A, P-1, P-11, and C-1, and of the clay in its completely remolded state at the original water content. In Fig. 8, the data for boring P-1 indicate that in the upper, more sensitive part of the deposit, the compressive strength of the soil adjacent to the pile was decreased by pile driving to a value intermediate between the completely remolded and the original strength. If the remolded strength is considered to be the lowest value to which the compressive strength can be reduced, it may be said that the pile driving reduced the compressive strength slightly more than half of

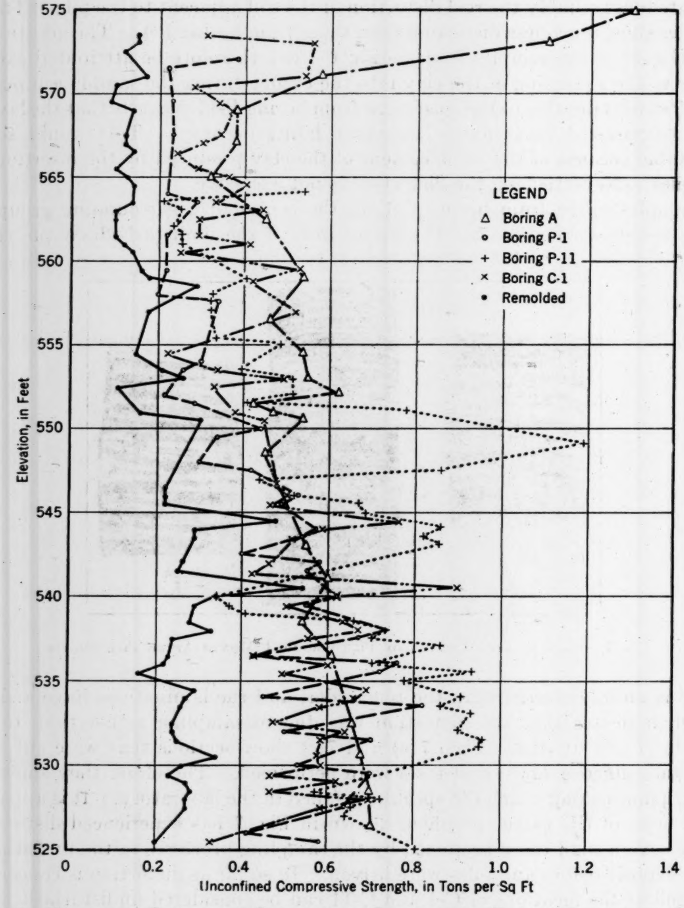

FIG. 8.—UNCONFINED COMPRESSIVE STRENGTHS OF SAMPLES FROM VARIOUS UNDISTURBED TEST BORINGS AND AS COMPLETELY REMOLDED

the maximum reduction possible. In the homogeneous clay layer constituting the lower part of the deposit the effect of pile driving appears to have been negligible. There is no essential difference between the average compressive strength of the soil before and after pile driving. The same information for boring P-11 shows that the compressive strength of the upper, laminated part of the deposit has increased somewhat since boring P-1 was made, and in the lower part of the deposit the compressive strength of the soil is generally much greater than it was before the piles were driven. Similar information for boring C-1, taken in the center of the pile group, shows that there was a small difference in the unconfined compressive strength of the soil in the middle of the group before and after pile driving. A similar conclusion can be drawn concerning the samples from boring C-11.

Because of the erratic nature of the laminated part of the deposit, no significant information can be obtained from the water content data. However, the unusually uniform lower layer provides an exceptional opportunity for studying the changes in water content which accompanied the changes in compressive strength previously described. Fig. 9 shows the variation in water content throughout the depth of this lower deposit at the three different times when borings were made. Originally the average value of the natural water content was 24.7%. A month after pile driving the water content adjacent to the pile (Fig. 9(*a*)) had decreased to 22.8%, and after 11 months it had decreased to 22.0%. Fig. 9(*a*) demonstrates clearly that the decrease in the average value of the water content was much greater than the statistical scattering of the individual values. Therefore, there can be no doubt that a significant, progressive decrease occurred. On the other hand, no trend can be discerned in the values of the water content in the soil at the center of a

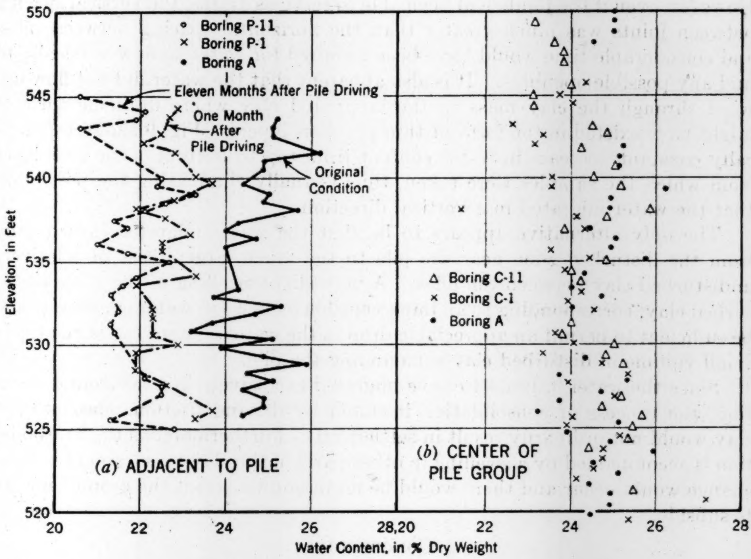

FIG. 9.—NATURAL WATER CONTENTS OF SAMPLES FROM VARIOUS TEST BORINGS

surrounding group of piles (Fig. 9(*b*)). For all practical purposes the water content at the middle of the group remained unchanged throughout the 11 months that observations were made.

INTERPRETATION OF RESULTS

The water content data demonstrate that a process of consolidation was initiated by pile driving. The water content adjacent to the piles decreased unmistakably with the passage of time, implying that the water must have migrated away from the disturbed region. It could not have gone into the piles, because the piles consisted of watertight steel shells with joints 12 ft apart in the vertical direction. The joints themselves were waterproofed. However, even if the joints had been able to serve as drains, the vertical spacing between joints was much greater than the horizontal spacing between piles, and considerable time would have been required for water to flow vertically to find any possible openings. It is also apparent that the water did not flow upward through the clay mass to the laminated clay where drainage surfaces might have existed in the form of thin pervious layers. Fig. 9 shows a practically constant decrease in water content from top to bottom of the 20-ft layer from which the samples were taken, thus virtually eliminating the possibility that the water migrated in a vertical direction.

The only alternative appears to be that the water migrated horizontally from the disturbed zone near the pile to the much larger mass of relatively undisturbed clay between the piles. A very slight swelling of the more undisturbed clay, corresponding to an imperceptible increase in water content, would be sufficient to permit an appreciable drop in the water content in the relatively small volume of disturbed clay surrounding the pile.

Since the water appears to have migrated exclusively in a horizontal direction, the process of consolidation initiated by driving friction piles into the clay would not ordinarily result in settlement. Furthermore, as the consolidation is accompanied by a swelling in other parts of the clay mass, no net volume change would occur and there would be no reason to expect the ground surface to subside.

CONCLUSIONS

The preceding information indicates that the driving of piles at the site where the investigation was made did not have the detrimental effects which were forecast on the basis of soil tests on undisturbed and remolded samples and the theory of consolidation as applied by Mr. Casagrande. The soil adjacent to the pile was not completely remolded but, on the contrary, was disturbed only to an extent that, at the most, might be called "half remolded." For all practical purposes the soil in the middle of the pile group was not remolded at all.

It is also evident that the type of consolidation produced by the pile driving did not involve a vertical flow of water out of the soil, but only a lateral flow. Therefore, if the bridge foundation had been supported by friction piles rather than point bearing piles, there is no reason to believe that the consolidation would have produced a settlement of the ground surface. The assumption that the settlement would proceed as a result of vertical migration of the water out of the clay, which formed the basis of the Casagrande settlement forecast, is not in agreement with the observations made in the investigation. Therefore, it is extremely doubtful that the disturbing effect of pile driving, even for friction foundations, would be detrimental in clay of the type encountered. On the contrary, it is quite probable that piles would be beneficial.

The clays at the site of the investigations varied considerably in sensitivity to disturbance, the stratified clays being much more sensitive than the unstratified clays. However, the evidence in the investigation does not warrant the conclusion that disturbance due to driving piles into soft clay is a source of detrimental decrease in the strength of these clays or a source of excessive settlement.

In the first place, the region of disturbance is limited to a zone within a few inches of the pile and the main volume of the clay mass is apparently undisturbed. In the second place, based on the unconfined compressive strength, the character of the changes in soil structure were apparently such as to increase the strength in the unstratified clay and a large portion of the decrease in the stratified clay was recovered with time. In the third place, the change in moisture content in the disturbed zone left no evidence of migration of moisture from the clay mass as a whole. Under these conditions it is difficult to see how any settlement of the surface could result.

ACKNOWLEDGMENTS

This investigation was a cooperative enterprise of the Raymond Concrete Pile Company (New York, N. Y.), the Michigan State Highway Testing Laboratory, University of Michigan (Ann Arbor), and the University of Illinois (Urbana). The authors are indebted to their colleagues in the several organizations for valuable assistance in the field and laboratory work and in the subsequent analysis of the test data.

DISCUSSION

Leonardo Zeevaert,[5] Assoc. M. ASCE.—The experience of the authors in connection with the remolding of clay due to pile driving has been checked in several details by the experience of the writer in observing the subsoil of Mexico City, Mexico. As an example, certain conclusions can be drawn from the construction of one particular foundation.

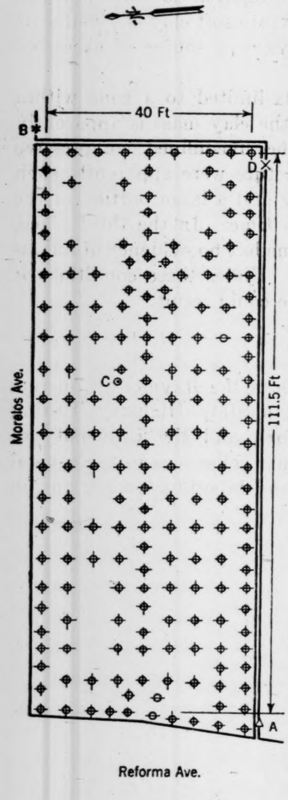

Fig. 10.—Pile Distribution

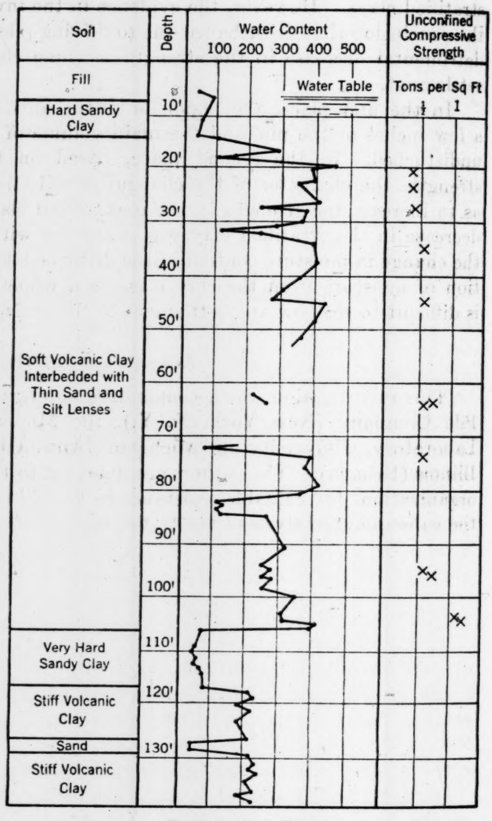

Fig. 11.—Soil Profile

General Information.—At the south corner of the intersection of Morelos Avenue and Reforma Avenue an 11-story concrete building was erected on wood piles driven in three sections to a depth of 110 ft. The site and the distribution of piles, and the location of reference points are shown in Fig. 10. The effective area covered by the piles is 4,460 sq ft and the number of piles is 187; therefore, the average area covered per pile is 23.8 sq ft and the average distance from center to center of piles is 4.88 ft.

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[5] Visiting Research Associate, Univ. of Illinois, Urbana, Ill.; Prof. of Soil Mechanics and Foundations, Univ. of Mexico (on leave of absence), and Cons. Engr., Mexico City, Mexico.

Before excavation or pile driving, four reference points A, B, C, and D, Fig. 10, were established on the surface of the ground. These points were referred to a fixed bench mark on the old monument of Cristobal Colon on Reforma Avenue, at a distance of about 150 ft from the building.

The site was first excavated to a depth of about 6.5 ft, and holes for the piles were pre-excavated to a depth of 14 ft. The piles were then driven through soft volcanic clay until they rested on a hard and well-packed sandy clay stratum at a depth of 110 ft from the surface of the ground. The soil profile and the variation of the water content and unconfined compressive strength with depth (corresponding to Figs. 2, 3, and 8) are given in Fig. 11. Representative compressibility curves of the soft volcanic clay are shown in Fig. 12.

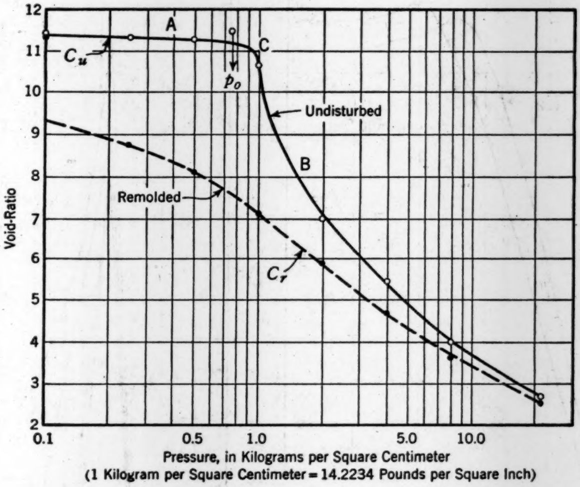

FIG. 12.—COMPRESSIBILITY CURVES

The curve C_u represents the compressibility characteristics of one of the "undisturbed" samples taken at the site by a Shelby tube of 5-in. diameter with a sharp cutting edge. Curve C_r represents the compressibility characteristics of the same material after it was completely remolded at its original water content. The computed overburden pressure at the depth of 45 ft from which the sample was taken is denoted in Fig. 12 by p_o.

These two compressibility curves show the usual pattern of undisturbed and completely remolded samples for Mexico City volcanic clay. The behavior of the material in the field is closely expressed by curve C_u. The section AC, Fig. 12, corresponds to the compressibility if the load is too small to produce a marked change in the natural structure of the clay, and the part CB, to the compressibility if the load is great enough to produce a change in the structure. At point C, designated as the "confined compressive strength," a transition occurs. The compressibility increases from values represented by branch A-C of the curve to those represented by branch C-B. For this sample the increase is at least tenfold.

The stress-strain relationship for unconfined compression specimens tested at constant water content is given in Fig. 13 for two samples at different depths. Under these conditions of deformation the material behaves almost elastically up to about 2% strain. Failure follows between strains of 2% and 3%. In the remolded state at the same water content the material bulges under its own weight to a strain of about 5%. The dotted line in Fig. 13 represents the approximate stress-strain relationship for a remolded sample.

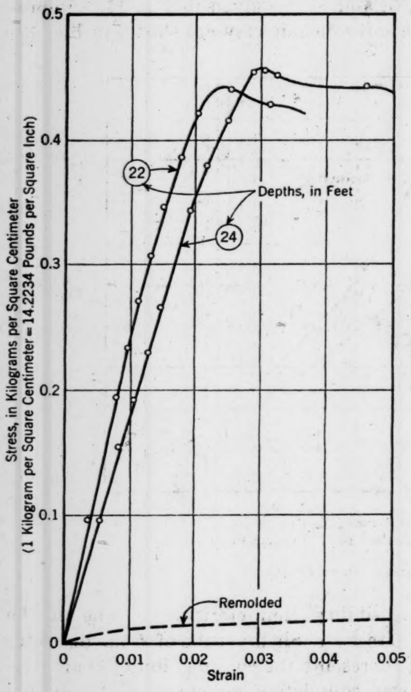

FIG. 13.—UNCONFINED COMPRESSION

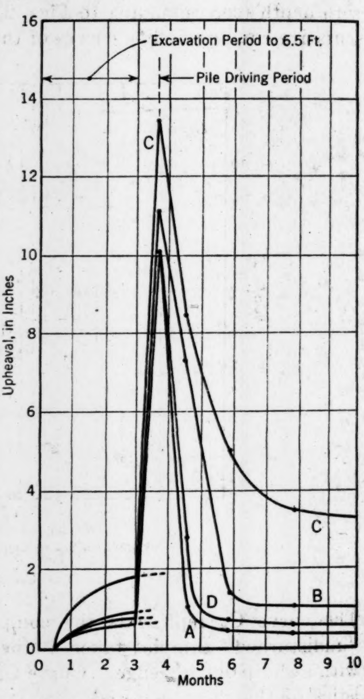

FIG. 14.—OBSERVATIONS DURING PILE DRIVING

Observations in the Field.—During pile driving the reference points on the surface of the ground moved upward considerably as shown in Fig. 14. Points close to the middle of the lot (curve C, Fig. 14) heaved as much as 13 in., those at the south boundary (curves A and D, Fig. 14), about 10 in., and those at the northeast corner (curve B, Fig. 14), about 11 in. The upward movement of the surface of the ground occurred as rapidly as the piles were driven.

After pile driving was finished, the surface of the ground settled back quickly almost to its original position. A tendency of the boundary points (A, B, and D) to subside slightly and the interior points (point C, Fig. 10) to remain heaved, with respect to their original position, may be noticed. The settlement had practically ended one month after the piles were driven. Further observations have shown that, during the succeeding 2 years, the surface of the ground did not settle any further with reference to the bench mark. The important fact that the soil heaved rapidly during pile driving and came back in a very short time to its original position, in spite of the presence of the piles, suggests the following, in support of the conclusions advanced in the paper:

1. The mass of clay did not adhere firmly to the piles by skin friction. No more than a very small dragging force could have developed as a result of the shearing resistance of the remolded soil close to the pile during the relative displacement between piles and soil.
2. After the soil came back to its original position, no appreciable settlement of the surface of the ground was observed with relation to the fixed bench mark. It appears therefore that the clay between the piles was not substantially altered during pile driving.

Deductions from the Observations.—To test the hypothesis that piles driven in the highly sensitive clay of Mexico City radically disturb that clay, it is first necessary to make a rough estimate of what should happen to the ground surface for different degrees of disturbance in the clay mass due to pile driving. For such an estimate it is necessary to know approximately the frictional force that develops along the piles as the heaved soil subsides to its original position. If the frictional force is larger than, or equal to, the weight of the clay surrounding the piles, the surface of the ground will remain at its heaved position regardless of the degree of disturbance of the clay. Depending on the value of this force, the surface of the ground will come back to a certain position which is a function of the degree of disturbance in the mass of clay and of the dragging force along the piles.

Dragging Force.—The dragging force exerted on the piles during the movement of the soil is small. This is substantiated by the fact that, in Mexico City, structures on point-bearing piles appear to raise above the ground surface, and the sidewalks appear to settle away from the buildings because of the self-consolidation that occurs in the volcanic clay deposit. The sidewalks settle to the very edges of buildings resting on piles, even if the piles are as close together as three to four diameters. If there were appreciable frictional drag, the settlement should decrease near the edges of the buildings.

When a pile is driven to the hard stratum, only a small effort is necessary to push it through the volcanic clay. Usually no vibrations are introduced in the soil because the weight of the pile and hammer, aided by very gentle blows of the hammer, is sufficient to force the pile down to the required elevation where the driving is difficult for a short distance. When the point of the pile is pushed into the clay, the clay is displaced from beneath the point possibly as indicated in Fig. 15. The high pressures imposed by the pile point transform the clay into a highly viscous material that flows upward and away from the point of the pile until it surrounds the shaft in the form of an annular ring. On some occasions, it has been observed that fluid clay has squeezed out to the surface of the ground at the shaft of the pile. Since the process occurs rapidly, the clay changes position at an unaltered water content. Analyzing a horizontal slice above the point of the pile one finds that the new state of the clay surrounding the pile may be classified approximately in three different zones as follows (see Fig. 15):

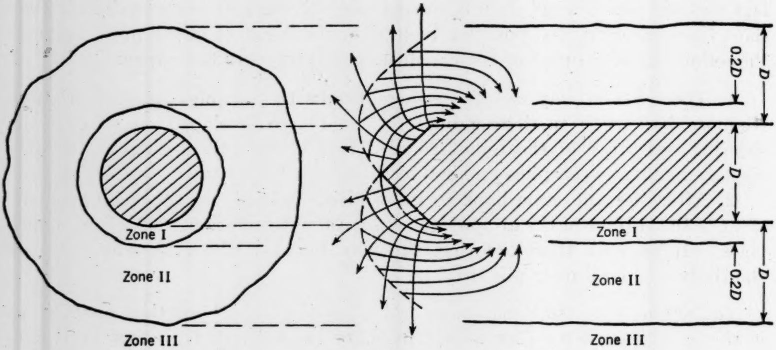

FIG. 15

In Zone 1 the material is probably in a perfectly remolded condition and keeps in movement during pile driving until the pile reaches its ultimate position. If no remolded clay is squeezed up onto the ground surface, the volume of material in the annular ring corresponds exactly to that displaced by the pile. The thickness of the ring is therefore 0.2 *D*.

In Zone II the material is probably considerably disturbed. In this region it is assumed that the clay moves as the point of the pile displaces the soil; afterward the material should remain almost motionless. The outside diameter of the annular ring corresponding to this zone probably extends to 3 *D*.

In Zone III the clay is assumed to remain unaltered during pile driving except for a temporary upward elastic deformation due to the pressures caused by the displaced volume of remolded clay during pile driving. Zone III constitutes the remainder of the mass of clay located within the piles. If the linear upward deformation should exceed about 2%, this zone would become somewhat altered (see Fig. 12 and Fig. 13). However, at the site observed, the level readings indicated a maximum deformation of only 1.2%. Therefore, the clay in Zone III retains its elastic, undisturbed character.

Because of the remolding effect the compressibility of the clay surrounding the piles to a distance of approximately *D* increases as much as ten times. The large hydrostatic excess pressures in the altered clay, caused by pile driving, make this portion consolidate rapidly. This produces a stress relaxation in the rigid, elastically distorted, undisturbed clay and allows it to regain its original position.

The circular boundary of the undisturbed clay moved inward toward the pile because of the consolidation. The movement cannot exceed about 0.12 *D*, a distance corresponding to the volume of the pile, because the ground surface settled approximately to its original position. Hence, the volumetric strain produced in the remolded clay is not more than 0.12. On the assumption that the initial void ratio of the remolded clay along the pile was equal to the average void ratio in the undisturbed clay before pile driving, and that it was equal to about 9.0, the change in water content in the annular ring of thickness D was approximately $120/2.4 = 50\%$, in which 2.4 is the specific gravity of the solids. Since the initial average water content was 375%, the reduction is of the order of $100 \times 50/375 = 13.3\%$. The usual sensitivity of this clay is greater than about 15. Therefore, a 13% reduction in the water content of the remolded clay could not appreciably increase its shearing strength.

According to the preceding reasoning, one should use the shearing strength of the perfectly remolded clay to estimate the magnitude of the dragging force on the piles due to settling of the clay between them. If an average value of 0.3 ton per sq ft is used (Fig. 11) for the shearing strength of the undisturbed clay, and a minimum value of the sensitivity is taken as 15, then the shearing resistance during the relative movement of the pile and the soil is of the order of $0.3/15 = 0.02$ ton per sq ft. At the site of the observations, the piles are embedded in clay for 90 ft and have an average diameter of 10 in. Hence, the computed total downward force per pile is of the order of $3.14 \times 0.833 \times 90 \times 0.02 = 4.7$ tons. Therefore, the soil would adhere firmly along the piles only if the weight of the surrounding soil were smaller than this quantity.

Test of Working Hypotheses.—Since the dragging force is known approximately and has an order of magnitude of 4.7 tons per pile, it is now possible to test (1) the validity of the thesis of this paper that the soil is not remolded appreciably by pile driving, as well as (2) the concept expressed diagrammatically by Fig. 15. These characteristics can be examined by computing the settlement of the ground surface and comparing the results with the observations. First, assume the clay mass to be completely disturbed during pile driving, and the mass of clay between the piles to act as a perfectly remolded material. The material will consolidate to a void ratio in accordance with that determined by a completely remolded specimen in the laboratory. Secondly, assume the clay to behave as a remolded material only to a distance from the shaft of the pile to one diameter, and the remainder of the clay between the piles not to change substantially.

The deformation of the surface of the deposit because of a change in volume compressibility and a change in effective stress may be expressed by

$$S = \frac{\Delta e}{1 + e} H + m_v H \Delta p \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

in which S is the surface deformation; H is the thickness of the deposit, equal to 90 ft; p is the average effective stress; and m_v is the average coefficient of unit volume change of the deposit in the unaltered condition, equal to 0.045 sq ft per ton.

Concept (1) will be tested by a computation for the center of the area. The average overburden pressure including the relief of stress due to excavation is 0.672 ton per sq ft. One pile, representing an average area of 23.8 sq ft, offers a frictional resistance of 4.7 tons. Hence, the approximate change in stress in the deposit will be $- 4.7/23.8 = - 0.197$ ton per sq ft. The initial void ratio from the compressibility curve C_u, Fig. 12, is 11.2 for a pressure of 0.672 ton per sq ft and the final void ratio taken for a pressure of $0.672 - 0.197 = 0.475$ ton per sq ft is found from curve C_r, Fig. 12, to be 8.2. Therefore, $\Delta e = 3.0$; and Eq. 1 gives $S = \frac{3.0}{1 + 11.2} \times 90 - 0.045 \times 90 \times 0.197 = 22 - 0.7 = 21.3$ ft; that is, the ultimate subsidence of the surface of the ground would be roughly 21 ft if settlement of the Mexico City volcanic silty clay occurred contrary to the Cummings-Kerkhoff-Peck concept. Drainage conditions are favorable in the deposit. Therefore, at least a large percentage of this subsidence should appear in the first months after pile driving. To the knowledge of the writer, there has never been reported a subsidence of the ground due to pile driving in Mexico City that amounts even to a small fraction of this value.

The second concept (that illustrated by Fig. 15) leads to the following values, in tons per square foot, for the center of the lot:

Description	Pressure
Overburden pressure considering excavation..........	0.672
Decrease in pressure due to frictional resistance of piles	−0.197
Increase in pressure in the undisturbed clay due to the area occupied by pile and remolded material (26%)	0.175
Final pressure..................................	0.650

Increment change in pressure is $0.650 - 0.672 = - 0.022$ ton per sq ft.

Since the central portion of the clay remains unaltered, $\Delta e = 0$, and Eq. 1 becomes: $S = 0.045 \times 90 \times (- 0.022) = - 0.089 \text{ ft} = - 1.07 \text{ in.}$

If similar computations are made for the corner of the lot, a value of + 0.25 in. is obtained for the subsidence of the surface of the ground.

Conclusion.—The results obtained in the second computation, although only a rough estimate, show with reasonable accuracy the general trend and order of magnitude of the movements that have been observed in the field, and are in agreement with general experience in Mexico City. The conclusion of the authors that the disturbance is limited to a zone within a few inches from the pile and that the main volume of the clay mass remains practically undisturbed during pile driving appears to be proved. Therefore, the writer believes that the real phenomena in the field are in close agreement with the second concept, illustrated in Fig. 15.

Finally, according to the concept of complete remolding, structures on spread footings or rafts adjacent to sites where pile foundations are driven should begin to settle because of the disturbance. The writer has never observed this phenomenon, although the compressibility of the clay is so great when remolded that any such occurrence would certainly have attracted his attention.

Jacob Feld,[6] M. ASCE.—Just about two hundred years[7] ago John Muller wrote—

> "As the foundations of all buildings in general, are of the utmost importance, in respect to the strength and duration of the work, we shall enter into all the most material particulars, which may happen in different soils, in order to execute works with all the security possible, because many great buildings have been rent to pieces, and some have fallen down, for want of having taken proper care in laying the foundation."

—and it is in this approach to the foundation problem that the authors have conducted their study to determine how a clay soil is altered by the driving of closed end piles.

A summary of the five recommendations and comments made by Muller seem very pertinent to the authors' conclusions:

(1) Use proper augers to bore holes from 10 ft to 15 ft deep in order to discover the nature of the soil and its hardness.

(2) If the soil is gravel or hard stiff clay, no sinking will occur.

(3) If the soil is not uniform, lay a grate of timbers crossed two ways, sometimes boarded over with 3-in. wood planks; sand needs such grates.

(4) If the soil is first soft and hard below, then drive piles and lay a grate above them; drive a few piles to determine the proper length before cutting all the piles.

(5) Sometimes a bed of gravel or clay has soft mud below, and it is dangerous to drive piles, then build on a timber deck and let it sink to bearing (a method used in Russia and Flanders with good results).

The authors have contributed to this centuries old search for the "proper care in laying the foundation." It does not seem that they intend to convey an impression that the clay was not altered, for such an impression would not be substantiated by the results.

Under the conditions of the test, the authors' conclusions that the soil was not weakened seem warranted.

The spacing diagram shows that a pile occupies 20 sq ft, of which the pile displaces 5%. Of the 65 ft of clay, between El. 505 and El. 570, the volume replaced by the pile is 65 cu ft. This volume displacement would indicate a rise of 3.40 ft in the earth's surface, if no change occurred within the soil mass. The paper reports that the "ground surface between the piles heaved about 2 ft during the driving operations." The remaining 1.40 ft indicates an average compressive strain of more than 2% in the 65-ft layer of clay. The intensity of the resulting stress is indicated by the squeezing of as much as 1 ft of the clay into the casing of test boring P-1 taken near the pile 1 month after driving. No such phenomenon is reported in the P-11 boring. Is it possible that the stress condition was dissipated in the 10-month interval? This would explain the difference in the distortion of layers shown in the samples in Fig. 6, but it is hardly proof that the clay is not affected by the pile driving, or that the pile may not be affected by this change in internal soil conditions.

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[6] Cons. Engr., New York, N. Y.

[7] "Treatise Containing the Practical Part of Fortification," by John Muller, A. Millar, London, 1755, p. 160.

Another difficult item to explain is the moisture loss, unless it is assumed that the solid fraction of the soil displaced is squeezed into the immediately adjacent soil and that the soil was not fully saturated, so that the gas content was compressed sufficiently to make room for the additional solid material. This change may not be equivalent to remolding, but it certainly changes the soil characteristics and may have unexpected effects on the piles. This type of soil stress condition has been known to cause heaving of piles.

The foregoing comments are not in criticism of the authors' study. The costs of such field and laboratory tests are quite high. The piles considered are definitely end bearing as far as load-carrying capacity is concerned. Had the piles been driven as friction piles, a record of settlements on the piles or on the structure carried by the piles could have been used to correlate changes in soil with change in vertical position. That is, after all, the subject about which the paper should be chiefly concerned. As Muller pointed out, if the soil is "first soft and hard below," then drive piles and place the load on them. However, when the soft mud is below, it is dangerous to drive piles. It would be worthwhile to have the authors' conclusions as to whether the alteration of soft clay resulting from pile driving can be depended upon as an improvement of the soil bearing value at the depths where the piles transfer load to the soil. Even Muller knew that piles in such soils increased the soil bearing value near the surface; and he recommended a French method consisting of a pile enclosure with the soft soil dug out with a "machine scoop," not deeper than 6 ft and surfaced with a crib if the soft muck ran deeper, and also a surface of cement concrete "which sets under water."

It is difficult to believe that this expensive research program was undertaken only to disprove the statement that soft clay is completely remolded by pile driving. It is hoped that the authors will publish any other practical conclusions observed in this work.

JOSEPH C. FEGAN,[3] JUN. ASCE.—This paper contains the first experimental data that can be used to evaluate the analysis of clay structures presented[4] by A. Casagrande, M. ASCE, in 1932. The fundamental contentions of Mr. Casagrande that—"* * * pile driving in consolidated or partly consolidated clay remolds the clay and thereby increases its compressibility"—are substantiated by the experimental data in the paper. The authors' interpretation of the data, and their conclusions, conflict with the theory developed and the assumptions made by Mr. Casagrande. The principal variance of the two papers is in the estimate of the extent of the zone of remolding. The paper uses unconfined compression test strength as the sole quantitative basis of determining the amount of remolding. However, a measure of the increase in compressibility is the dominant need, since failure in the type of foundation being considered is generally caused by large settlements. Confined consolidation tests most accurately simulate the condition of the soil in the ground and directly measure the consolidation of the sample. By plotting the consolidation, under a given load, of undisturbed samples, and samples at various distances from the pile, the extent of the zone in which detrimental consolidation takes place can be determined. This method is used by Mr. Casagrande, but he was limited by the fact that only the undisturbed sample and the completely remolded laboratory sample were available. If confined consolidation tests could be made on the samples taken by the authors, the missing information between these two extremes would be furnished. Until such tests are made, the results of Messrs. Cummings, Kerkhoff, and Peck cannot be used with confidence to evaluate the accuracy of the conclusions of Mr. Casagrande.

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[3] Cons. Engr., New York, N. Y.

On the basis that dried samples, although severely distorted, retained evidence of their original stratification, the authors conclude that complete remolding did not occur. The significance of remolding in clay is that grain arches, which are the principal supporting elements of the material, are broken down so that the grains are free to move and consequently can consolidate partly. This process does not require the mixing of soil from the component strata, but can occur within each stratum so that the grain arches are broken down with only a small relative movement of adjacent grains and a destruction of the cementing material at grain contacts. In such a situation the strata of the dried sample as distinguished by color, texture, etc., would not be disarranged although reduced in thickness. Thus, the principal supporting capacity of the clay would be lost without the intermingling of component strata.

By eliminating all but a horizontal drainage path away from the pile, the authors observed that the process of consolidation initiated by pile driving would be arrested by the lack of suitable drainage. The possibility of a vertical drainage path at the surface of the pile was not considered, however. In 1927, Karl Terzaghi, Hon. M. ASCE, concluded[9] that, when piles are being driven in clay, displaced pore water is forced upward between the surface of the pile and the impervious clay to a free water surface above. Pore water forced from a stressed clay mass would similarly be forced vertically along the pile surface to the nearest free drainage surface. In this situation, the static load forces on the pile would be equivalent to the dynamic force of pile driving considered by Mr. Terzaghi. From observation of Fig. 4, it can be noted that the clay at the surface of a pile is so remolded that numerous cracks and groves exist. These combine to form an open channel flow path around the pile, that offers less resistance to flow than does the geometrically shorter route to the center of the impervious clay mass. Thus the process of consolidation initiated by pile driving can continue through drainage surfaces provided by the piles themselves.

Conclusions.—Although the paper provides much needed experimental information, there is still need for a confined consolidation compression test on the samples obtained and for a settlement calculation for a typical friction pile foundation at the site of the samples. Interpretations and conclusions loosely considered as observations may mislead the engineering profession into neglecting the valuable analysis of the effect of driving friction piles in clay presented by Mr. Casagrande.[4]

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[9] "The Science of Foundations—Its Present and Future," by Charles Terzaghi, *Transactions*, ASCE Vol. 93, 1929, p. 270.

Gregory P. Tschebotarioff,[10] M. ASCE.—Interesting and valuable data on the effect of pile driving on the water content and on the unconfined compressive strength of a clay layer are presented in this paper. However, it is to be noted that the authors did not offer any direct data to support the thesis that the ground surface between the piles did not subside. It is comparatively easy to obtain this information in conclusive form by direct measurements of settlement provided they are undertaken simultaneously with other forms of study at the start of a project. The writer is familiar with a case in which such direct measurements were made somewhat later but nevertheless produced irrefutable evidence in support of conclusions diametrically opposed to the conclusions submitted by the authors on the important point of the ground surface settlements.

This case was as follows: The structural framework of a large 4-story building rested on 120-ft steel pipe piles which had been driven with closed end through laminated soft varved clay of glacial origin to a good point bearing on an underlying hard stratum. The building had a subbasement, which involved excavating from 3 ft to 4 ft of the soil and reducing the load on the ground by a corresponding amount. Therefore, the original designers of the building deemed it safe to let the basement floor and partitions rest in part on the ground itself, rather than to have them entirely supported by girders connected to the footings that capped the piles. Soon after the building was completed and occupied, signs of distress appeared in the basement in the form of structurally unimportant but unsightly cracks in the floor and in the partitions. The nature of these cracks indicated the probability that part of the concrete floor slab which rested directly on the ground between the pile groups was losing its support there. Holes were then drilled in the basement floor slab and reliable bench marks were established in the upper few feet of the clay. Precise measurements were performed which indicated that the surface of the clay was settling in respect to the elevation of the pile capping beams. The settlement was still continuing at a decreasing rate after several years had elapsed. All evidence pointed to a gradual process of reconsolidation of the clay under its own weight as a result of its partial remolding by the pile driving. The steel pipe piles were driven in one single welded section. Thus drainage of the clay was possible only upward and downward.

In the case just described the distance between pile groups was four to five times greater than the distance between the piles in the case described by the authors. A lesser degree of remolding should have been expected; nevertheless ground surface settlements were actually measured between the pile groups. Otherwise the conditions were similar. Therefore the contrary conclusions of the authors concerning ground surface subsidence are subject to question unless direct and quantitative evidence is produced to support them. It is too late now to start settlement measurements since equilibrium has probably already been reached in this case. The writer has an hypothesis to offer which may explain, in a manner quite different from that of the authors, the observations concerning water content changes. This hypothesis is based on the results of tests performed in the writer's Soil Mechanics Laboratory at

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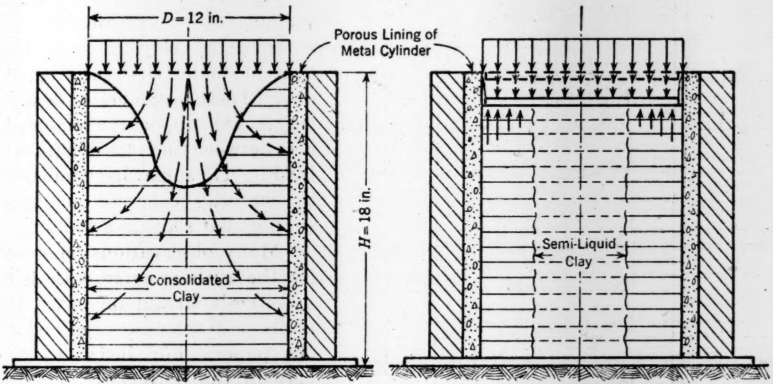
[10] Associate Prof. of Civ. Eng., Princeton Univ., Princeton, N. J.

Princeton University in Princeton, N. J., with an apparatus termed "lateral earth pressure meter." [11,12]

The lateral earth pressure measurements themselves, as well as the testing techniques which were developed in order to meet the special conditions created by the deformations of the surface of the consolidating clay, are not pertinent to the present topic and therefore will not be discussed. The observed deformations of the surface of the originally fluid clay, however, are of direct interest and are illustrated by Fig. 16. When the vertical pressure was applied by compressed air to the originally plane surface of the clay through a flexible rubber membrane at all stages of consolidation, as well as after its completion, the surface of the clay was found to show a pronounced dip at the center of the cylindrical apparatus. This dip could be explained only by the restraining effect of shearing stresses developed along the rigid but pervious vertical walls of the device. Thus the rate of the settlements of points at the clay surface was governed not by the rate of consolidation of the clay beneath each point of the surface, but by plastic deformations of the clay mass which must have followed trajectories approximately indicated by the dash-dotted lines and arrows in Fig. 16(*a*).



(*a*) VERTICAL PRESSURE APPLIED THROUGH A FLEXIBLE RUBBER MEMBRANE

(*b*) VERTICAL PRESSURE APPLIED THROUGH A RIGID PLATE

FIG. 16.—THE EFFECT OF BOUNDARY CONDITIONS ON THE SURFACE DEFORMATIONS AND ON THE CHANGE OF WATER CONTENT OF A CLAY CONSOLIDATED IN THE LABORATORY

An entirely different condition was observed when a rigid steel plate was interposed between the clay surface and the rubber membrane. It was found, after completed consolidation, that the steel plate had been supported by a hollow outer cylinder of consolidated clay with a central shaft of semi-liquid clay of the original consistency, as illustrated by Fig. 16(*b*).

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[11] "Special Features of Large-Scale Earth Pressure Tests," by Edward R. Ward, John R. Bayliss, and Philip P. Brown, in "Lateral Earth Pressures on Flexible Retaining Walls: A Symposium," *Transactions*, ASCE, Vol. 114, 1949, pp. 455–463.

[12] "Effects of Boundary Conditions on Lateral Earth Pressures," by Gregory P. Tschebotarioff and John David Welch, *Proceedings*, 2d International Conference on Soil Mechanics and Foundation Eng., Rotterdam, 1948, Vol. III, pp. 308–313.

The foregoing findings and observations, which were made in the laboratory, will now be extended to illustrate a somewhat similar hypothetical condition in the field, which may arise when concrete piles are more pervious than the surrounding clay and can serve as rigid vertical drains for that sensitive clay remolded by the pile driving and reconsolidating under its own weight. This hypothetical condition is illustrated by Fig. 17. The piles would be surrounded by cylinders of reconsolidated clay of a water content smaller than the original one. These cylinders of denser clay would support the remolded clay between them (as illustrated by Fig. 17) somewhat similarly to the way concrete columns support flat slabs in a building.

Many observations reported by the authors agree with this hypothesis should one wish to apply it to the case described in the paper. For instance, the clay in the borings P next to the piles was under such high pressure that it squeezed into the casing. On the other hand, this did not happen in the C-borings midway between the piles. This reaction should be expected from Fig. 17 since, according to the hypothesis which it illustrates, the remolded and not reconsolidated clay in that region would be partly supported by the cylinders of denser clay around the piles.

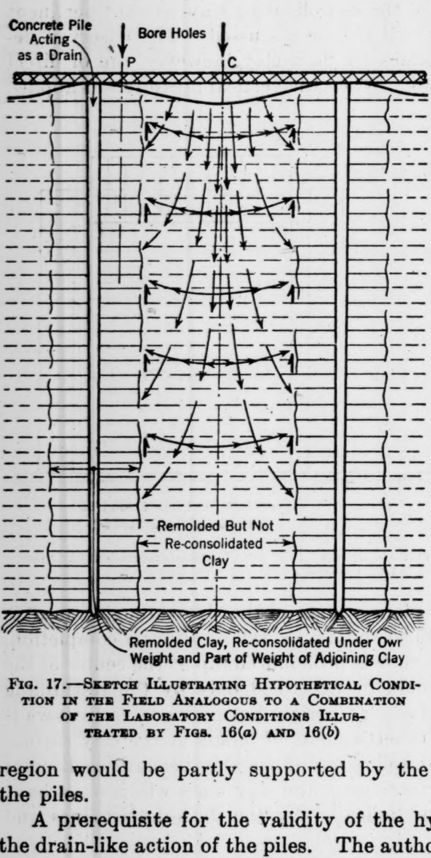
Concrete Pile Acting as a Drain
Bore Holes
P
C
Remolded But Not Re-consolidated Clay
Remolded Clay, Re-consolidated Under Owr Weight and Part of Weight of Adjoining Clay

FIG. 17.—SKETCH ILLUSTRATING HYPOTHETICAL CONDITION IN THE FIELD ANALOGOUS TO A COMBINATION OF THE LABORATORY CONDITIONS ILLUSTRATED BY FIGS. 16(*a*) AND 16(*b*)

A prerequisite for the validity of the hypothesis illustrated by Fig. 17 is the drain-like action of the piles. The authors claim that this was not possible in the case reported by them, but their reasoning is not entirely convincing. The waterproofing of the joints of the corrugated metal shell might have appeared fully effective, it is true, during the few hours (or even minutes) that elapse between the driving of the shell and its filling with concrete; but the clay itself would help to produce that impression since the visible discharge through it would be very slight within such a short time even in an open and only partly lined pit. These same joints, on the other hand, might permit an appreciable leakage through them over a period of a month. The condition of a thin layer of soil in immediate contact with the skin of the corrugated metal shell of the pile might be such that some seepage in a vertical direction toward the joints could be facilitated there. For instance, it is conceivable that some of the more pervious fill material from the surface filled the hollows of the corrugations of the shell during the pile driving and was carried down with the pile.

For all the aforementioned reasons, the writer cautions against the unreserved acceptance of the thesis of the authors according to which a clay remolded by pile driving will not reconsolidate and settle under its own weight.

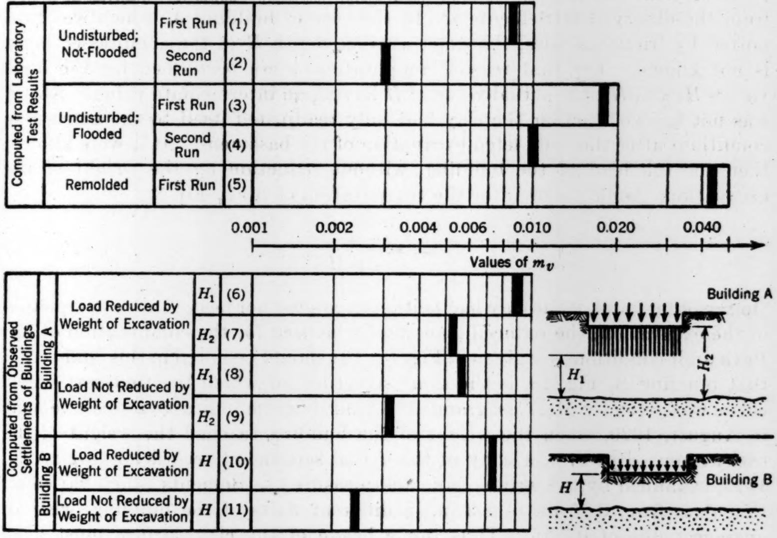

FIG. 18.—COMPARISON OF VALUES OF THE COMPRESSIBILITY COEFFICIENT m_v COMPUTED FROM THE AVERAGE RESULTS OF DIFFERENT LABORATORY CONSOLIDATION TEST PROCEDURES AND FROM SETTLEMENT OBSERVATIONS OF BUILDINGS A AND B

On the other hand, the writer fully agrees with the contention of the authors that A. Casagrande, M. ASCE, exceeded reasonable limits in his original 1932 forecast of the extremely detrimental effect of pile driving into clays; and especially so in his comparison with other possible sources of disturbance. This opinion is based on the results of a combined field and laboratory study which the writer performed in 1946.[13] Fig. 18 compares the values of the coefficients of volume change m_v as obtained from different procedures of laboratory consolidation tests and from the observed settlements of two buildings. The laboratory m_v-values were obtained from the formula—

$$m_v = \frac{a}{1 + e_1} = \frac{e_1 - e_2}{\Delta p \, (1 + e_1)} \dots\dots\dots\dots\dots\dots \quad (2)$$

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[13] "Comparison of the Extent of Disturbance Produced by Driving Piles Into Plastic Clay to the Disturbance Caused by an Unbalanced Excavation," by Gregory P. Tschebotarioff and James R. Schuyler, *Proceedings*, 2d International Conference on Soil Mechanics and Foundation Eng., Rotterdam, 1948, Vol. II, pp. 199–205.

in which a is the coefficient of compressibility $= \Delta e/\Delta p$; e_1 is the initial void ratio; e_2 is the final void ratio; and Δp is the increment of unit vertical pressure. The laboratory m_v-values marked "second run" were obtained by the procedure described by the writer elsewhere[14] in connection with studies of swelling clays. The preconsolidation load of such clays was determined first. Then a new specimen of the clay was loaded up to the preconsolidation load, unloaded to the value of the weight of the overburden, and loaded again—this second loading being then referred to in Fig. 18 as the "second run."

Several alternatives had also to be considered when computing the m_v-values from the observed settlements s. In the case of building A, which was supported by frictional piles, the exact average depth H of the compressed layer is not known. For that reason computations were performed for the limit values H_1 and H_2, the actual value of H having an intermediate value. Also it was not known whether the clay had fully readjusted itself to the new stress conditions after the completed excavation of the basement. If it were known, then the full load of the building, without reduction by the weight of the excavation, should be used for the computation of Δp in Eq. 3:

$$m_v = \frac{s}{\Delta p \, H} \quad \text{......................} \quad (3)$$

On the other hand, if the clay had neither expanded nor heaved after completion of the excavation, the reduced load should be used for the computation of Δp. Both limit conditions are given in Fig. 18. It should be noted in this connection that building B, Fig. 18, is the same structure for which the time-settlement curve was given by Mr. Casagrande.[15] That diagram indicates zero settlement in August, 1929, when the weight of the building reached the weight of the excavation. However, a copy of the actual settlement record for August 26, 1929, examined by the writer, indicated measured settlements which varied on that date from 0.25 in. to 1.3 in. in different parts of the building, with an average value of 0.7 in. Thus, some heave of the clay surface must have occurred previously during excavation, and the actual condition in the field would lie between the two limits indicated in Fig. 18.

With this in mind, it can be noted from Fig. 18 that the value of the m_v-coefficient of a completely remolded laboratory sample (in line (5), $m_v = 0.0430$) is approximately five times greater than the largest possible value based on actually measured settlements (in line (6), $m_v = 0.0090$) of that coefficient for building A which was supported by piles. It is approximately seven times greater than the average value of lines (6), (7), (8), and (9) ($m_v = 0.0056$) which the writer believes to correspond more closely to actual conditions. At the same time this latter value is almost identical with the average value (lines (10) and (11), Fig. 18 ($m_v = 0.0050$)) for building B which was not supported by piles.

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[14] "Comparison Between Consolidation, Elastic and Other Soil Properties Established from Laboratory Tests and from Observations of Structures in Egypt," by Gregory P. Tschebotarioff, *Proceedings*, 1st International Conference on Soil Mechanics and Foundation Eng., Cambridge, Mass., 1936, Vol. I, pp. 33–36.

[15] "The Structure of Clay and Its Importance in Foundation Engineering," by A. Casagrande, *Journal*, Boston Soc. of Civ. Engrs., April, 1932, Fig. 28.

All laboratory tests on which Fig. 18 is based were performed on samples obtained by Shelby tube samplers 3.5 in. in diameter. A detailed discussion of the significance of the results is beyond the scope of the present paper. It should be mentioned, however, that only the samples for which the photographs showed no distortion at all were used for final evaluations of undisturbed values.[16] Their average strain at unconfined failure was 9.3% for a 10-min to a 20-min duration of test. On the other hand, samples of sensitive clay of the type shown in Fig. 5, although claimed by the authors to be "practically undistorted," in the writer's experience would have to be considered partly remolded with much higher strains at failure, 30% to 40% lower strengths, and greater compressibility. It is to be hoped that the closing discussion will give the strains of specimens at failure. Presumably the use of only 2-in.-diameter tubes is responsible for the condition of the samples illustrated by Fig. 5.

The writer surmised that the almost identical values of the m_v-coefficients of buildings A and B (Fig. 18) might be explained by the remolding effects accompanying the distortions of all underlying layers, which distortions could be induced by a heave at the bottom of the excavation, such as appears to have occurred in the case of building B, whereas the remolding caused by pile driving might be much more localized. This hypothesis was confirmed in a qualitative manner by a three-dimensional model study with gelatine performed in 1947 by James R. Schuyler.[13]

To summarize, the writer feels that, although the authors are right in pointing out that Mr. Casagrande originally exaggerated the detrimental effects of pile driving on clay, the authors themselves have now gone too far in the opposite direction.

Philip C. Rutledge,[17] Assoc. M. ASCE.—Effort, money, and time have been spent to obtain the factual data in this paper, on a subject that has long been controversial. The writer commends the authors for securing and presenting the data completely to the engineering profession in their paper, but he cannot agree with the conclusions drawn from these data. A fortunate circumstance of this investigation was the existence of an extremely uniform deposit of soft blue clay below El. 550 from which the authors have plotted extensive test data between El. 550 and El. 525. This discussion will be concerned primarily with this soil layer and these data.

The authors describe this layer as a homogeneous soft blue clay, with the exception that it includes "an appreciable number of small shale pebbles up to ¼ in. in diameter," and as not particularly sensitive to remolding. The authors have included statements of average water contents before and after pile driving from the data shown in Fig. 9(*a*). The writer has used these averages and has abstracted compressive strength data from Fig. 8 and water content data for borings C-1 and C-11 from Fig. 9(*b*) and computed similar averages. These average test results, all taken directly from the authors' data, are summarized in Table 1. Because of the uniformity of the clay layer in question, it is assumed that averages of test data over a 25-ft depth equalize the minor natural variations from sample to sample and represent acceptably the average conditions of the entire clay layer.

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[16] "The Determination of the Shearing Strength of Varved Clays and of their Sensitivity to Remolding," by Gregory P. Tschebotarioff and John R. Bayliss, *Proceedings*, 2d International Conference on Soil Mechanics and Foundation Eng., Rotterdam, 1948, Vol. I, Fig. 1.

[17] Prof., Civ. Eng., Technological Inst., Northwestern Univ., Evanston, Ill.

The data in Table 1 do not tell the entire story on what happened to the homogeneous clay during the pile driving because considerable periods of time elapsed between the pile driving and the sampling and testing, and both the water content and the compressive strength changed progressively with time. The significance of the data is shown in Fig. 19, in which a part of a consolidation test compression curve for a Detroit clay, with approximately the same initial water content and plasticity characteristics, is superposed on a plot of points representing the authors' average water content and compressive strength data. Previous work[18,19] has shown that on a semilogarithmic plot of test data for saturated natural clay, the compressive strength-water content relation is a curve parallel to the virgin (or essentially straight-line) part of the consolidation test curve. In this case it seems more than a coincidence that the points for the average strengths of completely remolded clay and of clay immediately adjacent to the pile 1 month and 11 months after driving should fall on a single straight line approximately parallel to a consolidation test curve that is reasonable for this clay.

TABLE 1.—AVERAGE SOIL TEST RESULTS (HOMOGENEOUS SOFT BLUE CLAY, EL. 550 TO EL. 525; LIQUID LIMIT, 30.5%; PLASTIC LIMIT, 17.5%; AND PLASTICITY INDEX, 13.0%)

Description	Boring A	Boring P-1	Boring P-11	Boring C-1	Boring C-11	Remolded
Elapsed time after driving, in months.....	Before	1	11	1	11	Before
Water content (per cent of dry weight)...	24.7	22.8	22.0	23.8	24.4	24.7
Compressive strength (tons per sq ft).....	0.61	0.56	0.75	0.56	....[a]	0.28

[a] Not shown.

From Fig. 19 the writer's interpretation of the authors' data follows. It is first assumed that the degree of sample disturbance due to sampling operations is approximately the same for all borings because all samples were taken by the same procedure. With this limitation, point A in Fig. 19 for boring A represents the condition of the clay before pile driving. During pile driving the condition of the clay immediately adjacent to the piles moved from point A to point B or to a condition of complete remolding. In the first month after pile driving, this clay consolidated from point B to point C with a consequent increase in strength. In the following 10 months, the consolidation and increase in strength continued to point D, where the strength is greater than the original natural strength of the clay. The one test point, E, on the center of a pile group (about two and one-half diameters from the surfaces of the piles) indicates that the clay in this location was partly remolded and after 1 month had consolidated about one half as much as clay immediately adjacent to the piles.

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[18] "Review of the Cooperative Triaxial Research Program," by P. C. Rutledge, in "Triaxial Shear Research and Pressure Distribution Studies on Soils," Soil Mechanics Fact-Finding Survey, U. S. Waterways Experiment Station, Vicksburg, Miss., April, 1947, pp. 20-26 and 79-100.

[19] "Strength of Natural Clays," by P. C. Rutledge, paper presented at ASCE Meeting, Boston, Mass., October, 1948, *Civil Engineering*, November, 1948, p. 23 (abstract).

The data in Fig. 19 show why, when piles are pulled out of clay several months after driving, a cigar-shaped mass of clay sometimes adheres to the pile. The strength of the clay immediately adjacent to the pile (point D, Fig. 19) has become greater than the strength of the clay some distance from the pile (points A or E). On pulling, the clay shears through its weakest zone, which is either at the interface between pile and clay or at some distance away from the pile surface.

It is unfortunate for the authors' arguments concerning vertical consolidation settlements of the clay that the piles in this investigation were driven to point bearing in hardpan. With the piles penetrating through the clay mass as fixed vertical reinforcement 4.5 ft on centers, little vertical consolidation should be expected even if all the clay between the piles is completely remolded and tending to consolidate under its own weight. If an area is considered bounded by the center lines of a four-pile group, there is a mass 4.5 ft by 4.5 ft by 60 ft deep in which 5% of the total volume is perfectly rigid, consisting of four quadrants of a pile at the four corners of the mass. According to the authors' observations, the top of the clay heaved about 2 ft during driving, representing a total volume increase of about 3% as compared with 5% of the total volume occupied by the piles. From these rough computations some decrease in volume of the soil during driving is indicated.

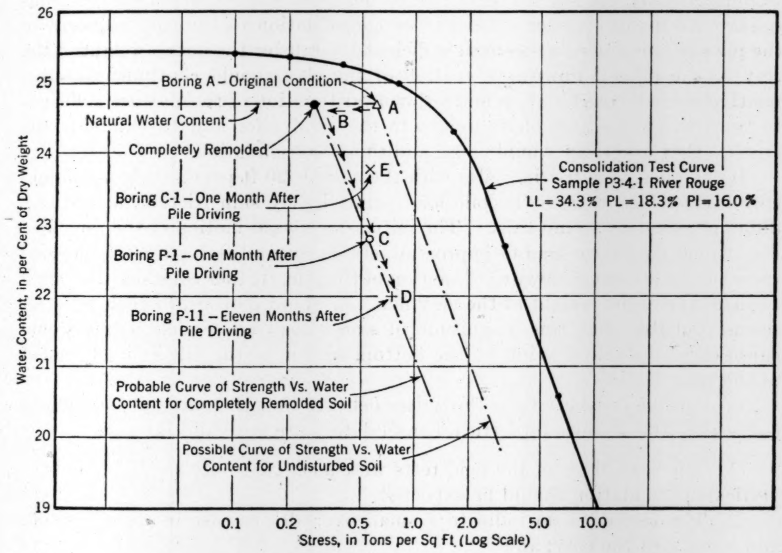

FIG. 19.—PLOT OF AVERAGE DATA STRENGTH VERSUS WATER CONTENT

The 95% volume of clay, assumed to be completely remolded, is no longer in equilibrium with the stresses resulting from the weight of overlying material and tends to consolidate and to move downward. However, 22% of the surface area of the clay mass is bounded by nonmoving pile surfaces on which downward motion will develop shear stresses, while the remaining 78% of the surface area is in contact with clay in the same condition as the mass under consideration. Ignoring the possibility of arching effects, a shear stress of 0.26 ton per sq ft at the interface between piles and clay is sufficient to carry completely the weight of the clay mass. From the authors' tests, the shear strength at this point immediately after driving was 0.14 ton per sq ft, and it increased to 0.28 ton per sq ft in the first month after driving. In other words, downward motion at the interface between pile and clay could develop a shear stress sufficient to transfer half the total weight of the clay to the piles immediately after driving. A stress transfer of such magnitude in this clay would certainly greatly decrease the tendency of the remolded clay to consolidate vertically. After 1 month the consolidation of the clay adjacent to the piles has produced a strength sufficient to transfer the entire weight of the clay to the piles if downward motion of the clay should continue. Thus a relatively small consolidation and a downward motion of the clay are sufficient to transfer a large part of its weight to the rigid piles and to eliminate the stresses that otherwise would cause additional consolidation.

If the piles were friction piles with their lower 20 ft penetrating a nonrigid clay layer which carried the pile load primarily by skin friction, an entirely different situation would exist. The initial downward motion of the remolded clay would transfer a load of approximately 30 tons to each pile. If this load increase did not cause downward motion of the pile, further consolidation could transfer the entire weight of the clay mass, or about 58 tons, to each pile. It seems probable that this magnitude of excess load on a friction pile would cause excessive skin friction on the bottom section of the pile and settlement of the pile.

From these considerations the writer believes that the authors' contentions concerning the absence of vertical consolidation are not valid because:

(1) The conditions of the field tests were such that only small amounts of vertical consolidation should be expected;

(2) The test data do indicate a small over-all decrease in water content and volume of the clay; and

(3) The conditions of this investigation cannot be extended to the entirely different conditions of friction piles.

In general, the authors' test data seem to provide a remarkable confirmation of the predictions by A. Casagrande,[4] M. ASCE, of the remolding effect of pile driving on clay which were made in 1932 on the basis of hypothetical evidence. Finally, the writer again expresses his appreciation to the authors for their efforts in obtaining these factual test data and publishing them in full so that all engineers can analyze and draw conclusions from the observations.

A. W. **Skempton.**[20]—A comparatively small number of case records are available describing field investigations conducted with the object of obtaining data on the behavior of clays in relation to piling. This paper is an interesting and valuable addition to that literature. In a problem such as that of the effect of driving piles into a stratum of clay an answer can be obtained in any given case only by reference to past records relating to a similar clay, since the difficulties of predicting the influence of pile driving on the clay properties, from laboratory tests alone, are very great.

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[20] Univ. Reader in Soil Mechanics and Asst. Prof., Imperial College, Univ. of London, London, England

As knowledge on this problem, therefore, is dependent upon the collection of reliable field data, and as the amount of the data is at present very limited, it is all the more important to examine in detail the results given in the present paper. Consequently, the writer feels justified in raising several points on which he does not find himself in full agreement with the authors, and he will also give a brief description of a job in which the driving of piles in a bed of clay of moderately high sensitivity was instrumental in leading to disastrous consequences.

As a basis for examining the results of the authors' investigation, the writer has evaluated the average strength in both the upper and lower clay strata, as found in borings A, P-1, P-11, and C-1 (Figs. 8 and 9). These average values are given in Table 2, from which it would seem that the following five conclusions may be drawn:

TABLE 2.—AVERAGE COMPRESSIVE STRENGTHS OF SAMPLES REPORTED BY AUTHORS

Strata	Elevation (ft)	COMPRESSIVE STRENGTHS (Kg per Sq Cm)[a]				ADJACENT TO PILE						CENTER OF PILE GROUP, 3 WEEKS AFTER DRIVING		
						3 Weeks After Driving			11 Months After Driving					
		Undisturbed	Fully remolded	Sensitivity[b]	Remolding loss[c]	Strength (kg per sq cm)	Loss[d] (%)		Strength (kg per sq cm)	Loss[d] (%)		Strength (kg per sq cm)	Loss[d] (%)	
							Undisturbed	Remolding		Undisturbed	Remolding		Undisturbed	Remolded
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
Upper	570 to 550	0.48	0.15	3.2	0.33	0.25	48	70	0.39	19	27	0.36	25	36
Lower	550 to 525	0.61	0.27	2.3	0.34	0.56	8	15	0.72	−18	−32	0.56	8	15

[a] 1 kg per sq cm = 14.2 lb per sq in. [b] Sensitivity (dimensionless) = $\frac{\text{undisturbed strength}}{\text{remolded strength}}$. [c] Loss in strength on full remolding. [d] Loss in strength as percentages, respectively, of undisturbed strength and full remolding loss.

(1) The upper clay has a moderate sensitivity, the average value for the sensitivity of the clay between El. 570 and El. 550 being 3.2. In the light of this result the writer does not agree with the statement (under the heading, "General Soil Conditions") that:

> "In their undisturbed state, the samples had an unconfined compressive strength approximately eight times that of the same samples when remolded in the laboratory at their natural water content."

This appears to be true only for the two samples at El. 571 and El. 575, and it is clear from Fig. 8 that these two samples are not characteristic of the upper clay.

(2) The upper clay adjacent to the pile, 3 weeks after driving, suffered a loss in strength of 48%, which amounts to 70% of the full remolding loss. At the center of the pile group, at a distance of about two and a quarter diameters from the surface of the pile, the corresponding losses were 25% and 36%. Thus, in this clay of moderate to low sensitivity the strength 3 weeks after driving varied from one half to three quarters of the original strength and, moreover, this loss amounted to between one third and two thirds of the maximum possible reduction. These results are not so drastic as those suggested by A. Casagrande[4] M. ASCE (and he was certain to err on the side of safety in making a generalization); but they appear, nevertheless, to be rather substantial losses and, at least as far as the upper clay is concerned, the writer cannot altogether agree with the conclusion, drawn by the authors (under the heading, "Conclusions") that the effect of pile driving, even for friction foundation, would quite probably be beneficial.

(3) In the same connection it may be noted from Table 2 that the strength of the upper clay, adjacent to the pile, 11 months after driving is still only about 80% of its original value.

(4) Corresponding to a loss in strength of between 25% and 48% there would presumably be an appreciable increase in compressibility and a decrease in modulus of compression. These effects, again, cannot be considered as beneficial from the point of view of foundation behavior.

(5) Turning now to the lower clay between El. 550 and El. 520 the writer is in substantial agreement with the authors in their conclusion that the effect of pile driving is comparatively unimportant. It has to be remembered, however, that the lower clay is one of low sensitivity—namely, 2.3 (average), and that many geologically recent estuarine and marine clays have sensitivities of between twice and five times this value.

A case in which the effect of pile driving appears to be one of the main causes of an engineering disaster may now be described. In 1922 an attempt was made to provide a friction pile foundation for a grain silo in Durban Harbour, Union of South Africa.[21] A cross section of the foundation and of the underlying deposits is shown in Fig. 20, from which it will be seen that the piles were driven through a layer of sand into a stratum of clay about 42 ft thick overlying (at a depth of 80 ft below surface) a second layer of sand. Rock, consisting of thinly bedded limestones and shales of Cretaceous age, was found at a depth of 92 ft. Borings showed that the clay was "tough" and it was decided to use it as a foundation material for the piles. Seventeen widely spaced test piles were then driven, from which it was concluded that a working load of about 45 tons per pile could be obtained. These tests were not altogether satisfactory since the piles were driven in the storage annex site (see Fig. 20) when the excavation had reached about only one half of its final depth, and the piles were therefore more deeply embedded in the sand than was to be the case in the actual foundation.

Pile driving was commenced in the annex site in August, 1921, by which time the cofferdam of sheet piling had been completed, and it appears that the sets became progressively worse. The specified set had been 2½ in. for ten blows but the final average set for the annex piles was 8.6 in. This would seem to be a result of remolding in the clay, but a contributory cause may also have been the excavation for the working house which, by December, 1921 (when the annex piling was completed), was well advanced.

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[21] "Report of the Durban Grain Elevator Commission," Govt. Printers, Cape Town, Union of South Africa, 1924.

At that time the sheet piling separating the two excavations was seen to be bulging between the tie bars. This indicates lateral displacement of the ground, which is not surprising since the piles were driven at about 4-ft centers over the entire area of the annex (200 ft by 100 ft in plan) and since photographs taken at the time show that the waling used in the separation cofferdam was inadequate.

In spite of the ominous progressive decrease in driving resistance the annex raft was cast, and additional ties were provided to strengthen the cofferdam.

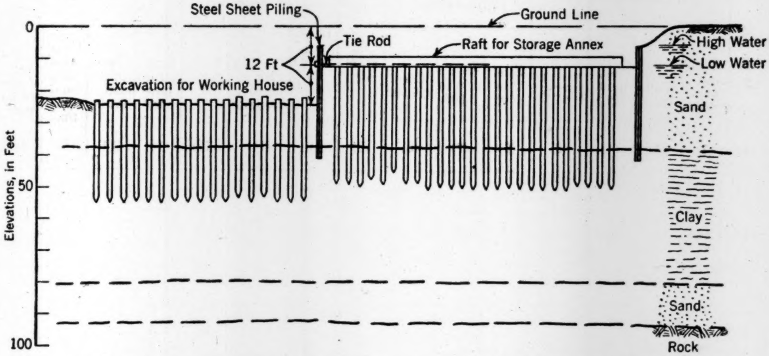

FIG. 20.—CROSS SECTION THROUGH FOUNDATIONS; GRAIN ELEVATORS, DURBAN, UNION OF SOUTH AFRICA, 1922 (CONCRETE PILES, 15 IN. SQUARE)

Meanwhile, excavation in the working house site continued and in February, 1922, pile driving was begun in this area. From the start the sets were worse than the specified value, and when the working house piling was completed in April, 1922, the average set for ten blows being obtained was of the order of 25 in. It may be noted from Fig. 20 that these piles are embedded for more then half their length in the clay, and the most obvious reason for the extremely bad set is that the clay was suffering a progressive loss in strength due to remolding, especially since the working house pile driving was the only operation under way during the period from February to April.

During the next 6 weeks twenty-eight extra piles were driven, and construction of the reinforced concrete structure on the annex raft was commenced. By the middle of June, however, the raft had settled about 1 in., and voids were discovered beneath it. After discussion it was concluded that the working house piles were entirely inadequate and it was decided to sink cylinders to rock for this foundation and to grout beneath the annex raft.

Unfortunately the cylinders were not driven with sufficient care to avoid an excessive loss of ground, and the raft settled many inches, with consequent severe cracking. Eventually the entire foundation was abandoned, and entirely new foundations were made by sinking (with careful control) a series of compressed air cylinders, 9 ft and 12 ft in diameter, to the rock. The loss involved on the foundations was estimated at rather more than £200,000.

The writer can give the results of some laboratory tests on the clay since, in 1946, he was requested to act as consultant for the foundations of a large structure which was to be built at a distance of about one quarter of a mile from the grain silo.

Borings showed a succession of strata similar to those beneath the grain silo, and the clay at the new site was not only placed at approximately the same elevation but corresponded with the visual description when the 1921 borings were being made. The relevant laboratory tests are summarized in Table 3, from which it will be seen that the clay has a shear strength of about 1,200 lb per sq ft in its undisturbed strength and a sensitivity of about 4.0 (Cols. 11 and 12, Table 3). Typical stress-strain curves are shown in Fig. 21 and they are characteristic of a clay of moderately high sensitivity. The laboratory tests

TABLE 3.—RECORD OF BORINGS AT DURBAN HARBOR, UNION OF SOUTH AFRICA, MADE IN 1946

Depth (ft)	CONSISTENCY				PARTICLE SIZE (MILLIMETERS)			Specific gravity	Density (lb per cu ft)	SHEAR STRENGTH		CONSOLIDATION			
	Water content (w)	Liquid limit (LL)	Plastic limit (PL)	$\frac{w-PL}{LL-PL}$	Sand 2 to 0.06	Silt 0.06 to 0.002	Clay <0.002			One half compressive strength[a]	Sensitivity	Overburden pressure[b]	Preconsolidation load[a]	Compressibility (sq ft per ton)	Coefficient of consolidation[d]
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)
BORING A															
45	58	111	28	0.36	....	....	....	....	100	1,210	3.9	1.5	....	....	....
48	63	112	33	0.38	....	....	....	2.66	99	1,270	....	1.55	3.2	0.023	0.021
BORING B															
46	56	....	....	....	....	....	....	....	102	1,180	4.5	1.5	....	....	....
50	64	115	32	0.38	....	....	....	....	101	....	....	1.65	....	....	....
53	64	113	35	0.37	11	47	42	2.64	99	1,270	....	1.7	3.4	0.018	0.037
BORING C															
49	65	109	31	0.43	....	....	....	....	99	1,110	....	1.6	3.3	0.025	0.020
BORING D															
48	46	88	27	0.31	19	48	33	....	107	1,330	3.6	1.55	....	....	....
AVERAGE BORINGS, A TO D															
....	60	108	32	0.37				2.65	101	1,230	4.0	1.6	3.3	0.022	0.026

[a] Pounds per square foot. [b] Tons per square foot. [c] Compressibility. [d] Square centimeters per minute.

indicate that the clay is lightly overconsolidated. This is probably caused by partial drying subsequent to its formation but prior to the deposition of the overlying sand, since in one boring the clay was seen to be oxidized to a brown color in the top 2 ft of the stratum.

In conclusion, the writer wishes to emphasize again the great value of observations such as those recorded by the authors, but he feels that experience such as that at Durban and, to a more limited degree, the behavior of the upper clay at Detroit, Mich., indicate that in some clays the effect of pile driving can be detrimental. The lower clay of Detroit, on the other hand, shows that this is not always the case. More field research is required before engineers can be reasonably certain as to the border line separating the harmful from the beneficial effects of piling in clays.

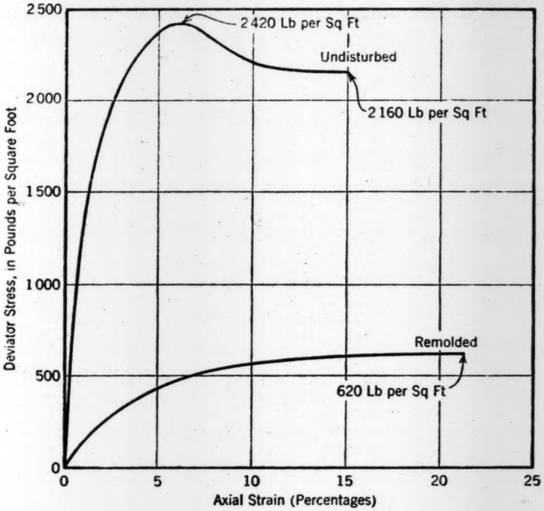

FIG. 21.—TYPICAL STRESS-STRAIN CURVES (ANGLE OF SHEARING RESISTANCE ϕ EQUALS ZERO IN UNDRAINED TRIAXIAL TESTS)

JAMES R. SCHUYLER.[22]—Valuable comparative soil strength data related to the effect to driving piles into soft clay are recorded in this paper. Qualitative information derived by the writer from model tests at the Soil Mechanics Laboratory of Princeton University at Princeton, N. J., confirms the authors' conclusion that the original forecast of A. Casagrande, M. ASCE, in 1932, concerning the detrimental effect of pile driving into clays, was extreme, especially when other possible sources of disturbance are analyzed. However, the thesis advanced by the authors that a soft clay stressed by pile driving will not reconsolidate and settle under its own weight is unduly magnified in view of the following information, unless settlement observations between piles can be presented to support that thesis.

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[22] Prin. Engr. (Soils), New Jersey State Highway Dept., Trenton, N. J.

As a result of a combined field and laboratory study, Gregory P. Tschebotarioff, M. ASCE, advanced a hypothesis to the effect that the remolding caused by pile driving was smaller in magnitude and more limited in extent than was the remolding accompanying the distortions induced by a heave in the bottom of an excavation.[13] Model studies in gelatine were performed to check the disturbance produced by driving piles into plastic clay as compared to the disturbance caused by an unbalanced excavation. The model studies confirmed this hypothesis in a qualitative manner.

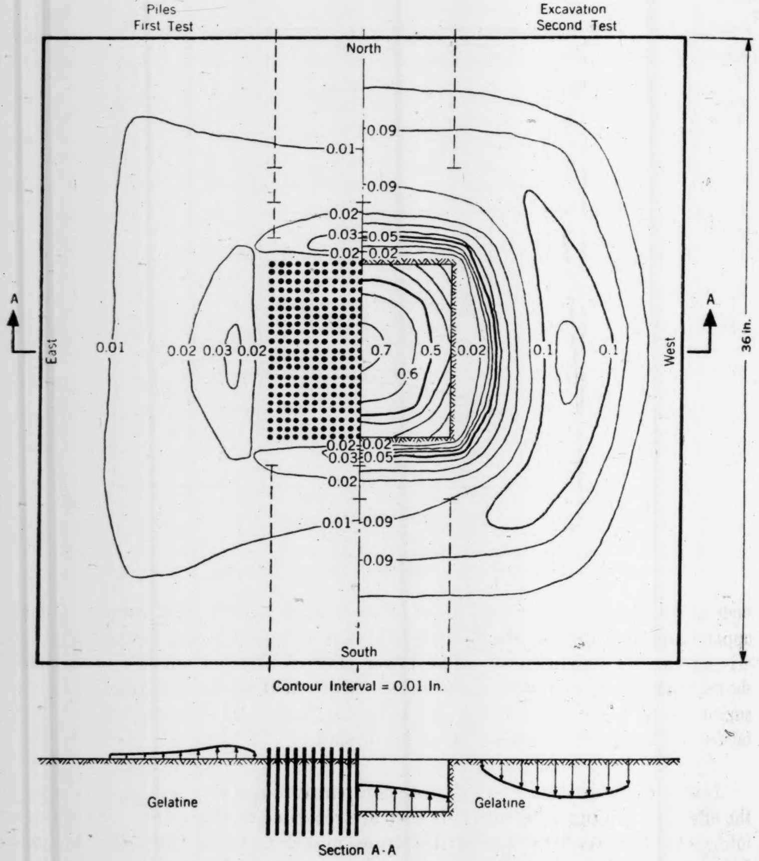
Piles
First Test
Excavation
Second Test
North
0.09
0.01
0.09
0.02
0.03
0.05
0.02
0.02
A
A
East
0.01
0.02
0.03
0.02
0.7
0.5
0.02
0.1
0.1
0.6
West
36 in.
0.02
0.02
0.03
0.05
0.02
0.01
0.09
0.09
South
Contour Interval = 0.01 In.
Gelatine
Gelatine
Section A-A

FIG. 22.—COMPARISON OF SURFACE DISPLACEMENTS

A 3-ft by 3-ft metal tank, 1 ft deep, was constructed and filled with gelatine for each test after devices were inserted in the tank to measure displacements within the mass of gelatine. The comparative study was performed: (1) By measuring the deformations of the gelatine surface and the displacements within the mass of gelatine when 441 piles, 4.25 in. long and 0.125 in. thick, were driven into the gelatine, outward from the center of the gelatine surface in a spiral sequence; and (2) by taking measurements at the same locations when an excavation was simulated in the same area to which the piles were confined. All displacements were measured to accuracies of 0.001 in. in relation to a rigid frame. Mechanical extensometers were adjusted by a micrometer screw until contact with ultrathin metallic plates on the surface was signaled by the glow of a neon light. Within the mass of gelatine thin metal disks were connected to light metal rods, which extended through openings in the walls and bottom of the tank. These rods were incased in vaseline-filled glass tubes which were extracted after the gelatine had set. Horizontal displacements were measured only along the north-south axis of the tank (Fig. 22), and a comparison with the displacements along the east-west axis of the tank indicated that these devices did not seem to restrain to any appreciable extent the freedom of gelatine displacements in the vertical direction. The strength characteristics of the gelatine were the same for both tests.

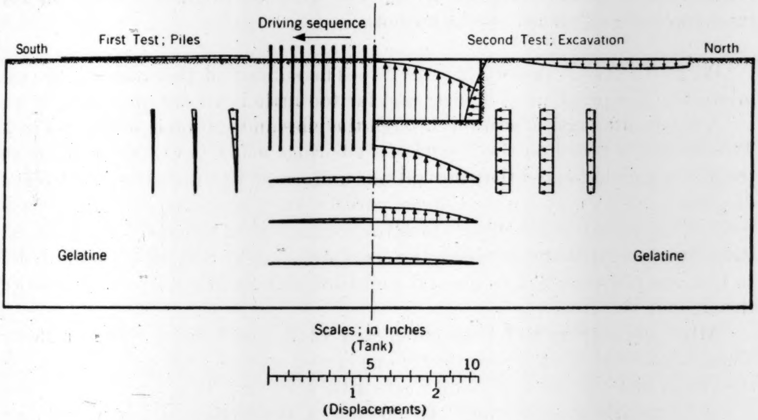

FIG. 23.—COMPARISON OF DISPLACEMENTS BY MODEL TESTS

The results of both tests are compared graphically in Figs. 22 and 23. The heave between piles reported by the authors is synonymous with the results of this model test. Furthermore, Fig. 23 shows that, if a large number of piles is driven, the gelatine is stressed for an appreciable distance surrounding the volume occupied by the pile group.

Since the authors report that the pile driving reduced the compressive strength of the clay slightly more than half of the maximum possible reduction, the writer contends that this weakening of the soil structure will be prevalent throughout the zone of clay stressed by pile driving, as indicated by Fig. 23, and that this clay will consolidate under its own weight. Such consolidation would not be detrimental to the structure described by the authors; but, in cases where the soil between piles is utilized for the support of a floor and curtain walls, the authors' hypothesis concerning the process of consolidation initiated by the degree of disturbance reported in the paper to cause no settlement to the ground surface should be accepted with reservations until more certain evidence is presented to substantiate their speculative assumption. Mr. Tschebotarioff has reported consolidation of a glacial clay deposit which was injurious to a floor and partitions in the basement of a building which was founded on long steel piles driven to a firm stratum.

For the present a moderate theory concerning the increase in compressibility which some clays undergo when structurally weakened should fit the category midway between the extremes of Mr. Casagrande's original assumption and the hypothesis advanced by the authors.

B. A. KANTEY,[23] Assoc. M. ASCE.—The subject of this paper is of vital interest to the practicing engineer and far too little is known about it.

A composite vertical section through the piles and subsoil is shown in Fig. 2. This section is based on the results of six borings made at various locations on the site, which indicated that the soil was uniform in its properties in horizontal directions over the entire area investigated. In view of the very small variations in moisture content upon which the conclusions of the authors were based, the writer feels that the details of the six borings might well have been included to indicate the variation in natural moisture content in a horizontal direction throughout the area.

After pile driving had been completed additional borings were put down. These additional borings consisted of one boring next to a pile and one boring in the center of a pile group about 3 weeks after driving. This cycle was repeated about 11 months after driving. In effect then, the original boring A, and these four additional borings, combine to form one set of results which show the variation in moisture content with time and with distance from a pile. The writer feels that the conclusions drawn in this paper are so important that additional sets of results would have been of great value. Could these additional sets of results have been augmented with the extra borings suggested subsequently in this discussion, the data presented might have had even greater value.

Basically the paper undertakes to show that pile driving in the material investigated does not cause such complete and deleterious remolding as has been assumed since the publication of a paper by A. Casagrande,[4] M. ASCE, in 1932. Consequently, the disturbing effect of pile driving in this material would not be so detrimental and therefore would not cause harmful settlement of the surface.

The authors point out that, in spite of Mr. Casagrande's warning that his tentative conclusions should not be accepted without large-scale field research to back them up, these tentative conclusions have been widely accepted. The conclusion with which the authors mainly disagree is that the clay immediately surrounding a pile would be completely remolded to a distance of one half the diameter of the pile, and that it would be disturbed sufficiently to cause a large increase in compressibility to a distance of one and a half diameters.

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[23] Research Officer, Soil Mechanics Div., National Bldg. Research Inst., Pretoria, Union of South Africa.

From the results produced by the authors it seems clear that this is not the case for the material discussed. In fact, the evidence produced shows only partial remolding adjacent to the pile and practically no effect in the center of the pile group. Since this result was also found by W. S. Housel, M. ASCE, and J. R. Burkey,[24] working in the same area, it seems logical to accept the conclusion that driving piles into the material under discussion would not have the detrimental effects that could be forecast on the basis of the work reported by Mr. Casagrande.

It seems clear, then, that there is a great need for more research into the effect of remolding. What appears to be particularly needed is some tie-up between the sensitivity ratio as expressed by the ratio between the unconfined compression test on undisturbed and on remolded samples, the Atterburg limits, the natural moisture content, and the change in void ratio with pressure for undisturbed and remolded conditions.

Tentative examination of results available in various papers seems to indicate the possibility of some kind of correlation between sensitivity to remolding and the Atterburg limits where the natural moisture content is below the liquid limit. A clay with a low liquid limit and a low plasticity index generally appears to be less sensitive than one with higher limits, and this assumption does not appear to be unreasonable when considering Mr. Casagrande's suggested explanation of the cause of remolding.[4]

Mr. Casagrande also suggests that a soil with a simple honeycomb structure will show a smaller difference in compressibility between the undisturbed and remolded states than one with the complex structure described in his paper, and that the larger the plasticity index, the more compressible the soil. Frank A. Marston, M. ASCE, has also pointed out that certain clays appear to have a form of structure permitting a high void ratio yet offering considerable resistance to change in volume.[25] When the structure is broken down, the water in the larger voids is released and the once stiff clay becomes soft and plastic. Accepting Mr. Casagrande's explanation of the cause of this behavior leads to the hypothesis that it is essentially a colloidal phenomenon. This may then indicate that a measure of the colloidal clay content might be a measure of a clay's sensitivity to remolding. It also appears possible that low colloidal clay content would lead to a low liquid limit and, hence, a low plastic limit which could then be correlated with sensitivity.

All the examples found have dealt with clays having a natural moisture content close to, but below, the liquid limit. In Durban, Union of South Africa, there is a clay that has a liquid limit of between 50 and 60 and a natural moisture content generally slightly above the liquid limit. This clay has a rather low sensitivity ratio based on the unconfined compression test but it is very much more compressible when remolded than when it is in the natural state. There is a possibility, therefore, that the criteria for sensitivity may have to be correlated with the natural moisture content and its relation to the liquid limit.

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[24] "Investigation to Determine the Driving Characteristics of Piles in Soft Clay," by W. S. Housel and J. R. Burkey, *Proceedings*, 2d International Conference on Soil Mechanics and Foundation Eng., Rotterdam, Vol. V, 1948, pp. 146–154.

[25] *Journal*, Boston Soc. of Civ. Engrs., April, 1932, p. 210.

As regards the Durban clay, some years ago a serious failure occurred in a grain elevator erected on this clay. Although a possible cause for this failure could be the remolding effect of pile driving, this has not yet been definitely established. In the near future, however, it is the intention to conduct a complete investigation and it is hoped that this will lead to more information on the relation between disturbed and undisturbed samples.

The writer feels that there is not sufficient evidence to warrant the conclusion of the authors that the water must have migrated horizontally from the disturbed zone near the pile to the much larger mass of relatively undisturbed clay between the piles. In Fig. 9(*a*) curves are given for the original moisture content, the moisture content about 3 weeks after driving, and about 11 months after driving.

Using the average values quoted, it can be seen that during the first 3 weeks the moisture content dropped by 1.9% whereas 11 months later it had only dropped a further 0.8%. Although this does show a progressive decrease in moisture content with time, it would have been of interest to have obtained several intermediate readings to see when most of the change took place, to have obtained a curve of change in moisture content with time, and to have found out how much of the change in moisture content was a result of the actual pile driving and how much was a result of the consolidation caused by driving. This information might well have been supplemented with void ratio versus load curves for the undisturbed and remolded states.

The writer cannot agree entirely with the finding that the water did not flow upward through the clay mass. Examination of the curve for boring P-11 in Fig. 9(*a*) can be interpreted to indicate a perceptible decrease in moisture content (of 22.2% to 21.4%) from the lower layers upward. The variations in moisture contents are too small, however, to warrant any definite conclusions and the writer hesitates to suggest that the evidence indicates a definite upward flow of moisture.

The authors suggest that the water must then have migrated horizontally; but this too is not generally borne out by the results. A slight indication of this trend is seen in Fig. 9(*c*) where, below El. 530, boring C-11 appears to have a slightly higher moisture content than boring C-1; but this does not appear to be a general characteristic. Even though the slight heaving of the ground during driving may have caused a very slight swelling of the undisturbed clay, which would have been sufficient to take up the reduction in moisture content of the clay next to the pile, borings might have disclosed this fact and a series of borings from the center of a pile group toward the piles soon after driving might have given valuable additional data.

In summary, then, the writer feels that, although the evidence presented in the paper supports the conclusions drawn, the evidence is not sufficiently clear cut to obviate any argument. Additional information along the lines indicated herein may well have served to clarify the position and it is regretted that more borings were not made.

D. P. Krynine,[26] M. ASCE.—Subsidence of pile foundations driven through unconsolidated soft deposits is one of the most troublesome of engineering problems, particularly along the west coast of the United States. The question as to whether the pile driving itself may artificially convert a consolidated clay into a nonconsolidated clay and hence cause a drag on the piles with subsequent settlement of the structure is of vital interest. Hence the publication of the present paper is timely; and, although end bearing piles are considered, all conclusions of the paper as to the remolding of the clay by the piles are valid in the case of friction piles as well.

Support of a Friction Pile.—The well-known fact that, when a pile driven through clay is pulled out, there is generally some clay on it suggests that the adhesion of the clay to the pile is greater than the shearing strength of the clay material. Thus, it may be safely admitted that a friction pile in a state of equilibrium is kept in place by the vertical component of the shearing force acting along the surface of the pile in its final position. The maximum of this component, or the bearing capacity of a friction pile, corresponds to the stage at which the average shearing stress at the surface of the pile equals the average shearing strength of the surrounding clay material (case of limit equilibrium). This condition of equality of the average shearing stress and the average shearing strength along a failure surface at the stage of limit equilibrium is a generally accepted assumption and as such it will be used in this discussion.

Stability of the Experimental Piles.—The writer uses the term "experimental piles" to designate the piles shown in Fig. 2. Consider the limit equilibrium of these piles 11 months after the pile driving, when the average shearing strength of the clay between El. 550 and El. 525 was roughly one half of the unconfined compressive strength (0.75 ton per sq ft). Assume that this value may be applied to the shearing strength of the soil material below El. 550 (in other words, assume that sand and gravel are replaced by weaker clay). The average circumference of a pile is about 3.3 ft and the length of a pile below El. 550 is about 60 ft. Therefore, the bearing capacity of an experimental pile, neglecting the support of the remolded clay above El. 550 and the bearing on the hardpan, would be $\frac{1}{2} \times 0.75 \times 3.3 \times 60 = 75$ tons. Certainly it is difficult to believe that this pile may settle under ordinary loading after the final shearing strength is acquired by the clay; and it may be stated that at the age of 11 months the experimental piles were substantially supported by the lower part of the clay deposit (below El. 550), and that in all probability very little pressure reached the hardpan (see Fig. 2).

Increase of the Bearing Power of the Experimental Piles Due to the Pouring of Footing.—The unconfined compressive strength, and hence the shearing strength of the lower part of the clay deposit (below El. 550), did not change very much after pile driving (boring P-1, Fig. 8) but increased considerably (boring P-11, Fig. 8) after the footing was poured (between 1-month and 11-month borings, as in Fig. 2). Thus, the increase in shearing strength of the clay, between 1-month and 11-month borings, is apparently a result of the application of the load to the pile—that is, a result of the increase of normal stress in the mass. Obviously, the shearing strength (boring P-11, Fig. 8) increased gradually, and immediately after pouring concrete the load was taken up by the pore pressure thus causing hydrostatic excess next to the pile and flow of moisture away from it.

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[26] Cons. Engr.; Visiting Prof. of Eng., Univ. of California, Berkeley and Los Angeles, Calif.

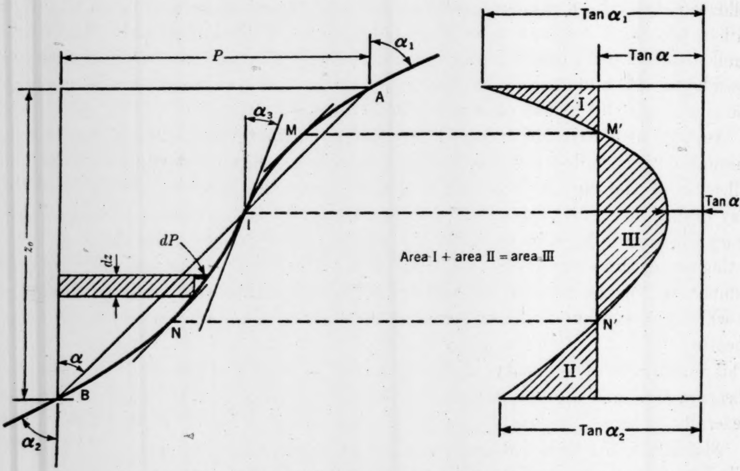

FIG. 24.—DIAGRAMMATIC COMPARISON OF THE LOADS CARRIED BY A FRICTION PILE AT DIFFERENT DEPTHS

Remolding of the Upper Part of the Clay Deposit.—The upper part of the clay deposit certainly was remolded, and the application of the load from the footing did not increase its shearing resistance appreciably—because much of the material was squeezed from the upper part of the clay deposit during the pile driving, and there was not sufficient material to be compacted. At the lower part of the deposit, however, because of the higher shearing resistance of the material and its deep location, there was sufficient confined clay to be strongly compacted, with a corresponding increase in shearing strength.

Graphic Representation of the Bearing Power of a Friction Pile.—In June, 1948, L. LeRoy Crandall, Assoc. M. ASCE, reported the results of a measurement of the bearing capacity of a Raymond concrete pile, cast in place in a corrugated steel shell, at Los Angeles, Calif.[27] At the center of the pile there was a steel pipe with SR-4 electric strain gages fixed at its surface. From the strains measured at different elevations and for different loads (and using the moduli of elasticity of the materials), the load carried by the pile was computed and represented graphically in the form of a family of curves, each curve corresponding to a load. Fig. 24 represents a similar curve in a simplified and idealized form. This curve may be analyzed as follows: Let P equal the total load carried by the friction pile; d, its diameter; and z, the depth measured from the top of the pile, the total length of the pile being z_o. Then the unit friction f (or shearing resistance of the soil at a given point) is

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[27] "Electrical Resistance Gauges for Determining the Transfer of Load from Driven Piling to Soil," by L. LeRoy Crandall, *Proceedings*, 2d International Conference on Soil Mechanics and Foundation Eng. Rotterdam, 1948, Vol. IV, p. 122.

$$f = \frac{1}{\pi d} \frac{dP}{dz} \quad \text{(4)}$$

It is obvious that the derivative $\frac{dP}{dz}$ is tangent of the angle made by the tangent to the curve at a given point with the vertical. If the shearing resistance is constant along the entire length z_o of the pile, the curve in question would be the straight line AB making an angle α with the vertical.

In this case,

$$\frac{dP}{dz} = \frac{P}{z_o} = \tan \alpha \quad \text{(5)}$$

At the points where the tangent to the curve is parallel to line AB (points M and N) the shearing resistance of the material is

$$f = \frac{1}{\pi d} \frac{P}{z_o} = \frac{1}{\pi d} \tan \alpha \quad \text{(6)}$$

Finally, the minimum "friction" corresponds to the inflection point I of the curve; thus:

$$f_{\min} = \frac{1}{\pi d} \tan \alpha_3 \quad \text{(7)}$$

In Fig. 24, sum of the cross-hatched areas I and II is equal to area III. Figs. 25 and 26 represent such curves for the experimental piles between El. 565 and El. 525. Fig. 25 corresponds to the time before pile driving and Fig. 26 to a time 11 months after pile driving. In both cases the experimental piles have been considered friction piles. With the pile as a unit the influence of the remolding in this case is insignificant; but Fig. 26 should clearly show that the upper part of the clay deposit (above El. 550) contributed only slightly to the stability of the pile.

To reduce numerical computations, Figs. 25(a) and 26(a) have been prepared on the basis of simplified unconfined compressive strength diagrams. It is interesting to notice from the diagrams in Fig. 8 that, apparently at the top of each of the two parts of the clay deposit under consideration (the upper and the lower), there was a crust of about 5 ft, as in the case of Chicago or Boston clays.

Flow of Moisture.—Moisture in an earth mass moves under the influence of some difference of potentials—in this particular case, pore pressure difference. The flow problem in this case is quite analogous to that of the motion of a water drop on an oblique plane. The movement in the latter case occurs along the direction of the maximum slope, and in the former along the direction of the maximum hydraulic gradient offered to the moisture at a given point. At the extreme upper part of the clay deposit moisture probably moved vertically, whereas at deeper strata lateral motion from a pile (zone of large pore pressure) to the center of the cluster or toward the outside (zone of smaller pore pressure) was both a possibility and a probability.

Conclusions.—In this paper two different clay materials are considered—the upper and the lower parts of the clay deposit. The ratio of their thicknesses and their relative position made any considerable settlement impossible, although there was probably some insignificant consolidation of the upper part. If the total soil deposit consisted exclusively of the material in the upper part, there would have been remolding and consolidation of clay combined with settlement. However, if the total soil deposit consisted exclusively of the material in the lower part of the deposit, probably there would have been

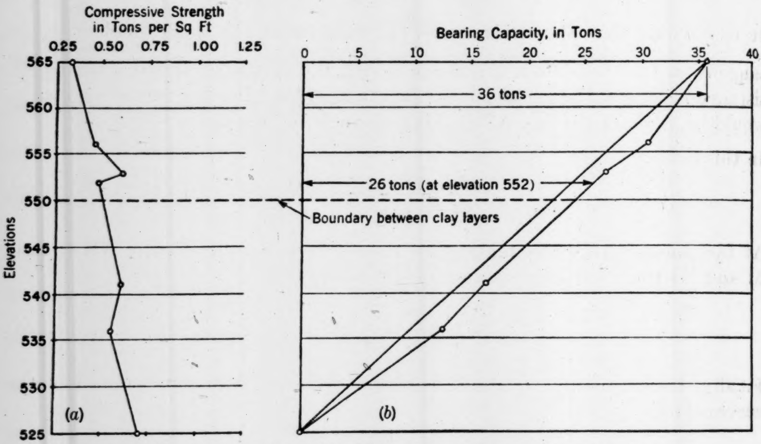

FIG. 25.—LOAD THAT WOULD BE SUPPORTED BY THE "EXPERIMENTAL PILES" BETWEEN EL. 565 AND EL. 525 IF THE CLAY WERE NOT REMOLDED

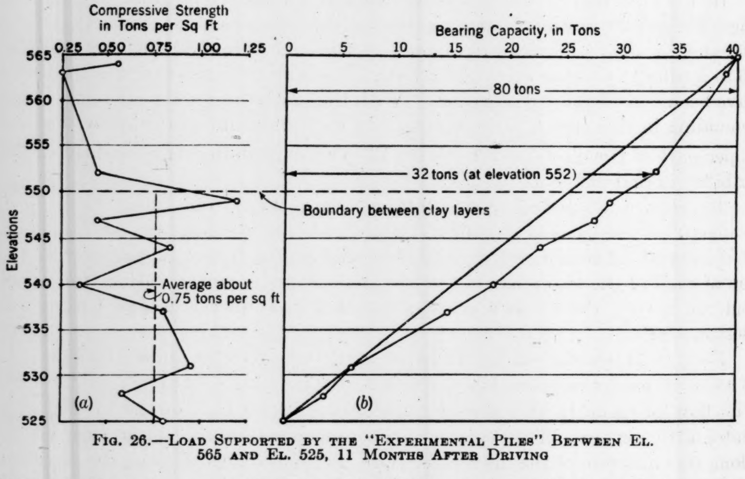

FIG. 26.—LOAD SUPPORTED BY THE "EXPERIMENTAL PILES" BETWEEN EL. 565 AND EL. 525, 11 MONTHS AFTER DRIVING

practically no settlement, except perhaps a brief period after pouring the footing.

A reader concludes from this paper that in a general case pile driving in clay may or may not cause consolidation and settlement, according to the properties of the clay. This conclusion does not contradict the opinion of A. Casagrande, M. ASCE. Nevertheless, the paper has destroyed forever the prejudice, very common among engineers, that the clay always loses much strength at points adjacent to the pile. As demonstrated by the paper, this loss of strength may or may not occur.

The paper would have furnished more topics for thinking if the water content of the upper part of the deposit (Fig. 9), and especially the void ratios of the material through the deposit before and after pile driving, were known to the reader. However, it is a valuable contribution to the knowledge of clay properties, and the authors are to be commended for their fine work.

Robert F. Legget,[28] M. ASCE.—So many questions with regard to piles and pile driving remain unanswered that this paper is a most welcome addition to the literature of foundation engineering. The authors are to be congratulated on their initiative in arranging such useful field tests and in presenting their findings so concisely and clearly. The excellence of the presentation leaves the impression that the conclusions are logical and well founded. This was the writer's feeling after a first reading of the paper. Several subsequent re-readings, and discussion of the paper with colleagues, however, have raised such important questions that it seems only right to place these on record. The closing discussion should further enhance the value of the paper and may point the way for other investigators to develop still further the work which the authors have so well pioneered.

Possibly the most important comment to be made is that all the conclusions of the authors are based on the interpretation of results from only one set of samples for each condition under review (that is, the P and C positions, 1 month and 11 months after driving). In this particular case, the lack of symmetry in the layout of the piles, the known variations in pile driving operations, and the fact that the problem is one in soil mechanics combine to make the absence of a duplicate set of bore holes unusually unfortunate. Knowing the care with which the sampling and testing must have been done, the writer would expect that duplicate samples would confirm the results obtained. In view of the importance of the results, however, such confirmation would have been very valuable particularly since all conclusions are based on the assumed accuracy of bore hole A as typical of the local soil conditions.

One interesting omission from the paper is any reference to the actual movement of soil which must accompany pile driving, with the exception of the statement (under the heading, "Introduction") that "* * * the ground surface between the piles heaved about 2 ft during the driving operations." A simple computation will show that this amount of heave accounts for about one half of the volume of soil actually displaced by the pile shells, allowing for a gradual tapering off of the heave outside the outer row of piles. As a consequence, some lateral flow of the clay, or some sudden escape of water, must have taken place. The direction, nature, and magnitude of this action would be dependent on the order in which the piles were driven. Such information is not given in the paper Do the authors consider that the order of pile driving is of significance in relation to their studies? Would they relate their answer, whether positive or negative, to the lateral displacement of soil or to the release of water during driving?

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The moisture contents of the clay as recorded by the authors are singularly interesting. (Detailed study, however, is hindered by the fact that the values for samples from hole A are plotted differently in Figs. 9(*a*) and 9(*b*).) The authors state (under the heading, "Results of Tests") that "* * * no trend can be discerned in the values of the water content in the soil at the center of a surrounding group of piles * * *." The data plotted in Fig. 9(*b*) seem to show some slight reduction in the soil moisture content after driving, thus emphasizing how valuable a duplicate set of records would be. Possibly the authors have more extensive records than those plotted, on which their comment was based.

The authors state (under the heading, "Interpretation of Results") that "* * * the water migrated horizontally from the disturbed zone near the pile to the much larger mass of relatively undisturbed clay between the piles." If this conclusion is correct, it should be possible to correlate the rate of flow presented by the authors' records with the known consolidation characteristics of the blue clay, since all the relevant factors are available. Have the authors investigated this correlation? It would be more than interesting to know if such an investigation confirms what the authors regard as the only explanation of the water loss.

The first borings taken after the pile driving are stated by the authors (under the heading, "Introduction") to have been put down 3 weeks after the piles were driven. In view of the common experiences when piles are "set up" within a relatively few hours after having been driven, one is forced to the conclusion that soil moisture contents taken 3 weeks after driving do not give a complete picture of the soil disturbance in question. The "practical" explanation of "setup" is that water is released by the sudden disturbance of displacement of clay soil, during driving, and that it tends to lubricate the pile as long as driving continues. Subsequent readjustment of the disturbed clay, within the first day or two after driving, explains the usual difficulty encountered if redriving must be attempted.

"Practical" explanations are often wrong but the writer has been able to observe the accuracy of this particular explanation in several locations. Fig. 27 shows some creosoted timber piles that were driven into hard Leda clay at a remote location on the north shore of the Gulf of Saint Lawrence, in Canada. Because of tidal conditions, it was possible to observe, at periods of low spring tides, the action of pile driving on the clay. Fig. 27 shows, clearly, how the ground heaved around the piles. The heaving around the pile on the right is typical of all that observed, the lumps of clay around the center pile having been caused by driving on to a small boulder buried just below the surface. In almost all cases, such as in the case of the pile on the right, in addition to the ground heaving to a distance about 2 ft from the pile, clay was squeezed up, in the form of a soft cream, around the pile in an annular ring not more than 1 in. wide. The clay is very stiff in its natural state so that in this case, at least, much of the water escaped vertically, an explanation which the authors discard in the case they studied. Since the piles described in the paper were driven on dry land, it may be that the authors were able to watch the surface around the pile casings during driving; it would be interesting to know if this procedure were followed and, if so, what observations were made. Can the authors state explicitly that no water, or softened clay, escaped at the surface?

FIG. 27.—DISTURBANCE OF HARD CLAY BY PILE DRIVING

Accurate moisture contents are so essential a part of the investigation described that one is forced to question the statement (under the heading, "Method of Sampling and Testing") to the effect that "A large part of the remolded sample was then used for the determination of the natural water content * * *." In view of this procedure, and the common laboratory experience with changing moisture contents, even when clays are handled in humid rooms, can this procedure be regarded as perfectly satisfactory? A further detail is the possibility of a change in moisture content between the two sides of the P samples. Since these samples were taken so very close to the pile shells, it is conceivable that a change in moisture content might have been observed between the sides of the samples adjacent to the casings and those opposite. Was any such difference observed?

Finally, the authors conclude by stating that their investigation

> "* * * does not warrant the conclusion that disturbance due to driving piles into soft clay is a source of detrimental decrease in the strength of these clays or a source of excessive settlement."

This statement is literally correct, of course, but it is almost inevitable that, in combination with the general title of the paper, this conclusion will tend to be applied to clays other than those from the Detroit (Mich.) area. Other studies may show similar results for other clay soils but until they do, and until some confirmation of the authors' results—even for the Detroit clay—has been obtained, it would seem advisable for the authors to be very much more specific in the defining of the rigid limitation of their conclusions. In all his experience, the writer has never encountered a clay soil with its natural moisture content so symmetrically fixed between its plastic and liquid limits. Many Canadian clays have natural moisture contents close to, if not exceeding, the liquid limit. The almost "ideal" character of the Detroit clay may possibly account for the findings of the authors and, if true, this fact would tend to limit the general applicability of the conclusions. However, it would in no way minimize the value of the findings in the paper to which the writer would pay full tribute. The very questions raised in this discussion show clearly the stimulation to critical thinking about the "pile problem" provided by this most interesting and valuable paper.

Stuart B. Avery, Jr.,[29] **and Stanley D. Wilson,**[30] **Assoc. M. ASCE.**—Factual data, on a subject that has long been controversial, are contained in this paper; and the authors are to be commended for initiating a type of investigation that, if conducted systematically on various types of clay, will furnish much needed information.

The investigation reported was made at the site of a structure supported by point-bearing piles driven through a clay of low plasticity and slight sensitivity, to hardpan. Under these conditions it is difficult to conceive detrimental effects caused by pile driving. The writers' interpretation of the data differs from that presented by the authors.

Effect of Disturbance on Consolidation and Strength Characteristics of Clays.—It is well established that structural disturbance of a mass of undisturbed, sedimentary clay changes its consolidation characteristics. This is illustrated by Fig. 28. Typical void ratio-pressure curves and corresponding shear strength-pressure curves for a sensitive clay in undisturbed, badly disturbed, and laboratory remolded states are shown in Fig. 28(*a*).

Driving piles into a fully saturated clay, without pre-excavation, causes the total foundation volume to increase by the volume of the piles. In addition, the following effects will result in a clay having a sensitive structure:

(1) At least a portion of the clay is badly disturbed at unchanged water content. This means that the effective stress in the clay moves from point A to point B, Fig. 28(*a*). In this condition the clay is only partly consolidated under its own weight.

(2) The shear strength is correspondingly reduced from point C to point D.

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(3) The badly disturbed clay immediately begins to reconsolidate under its own weight and would reach full consolidation at point E. However, if a part of the weight of the overlying clay is supported by negative skin friction, between the piles and the clay, the consolidation would stop at an intermediate point F, corresponding to full consolidation under the decreased load. (The term "negative skin friction" was first introduced by Karl Terzaghi,[31] Hon. M. ASCE, in 1935.)

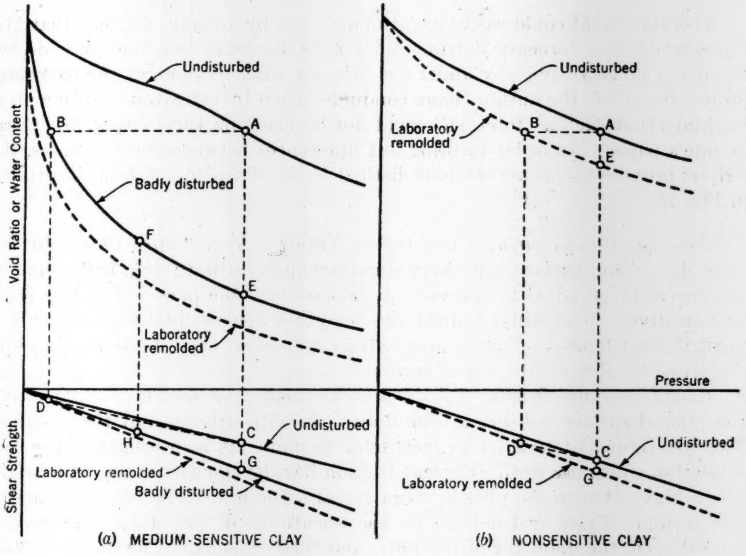

FIG. 28.—LABORATORY TESTS ILLUSTRATING EFFECT OF DISTURBANCE ON CONSOLIDATION AND STRENGTH CHARACTERISTICS OF SEDIMENTARY CLAYS

(4) The shear strengths corresponding to points E and F are shown graphically by points G and H, respectively. It should be noted that the disturbed clay, after 100% consolidation under its own weight, has a higher shear strength than the original undisturbed clay.

The same relationships for a nonsensitive clay are shown in Fig. 28(*b*), from which it can be seen that even complete laboratory remolding would have little effect on either the strength or the consolidation characteristics of such a material.

It was from an analysis of such laboratory test curves that A. Casagrande,[4] M. ASCE, in 1932, warned that increased settlements may result from driving friction piles, closely spaced, into a sensitive clay. His conclusion applied only to the case of floating, or friction, piles, and the writers question the applicability of the following statement of the authors (under the heading, "Conclusions"):

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[31] "Die Setzung der Fundierungen und ihre Wirkung auf den Oberbau," by Karl Terzaghi, *De Ingenieur*, 1935, Nos. 50 en 51.

> "The preceding information indicates that the driving of piles at the site where the investigation was made did not have the detrimental effects which were forecast on the basis of soil tests on undisturbed and remolded samples and the theory of consolidation as applied by Mr. Casagrande."

This statement could easily be misunderstood by a reader to mean that Mr. Casagrande had forecast detrimental effects for point-bearing piles driven through a nonsensitive clay under conditions similar to those at the particular project for which the authors have conducted their investigation. The writers maintain that such a "forecast" could not be made on the basis of Mr. Casagrande's paper. In order to bring out differences between the two cases, the writers propose to discuss the four distinct types of pile foundations illustrated in Fig. 29.

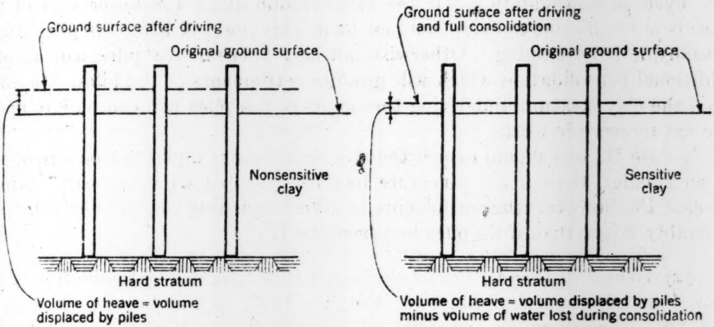
Case I. Point-Bearing Piles Driven Through Nonsensitive Clay.—In this case, the ground surface will heave simultaneously with driving, the volume of this heave being equal to the volume displaced by the piles. The clay being nonsensitive, there will be little tendency for reconsolidation, and only a negligible settlement of the ground surface will occur even if the clay is badly disturbed by the pile driving operation.

Case II. Point-Bearing Piles Driven Through Sensitive Clay.—As in case I, the ground surface will heave simultaneously with driving, but the disturbed clay will immediately start to reconsolidate under its own weight. The consolidation continues until sufficient friction has developed to carry the weight of the clay between the piles by negative skin friction. The final elevation of the ground surface will depend on the sensitivity of the clay, the degree of disturbance, the diameter of the piles, and their spacing. The more sensitive the clay, the greater will be the volume of water lost as a result of complete reconsolidation under the existing overburden. Although with increased pile spacing the disturbance is less, the percentage of the soil weight between the piles that can be carried by negative skin friction is also reduced. Therefore, the greater the pile spacing the greater will be the effective load under which the disturbed clay must reconsolidate. The writers' experience indicates that even in extreme cases the settlements due to consolidation are less than the heave that had resulted from the volume displacement of the piles. Hence, a net heave is to be expected in any case.

In cases I and II no detrimental effects can result, unless in case II the soil between the piles is supporting a part of the structure, or the piles are designed with so low a factor of safety that they settle or fail under the additional load of the weight of some of the soil between the piles.

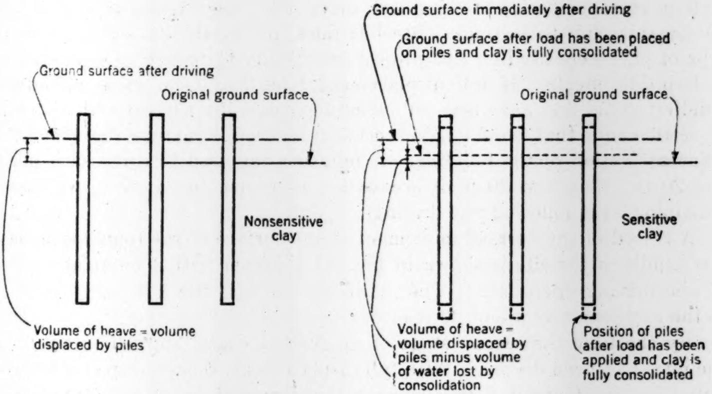
Case III. Friction Piles Driven Into Nonsensitive Clay.—As in case I, the ground surface will heave simultaneously with driving and the volume of heave will be equal to the volume displaced by the piles. When the load is applied, the piles and the ground surface will settle together because of the consolidation of the lower part of the clay stratum under the additional load of the structure.

Case IV. Friction Piles Driven Into Sensitive Clay.—As in case II, the ground surface will heave simultaneously with driving and the disturbed clay will begin to reconsolidate under its own weight. However, the skin friction will soon build up to a value which will enable it to carry the weight of the clay between the piles by negative skin friction. In case IV this condition will develop only along the upper part of the pile, whereas along its lower part


Ground surface after driving
Original ground surface
Nonsensitive clay
Hard stratum
Volume of heave = volume displaced by piles
Ground surface after driving and full consolidation
Original ground surface
Sensitive clay
Hard stratum
Volume of heave = volume displaced by piles minus volume of water lost during consolidation

Case I.- Point-bearing Piles Driven Through Nonsensitive Clay

Case II.- Point-bearing Piles Driven Through Sensitive Clay


Ground surface after driving
Original ground surface
Nonsensitive clay
Volume of heave = volume displaced by piles
Ground surface immediately after driving
Ground surface after load has been placed on piles and clay is fully consolidated
Original ground surface
Sensitive clay
Volume of heave = volume displaced by piles minus volume of water lost by consolidation
Position of piles after load has been applied and clay is fully consolidated

Case III.- Friction Piles in Nonsensitive Clay

Case IV. Friction Piles in Sensitive Clay

FIG. 29.—SETTLEMENT EFFECTS IN FOUR CASES OF PILE FOUNDATIONS

this clay load will be transferred into the adjacent clay by means of positive skin friction. Eventually a condition of equilibrium will be established such that the upper part of the clay will carry only a small part of its own weight. In this condition, if the piles are loaded, the load will be transmitted by skin friction into the clay along the lower part of the piles, which then must start a new cycle of consolidation. If the skin friction along the lower part of the piles is not sufficient to carry the new load, plus the clay load, the piles would actually fail by slipping. Otherwise, all clay between the piles will develop additional consolidation which will produce settlements of the piles. In addition, the clay stratum underlying the points of the piles will consolidate under the net increase in load.

In case III one should expect that the settlements would not be larger, and often smaller, than if the structure had been supported by spread footings. In case IV, however, the settlements in a very sensitive clay may well be considerably larger than if no piles had been used.

Experiences in Driving Piles Into Boston Blue Clay.—The foundation of the new John Hancock Building[32] in Boston, Mass., is an example of case II, Fig. 29. The soil conditions at the site are as follows: 20 ft of sand and gravel fill, 20 ft of organic clay, 13 ft of medium stiff clay, 80 ft of soft clay, 10 ft of hardpan, and then bedrock. The organic clay is known locally as Charles River silt and, with a liquid limit of about 50, has physical properties comparable to those of a very soft clay. The soft clay is typical Boston Blue clay with a medium sensitivity. The foundation consisted of 14-in., 117-lb H-piles, with an average spacing of about 5 ft, driven from slightly above the surface of the organic clay to bedrock. Steel H-piles, rather than a full displacement type of pile, were used on this project specifically to reduce to a minimum all undesirable effects. If full displacement piles had been used, an adjacent building would first have been subjected to sharp differential heaving and later to settlements that would have certainly caused structural damage to the adjacent section of the building. A follower was used to drive each pile the last 20 ft. The first 20 ft of excavation preceded pile driving, whereas the remaining 25 ft followed pile driving.

A record of the vertical movement of the surface of the foundation clay at the middle of the site is shown in Fig. 30, together with a cumulative record of pile driving operations. The parabolic curve is the estimated heave due to the excavation of about 20 ft of soil.

By the time the second stage of excavation began approximately 80% of the piles had been driven. With full displacement, this corresponds to a computed heave of about 6 in., whereas the observed heave attributed to pile driving amounted to only about 3¾ in. The main reason for this difference is believed to be the consolidation of the clay caused by the disturbance of the clay structure during pile driving. This is indicated by the settlement Δ_1 of the observation points (see Fig. 30) from August 1 to August 13 when the driving operations were suspended, by which time no piles had been driven

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[32] "The Pile Foundation for the New John Hancock Building in Boston," by A. Casagrande, *Journal*, Boston Soc. of Civ. Engrs., October, 1947, p. 297.

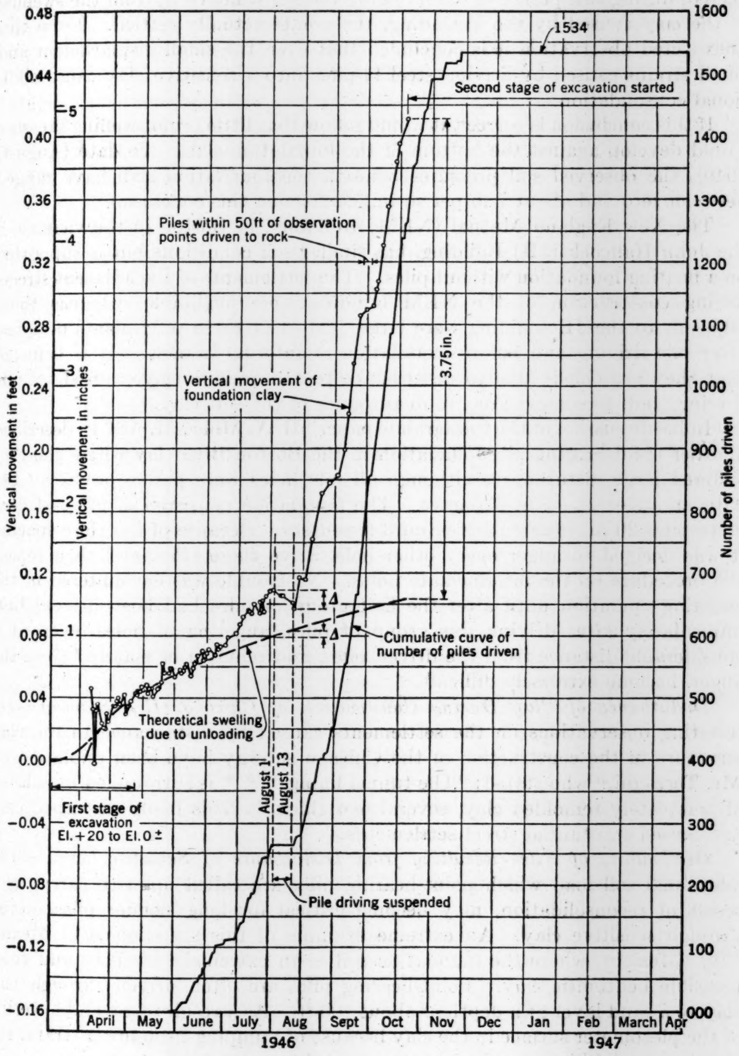

FIG. 30.—VERTICAL MOVEMENT OF FOUNDATION CLAY COMPARED WITH NUMBER OF PILES DRIVEN

closer than 50 ft from the observation points. Whereas one should have expected, during this period to observe only the slight heave Δ_2 from the swelling of the clay caused by the unloading, the points actually settled. From this unexpected observation it is concluded that even the small displacement and small strains caused by driving steel H-piles into a sensitive clay cause additional consolidation.

If this conclusion is correct, it would follow that little or no swelling pressure would develop against the bottom of the foundation slab. To date (August, 1949), the observed soil pressures beneath the foundation slab have ranged between zero and about 7 lb per sq in., confirming this conclusion.

The New England Mutual (NEM) Building[33] in Boston in the vicinity of the John Hancock (JH) Building, has similar soil conditions but is supported on a floating foundation without piles. The settlements of the adjacent streets during construction of the NEM building were neglibible, whereas those adjacent to the JH building were large. Most of these settlements occurred after pile driving and before installation of internal bracing. It is believed that they are chiefly due to disturbance to the organic clay, caused by pile driving, and, to some degree, to disturbance of the soft clay.

In his discussion of the Casagrande paper,[4] H. A. Mohr, M. ASCE, described another point-bearing pile foundation in the Boston Blue clay where the clay suffered large disturbances although all pile holes were partly pre-excavated with an auger 22 in. in diameter. The foundation consisted of pre-cast concrete piles 20 in. square, driven on 3-ft centers. Because of the close spacing it was decided to auger every other hole, drive these piles, and then repeat the procedure on the intermediate holes. No trouble was encountered in the augering operation until after the first group of piles had been driven; but, immediately after driving this group of piles, augering of holes, even at a considerable distance from the driven piles, and removal of material from the auger, became extremely difficult.

Disturbance of Clay During Construction of Chicago (Ill.) Subway.—Interesting observations on the settlements caused by disturbance to the clay structure in the construction of the Chicago subway have been published by Mr. Terzaghi,[34] who stated: "The tunnel lining * * * is surrounded by a layer of completely remolded clay, several feet thick." This badly disturbed clay has caused substantial street settlements.

Overloading of Piles Resulting from Disturbance of Sensitive Clays.—The additional soil load which point-bearing piles are called upon to carry, as a result of reconsolidation, may become critical for long slender piles driven through sensitive clay. An extreme example of this situation is in Mexico City (Mexico), where the foundation soil is an extremely sensitive and compressible bentonitic clay. Point-bearing piles are often driven through this clay to a sand layer at a depth of about 100 ft. As a result of general lowering of the piezometer surface in the clay because of pumping from lower strata, the clay is consolidating, causing continual subsidence of the ground surface.

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[33] "Application of Soil Mechanics in Designing Building Foundations," by A. Casagrande and R. E. Fadum, *Transactions*, ASCE, Vol. 109, 1944, pp. 383–490.

[34] "Shield Tunnels of the Chicago Subway," by Karl Terzaghi, *Journal*, Boston Soc. of Civ. Engrs., July, 1942, pp. 163–210.

In connection with a foundation project in Mexico City, pile pulling tests were performed to determine if negative skin friction would develop to a magnitude sufficient to carry the load of the clay, and also to determine the rate at which it would develop. The clay at the site of the test is of volcanic origin. It had a liquid limit of about 400, a plastic limit of about 100, and a natural water content at or slightly below the liquid limit (void ratio about 8.0). The unconfined compressive strength of undisturbed samples of this clay was about 0.3 ton per sq ft.

The test piles, in groups of four, on 3-ft centers, drove very easily, showing only a slight increase in resistance with increasing depth. The results of pulling tests showed that, only 2 hours after driving, the shearing force required to break the piles loose was at least 0.15 ton per sq ft, which is of the same order of magnitude as the shear strength of the undisturbed clay. This shearing strength is more than ample, at normal pile diameter and spacing, to carry the entire weight of the clay between the piles. Furthermore, the shearing strength of the reconsolidated clay, after the piles were broken loose, was found to be at least one half of its maximum value, in contrast to the negligible shear strength of the clay when freshly remolded.

These tests explain why the ground surface does not subside between piles, even though the ground surface in the immediate vicinity of a structure is subsiding. However, overloading of the piles may develop even if the clay between the piles is not in the least disturbed except for a few inches surrounding the piles. It is conceivable that under these conditions long, slender piles could actually fail in the course of time.

Disturbance of Clay at Detroit (*Mich.*) *Investigation.*—The clay described by the authors falls somewhere between cases I and II (Fig. 29). In view of the preceding discussion it is apparent that no detrimental effects could possibly occur, providing the piles are sufficiently strong to withstand the additional load of the clay between the piles as well as of the structure.

Clay with a liquid limit of about 30, a plastic limit of about 19, and a natural water content midway between these two values, together with the modest overburden, indicates two possibilities: (1) The clay is well overconsolidated, probably by drying; or (2) The clay has relatively little sensitivity to remolding. In either case one would expect that pile driving would have little effect in increasing the settlements from disturbance to the clay. Some settlement must have occurred due to reconsolidation of disturbed clay as evidenced by the fact that computed heave corresponding to full displacement is about 4 ft, whereas measured heave was only about 2 ft.

The authors state, under the heading, "Results of Tests":

> "In the homogeneous clay layer constituting the lower part of the deposit the effect of pile driving appears to have been negligible. There is no essential difference between the average compressive strength of the soil before and after pile driving."

This statement refers to the strengths obtained from boring P-1, taken 1 month after pile driving. From Fig. 9 it is obvious that the water content decreased appreciably during this period, and the writers have computed that more than 50% of the reconsolidation had already taken place. This would correspond approximately to points F and H in Fig. 28(*a*). During the following 10 months, the clay further consolidated to some point between F and E, resulting in an even higher shear strength than that of the original undisturbed material.

Under the heading, "Interpretation of Results," the authors state:

> "Since the water appears to have migrated exclusively in a horizontal direction, the process of consolidation initiated by driving friction piles into the clay would not ordinarily result in settlement."

The direction of the flow will depend largely on the relative permeabilities vertically and horizontally, but this has no effect on the ultimate settlement. Probably the excess water resulting from disturbance to the clay structure will first drain away from the vicinity of the pile in horizontal direction (unless the pile itself permits drainage). This lateral drainage may well cause slight swelling in the immediately surrounding mass of clay. However, eventually the excess water must be drained off to enable the grain structure to carry the loads, and this drainage must result in settlements, unless the piles are essentially point-bearing piles. Mr. Casagrande,[35] in discussing skin friction on floating piles, stated:

> "Unless the piles themselves facilitate the drainage, the consolidation will proceed from the surface downward, so that the shearing resistance of the clay, and therefore the skin friction along the upper portion of the piles, will increase before a similar increase can take place along the lower section of the piles."

In contrast to this, the reported results indicate that, in this case, the reconsolidation of the badly disturbed clay surrounding the pile proceeded at the same rate along the entire length of the pile. However, since the excess water must eventually find its way to the surface, or to a more pervious stratum, and since it is the escape of this water that causes the settlements, the writers disagree with the statement in the "Conclusions" that:

> "The assumption that the settlement would proceed as a result of vertical migration of the water out of the clay, which formed the basis of the Casagrande settlement forecast, is not in agreement with the observations made in the investigation."

In this connection it should be noted that in the early nineteen thirties it was still generally assumed that drainage during consolidation developed mainly in a vertical direction. The original, one-dimensional theory of consolidation by Mr. Terzaghi includes this as one of the basic assumptions. Experience has since shown that lateral drainage is in many cases more important than vertical drainage. That applies particularly to settlements of buildings having a width of the same order of magnitude as the thickness of the clay layers. If water can drain horizontally more readily than vertically, it will speed up the consolidation process, but it will of course have no influence on the final value of the settlements.

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[35] "The Structure of Clay and Its Importance in Foundation Engineering," by A. Casagrande, *Journal*, Boston Soc. of Civ. Engrs., April, 1932, p. 187.

Comparison of settlements under a relatively small loaded area with those under a large area, with identical foundation conditions and loads, was afforded in connection with settlement analyses of the Logan Airport[36] in Boston. These observations proved conclusively that, under an area comparable in size to an average building, consolidation proceeded much faster than when a similar load was distributed over a very large area.

Conclusion.—The writers believe that experience to date, including that reported by the authors, indicates that pile driving in a sensitive, sedimentary clay will badly disturb the clay immediately surrounding the pile and that even at a considerable distance from the piles, where the strain due to deformation is quite small, there can be sufficient disturbance to the clay structure to cause additional settlements.

In 1932 Mr. Casagrande visualized that large distortions are required to disturb clay structures sufficiently to cause additional settlements. Evidence which has accumulated since shows that even minor strains may cause sufficient disturbance to some sensitive, sedimentary clays so as to result in additional settlements. Thus, the effects of disturbance are in some cases more serious than one would be justified to assume on the basis of the Casagrande paper.[4] Certainly, a conclusion that the effects of disturbance are of small import and perhaps entirely nonexistent, as implied by the authors, is not warranted either on the basis of their investigation, or in the light of other available evidence.

Additional investigations are required to determine the limits of the zone of disturbance. The writers do not believe that the authors' investigation warrants the conclusion that "* * * the region of disturbance is limited to a zone within a few inches of the pile * * *." Until more data are available, designers would do well to make conservative assumptions regarding the possible effects of this disturbance on pile foundations in sensitive, sedimentary clay.

With this investigation the authors have pioneered in a new, but long overdue branch of foundation investigation. It is hoped that this work will be followed by similar tests in more sensitive clays of greater plasticity. In such investigations it would be well also to include both consolidation tests and pore pressure measurements in the clay before and after driving of the piles (perhaps the piezometers could be built into the sides of the piles) to furnish additional information on the amount of disturbance and on the additional consolidation that is taking place in the clay between the piles.

GEORGE L. FREEMAN,[37] M. ASCE, AND JAMES D. PARSONS,[38] ASSOC. M. ASCE.—Interesting information has been made available in this paper, although the writers are not in agreement with the conclusions derived from the data. They believe that the situation would have been entirely different had the piles not penetrated the uniform soft clay bed into the underlying sand and gravel.

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[36] "Analysis of Pore Pressure and Settlement Observations at Logan International Airport," by J. P. Gould, thesis presented to Harvard Univ., at Cambridge, Mass., in 1949, in partial fulfilment of the requirements for the degree of Doctor of Science.

[37] Partner, Moran, Proctor, Freeman & Mueser, New York, N. Y.

[38] Associate Partner, Moran, Proctor, Freeman & Mueser, New York, N. Y.

A. Casagrande,[4] M. ASCE, provided the profession with a sound explanation of what had been previously observed relative to the detrimental effects of driving piles into soft clays. His paper has done much to curb the indiscriminate use of friction piles as a structural support in such soils.

The conclusions of the authors are based largely on the laboratory results of tests made on samples recovered from the uniform soft blue clay, persisting from about El. +505 to El. +550, through which the point-bearing piles were driven. The average unconfined compression strengths and water contents of the bed as a whole for the several periods considered were as shown in Table 4.

An examination of Fig. 5 indicates that the 2-in. samples suffered some distortion during sampling, and a similar distortion is noted in the samples shown in Fig. 7. It is assumed that the 2-in. samples were entirely subjected to unconfined compression tests and that the "drag" which is evident around the periphery of the samples was not trimmed off before testing. It is further assumed that all the "undisturbed" samples tested had suffered a degree of disturbance similar to that evident in Figs. 5 and 7. Examination of the soil surrounding the pile, as shown in Fig. 4, indicates very substantial remolding of the soil extending away from the pile surface for at least the diameter of the pile itself. Although samples secured adjacent to the pile, shown in Fig. 6, did retain their "laminated structure," to some extent, they are visibly much disturbed and their strengths, as determined from tests, are the strengths of considerably remolded soils. Had the samples from boring A been truly undisturbed, their compressive strengths would no doubt have been substantially more than shown by the tests, and there should have been a greater difference between the strengths of undisturbed samples and samples recovered from the P borings and C borings.

TABLE 4.—RELATIVE CHARACTERISTICS OF UNIFORM SOFT CLAY, IN RELATION TO TIME OF PILE DRIVING

Months from driving	Average Water Content (Percentages)		Average Compressive Strength (Tons per Square Foot)	
	Adjacent to piles (P borings)	Midway between piles (C borings)	Adjacent to piles	Midway between piles
Before driving	24.7[a]	24.7[a]	0.61[a]	0.61
1 month after	22.8	23.8	0.56	0.56
11 months after	22.0	24.4	0.75	....[b]

[a] Boring A. [b] This value is mentioned by the authors but not given in the paper.

The degree of disturbance midway between the piles must have been nearly the same as at the pile periphery, because the compressive strengths 1 month after pile driving are the same in both cases. With regard to the upper laminated clay, the authors state (see "General Soil Conditions") that:

> "In their undisturbed state, the samples had an unconfined compressive strength approximately eight times that of the same samples when remolded in the laboratory at their natural water content."

This difference is not apparent in Fig. 8, except in the upper few feet of the laminated clay. The same reduction in soil strength after 1 month, in the samples adjacent to, and midway between, the piles, may be purely a coincidence; and the test results certainly do not warrant the assumption that the disturbed soil zone is limited to within a few inches of the pile.

The limited scope of the investigation, and the scattered results of the few tests made, do not warrant the conclusions presented in the paper. Certainly more than one boring of each type should have been made before and after driving piles. It would also be interesting to know the results of unconfined compression tests on the samples from boring C-11. The addition of consolidation tests to the test program might have yielded valuable information regarding the relative strengths and degree of disturbance of the soil in different locations.

The authors state that, as a result of the pile driving operations, the ground surface between the piles heaved about 2 ft. Piles driven in similar soil formations frequently cause adjacent piles to heave, sometimes enough to require redriving. When the piles are point bearing and closely spaced the resulting settlement of the soil between the piles may be negligible, even when significant heaving has been noted, because of the transfer of soil weight to the pile surfaces through friction. The resulting arching effect between piles is sufficient to restrain the natural tendency for the remolded or disturbed soil to reconsolidate. The foregoing observed phenomena, however, cannot properly be construed to indicate that, if the piles were not point bearing and were in effect friction piles, they, or the soil between the piles, would not settle as a result of the remolding effect and added load. On the contrary, settlements should be expected and in all probability would occur. The authors' conclusions would be far more convincing had the authors determined from actual level observations whether the soil did or did not settle after pile driving.

Although the degree of restraint against settlement afforded by the piles depends (among other things) on the physical properties of the clay penetrated, such restraint has been observed to occur in some of the softest of soils. At LaGuardia Field, in New York, N. Y., for example, where the soft organic silt is approximately 60 ft thick, settlements of the order of 5 ft have been observed since 1937 when about 20 ft of ash fill was placed over much of the site. On the other hand, in areas where structures had been supported on end bearing piles penetrating the fill and mud and supported in the underlying sand, observed settlements between the piles have been of the order of 1.5 ft for the same period. Obviously, it cannot be concluded that, if the structures had been built on friction piles supported in the organic silt, settlements of the lower order would have resulted, or that friction piles would be beneficial to the situation.

Neither the evidence of Fig. 4 nor the evidence of the writers' frequent observations supports the statement (see "Conclusions") that: "* * * the region of disturbance is limited to a zone within a few inches of the pile and the main volume of the clay mass is apparently undisturbed." As early as 1915, Karl Zimmerman[39] reported on the influence of pile size and shape on disturbance to the surrounding soil, and pointed out that both factors are influenced by the type of soil involved. In varved soils where the distortion of the soil may be readily seen it is not uncommon to observe severely disturbed zones extending at least a diameter beyond the face of the piles.

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[39] "Die Rammwirkung im Erdreich," by Karl Zimmerman, *Versuche auf neuer Grundlage*, Berlin, 1915.

In most instances the observed effects of driving piles into soft clay formations are more pronounced than stated by the authors. The disturbance exhibited in Fig. 31 was a matter of considerable concern in one such instance

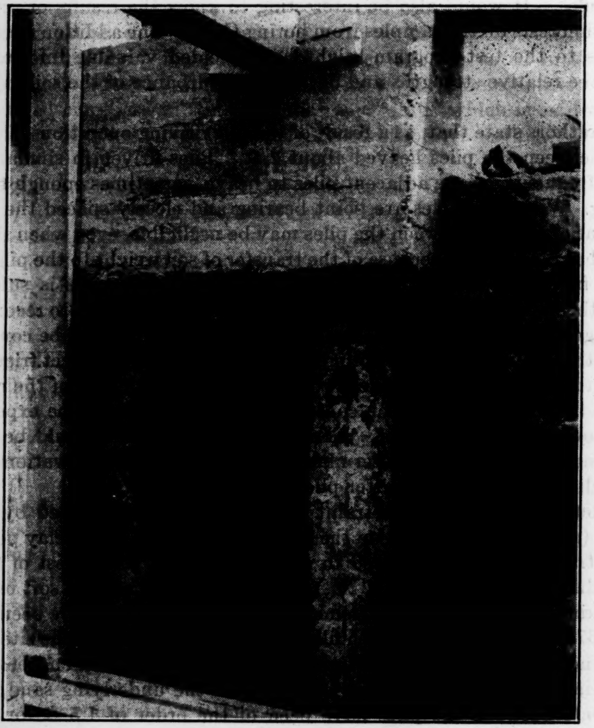

FIG. 31.—EXCAVATION ADJACENT TO PILES SHOWING VISUAL EVIDENCE OF DISTURBANCE DUE TO PILE DRIVING

and is offered as direct evidence against the conclusions submitted by the authors. The piles shown are point-bearing, 15-in., closed-ended, pipe piles, driven through approximately 100 ft of relatively soft varved clay, of glacial origin, into an underlying compact sand and gravel stratum. The structure supported by the piles is shown in plan in Fig. 32 and may be described as a heavy ornamental type of building. Its entire weight, with the exception of the ground floor partitions and slab, is carried by the piles. It was considered unnecessary to support either the ground floors or the partitions since they imposed no additional weight on the underlying formation because it was necessary to excavate more than their weight in soil to reach the grade established for the floors. The general grades of the ground surface and floor, together with the soil formations, are shown in Fig. 33. The piles are on a considerably wider spacing than those described by the authors; and the footings are spaced 10 ft on centers along the foundation walls.

Piles were driven from June to August, 1940, and the ground floor was laid in November and December, 1940. Shortly before the building was completed and ready for occupancy, cracks appeared in the ground floor partitions and slab and it was found that the ground between the pile clusters was settling. The settlements could be traced to no other cause than the disturbance of the soft varved clay produced by driving pipe piles through this formation.

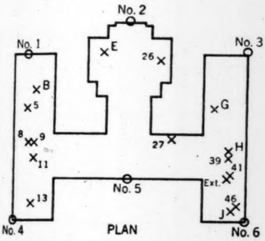

FIG. 32.—PLAN SHOWING LOCATION OF BORINGS AND SETTLEMENT OBSERVATION POINTS IN FLOOR

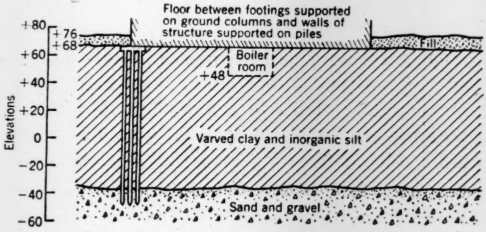
Floor between footings supported on ground columns and walls of structure supported on piles
+80
+76
+68
+60
+40
+20
0
-20
-40
-60
Elevations
Fill
Boiler room
+48
Varved clay and inorganic silt
Sand and gravel

FIG. 33.—COMPOSITE VERTICAL SECTION OF SOIL UNDERLYING BUILDING

As a temporary solution, underfloor grouting or "mud jacking" was done and holes were drilled through the floor in August, 1941, for the purpose of establishing settlement observation points in the surface of the underlying varved clay bed. The locations of these observation points are shown in Fig. 33, and the settlements that were observed are plotted against the square root of elapsed time in Fig. 34. Because of the straight-line relationship so established it was possible to estimate, for comparison with observed settlements in the floor, the amount of settlement previous to the installation of the settlement observation points. These settlements occurred in spite of an actual net decrease in load on the underlying clay formation that was responsible for the settlements.

Conclusions.—The writers believe that the authors' conclusions regarding the effect of driving piles into soft clays are consistent neither with the data presented, nor with current experience, for the following reasons:

1. The conditions investigated by the authors are not analogous to the situation in which friction piles have been supported in clay;
2. Had the piles in this case been supported by friction in the clay bed, settlement of both the piles and the surrounding ground would have occurred;

3. Laboratory test data submitted by the authors, and their own statements, indicate a loss of strength in the clay both adjacent to and between the driven piles;

4. Illustrations presented by the authors indicate some disturbance of the soil even in "undisturbed" samples, and great disturbance in the samples secured from the region adjacent to the piles;

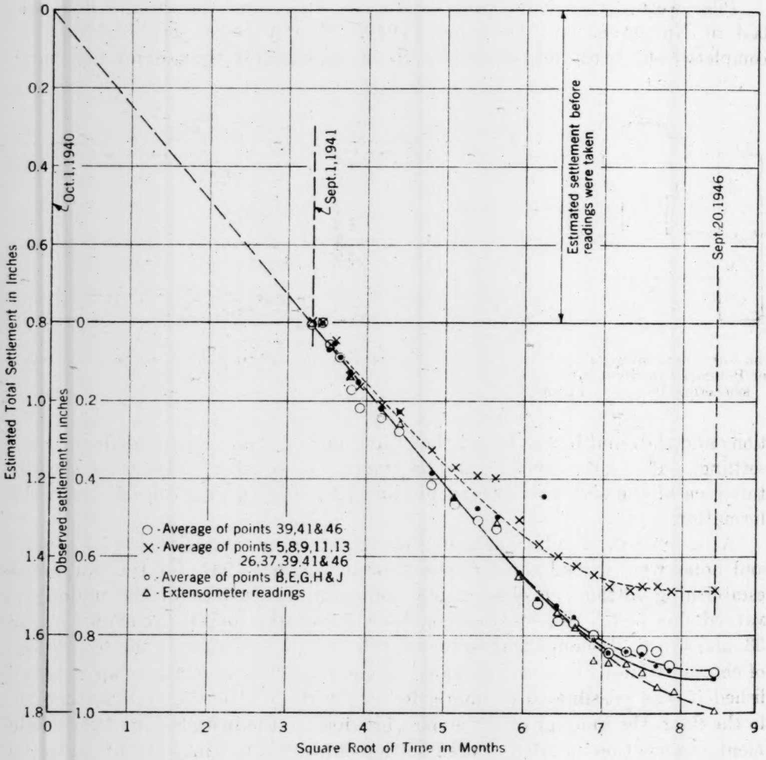

FIG. 34.—SETTLEMENTS INDUCED BY REMOLDING RESULTING FROM PILE DRIVING

5. The data presented in no way contradict the original contentions of Mr. Casagrande[4] regarding the remolding effect of piles driven into soft clay; and

6. To substantiate the authors' conclusions more borings and more laboratory tests are required, including tests of the soil midway between piles. The simple level observations required to prove the settlement or nonsettlement of the ground surface should be included in any further similar investigations.

L. LeRoy Crandall,[40] Assoc. M. ASCE.—The engineering profession is indebted to the authors for obtaining and presenting factual data on this controversial subject. Although the writer can think of additional tests and field measurements that would have been desirable, he realizes only too well the limitations which time and money place upon the scope of a research project in foundation engineering.

Within the writer's professional experience have come a number of projects on which piles were driven into soft clays without ill effects. Although data necessary to explain the changes in specific properties of the soft clay, with the passing of time, have not been compiled, the very fact of satisfactory performance is evidence, in some degree. Such evidence is to be found in the following examples, previously cited in another connection[41] by William W. Moore, M. ASCE.

Example 1.—A foundation investigation of the site of the Avon Refinery of the Tide Water Associated Oil Company at Avon, Calif., resulted in the selection of cast-in-place concrete piles for the support of a catalytic cracking unit. The foundation of the unit was 60 ft by 130 ft in plan and was required to support a dead load of approximately 8,700 tons. A total of 352 piles, spaced approximately 3 ft on centers, was required for a design load of 40 tons per pile; the dead load was slightly more than 24 tons per pile. The foundation soils were found to be predominantly clays interspersed with strata of fine sand. The moisture content of the clays varied from 20% to 30%, and the dry density ranged between 90 lb per cu ft and 110 lb per cu ft. The shearing strength of the clay, as determined by direct shear tests on undisturbed samples, varied from 640 lb per sq ft to 1,750 lb per sq ft. The computations of pile capacity indicated that the friction piles could support 40-ton design loads if driven to a depth of 30 ft with the pile caps at a depth of 5 ft.

As had been fully anticipated, the piles failed to achieve a driving resistance sufficient to indicate a capacity of 40 tons by a dynamic formula at the designed depth of penetration. The pile driving contractor insisted that the piles should be driven to greater depths and refused to accept responsibility if they were not. In order to disprove the capacity indicated by the driving formula, the piles were allowed to stand for a few days and driving was again begun. The redriving indicated a formula capacity of 60 tons, whereas the initial formula capacity was only 20 tons. Although the contractor still was unconvinced, the owner accepted the recommendations based upon the foundation investigation without requiring the performance of substantiating load tests.

Some heaving of the ground surface was observed, but there was no evidence to indicate that the soil settled under its own weight due to remolding. The settlements of the completed structure have been extremely small. Although no special studies were made to evaluate the effects of the pile driving on the strengths of the clays, such effects in this case must have been negligible or else the clays regained their full strength within a very short period. It may be reasoned that remolding of the clays caused the low driving resistance initially observed, and that the clays substantially regained their strength by the time driving was recommenced. However, it is the writer's belief that the increase in resistance was caused by the dissipation of excess hydrostatic pore pressures developed in the clay during driving. It seems reasonable to assume that such excess pressures would be dissipated, at least adjacent to the pile, much more rapidly than would the clay regain its strength if it had been seriously weakened by remolding.

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[40] Resident Partner, Dames & Moore, Foundation Engrs., Los Angeles, Calif.

[41] "Experiences with Predetermining Pile Lengths," by William W. Moore, *Transactions*, ASCE, Vol. 114, 1949, p. 351.

Example 2.—Pre-cast concrete piles deriving their support in friction from moderately firm clays were used to support a reinforced concrete pier at a shipyard in Richmond, Calif. The foundation investigation disclosed that the moisture content of the clay ranged from 28% to 69% and the dry density varied from 60 lb per cu ft to 97 ft lb per cu ft. Direct shear tests on undisturbed samples gave shearing strengths of from 640 lb per sq ft to 1,910 lb per sq ft. Tongue-and-grove sheet piles, 18 in. by 24 in., were computed to have a capacity of 90 tons when driven 25 ft below the surface of the bottom of the bay.

On this project, a test was performed to check the vertical load supporting capacity of a typical single sheet pile used as a bearing pile. The results of the load test fully confirmed the capacity computed from the tests on undisturbed samples. If by remolding of the clay severe weakening had occurred, it is difficult to imagine that the supporting capacity of the pile would not have been considerably less than the computed capacity. The driving resistances again yielded formula capacities which were generally much less than the static capacity of the piles. The settlements of the completed pier have not exceeded the amounts computed for the clays in an undisturbed state.

Example 3.—Another case in point is a pile loading test made at California Point on San Francisco Bay. The foundation investigation disclosed the soils to be soft, clayey bay deposits having high moisture contents, low densities, and low shearing strengths. The pre-cast concrete piles were computed to have a supporting capacity of 42 tons for a penetration of 60 ft into the soft clay, the surface of which was 30 ft below water level.

A test pile was installed and penetrated the full 60 ft under its own weight and the weight of the hammer; not a single blow of the hammer was required. The load test, which was begun 13 days after driving, confirmed the computed capacity as determined from the tests on undisturbed samples, in spite of the negligible resistance developed in placing the pile. It seems that any serious loss of strength by remolding of these extremely soft clays would certainly have seriously reduced the capacity of the pile.

The experiences of the writer and his associates on the aforementioned projects, as well as on a number of other projects which were not discussed, are in accordance with the findings of the authors. The results obtained on these projects support one of two theories—(1) either the remolding of the soft clay by a driven pile is not nearly so extreme as the remolding observed in the laboratory, or (2) the strength of the remolded soft clay recovers so rapidly that it becomes comparable to the strength of undisturbed clay in a relatively few days. In either event, the supporting capacity of piling driven on these projects has not been appreciably affected by any remolding nor have the settlements of groups of piling exceeded the settlements computed on the basis of laboratory tests on undisturbed samples. It may be that for some clays the use of driven piling will prove detrimental, but it is believed that the occurrence of such conditions is the exception rather than the general rule.

W. S. Housel,[42] M. ASCE.—An investigation, similar to that reported in this paper, was conducted at the Union Station in Detroit, Mich., in connection with the design of grade separation structures on the John C. Lodge Expressway.[24] It is pertinent to compare the conclusions drawn in that case with those presented by the authors. In some respects the later investigation confirms the authors' deductions and provides some additional data regarding this problem about which experimental results are still rather fragmentary. The primary objective of the tests was to study methods of controlling the volume displacement due to pile driving which sometimes has harmful effects on adjacent structures or utilities. With respect to this phase of the investigation it was found that the use of tapered, open end steel piles of the type used did eliminate a major part of the volume displacement. Subsequent loading tests indicated that those piles in which the clay core was not removed had an ultimate carrying capacity as controlled by the soil comparable to that measured on the piles filled with concrete.

Largely as a result of the publication of the earlier investigation made by the authors it was decided to conduct some parallel studies in connection with the Union Station pile tests to study remolding and moisture content changes due to pile driving. The tentative conclusions drawn from the investigation, and which may be compared with the authors' deductions, are as follows:

1. The Detroit clay deposit consists of at least two depositions of lacustrine clay showing a different reaction to pile driving in so far as loss of shearing resistance is concerned. The physical characteristics of these deposits, as determined by index tests, show some similarity to those reported by the authors.
2. Above approximate El. 530 at the Union Station, clay within 3 in. of the pile suffered loss in shearing resistance comparable to the value when remolded. Determination of recovery was based on two types of shear tests made approximately 2.5 months after pile driving. Below El. 530 there was no apparent loss in strength due to pile driving at a distance of approximately 3 in. from the pile. No tests were made to determine recovery of the soil with time. Shear values in the lower clay deposits 2.5 months after driving were substantially higher than the original shear strength found in the check boring but within the range of variation found in a series of exploratory borings. These results were confirmed by two similar borings at a distance of 3 in. from the piles.
3. In two borings at a distance of several feet from the piles there was no evidence of loss in shearing resistance due to remolding in either the upper or the lower clay deposit. In one case shear values in the lower deposit were greater than in the check boring but within the range found in the series of ex-

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[42] Associate Prof., Civ. Eng., Univ. of Michigan, Ann Arbor, Mich.

ploratory borings. This would tend to confirm that the higher shear values found in the lower clay were the result of a natural variation in the deposit rather than an alteration produced by pile driving.

4. There was no positive evidence of any change in moisture content in either the top or bottom clay deposit but at one depth where there was a particularly soft and highly plastic clay there were scattered results from which it might be concluded that a decrease in moisture content had taken place within 3 in. of the piles.

In presenting these data the writer wishes to point out that, although the results reported are based on duplicate borings and duplicate laboratory shear tests and while they indicate some trends, there is such a wide variation in shearing resistance and physical characteristics in the natural state that the statistical basis of comparison becomes extremely narrow, making the available data comparatively fragmentary. For example, in both investigations there are only data at two points with respect to distance from the piles, one at approximately 3 in. and one at several feet, giving little measure of the boundary of zones within which remolding is important. Perhaps more important is the possible variation with time, particularly in the rate of recovery of the loss in strength. On this point there is quite incomplete coverage in the results reported from both investigations, and considering the character of the tests and the expense involved it is not likely that such information will be made available under ordinary circumstances.

It would appear to the writer that there are other, and perhaps more important, aspects of the effect of driving piles in soft clay to which attention might be directed and on which observation and experience may provide more conclusive information. For one thing, it is a matter of common experience in pile driving that the load-carrying capacity of the piles as determined by conventional pile driving formulas may be very small at the time the driving is completed, but within a period of hours or a few days friction piles "set up" and offer much greater resistance to penetration. In fact, it is common practice to take advantage of this well-known phenomenon in meeting specifications that require a certain pile capacity as determined by commonly used formulas.

Some years ago the Detroit Edison Company conducted a rather extensive series of load tests on friction piles to determine the recovery of the soft clay from the detrimental effects of pile driving. These tests were made by direct loading on two 16-in. (outside diameter) steel piles, approximately 65 ft long, which were driven at the company's Delray Power Plant in Detroit. Soil conditions in the area are comparable to those under discussion by the authors, being in the River Rouge area of Detroit where the entire depth of clay to hardpan consists of a soft-plastic lacustrine deposit.

Immediately after the piles were driven, load testing equipment was set up and direct loading tests were conducted at intervals starting from about 2 hours after the piles were driven and continuing for about 60 days. The results of these tests have never been published but seem to be pertinent to the subject now under discussion and do provide evidence of the nature of the temporary loss in strength and subsequent recovery of such clays after pile driving. Observations of these phenomena indicate a rapid recovery in shearing resistance accompanied by parallel changes in elastic properties which are seldom associated with changes in moisture content and consolidation of the soil mass.

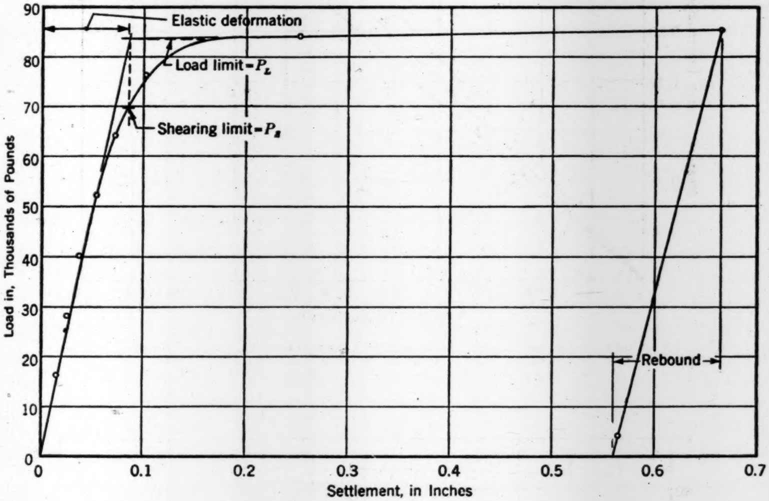

FIG. 35.—LOAD SETTLEMENT DIAGRAM, 16-IN. STEEL CYLINDER IN SOFT BLUE CLAY; TEST NO. 20, DELRAY POWERHOUSE, DETROIT, MICH.

A typical load settlement diagram taken from the series of thirty-six tests conducted on the two piles is shown in Fig. 35. This diagram indicates the resistance characteristics of the clay mass measured by loading a so-called friction pile. Even though the clay is soft or highly plastic it gives evidence of an extended range of elastic behavior shown by the linear relation between load and settlement extending close to the load limit. After the load limit has been exceeded there is a stage of rapid progressive settlement shown by an almost horizontal load settlement line followed by a rebound or elastic recovery after the release of the load. Throughout the series of tests it was found that the rebound was approximately equal to the total elastic deformation as determined by the intersection of the elastic load settlement line with the failure line, a point which is designated as the load limit, P_L. The shearing limit, P_S, has been selected as the load carried at a deformation equal to the total elastic deformation. The difference between the shearing limit and the load limit in

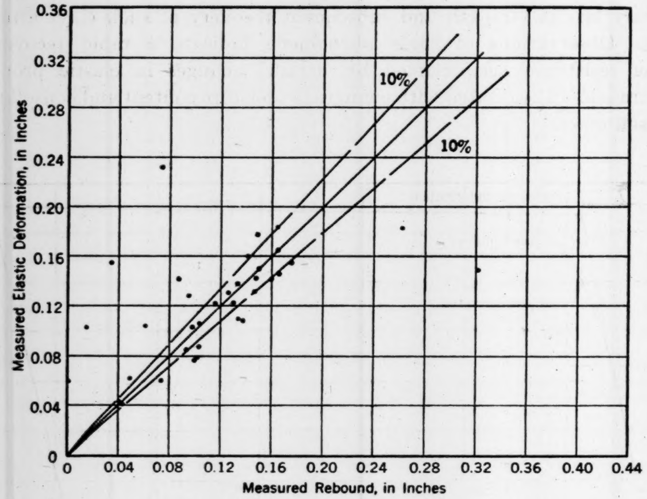

FIG. 36.—COMPARISON OF ELASTIC DEFORMATION AND REBOUND, 16-IN. STEEL CYLINDER IN SOFT BLUE CLAY, DELRAY POWERHOUSE, DETROIT, MICH.

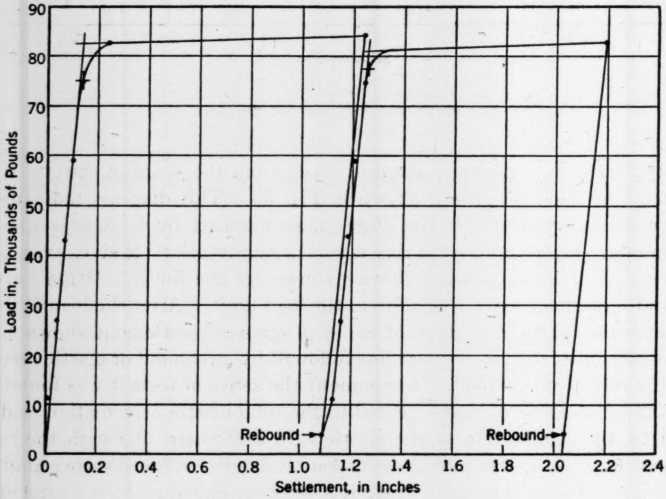

FIG. 37.—LOAD SETTLEMENT DIAGRAM, 16-IN. STEEL CYLINDER IN SOFT BLUE CLAY; TEST NO. 32, DELRAY POWERHOUSE, DETROIT, MICH.

the curved portion of the load settlement diagram is considered to be the point resistance. In this connection it should be pointed out that the friction pile in Fig. 35 was tested with the clay core inside the pipe extending almost to the top of the pile, a condition resulting immediately after driving. In Fig. 36 is shown a comparison between rebound and total elastic deformation for the series of tests on such a friction pile, supporting the conclusion that the rebound and total elastic deformation were equal.

In Fig. 37 is shown a load settlement diagram for the same pile after the clay core had been removed for the purpose of completely eliminating point resistance. In the first place it is evident that the difference between the shearing limit and the load limit has almost disappeared and the failure of the pile is quite abrupt. Of greater importance, however, is the comparison afforded by immediate repetitions of load shown in Fig. 37. This type of test was made a number of times throughout the series of tests demonstrating beyond any reasonable doubt that friction piles in such cohesive clays could be loaded to rapid progressive settlement without impairing the shearing resistance of the material.

The variation in load-carrying capacity of the pile, which may be assumed to be equivalent to the shearing resistance of the soil, provides the most important data available from these tests. This variation is shown in Fig. 38 in which the shear limit as previously defined has been plotted for one of the piles with the time in days since driving as the abscissa. A few hours after driving the shear limit was approximately 25,000 lb. This increased rapidly and in a period of 14 days the shear limit had increased to approximately 77,000 lb, or about three times its initial value. At this point the pile had developed a capacity approximately equal to that which had been estimated from the independent shearing resistance tests in which the average shear for this depth was approximately 275 lb per sq ft. A duplicate test on the second pile at this time provided a very satisfactory confirmation of the total shearing capacity.

One of the most interesting observations which this series of tests provided resulted from what might be termed an accident that occurred after the first 14 days had elapsed. Each of the preceding piles was tested to rapid progressive failure and at this point it became necessary to put an additional length of pipe on this pile in order to continue the tests. The construction superintendent who had been assisting on the project was very much interested and very anxious to do everything possible to bring the investigation to a successful conclusion. Consequently, in order to eliminate any possible settlement from incomplete seating of the new length of pipe at the joint, he brought the pile driver back on the job and drove the pipe about 3 in., possibly striking no more than six blows. The surprising result is shown on the curve in Fig. 38. The next test after redriving of the pile shows an abrupt decrease in shearing resistance which was substantial although not as great as the original loss in strength.

It seems obvious that the slight movement of the pile involved in redriving could not have produced any significant amount of remolding of the soil or change in the moisture content. It should also be pointed out that the displacement involved was only a little more than that which took place in each of the static load tests in which the soil was carried to rapid progressive displacement. It may be stated with considerable certainty that deforming the soil (by rapid shearing displacement under static load) did not impair its strength; but the shock of just a few blows of the hammer produced a marked decrease in strength. Furthermore, the decrease in shearing resistance was accompanied by a parallel decrease in elastic deformation as recorded in both the rebound and the total elastic deformation. These data are shown in Fig. 39 and, in the writer's opinion, present very strong evidence that the primary source of the loss in strength and its subsequent recovery can be traced back to the very source of cohesion in adsorption or molecular orientation of soil moisture. The phenomena may be called a thixotropic effect, which is characteristic of colloidal systems and has been observed in clays of the character under discussion.

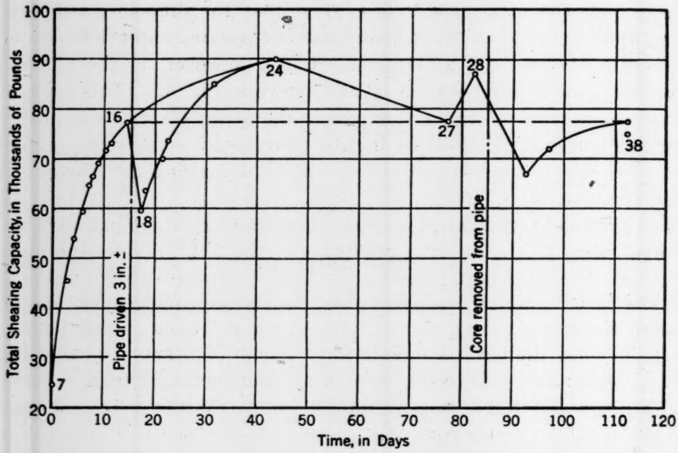

Fig. 38.—Variation in Total Shearing Capacity with Time Elapsed After Driving 16-In. Steel Pipe (Friction Length, 65.75 Ft to 67.86 Ft); Delray Powerhouse, Detroit, Mich.

This series of tests produced evidence on at least one other interesting phenomenon. It may be mentioned in passing even though its basis is somewhat speculative. As the tests were conducted the shearing resistance increased with time up to a maximum of 43 days as shown by test No. 24 (Fig. 38) where the shearing limit is approximately 90,000 lb. This total capacity was substantially greater than had been anticipated from previous shearing resistance tests and greater than the shearing resistance on the duplicate pile that was being tested. At the same time the test data gave evidence of a disparity between elastic deformation and shearing resistance which led to the speculation that a dynamic point resistance was being built up as a result of repeated tests which produced rapid progressive penetration of the pile. The magnitude of this dynamic point resistance was quite high, approaching the passive pressures that might be anticipated under the static head of some 65 ft.

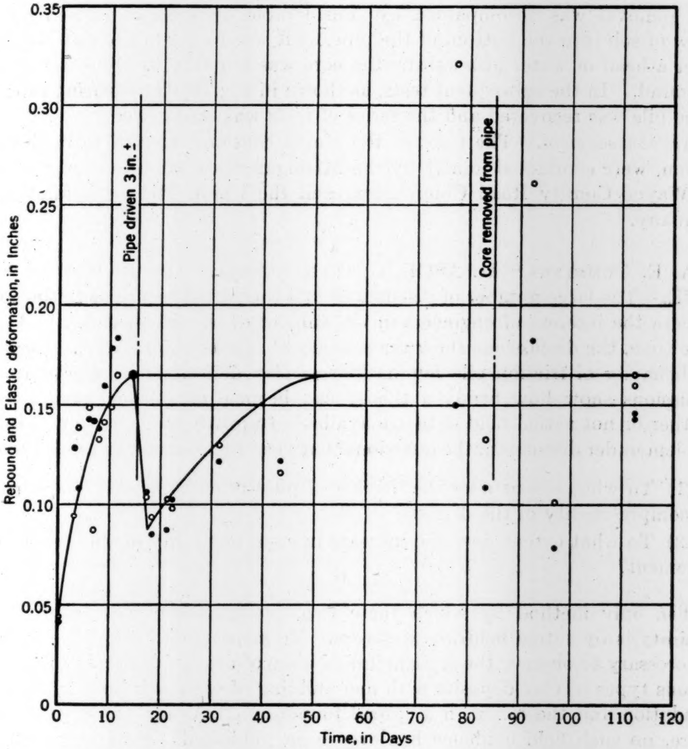

FIG. 39.—VARIATION IN ELASTIC DEFORMATION AND REBOUND WITH TIME ELAPSED AFTER DRIVING, 16-IN. STEEL PIPE (FRICTION LENGTH 65.75 FT TO 67.86 FT); DELRAY POWERHOUSE, DETROIT, MICH.

It was then decided to do two things to obtain further evidence on this phenomenon. In the first place, the pile was left without load repetition for approximately 30 days and then tested as shown in test No. 27 (Fig. 38). The total shearing capacity at that time had dropped back to approximately 77,000 lb. This was fairly close to that which had been anticipated from other test data, indicating that the high intensity of pressure developed at the point of the pile had been dissipated. In the succeeding test No. 28, however, this dynamic resistance was substantially recovered. In the next test the core was removed from the pipe in order to eliminate point resistance completely. This removal was accompanied by considerable disturbance caused by the inflow of soil into the bottom of the pipe, as it was found in the soft clay that either a head of water or a substantial core was required to prevent such loss of ground. In the subsequent tests, as shown in Fig. 38, the shearing capacity of the pile was recovered and the series of tests was terminated.

Acknowledgment.—The tests at the Union Station, referred to in this discussion, were conducted jointly by the Michigan State Highway Department, the Wayne County Road Commission, and the Union Metal Manufacturing Company.

A. E. Cummings,[43] M. ASCE, G. O. Kerkhoff,[44] and R. B. Peck,[45] M. ASCE.—The large number of discussions and the varied reactions to the paper indicate the interest of engineers in the subject of the investigation. At the same time, the discussions show the scarcity of factual information concerning the behavior of friction pile foundations in clay deposits. Furthermore, the discussions show how firmly a theory can become established regardless of whether or not actual field data are available to prove or disprove it. In the problem under discussion, the questions that need to be answered are as follows:

(1) To what extent does the driving of piles remold the clay and increase the compressibility of the deposit?

(2) To what extent does the increase in compressibility of the clay lead to settlement?

The only method by which these two questions can be answered with certainty is by actual field investigation. To answer both questions it would be necessary to observe the settlements of pairs of nearly identical buildings on various types of clay deposits with one building of each pair on a friction pile foundation and the other on a spread foundation. As far as the writers are aware, no such field evidence has ever been published. Since opportunities to make observations of this kind are rare, it seems that the problem must be approached by answering the two questions separately. The investigation described by the writers furnished field evidence concerning the first question. Apparently this evidence and that collected by Mr. Housel represent the only information of this kind yet published. Most of the discussions are concerned with theoretical analyses, model tests, or speculation based on secondary effects.

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Several discussions deal with observations of upheaval and the subsequent subsidence of the ground surface between the pile heads. In the opinion of the writers, the basic questions in this problem cannot be answered by such observations. The fact that in some cases the ground surface subsides between the piles does not necessarily lead to the conclusion that a friction pile foundation in the soil would have settled. The subsidence may reflect merely the intense consolidation of a thin shell of clay around each pile, as indicated by the observations of the writers and the instructive measurements and computations by Mr. Zeevaert.

Mr. Rutledge presents convincing evidence that the clay adjacent to the piles in the writers' tests was actually in a completely remolded state immediately after pile driving and that it acquired a strength amounting to almost twice its remolded value at the end of 1 month. The writers are inclined to agree with this interpretation of the data. However, the fact that the clay adjacent to the pile may have been completely remolded at the time of pile driving does not necessarily lead to the conclusion that the settlement of a friction pile foundation on such a clay deposit would be excessive. As a practical matter, it is always at least a month, and often many months, before a pile foundation is completely loaded with the structure it is intended to support. If, for any reason, the clay acquires strength and rigidity during this period, the piles will be prepared to support their loads by the time the loads are actually applied. Even if the gain in soil strength is accompanied by consolidation and settlement of the piles, no harm is done to the structure.

The fact that an increase of strength may occur is amply demonstrated for the Detroit clay by the load tests reported by Mr. Housel. Similar results for a more sensitive material have been reported by Karl Terzaghi,[46] Hon. M. ASCE.

Mr. Skempton describes the construction of a grain elevator foundation in Durban, South Africa. Mr. Kantey has also discussed the Durban elevator. There appears to be some discrepancy among the test results for the clay soil believed to underlie the site of this structure. According to Mr. Skempton, the liquid limit of the clay is about 100%; the water content, about 60%; and the plastic limit, about 30%. According to Mr. Kantey, the liquid limit is about 55% and the water content is slightly higher than the liquid limit. Both writers describe the failure of this elevator foundation and attribute the difficulty to the remolding effect of the piles which were driven into the clay below the sand bed. The writers believe that the observed events can be explained in a much simpler manner and that it is very doubtful that the remolding effect of pile driving had any influence on the failure.

This grain elevator is owned and operated by the South African Railways and the very extensive and costly failure of the foundations led to the appointment, by the government, of a Commission of Inquiry. The piles on the upper level were driven first and the concrete mat was cast on top of the piles. During the driving of the upper piles, the steel sheet piling bulged toward the excavation. As pile driving proceeded in the lower level, the sheet piling continued to bulge and sag toward the excavation. By the time the lower level pile driving was practically completed, the edge of the slab nearest the excavation had sunk almost 2 ft. The sheet piling separating the two levels of excavation was not designed to provide restraint to the soil but merely to keep the working area dry.

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[46] *Proceedings*, ASCE, February, 1942, p. 317.

The much larger penetrations per blow in the low area were probably due to the fact that the amount of sand through which these piles were driven was considerably less than that through which the piles on the upper level were driven. In addition, in order to excavate the pit for the Working House area it must have been necessary to lower the water table at least 15 ft. This lowering of the water table would have increased the effective pressure on the bed of underlying clay and led to settlement due to consolidation.

The subsequent installation of the cylinder foundations, as described by Mr. Skempton, produced a loss of ground and resulted in the final collapse of the upper slab and the sheet piling. At that stage of the construction, the contractor was discharged and the work was taken over by the South African Railways. In view of these facts, the failure of this elevator foundation can hardly be attributed to the remolding effect of pile driving.

Mr. Legget inquires as to whether there was any significance in the order in which the piling at Detroit was driven. The writers believe that the order of the pile driving had no effect on the results of the investigation. In this connection, it should be pointed out that the reported upheaval of 2 ft is merely an approximate value not obtained by measurement. However, it may be stated definitely that no water and no softened clay escaped to the ground surface after flowing alongside the piles.

Mr. Legget also questions the accuracy of the moisture contents because of the possibility that the samples may have dried during the compression tests. Before the investigation was started, a series of tests was made to explore this possibility. Nine identical samples of remolded clay were made of the same dimensions as the specimens tested in the investigation. They were allowed to stand in the laboratory at a room temperature of 80° and a relative humidity of 56% and were tested after various intervals of time. The first sample was tested immediately after remolding and the last approximately 2 hours afterward. Neither the compressive strength nor the water content showed significant alterations within a period of 45 min. The actual tests reported in the paper were all performed in less than 10 min.

Several discussers have suggested that the soil test data should be supplemented by stress-strain curves. Sets of typical curves for the upper and the lower clay are shown in Fig. 40.

Messrs. Freeman and Parsons have referred to the work of Karl Zimmerman[39] but the writers are of the opinion that Mr. Zimmerman's work is in no way related to the problem under discussion. All his experiments were made with model piles which were driven from 28 in. to 30 in. into a box of artifically compacted sand. The piles were of various shapes and sizes, with diameters varying from about 2.5 in. to 3 in. In other words, the ratio of length to diameter of the model piles was approximately 10 to 1. For a 40-ft pile in the field, the length-diameter ratio in Mr. Zimmerman's tests would correspond to a pile 4 ft in diameter. As the title of his book[39] indicates, Mr. Zimmerman's work was done primarily to investigate the driving characteristics of model piles with various cross-sectional shapes and with varying tapers. The conditions under which his experiments were made, as well as the fact that he worked only with sand, make any reference to his work irrelevant with respect to the subject under discussion.

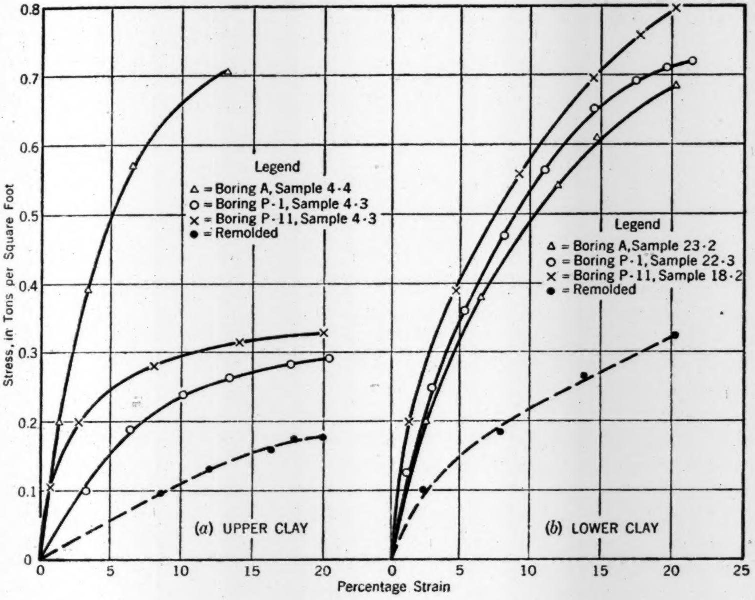

FIG. 40.—STRESS-STRAIN CURVES

Mr. Crandall has presented four examples of projects on which friction piles were driven in beds of clay. Although no factual data are presented as to the properties of the clay after pile driving, settlement records indicate that there was no excessive settlement on any of the projects described. All over the world there are many similar examples of satisfactory foundations consisting of friction piles driven into clay beds.

Mr. Housel has presented field data similar to those of the writers. His laboratory test results on samples taken before and after pile driving confirm the results reported in the paper.

In closing, the writers wish to thank the discussers for the time and effort that was used in preparing the discussions. The writers are well aware that the data presented in the paper are meager. Nevertheless, most of the discussions were based on even less field data. The paper and its discussions have helped to clarify the problem under investigation. It is hoped that engineers will be encouraged to make similar investigations to obtain factual data under varying conditions of soils and pile driving.

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Paper No. 2401

ANALYSIS OF PILE FOUNDATIONS WITH BATTER PILES

BY A. HRENNIKOFF,[1] ASSOC. M. ASCE

WITH DISCUSSION BY MESSRS. JACOB KAROL, J. OWEN LAKE, GEORGE S. MURPHY, AND A. HRENNIKOFF.

SYNOPSIS

This paper presents a method of analysis of pile loads in foundations involving nonparallel piles, when both longitudinal and lateral resistances of piles are taken into account. Although the basic ideas underlying this method are substantially the same as in the other similar methods, the manner of applying these ideas is different in so far as a clear separation is made of the aspects of the problem that are reasonably definite and generally accepted from those that are uncertain. The element of uncertainty is present in the form of three coefficients, characterizing deformability of the piles, whose values can only be roughly estimated. By way of approximation, the two less important of these coefficients may be ignored and the third may be expressed in terms of data possessed by the designer, which lead to workable formulas suitable for practical use. A method of estimating the pile coefficients is also given, and by varying the values of these coefficients over a wide range, it is shown by an example that the error introduced by ignoring the minor coefficients is only moderate.

PILE FOUNDATIONS

Batter piles are not used in foundations of structures carrying horizontal loads as often as their advantages would justify. Perhaps one of the reasons for this anomaly is the absence of a workable method of analysis.

In this connection one cannot fail to notice a conspicuous inconsistency in the usual design practice. On the one hand, many foundations carrying horizontal loads are provided with only vertical piles, so that all the horizontal restraint is caused by the lateral pile resistances alone. On the other hand, when the arrangement is such that two or more groups of nonparallel piles are present, the lateral pile resistances are completely ignored, as if they were of

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NOTE.—Published in February, 1949, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

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no consequence. This approach is out of line with the one adopted in the case of parallel piles.

It may be pointed out that pile foundations very seldom, if ever, fail because the piles break, but almost invariably fail because the movement of the foundation becomes excessive. This statement is in full agreement with the recognized method of determining the allowable loads on piles both in longitudinal and in transverse directions by loading tests. Thus, most building codes specify that the safe axial load on a pile must be such that the settlement produced does not exceed a definite small value, such as $\frac{3}{8}$ in. With regard to the allowable lateral loads most of the codes are silent, but here again resistances specified by some authorities are based on definite small displacements of the pile heads.[2]

In the foregoing it was shown that, from the viewpoint of design, the pile load is considered merely as a symbol for deflection. If a specification limits the pile load to 20 tons, it is not because a 25-ton load is believed to break it, or at least bring it too close to failure, but because the settlement corresponding to it is expected to be excessive.

This symbolic use of the allowable load is not limited to the piles alone, but is common in other cases as well. For example, the allowable load on soil is governed largely by the expected settlement of the structure.

The close relation existing between the allowable load on a pile and the resultant displacement has an important bearing on the validity of the generally accepted method of pile analysis, previously referred to, which consists of disregarding the lateral pile resistances. One might think that, once the lateral pile resistance is omitted, the total pile resistance is underestimated, and the design is conservative. This conclusion, however, generally is not correct, as is shown by the following.

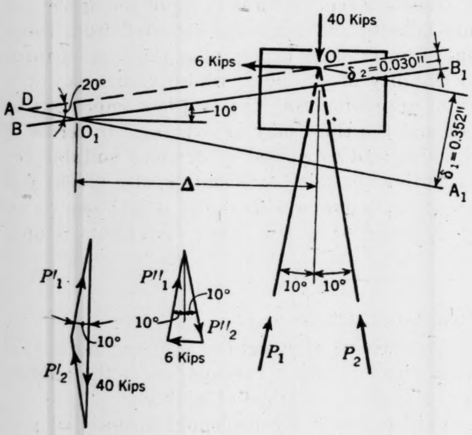

Fig. 1

Fig. 1 represents in a simplified form a foundation supported by two piles inclined in opposite directions at 10° with the vertical. The foundation carries a vertical load of 40 kips and a horizontal load of 6 kips, applied as shown. The lateral pile resistance is disregarded. It is assumed that the allowable axial load on each pile is 40 kips, and that the axial displacement of the pile head produced by this load is $-\frac{3}{8}$ in. The latter statement implies that displacement of the foundation in excess of $\frac{3}{8}$ in. is objectionable.

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[2] "Lateral Pile-Loading Tests," by L. B. Feagin, *Transactions*, ASCE, Vol. 102, 1937, p. 236.

By constructing two separate force triangles, one for horizontal loads and one for vertical loads, and then by combining the forces, the two pile loads are

$$P_1 = \tfrac{1}{2} \times 40 \times \sec 10^\circ + \tfrac{1}{2} \times 6 \times \sec 80^\circ = 37.6 \text{ kips}$$

and

$$P_2 = \tfrac{1}{2} \times 40 \times \sec 10^\circ - \tfrac{1}{2} \times 6 \times \sec 80^\circ = 3.2 \text{ kips}$$

Inasmuch as both P_1 and P_2 are compression loads and are less than 40 kips, the pile loads are satisfactory.

Consider now the horizontal displacement of the foundation. Assuming that the deformation is proportional to the load, the axial displacements of the pile heads are found to be, for pile 1, $\delta_1 = \frac{3}{8} \times \frac{37.6}{40} = 0.352$ in., and, for pile 2, $\delta_2 = \frac{3}{8} \times \frac{3.2}{40} = 0.030$ in.

The deflected position O_1 of the point of intersection of the piles (Fig. 1) is found at the intersection of the line AA_1, normal to pile 1 and located at 0.352 in. below O, and the line BB_1, normal to pile 2 and situated at 0.030 in. below O.

Evidently, $OD = 0.352 \operatorname{cosec} 20^\circ$ and $DO_1 = 0.030 \operatorname{cosec} 20^\circ$, so that the horizontal displacement of the foundation is $\Delta = (0.352 - 0.030) \times \operatorname{cosec} 20^\circ \times \cos 10^\circ = 0.928$ in., which is much greater than $\frac{3}{8}$ in. and therefore is inadmissible.

The large value of this displacement is, of course, the direct result of the small batter of the piles; it would decrease with an increase in the batter. However, practical considerations of pile driving preclude too great a batter, and the situation presented herein appears typical. Thus, the fact that the axial pile loads are moderate is no guarantee that the displacement of the foundation is small and the design adequate, unless the lateral movement is small also, or unless the lateral pile loads are moderate in comparison with the allowable values, based on small displacements.

EXISTING METHODS OF ANALYSIS

Before presenting the method proposed in this paper it is desirable to outline briefly predecessors of the method.

In the method developed by Swedish engineers and presented in the United States by C. P. Vetter,[3] M. ASCE, the lateral and rotational pile resistances R and M, acting on the ends of piles (Fig. 2), are determined by assuming the pile to act as a free beam fixed at some unknown depth h. This assumption has two weaknesses: (1) No method is suggested for determining the unknown depth h; and (2) the pile is actually loaded along its entire length by the side pressure of earth, and it does not resemble a free beam of span h. Also, no section of the pile is fixed both in position and direction, as is assumed with regard to the section at A. The effect of these erroneous assumptions is that no single value of h can result in true values of both end resistances M and R corresponding to different displacements of the pile head. No estimate of the error resulting from these incorrect assumptions seems to be possible.

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[3] "Design of Pile Foundations," by C. P. Vetter, *Transactions*, ASCE, Vol. 104, 1939, p. 758.

The method of pile design[4] of Carl Culmann is based on the resolution of P_{R0}, the resultant force acting on the foundation, into three component forces (Fig. 3), acting in the directions of the piles and passing through the centers of the respective pile groups. This method possesses the following weaknesses:

a. By assumption, the foundation must be supported by three nonparallel groups of piles, with one or several parallel piles in each group. From a practical point of view the requirement of the three nonparallel groups of piles may be an unnecessary complication.

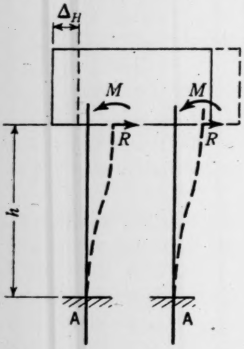

FIG. 2

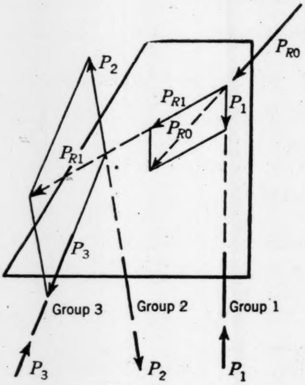

FIG. 3

b. The piles are assumed to develop axial forces only, and since no analysis of lateral deformations is made, the method is subject to the weakness previously discussed in connection with Fig. 1.

c. Each pile group is replaced in analysis by a resultant pile located at the center of gravity of the pile group. When several piles are present in each group, this assumption ignores the rotational resistances of the pile groups, developed by virtue of spreading the piles over an area, and thereby underestimates the strength of the foundation.

The method of H. M. Westergaard,[5] M. ASCE, also ignores the lateral pile resistances, and is thus subject to the same criticism.

ASSUMPTIONS OF PROPOSED METHOD

The method presented in this paper is based on the following assumptions:

1. The load carried by each pile is proportional to the displacement of the pile head. In the most general case this displacement consists of three components: The axial (in relation to the pile) displacement, δ_l; the transverse displacement, δ_t; and the rotational displacement, α. Although the displace-

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[4] "Theoretical Soil Mechanics," by K. Terzaghi, John Wiley & Sons, Inc., New York, N. Y., 1943, p. 363.

[5] "The Resistance of a Group of Piles," by H. M. Westergaard, *Journal*, Western Soc. of Engrs., Vol. 22, 1917, p. 704.

ments thus defined are proportional to the forces producing them, they need not necessarily be considered as elastic.

2. All piles behave alike with regard to the load-deformation relation.
3. The footing, in which the pile heads are embedded, is absolutely rigid.
4. The problem is two-dimensional—that is, the piles, as well as the external forces, are arranged in planes transverse to the length of the foundation, and they are symmetrical with regard to the transverse middle plane. Under these conditions all the pile movements take place in transverse planes.
5. The footing movements are small.

It is of course fully realized that none of these assumptions is exactly true, but it is believed that in most cases they do not misrepresent the actual behavior of the piles greatly, and, at the same time, no reasonably workable analysis can be developed without their assistance. When physical conditions in the foundation manifestly disagree with any of these assumptions—for example, when the footing is not rigid—the theory, of course, fails. The same assumptions have been used by Mr. Vetter and Professor Westergaard.

THE PILE CONSTANTS

The pile constants are defined as the forces with which the pile acts on the foundation when the pile head is given a unit displacement. There are three sets of these constants, corresponding to three different kinds of displacements.

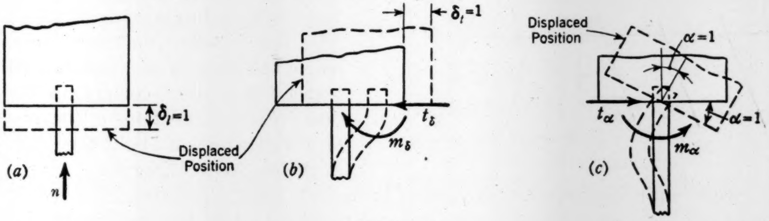

FIG. 4

Unit longitudinal displacement, $\delta_l = 1$ in. (Fig. 4(*a*)), brings into play a single constant n which is measured in pounds per inch and acts in the axial direction.

A unit transverse movement, $\delta_t = 1$ in. (Fig. 4(*b*)), produces a transverse resistance t_δ and a rotational resistance m_δ, measured in pounds per inch and pound-inches per inch, respectively.

A unit rotation, $\alpha = 1$ radian (Fig. 4(*c*)), is accompanied by a rotational resistance m_α and a transverse resistance t_α, measured in pound-inches per radian and pounds per radian, respectively. It may be observed that no axial resistances accompany either transverse or rotational displacements, and no moments or transverse forces are brought into play by the axial displacement.

It should be emphasized that the pile constants as defined herein represent the effect of the pile on the footing and not vice versa. Therefore, the directions of these constants must be carefully noted. As may be expected, the three primary constants (n, t_δ, and m_α), are directed opposite to the pile head displacements creating them, whereas the two secondary constants (m_δ and t_α), which are only incidental to the displacements produced, stand in such a relation to their primary partners (that is, to t_δ and m_α, respectively) that, if the t constant is directed to the right, the m constant is directed counterclockwise, and vice versa.

By the Betti theorem,[6] $t_\alpha = m_\delta$, so that in the most general case—that is, when the pile head is deeply embedded in the footing—there are only four constants characterizing the load-deformation relation of a pile—n, t_δ, $m_\delta = t_\alpha$, and m_α.

When the footing barely engages the piles, their heads may be considered as hinged instead of fixed, and $m_\delta = 0$. Likewise, the two α constants need not be considered in this case, and only n and t_δ remain.

THE FOUNDATION CONSTANTS

Fig. 5 represents the foundation of a structure supported by piles and referred to rectangular axes. For convenience, the x-axis is made coincident with the footing base; the positive directions of the axes are taken to the right and downward, and the origin is chosen arbitrarily. The foundation constants are defined as the resultant forces with which all piles act together on the footing, when the footing is given a unit translational displacement in the positive direction of one of the axes or a unit rotation about the origin in the clockwise direction.

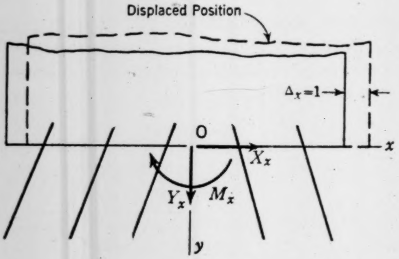

FIG. 5

Each unit displacement in general brings into play a resisting force, which may be represented by its two components acting along the coordinate axes and the moment about the origin. Thus, unit displacement along the x-axis gives rise to the constants X_x, Y_x, and M_x, in Fig. 5. A unit displacement along the y-axis creates the constants X_y, Y_y, and M_y, whereas a unit rotation about the origin produces the constants X_α, Y_α, and M_α. By the Betti law,[6] three pairs of these constants are equal, thus leaving only six independent values: X_x, $Y_x = X_y$, $M_x = X_\alpha$, Y_y, $M_y = Y_\alpha$, and M_α. The positive signs of these functions correspond to the positive directions of the axes and to the clockwise direction for the moments.

These foundation constants can be evaluated in terms of the pile constants and the geometry of the foundation. In order to avoid errors it is necessary to adhere strictly to the sign convention previously explained, keeping in mind that the foundation constants represent the action of the piles on the foundation, and not the opposite effect.

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[6] "Theory of Statically Indeterminate Structures," by W. M. Fife and J. B. Wilbur, McGraw-Hill Book Co., Inc., New York, N. Y., 1937, p. 39.

ANALYSIS OF PILE FOUNDATION

In order to determine the pile loads produced by the external forces acting on the structure it is necessary to find the three component displacements of the foundation: The displacement Δ_x along the x-axis; the displacement Δ_y along the y-axis; and the angle of rotation, α. These displacements are found by setting up three simultaneous equations expressing the equilibrium of the foundation under the action of the external forces and the pile forces induced by these displacements.

Let the components of the external forces, acting in the positive directions of the axes, and their moment about the origin, be X, Y, and M. The forces induced by the displacement Δ_x are then $X_x \Delta_x$ and $Y_x \Delta_x$ along the x-axis and the y-axis, respectively; and $M_x \Delta_x$, a moment about the origin.

The forces produced by the displacements Δ_y and α are expressed in a similar manner. The equations of equilibrium of the footing are then

$$X_x \Delta_x + X_y \Delta_y + X_\alpha \alpha + X = 0 \ldots\ldots\ldots\ldots\ldots (1a)$$

$$X_y \Delta_x + Y_y \Delta_y + Y_\alpha \alpha + Y = 0 \ldots\ldots\ldots\ldots\ldots (1b)$$

and

$$X_\alpha \Delta_x + Y_\alpha \Delta_y + M_\alpha \alpha + M = 0 \ldots\ldots\ldots\ldots\ldots (1c)$$

from which the component displacements of the footing—Δ_x, Δ_y, and α—are determined. Once these component displacements are known, the displacements of the individual pile heads are found by geometry, and from them the axial, lateral, and rotational pile loads are easily computed.

PRINCIPAL AXES

In a possible modification of the procedure the coordinate axes, together with their origin, may be moved to a position outside the footing base, so located that a displacement of the footing along one of the axes would produce a force on the footing acting exactly along that axis, and the rotation about the origin would produce a moment without any force. The similarity of this approach to the analysis of fixed-ended arches by means of the elastic center may be easily recognized. In this case, the three displacements of the foundation may be found directly without solving the simultaneous equations. Although apparently simple in principle, this approach has been found quite laborious in practice and for this reason is not pursued herein.

EXPRESSIONS FOR THE FOUNDATION CONSTANTS

Fig. 6 represents a foundation supported by several nonparallel groups of piles. Group 1 includes N_1 piles making an angle ϕ_1 with the foundation base; group 2 includes N_2 piles at an angle ϕ_2, and so on—the total number of pile groups being g. The angle ϕ is measured from the positive direction of the x-axis clockwise to the given pile.

The footing is displaced a unit distance parallel to itself in the positive direction of the x-axis. In view of the rigid embedment, the piles retain their original inclinations at the ends and the head of a typical pile, A, has moved to

A_1, a distance $\cos \phi_1$ along the pile and $\sin \phi_1$ across the pile away from point A. These displacements induce pile A to act on the foundation with the following forces, applied at the point A_1 or A, and shown in their proper directions in Fig. 6: A force, $n \cos \phi_1$, along the pile; a force, $t_\delta \sin \phi_1$, across the pile; and a moment, $m_\delta \sin \phi_1$. Similar forces are evidently applied at the heads of the other piles, and the only difference in their algebraic expressions is in the subscript of the angle ϕ.

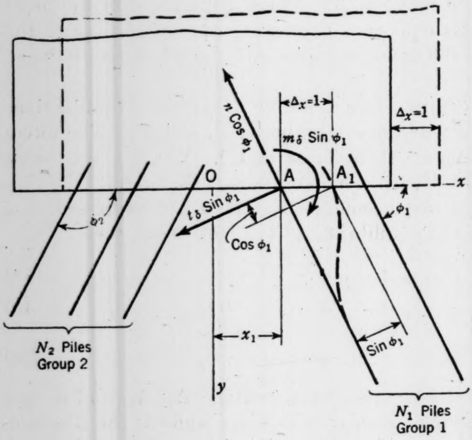


FIG. 6

The sums of the components of all the induced pile forces in the directions of the axes represent the foundation constants X_x and Y_x. The part of X_x contributed by the two resistances of pile A is, with its proper sign, $-(n \cos^2 \phi_1 + t_\delta \sin^2 \phi_1)$. For all N_1 piles of the same group this contribution is N_1 times greater. Summing up all g groups, the corresponding foundation constant is

$$X_x = -\sum_g [N\ (n \cos^2 \phi + t_\delta \sin^2 \phi)] \ldots\ldots\ldots\ldots\ldots (2a)$$

in which the symbol $\sum_g$ covers g terms of the form expressed by the brackets.

The contribution of pile A to Y_x is $t_\delta \sin \phi_1 \cos \phi_1 - n \cos \phi_1 \sin \phi_1$, or $-\frac{1}{2}(n - t_\delta) \sin 2\phi_1$. Thus,

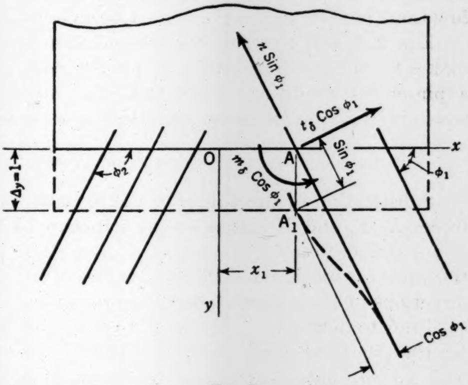
$$Y_x = -\tfrac{1}{2}(n - t_\delta) \sum_g (N \sin 2\phi) \ldots\ldots\ldots\ldots\ldots\ldots (2b)$$

The moment produced by the resistances of pile A about the origin is $Y_x x_1 + m_\delta \sin \phi_1$, or $-\frac{1}{2}(n - t_\delta) \sin 2\phi_1 (x_1) + m_\delta \sin \phi_1$. The moment of all the piles of group 1 about the origin is $-\frac{1}{2}(n - t_\delta) \sin 2\phi_1 (x_1 + x_2 + \cdots + x_{N1}) + N_1 m_\delta \sin \phi_1$, or $-\frac{1}{2}(n - t_\delta) N_1 \bar{x}_1 \sin 2\phi_1 + N_1 m_\delta \sin \phi_1$, in which $\bar{x}_1$ is the abscissa of the center of gravity of all the pile heads in group 1. Summing up these moments for all g groups,

$$M_x = -\tfrac{1}{2}(n - t_\delta) \sum_g (N \bar{x} \sin 2\phi) + m_\delta \sum_g (N \sin \phi) \ldots\ldots\ldots (2c)$$

in which $\bar{x}$ is the coordinate of the center of the group of N piles having the angle ϕ.

Fig. 7 presents the diagram needed for determining the Y constants of the foundation. The footing is moved a unit distance parallel to itself in the positive direction of the y-axis, so that the head of a typical pile of group 1 is moved from A to A$_1$, a distance $\sin \phi_1$ along the pile and $\cos \phi_1$ across the pile. The forces induced in the pile by these displacements are shown in their proper directions in Fig. 7.



$n \operatorname{Sin} \phi_1$
$t_\delta \operatorname{Cos} \phi_1$
$\operatorname{Sin} \phi_1$
$m_\delta \operatorname{Cos} \phi_1$
O
A
A_1
x
ϕ_2
ϕ_1
$\Delta_y=1$
x_1
y
$\operatorname{Cos} \phi_1$

FIG. 7

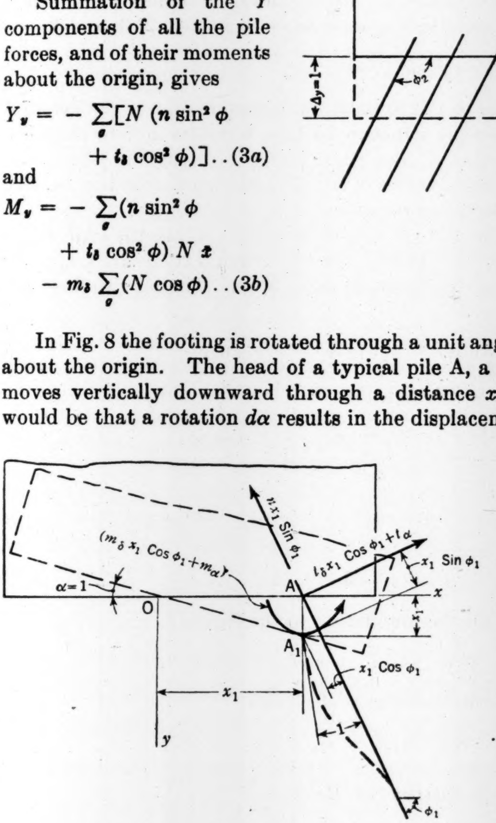
Summation of the Y components of all the pile forces, and of their moments about the origin, gives

$$Y_y = -\sum_g [N(n \sin^2 \phi + t_\delta \cos^2 \phi)] \quad ..(3a)$$

and

$$M_y = -\sum_g (n \sin^2 \phi + t_\delta \cos^2 \phi) N \bar{x} - m_\delta \sum_g (N \cos \phi) \quad ..(3b)$$

In Fig. 8 the footing is rotated through a unit angle in the clockwise direction about the origin. The head of a typical pile A, a distance x_1 from the origin, moves vertically downward through a distance x_1. (The correct statement would be that a rotation $d\alpha$ results in the displacement $x_1\, d\alpha$.) This displacement is resolved into the components $x_1 \sin \phi_1$ along the pile and $x_1 \cos \phi_1$ across the pile. Because of these displacements the pile exerts forces on the foundation—$n\, x_1 \sin \phi_1$ along the pile, and $t_\delta\, x_1 \cos \phi_1$ across the pile—in addition to the moment, $m_\delta\, x_1 \cos \phi_1$, about the origin. The pile head also undergoes a unit rotation, which brings into play the transverse force t_α and the moment m_α in the directions shown in Fig. 8. Taking the sum of the contributions of all piles to the moment about the origin, and keeping in mind that $m_\delta = t_\alpha$, the remaining foundation constant is found to be



$n x_1 \operatorname{Sin} \phi_1$
$(m_\delta x_1 \operatorname{Cos} \phi_1 + m_\alpha)$
$t_\delta x_1 \operatorname{Cos} \phi_1 + t_\alpha$
$x_1 \operatorname{Sin} \phi_1$
$\alpha=1$
O
A
A_1
x
x_1
$x_1 \operatorname{Cos} \phi_1$
1
y
ϕ_1

FIG. 8

$$M_\alpha = -\sum_g [(n \sin^2 \phi + t_\delta \cos^2 \phi) \sum_N (x^2)] - 2\, m_\delta \sum_g (N \bar{x} \cos \phi) - m_\alpha \sum_g (N) \quad ..(4)$$

The foundation constants determined by Eqs. 2, 3, and 4, with their proper signs, represent forces and moments exerted by the piles on the footing and not by the footing on the piles. Positive signs of these functions indicate forces acting in the positive directions of the coordinate axes or, in the case of moments, in the clockwise direction. The x coordinates and the trigonometric functions in their expressions must be taken with proper algebraic signs.

Eqs. 2, 3, and 4 presuppose full embedment of the piles in the footing, which makes them behave as if they had fixed ends. When the embedment is small, a pin-ended condition ensues, and the corresponding formulas for the pile constants may be found by substituting $m_s = m_\alpha = 0$ in Eqs. 2, 3, and 4.

Pile Forces

Substitution of the values of the foundation constants and of the external forces X, Y, and M acting on the structure in Eqs. 1 results in determination of the foundation displacements Δ_x along the x-axis, Δ_y along the y-axis, and the angle of rotation α. Positive values of Δ_x, Δ_y, and α correspond to positive directions of the axes and clockwise rotation.

Prior to determining the pile forces it is necessary to obtain the expressions for the pile head displacements. This is done in relation to an arbitrary pile with a coordinate x_1 (see Figs. 6, 7, and 8) as follows: The longitudinal displacement downward,

$$\delta_l = \Delta_x \cos \phi_1 + \Delta_y \sin \phi_1 + \alpha\, x_1 \sin \phi_1 \;\ldots\ldots\ldots\ldots\; (5a)$$

the transverse displacement to the right,

$$\delta_t = \Delta_x \sin \phi_1 - \Delta_y \cos \phi_1 - \alpha\, x_1 \cos \phi_1 \;\ldots\ldots\ldots\ldots\; (5b)$$

and the rotation clockwise, α.

The expressions for the pile forces are as follows: The longitudinal force (compression),

$$P = n\, \delta_l \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (6a)$$

the transverse force, acting on the foundation to the right,

$$Q = -\, t_s\, \delta_t + m_s\, \alpha \;\ldots\ldots\ldots\ldots\ldots\ldots\; (6b)$$

and the moment, acting on the foundation clockwise,

$$S = m_s\, \delta_t - m_\alpha\, \alpha \;\ldots\ldots\ldots\ldots\ldots\ldots\; (7)$$

Solution by Ratios

Although the expressions for the pile loads (Eqs. 6 and 7) are functions of the pile constants, it is important to realize that it is not the constants themselves but their ratios that determine the pile loads, as may be made clear from the following. Suppose that all four pile constants decrease in the same ratio q. Then, as may be seen from Eqs. 2, 3, and 4, all the foundation constants are reduced q times. The result is that the foundation displacement, determined from Eqs. 1, and the pile head displacements δ_l, δ_t, and α (Eqs. 5), all become q times greater. Turning now to the pile forces, it may be seen that Eqs. 6 and 7 are made up of the products of the pile constants and the pile displacements, and because the former functions decrease q times and the latter increase q times, the pile forces remain unchanged.

Thus, in the general case of fixed-ended piles, their loads are fully determined by the three ratios,

$$r_1 = \frac{t_\delta}{n} \qquad (8a)$$

$$r_2 = \frac{m_\delta}{n} \qquad (8b)$$

and

$$r_3 = \frac{m_\alpha}{n} \qquad (8c)$$

When the piles behave as having pinned ends, only one ratio, r_1, remains, r_2 and r_3 both becoming zero.

It is preferable to approach the problem by the method of ratios rather than that of pile constants, not only because of the smaller number of coefficients involved, but also because r_1, the most important of the ratios, is more easily ascertainable than the pile coefficients, as made clear later. For the purpose of explaining the ratio approach new quantities, referred to as the reduced foundation constants, are introduced. Their expressions are as follows (see Fig. 9):

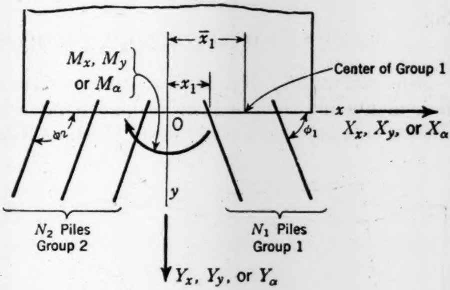


FIG. 9

$$X'_x = \frac{X_x}{n} = -\sum_g [N (\cos^2 \phi + r_1 \sin^2 \phi)] \qquad (9a)$$

$$Y'_x = \frac{X_y}{n} = \frac{Y_x}{n} = -\tfrac{1}{2}(1 - r_1) \sum_g (N \sin 2\phi) \qquad (9b)$$

$$M'_x = \frac{M_x}{n} = \frac{X_\alpha}{n} = -\tfrac{1}{2}(1 - r_1) \sum_g (N \bar{x} \sin 2\phi) + r_2 \sum_g (N \sin \phi) \qquad (9c)$$

$$Y'_y = \frac{Y_y}{n} = -\sum_g [N (\sin^2 \phi + r_1 \cos^2 \phi)] \qquad (9d)$$

$$M'_y = \frac{M_y}{n} = \frac{Y_\alpha}{n} = -\sum_g [(\sin^2 \phi + r_1 \cos^2 \phi) N \bar{x}] - r_2 \sum_g (N \cos \phi) \qquad (9e)$$

and

$$M'_\alpha = \frac{M_\alpha}{n} = -\sum_g [(\sin^2 \phi + r_1 \cos^2 \phi) \sum_N (x^2)] - 2 r_2 \sum_g (N \bar{x} \cos \phi) - r_3 \sum_g (N) \qquad (9f)$$

Although the foundation constants proper are unknown, the reduced constants can be easily determined, because they are expressed in terms of the pile ratios and not in terms of the pile constants. By rearranging to the form:

$$\frac{X_x}{n}(n\,\Delta_x) + \frac{Y_x}{n}(n\,\Delta_y) + \frac{M_x}{n}(n\,\alpha) + X = 0 \quad \text{(10)}$$

and substituting

$$\Delta'_x = n\,\Delta_x \quad \text{(11a)}$$

$$\Delta'_y = n\,\Delta_y \quad \text{(11b)}$$

and

$$\alpha' = n\,\alpha \quad \text{(11c)}$$

for the expressions in the brackets, named the reduced foundation movements, Eqs. 1 become

$$X'_x\,\Delta'_x + Y'_x\,\Delta'_y + M'_x\,\alpha' + X = 0 \quad \text{(12a)}$$

$$Y'_x\,\Delta'_x + Y'_y\,\Delta'_y + M'_y\,\alpha' + Y = 0 \quad \text{(12b)}$$

and

$$M'_x\,\Delta'_x + M'_y\,\Delta'_y + M'_\alpha\,\alpha' + M = 0 \quad \text{(12c)}$$

—from which Δ'_x, Δ'_y, and α' can be determined. The actual foundation movements and the pile head displacements cannot be found by this approach, but the pile loads are easily determined as follows:

From Eqs. 5, the reduced pile head displacements are

$$\delta'_l = n\,\delta_l = \Delta'_x \cos\phi_1 + \Delta'_y \sin\phi_1 + \alpha' x_1 \sin\phi_1 \quad \text{(13a)}$$

$$\delta'_t = n\,\delta_t = \Delta'_x \sin\phi_1 - \Delta'_y \cos\phi_1 - \alpha' x_1 \cos\phi_1 \quad \text{(13b)}$$

and α'.

From Eqs. 6 and 7 the pile loads (see Fig. 10) are given by

$$P = \delta'_l \quad \text{(14a)}$$

and

$$Q = -r_1\,\delta'_t + r_2\,\alpha' \quad \text{(14b)}$$

$$S = r_2\,\delta'_t - r_3\,\alpha' \quad \text{(15)}$$

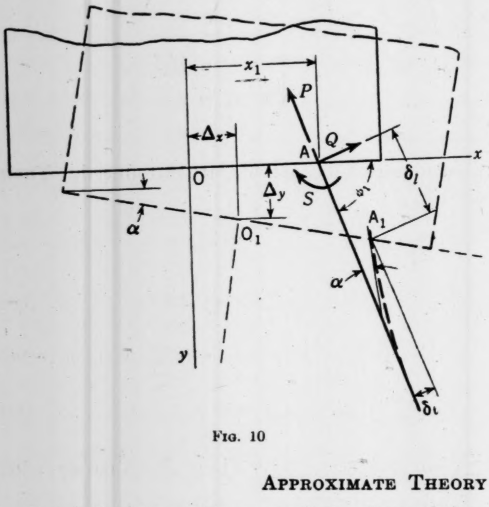


FIG. 10

APPROXIMATE THEORY

Determination of the numerical values of the pile ratios, as well as of the pile constants, is at best a rather uncertain procedure. Fortunately, however, the difficulty arising because of this uncertainty can be overcome by means of a moderate approximation, which at the same time adds to the practicability of the method. Comparative calculations have shown that it is possible, without serious error, to ignore completely the ratios r_2 and r_3, and to express the remaining ratio, r_1, in terms of the data possessed by the designer.

It is interesting to note that the ratios r_2 and r_3 (or, which is just the same, the constants m_δ and m_α) are disregarded in the conventional method of analysis of vertical piles (Fig. 11). According to this method, the axial load on the extreme pile (at a distance x_1 from the origin to the pile center) is

$$P = \frac{Y}{N} + \frac{M\,x_1}{\sum x^2} \qquad (16)$$

which clearly omits the effects of m_δ and m_α. Actually, the influence of m_δ is manifested in the end moments on the piles, due to the horizontal displacement of the foundation produced by the horizontal force X, thus augmenting the external moment M in its effect on the values of the longitudinal pile loads. At the same time the rotation of the footing caused by inequality of the longitudinal pile loads results in formation of the end moments, which are dependent on m_α, and somewhat reduce the vertical pile loads. Thus, the usual design practice, when applied to foundations supported by the vertical piles alone, definitely ignores the effect of r_2 and r_3 without explicitly stating so.

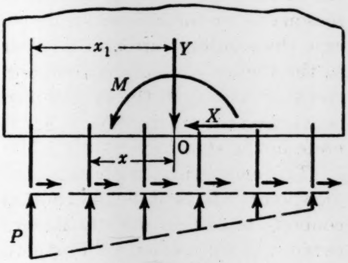


FIG. 11

Analysis of pile foundations by means of a single ratio r_1 involves omitting from the equations concerned terms containing the ratios r_2 and r_3. The moment on the end of the pile (Eq. 15) is, of course, eliminated. The complete set of equations required for the pile load analysis by the approximate theory includes, in addition to Eqs. 9*a*, 9*b*, 9*d*, 11, 12, and 14*a*.

$$M'_x = X'_\alpha = -\tfrac{1}{2}(1 - r_1)\sum_g (N\,\bar{x}\sin 2\phi) \qquad (17)$$

$$M'_y = Y'_\alpha = -\sum_g [(\sin^2\phi + r_1\cos^2\phi)\,N\,\bar{x}] \qquad (18)$$

$$M'_\alpha = -\sum_g [(\sin^2\phi + r_1\cos^2\phi)\sum_N (x^2)] \qquad (19)$$

and

$$Q = -r_1\,\delta'_t \qquad (20)$$

Eqs. 17, 18, 19, and 20, as well as Eqs. 9*a*, 9*b*, 9*d*, 11, 12, 13, and 14*a*, involve no approximation when the piles behave as hinged at the top ends, but the values of r_1 for the pin-ended condition and the fixed-ended condition are quite different.

If the allowable loads on the pile in the axial and transverse directions are P_a and Q_a, respectively, and the pile head displacements caused by these loads are $(\delta_a)_l$ and $(\delta_a)_t$,

$$n = \frac{P_a}{(\delta_a)_l} \qquad (21a)$$

and

$$t_t = \frac{Q_a}{(\delta_a)_t} \quad \text{(21}b\text{)}$$

Substituting Eqs. 21 in Eq. 8*a* and assuming that for properly established allowable loads $(\delta_a)_l = (\delta_a)_t$ results in

$$r_1 = \frac{Q_a}{P_a} \quad \text{(22)}$$

Inasmuch as the lateral pile resistances, as well as the axial resistances, are constantly relied upon in the usual design, the values of both P_a and Q_a must be known to the designer, whether by code, by test, or by guessing. Therefore, the ratio r_1 must also be known to him. This ratio can be wrong only if unreasonable or inconsistent values have been chosen for Q_a and P_a, in which case the resultant error in all fairness cannot be charged to the method, just as the theory of reinforced concrete cannot properly be charged with the discrepancy arising in the case of a reinforced concrete structure when the design is based on the assumption of 3,000-lb concrete and the actual strength of concrete in the structure is only 1,500 lb per sq in.

The comparison with reinforced concrete is instructive also from another viewpoint. It is well known that the relation between stress and strain in concrete is neither strictly linear, even at the beginning of the stress-strain curve, nor fully elastic. Furthermore, the so-called secant modulus, even in concrete of the same strength characteristics, is a quantity quite variable for different specimens and much different from the generally accepted value of $E = 1{,}000 f'_c$, yet the conventional design of reinforced concrete is seldom questioned with regard to the bending stresses in concrete and steel. It seems reasonable to adopt a similar attitude in the design of pile foundations. Then r_1 may be looked upon as a relative measure of pile resistances, even when the dependence of the allowable pile loads on the displacements becomes obscure.

It appears that a moderate variation in the assumed value of r_1 does not alter the values of the pile loads a great deal. If for some special reasons the assumed allowable pile loads Q_a and P_a are based on different values of the displacements δ_{at} and δ_{al}, a simple adjustment can be made in Eq. 22.

The actual values of the pile loads P and Q, found by Eqs. 14*a* and 20, should not exceed the allowable values so that

$$P \leqq P_a \quad \text{(23}a\text{)}$$

and

$$Q \leqq Q_a \quad \text{(23}b\text{)}$$

Substituting in Eq. 23*b* the expression for Q from Eq. 20, and eliminating the minus sign as immaterial, it is found that

$$r_1 \, \delta'_t = \frac{Q_a}{P_a} \, \delta'_t \leq Q_a \quad \text{(24}a\text{)}$$

or

$$\delta'_t \leqq P_a \quad \text{(24}b\text{)}$$

Compliance with Eq. 24*b*, which is equivalent to Eq. 23*b*, insures that the lateral pile load and the lateral displacement of the pile head are not excessive.

When stated in the form:

$$\delta_t \leqq \frac{P_a}{n} \dots\dots\dots\dots\dots\dots\dots\dots (24c)$$

Eq. 24*b* obviously means that the lateral pile head movement must not exceed the axial movement caused by the allowable longitudinal load. Evidently, this condition must be satisfied by piles in all cases, whether they are analyzed by a single ratio, r_1, or by all three ratios.

When the lateral pile resistance is negligible, r_1 may be assumed to be zero in the expressions for the foundation constants and for the pile loads. However, the condition expressed by Eq. 24*b* still holds in this case and warns against an excessive lateral pile movement, similar to that discussed under the heading "Pile Foundations."

In most cases the piles are nearly vertical, with the result that the value of the horizontal movement of the foundation Δ_x is fairly close to the value of the lateral pile movement δ_t; for this reason the condition:

$$\Delta'_x \leqq P_a :: \dots\dots\dots\dots\dots\dots\dots\dots (25)$$

may be used as approximately equivalent to Eq. 24*b*. Thus, Eqs. 23*a* and 25, or Eqs. 23*a* and 24*b*, may be used in all cases as the criteria of adequacy of the design, in lieu of Eqs. 23.

APPROXIMATE FORMULAS FOR THE PILE CONSTANTS

Determination of the pile constants n, t_δ, m_δ, and m_α or of the three ratios r_1, r_2, and r_3 need not be considered as a necessary part of the present investigation. However, in order to check the agreement of the approximate and the more exact analyses of the pile foundation, it has been found desirable to make a rough computation of these constants. The method used is admittedly crude, being based on a number of rather arbitrary assumptions. This circumstance, however, should not detract from the usefulness of the numerical investigation founded on it, because its main purpose lies not in determining the true values of the coefficients, but rather in establishing their order for different foundation conditions, with the view of demonstrating the closeness of the approximate and the more exact pile analyses under virtually all possible combinations of piles and soils. The range of the primary physical coefficients determining the values of the pile constants will be chosen so wide that the true values of the latter are not likely to go beyond the limits assumed in the investigation.

VALUES OF *n*

It follows from Eq. 21*a* that small values of *n* occur with lightly loaded piles which settle a great deal, whereas large values of *n* occur when the piles are loaded heavily but their settlements are small. Allowance must be made for both the deformation of the soil under the foot of the pile and the elastic shortening of the pile. A ½-in. axial displacement of the head under the allowable load will be considered a large settlement; the small settlement will be taken equal to the elastic shortening of the pile alone. In the latter case a distinction will be made between a very weak soil (so soft that it offers no axial support to the pile), in which the pile is assumed to be driven to a firm stratum below so that all its axial support comes from the foot reaction, and other soils, in which the pile is assumed to be supported by skin friction alone. The pile shortenings in these two types of soils are found by the formulas:

For very weak soil,

$$(\delta_a)_l = \frac{P_a\, L}{A\, E} \quad \text{(26a)}$$

and for other soils,

$$(\delta_a)_l = \frac{P_a\, L}{2\, A\, E} \quad \text{(26b)}$$

Two rather extreme classes of piles are used in the following analyses—a 30-ft timber pile, 9 in. in average diameter, and a 100-ft reinforced concrete pile, 20 in. square. The data pertaining to these piles, together with the computed values of n, are given in Table 1.

TABLE 1.—DATA AND COMPUTED VALUES OF n FOR SPECIMEN PILES

Pile	Section[a] (in.)	Area[a] (sq in.)	Modulus of elasticity, E (kips per sq in.)	Load, P_a (kips)	$(\delta_a)_l$ (IN.)			n (KIPS PER IN.)		
					Maximum	Minimum		Minimum	Maximum	
						Very weak soil	Other soil		Very weak soil	Other soil
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
30-ft timber...........	9[b]	63.5	1,500	30	0.5	0.1134	0.0567	60	265	530
100-ft reinforced concrete	20[c]	400	3,000	140	0.5	0.14	0.07	280	1,000	2,000

[a] Average. [b] Round. [c] Square.

CONSTANTS t_δ, m_δ, AND m_α

Approximate values of the constants t_δ, m_δ, and m_α can be determined by considering the pile as a beam on an elastic foundation of infinite length, loaded at the free end.[7] The end conditions necessary for determining the constants t_δ and m_δ are shown in Fig. 12(a), in which the transverse force is t_δ and the end moment m_δ, whereas the deflection at the end is unity, and the angle change is zero. For determination of m_α, the end conditions (Fig. 12(c)) are: A transverse force $t_\alpha = m_\delta$, a moment m_α, zero end deflection, and unity

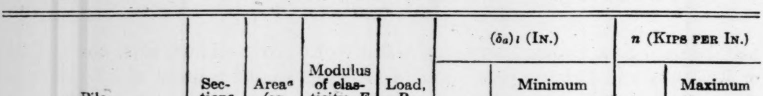


FIG. 12

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[7] "Strength of Materials," by S. Timoshenko, D. Van Nostrand Co., Inc., New York, N. Y., Vol. 2, 1941.

angle change. Translating the notation of S. Timoshenko into that of Fig. 12,

$$t_s = \frac{k}{\beta} \qquad \text{(27a)}$$

$$m_s = \frac{k}{2\,\beta^2} = \frac{t_s}{2\,\beta} \qquad \text{(27b)}$$

and

$$m_\alpha = \frac{k}{2\,\beta^3} = \frac{m_s}{\beta} \qquad \text{(27c)}$$

in which k is the elastic constant of the soil, defined as the ratio of the load per unit length of pile and the corresponding deflection of the pile at the same point, that is,

$$k = \frac{p}{\delta_t} \qquad \text{(28)}$$

Also, the coefficient

$$\beta = \sqrt[4]{\frac{k}{4\,E\,I}} \qquad \text{(29)}$$

in which I and E are, respectively, the moment of inertia of the cross section and the modulus of elasticity of the pile. When the pile behaves as having pinned ends (Fig. 12(*b*)),

$$t'_s = \frac{k}{2\,\beta} = \tfrac{1}{2}\,t_s \qquad \text{(30)}$$

This method involves the approximations and assumptions that: (*a*) The intensity of the earth reaction at each point of the pile is proportional to the deflection of the pile at the same point; (*b*) the cross section of the pile is constant throughout its full length; and (*c*) the pile is infinitely long. Under the last condition the greatest error occurs when the pile is short and rigid and the soil is weak. Correction for the finite pile length is possible but laborious, and calculations indicate that this error is moderate even in the extreme cases.

ESTIMATION OF THE SOIL PRESSURE CONSTANT k

The value of the soil pressure constant depends on the stiffness of the soil, the size of the pile, and its cross-sectional shape. Three widely different classes of soil are considered: (1) Very weak, compressing 1 in. under a pressure of 100 lb per sq ft; (2) weak, compressing 1 in. under a pressure of 1 kip per sq ft; and (3) medium, compressing 1 in. under a pressure of 10 kips per sq ft. Generally speaking, soil firmer than medium is not likely to require the presence of piles, and therefore is not considered in connection with the pile constants.

A round pile, by virtue of its shape, is believed to produce a sort of wedge action on the soil. This condition is allowed for by assuming that for a round pile the deformation is double that of a square pile of the same diameter.

The computation of k may be illustrated by the following examples, both of which assume very weak soil: 20-in. square pile, $k = \frac{100 \times 20}{12 \times 12} = 13.88$ lb per sq in.; and 9-in.-diameter round pile, $k = \frac{100 \times 9}{2 \times 12 \times 12} = 3.12$ lb per sq in.

The values of the pile constants t_δ, m_δ, and m_a, together with values of k and β, corresponding to different piles and soils are given in Table 2.

It is interesting to present at this point the available result of an independent determination of the pile constants. In a rather elaborate theoretical investigation, based on the data of lateral pile loading tests performed by L. B. Feagin, M. ASCE, A. E. Cummings,[8] M. ASCE, arrives at the following values (in this paper called the pile constants) for a round timber pile of 12-in. average diameter, with $E = 1.878$ kips per sq in., in the soil described as medium sand—$t_\delta = 32$ kips per in. and $m_\delta = 1{,}200$ in-kip per in. Reduction of the constants of the 9-in.-diameter timber piles in Table 2 to comparable sizes and stiffnesses gives resultant values of $t_\delta = 27.3$ kips per in. and $m_\delta = 895$ in-kip per in. With all the uncertainties of the underlying theories and the indefiniteness of the soil designation, the agreement of these two sets of constants must be considered as close.

TABLE 2.—VALUES OF t_δ, m_δ, m_a, k, β, n, r_1, r_2, AND

Pile (1)	Section[a] (in.) (2)	I (in.^{4})[a] (3)	E (kips per sq in.) (4)	Type of soil (5)	k (lb per sq in.) (6)	1,000 β (in.$^{-1}$) (7)	t_δ (kips per in.) (8)	m_δ (in-kip per in.) (9)
30-ft timber	9-in. round	322	1,500	Very weak	3.12	6.34	0.493	38.9
				Weak	31.2	11.28	2.77	123
				Medium	312	20.03	15.59	389
100-ft reinforced concrete	20-in. square	13,333	3,000	Very weak	13.88	3.05	4.55	745
				Weak	138.8	5.43	25.6	2,360
				Medium	1,388	9.65	143.7	7,450

The values of the three r ratios are also given in Table 2, two sets of ratios being given for each class of soil and each type of pile. The maximum values correspond to the least values of n in Table 1; the minimum values, to the greatest values of n.

NUMERICAL EXAMPLE

The theory presented herein is applied to the analysis of the pile foundation under the retaining wall shown in Fig. 13. The wall is supported by transverse rows of piles at 3-ft centers along the wall, five piles per row, the two back piles being vertical and the three front ones inclined at a batter of 3 on 1. The earth pressure is found by Rankine's theory, using the following data: The unit weight of concrete is 150 lb per cu ft; w, the unit weight of soil, is 100 lb per cu ft; and θ, the angle of friction of the soil, is 30°. The earth pressure at the back is represented by the active pressure:

$$E_a = \tfrac{1}{2}\, w\, h^2 \frac{1 - \sin\theta}{1 + \sin\theta} \qquad (31a)$$

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[8] Discussion by A. E. Cummings of "Lateral Pile-Loading Tests," by L. B. Feagin, *Transactions*, ASCE, Vol. 102, 1937, p. 255.

At the front it is represented by one half of the passive pressure or

$$\tfrac{1}{2}\, E_p = \tfrac{1}{2}\,(\tfrac{1}{2})\, w\, h^2_1\, \frac{1 + \sin\theta}{1 - \sin\theta} \quad \ldots\ldots\ldots\ldots\ldots\ldots (31b)$$

Both these pressures are horizontal and, to avoid complicating the problem by

r_3 FOR SPECIMEN PILES IN VARIOUS TYPES OF SOILS

m_α (in-kip per radian)	n (KIPS PER IN.)		r_1		r_2 (IN.)		r_3 (IN.^{3})	
	Maximum	Minimum	Minimum	Maximum	Minimum	Maximum	Minimum	Maximum
(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)
6,140	265	60	0.00186	0.00821	0.1468	0.648	23.16	102.3
10.900	530	60	0.00523	0.0462	0.232	2.05	20.57	181.9
19,400	530	60	0.0294	0.260	0.734	6.48	36.6	323.3
244,000	1,000	280	0.00455	0.01625	0.745	2.66	244	872
435,000	2,000	280	0.0128	0.0913	1.180	8.43	217.5	1,555
772,000	2,000	280	0.0719	0.513	3.725	26.6	386	2,760

[a] Average.

further uncertainties of soil mechanics, are assumed to have the same values in all cases of pile-soil combination, irrespective of the displacements of the foundation and structure. The origin of the coordinates is placed at the central pile and the external forces acting on the 3-ft section of the wall, referred to the origin, are shown in Fig. 13.

The problem is solved repeatedly using all twelve sets of the pile ratios and considering first that all the ratios r_1, r_2, and r_3 are effective (cases 2 to 13, Table 3). Then the ratio r_1 alone is considered to be effective (cases 2a, to 13a). In addition, the problem is solved by disregarding all pile resistances other than axial (case 1).

Employment of large and closely spaced reinforced concrete piles in a structure of this size may be criticized as incongruous. Nevertheless, although such piles would admittedly be too large for the conditions indicated, the analysis based on the ratios peculiar to them may be looked upon as the one

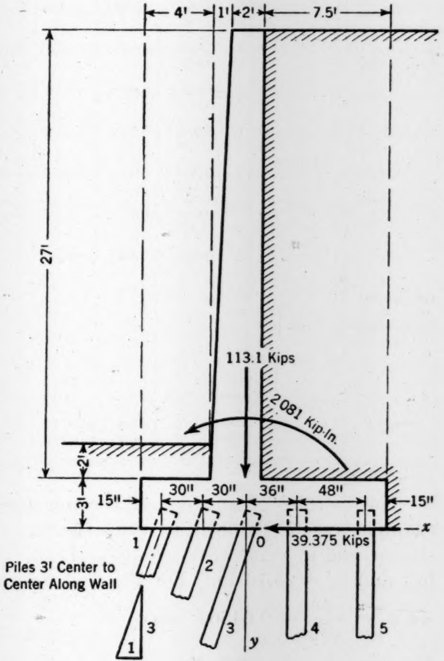

FIG. 13

presenting the picture of mechanical behavior of these piles in larger and more heavily loaded structures. As far as the structure in question is concerned, the piles endowed with the ratios corresponding to the large reinforced concrete piles will be considered as having the same moderate allowable load as the timber piles. This load in all cases is taken as $P_a = 40$ kips.

TABLE 3.—PILE LOADS FOR WALL OF FIG. 13

Pile	Type of soil	Case No.	PILE RATIOS			REDUCED FOUNDATION MOVEMENTS			PILE NO. 1		
			r_1	r_2 (in.)	r_3 (in.^{3})	Δ^1_x (kips)	Δ^1_y (kips)	α^1 (kips per in.)	P (kips)	Q (kips)	S (in-kip)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
		1	0	0	0	−311.1	−40.8	+0.638	+23.5	....	....
Timber, 9 in. round 30 ft long	Very weak	2	0.00186	0.1468	23.16	−212.6	−19.32	+0.3384	+29.7	+0.45	−39
		2a	0.00186	....	....	−233.0	−23.84	+0.413	+27.5	+0.44	....
		3	0.00821	0.648	102.3	−113.6	+2.136	+0.0405	+35.6	+0.91	−74
		3a	0.00821	....	....	−129.9	−1.603	+0.1176	+32.9	+1.03	....
	Weak	4	0.00523	0.232	20.57	−149.6	−5.74	+0.1590	+32.8	+0.80	−37
		4a	00.0523	....	....	−162.3	−8.58	+0.2101	+31.2	+0.84	....
		5	0.0462	2.05	181.9	−42.4	+17.44	−0.1525	+38.6	+1.16	−37
		5a	0.0462	....	....	−39.0	+18.05	−0.1429	+37.6	+1.32	....
	Medium	6	0.0294	0.734	36.6	−54.8	+14.75	−0.1108	+37.6	+1.25	−29
		6a	0.0294	....	....	−54.9	+14.64	−0.0973	+36.8	+1.34	....
		7	0.260	6.48	323.3	−17.50	+22.55	−0.2043	+38.5	+0.13	+30
		7a	0.260	....	....	−12.65	+23.57	−0.2158	+38.6	+0.12	....
Reinforced concrete, 20 in. square 100 ft long	Very weak	8	0.00455	0.745	244	−135.5	−2.448	+0.0854	+35.7	+0.65	−118
		8a	0.00455	....	....	−173.2	−10.92	+0.2417	+30.7	+0.78	....
		9	0.01625	2.66	872	−78.0	+10.02	−0.0805	+38.7	+0.91	−114
		9a	0.01625	....	....	−83.3	+8.52	−0.0162	+35.4	+1.24	....
	Weak	10	0.0128	1.18	217.5	−87.4	+7.89	−0.0407	+37.4	+0.97	−85
		10a	0.0128	....	....	−97.8	+5.39	+0.0254	+34.6	+1.17	....
		11	0.0913	8.43	1,555	−36.58	+18.57	−0.1561	+38.0	+1.04	+24
		11a	0.0913	....	....	−23.98	+21.24	−0.1852	+38.3	+1.14	....
	Medium	12	0.0719	3.725	386	−34.65	+19.08	−0.1720	+38.8	+1.05	−21
		12a	0.0719	....	....	−28.12	+20.39	−0.1737	+38.1	+1.22	....
		13	0.513	26.6	2,760	−17.70	+21.92	−0.1582	+35.5	−0.68	+253
		13a	0.513	....	....	− 9.59	+23.91	−0.2196	+38.3	−1.35	....

The method of analysis of the pile loads is illustrated by the following example (case 2, Table 3), in which the minimum pile ratios are assumed for the timber pile in very weak ground. From Table 2, $r_1 = 0.00186$; $r_2 = 0.1468$ in.; and $r_3 = 23.16$ in. For the first group of piles: $N_1 = 3$; $\phi_1 = 108° 26'$; $\sin \phi_1 = \frac{3}{\sqrt{10}} = 0.94869$; $\cos \phi_1 = -\frac{1}{\sqrt{10}} = -0.31623$; and $\sin 2\phi_1 = -0.6$. For the second group of piles: $N_2 = 2$; $\sin \phi_2 = 1$; $\cos \phi_2 = 0$; $\sin 2\phi_2 = 0$; and $\phi_2 = 90°$.

Thus,

$$X'_x = -[3\,(0.1 + 0.00186 \times 0.9) + 2 \times 0.00186] = -0.30874$$
$$X'_y = Y'_x = -\tfrac{1}{2}\,(1 - 0.00186)\,3\,(-0.6) = 0.89833$$
$$M'_x = X'_\alpha = -0.89833\,(-30) + 0.1468\,(3 \times 0.94869 + 2) = -26.239 \text{ in.}$$
$$Y'_y = -[3\,(0.9 + 0.00186 \times 0.1) + 2 \times 1] = -2.7006 - 2 = -4.7006$$

UNDER VARIOUS COMBINATIONS OF PILE RATIOS

PILE LOADS											
PILE No. 2			PILE No. 3			PILE No. 4			PILE No. 5		
P (kips) (13)	*Q* (kips) (14)	*S* (in-kip) (15)	*P* (kips) (16)	*Q* (kips) (17)	*S* (in-kip) (18)	*P* (kips) (19)	*Q* (kips) (20)	*S* (in-kip) (21)	*P* (kips) (22)	*Q* (kips) (23)	*S* (in-kip) (24)
+41.6	....	....	+59.8	....	....	−17.8	....	....	+12.7	....	....
+39.4	+0.44	−39	+49.0	+0.44	−38	−7.1	+0.45	−39	+9.1	+0.45	−39
+39.3	+0.43	....	+51.0	+0.43	....	−9.0	+0.43	....	+10.9	+0.43	....
+36.7	+0.91	−74	+37.9	+0.90	−73	+3.6	+0.96	−78	+5.5	+0.96	−78
+36.3	+1.02	....	+39.6	+1.02	....	+2.6	+1.07	....	+8.3	+1.07	....
+37.4	+0.80	−37	+41.9	+0.79	−37	0.0	+0.82	−38	+7.6	+0.82	−38
+37.2	+0.83	....	+43.2	+0.82	....	−1.0	+0.85	....	+9.1	+0.85	....
+34.3	+1.23	−40	+30.0	+1.29	−43	+12.0	+1.65	−59	+4.6	+1.65	−59
+33.5	+1.39	....	+29.5	+1.45	....	+12.9	+1.74	....	+6.1	+1.74	....
+34.5	+1.28	−30	+31.3	+1.31	−31	+10.8	+1.53	−36	+5.5	+1.53	−36
+34.0	+1.37	....	+31.2	+1.39	....	+11.1	+1.61	....	+6.5	+1.61	....
+32.7	+0.63	+17	+26.9	+1.13	+5	+15.2	+3.22	−47	+5.4	+3.22	−47
+32.5	+0.65	....	+26.4	+1.18	....	+15.8	+3.29	....	+5.4	+3.29	....
+38.1	+0.65	−118	+40.6	+0.64	−117	+0.6	+0.67	−122	+4.7	+0.67	−122
+37.5	+0.77	....	+44.4	+0.76	....	−2.2	+0.79	....	+9.4	+0.79	....
+36.4	+0.92	−116	+34.2	+0.94	−118	+7.1	+1.05	−137	+3.3	+1.05	−137
+34.9	+1.24	....	+34.4	+1.24	....	+7.9	+1.35	....	+7.2	+1.35	....
+36.3	+0.98	−86	+35.1	+0.98	−86	+6.4	+1.07	−94	+4.5	+1.07	−94
+35.3	+1.17	....	+36.0	+1.15	....	+6.3	+1.25	....	+7.5	+1.25	....
+33.6	+1.18	+12	+29.2	+1.31	−1	+12.9	+2.02	−66	+5.5	+2.02	−66
+33.1	+1.30	....	+27.8	+1.46	....	+14.6	+2.19	....	+5.7	+2.19	....
+33.9	+1.17	−27	+29.0	+1.29	−34	+12.9	+1.85	−63	+4.6	+1.85	−63
+33.2	+1.34	....	+28.2	+1.45	....	+14.1	+2.02	....	+5.8	+2.02	....
+30.9	+0.09	+213	+26.5	+0.85	+174	+16.3	+4.87	−34	+8.7	+4.87	−34
+32.0	−0.28	....	+25.8	+0.79	....	+16.1	+4.90	....	+5.5	+4.90	....

$$M'_y = Y'_\alpha = -2.7006\,(-30) - 2 \times 60 - 0.1468 \times 3\,(-0.31623) = -38.843 \text{ in.}$$

$$M'_\alpha = -\{(0.9 + 0.00186 \times 0.1)\,[(-30)^2 + (-60)^2] + 36^2 + 84^2\} - 2 \times 0.1468 \times 3\,(-30)\,(-0.31623) - 23.16\,(3 + 2) = -12{,}527 \text{ in.}^2$$

The simultaneous equations are, therefore,

$$-0.30874\,\Delta'_x + 0.89833\,\Delta'_y - 26.239\,\alpha' - 39.375 = 0$$
$$0.89833\,\Delta'_x - 4.7006\,\Delta'_y - 38.843\,\alpha' + 113.1 = 0$$
$$-26.239\,\Delta'_x - 38.843\,\Delta'_y - 12{,}527\,\alpha' - 2{,}081 = 0$$

from which the reduced foundation displacements are found to be

$$\alpha' = +0.3384 \text{ kip per in.}, \ \Delta'_y = -19.32 \text{ kips, and } \Delta'_x = -212.6 \text{ kips.}$$

The reduced displacements of the head of pile 1 are $\delta'_l = -212.6\,(-0.31623) - 19.32 \times 0.94869 + 0.3384\,(-60)\,0.94869 = +29.72$ kips; $\delta'_t = -212.6 \times 0.94869 + 19.32\,(-0.31623) - 0.3384\,(-60)\,(-0.31623) = -214.43$ kips; and $\alpha' = +0.3384$ kips per in. The loads on this pile are $P = 29.72$ kips; $Q = -0.00186\,(-214.43) + 0.1468 \times 0.3384 = 0.45$ kips; and $S = 0.1468 \times (-214.43) - 23.16 \times 0.3384 = -39.3$ in-kip. Assuming the allowable longitudinal load on the pile to be 40 kips, the criteria of adequacy of the foundation design (Eqs. 23*a* and 25) show that the value of P is satisfactory, but that the value of Δ'_x is excessive.

All the data pertaining to the foundation movements and the pile loads, obtained by repeated solutions of the problem by different sets of the pile ratios, are presented in Table 3. The results have been verified by evaluating all the pile loads and by checking them by means of the three equations of statics.

DISCUSSION OF THE SOLUTION RESULTS

It may be observed at the outset that the range of relative stiffnesses of piles in axial and lateral directions, implied in the values of various sets of pile ratios in Table 3, is very wide, extending all the way from zero value of lateral resistance in case 1 to a lateral resistance value more than half as great as the axial resistance in case 13 ($r_1 = 0.513$). Some interesting and unexpected results become apparent from Table 3.

a. The most striking feature is the excessive amount of horizontal displacement of the foundation, as indicated by the value of Δ'_x, which quantity greatly exceeds $P_a = 40$ kips in case of very weak and partly weak soils. Therefore, according to the lateral displacement criterion, Eq. 25, the foundation of Fig. 13 must be deemed inadequate, except in medium soil. This result may appear unexpected to many engineers.

b. Increase in the lateral stiffness of piles always leads to an increase in the lateral pile loads Q, as may well be expected. These loads are quite small, however, usually of the order of 0.5 kip to 1.5 kips per pile, so that their sum amounts to from only 2.5 kips to 7.5 kips per bent, against the external lateral load of 39.375 kips. Although the effect of the lateral pile resistances on the carrying capacity of the bent is small, its effect on the lateral rigidity of the foundation is tremendous. Thus, when the piles have no lateral resistance (case 1, Table 3) $\Delta'_x = -311.1$ kips, but when (case 6*a*) $r_1 = 0.0294$ and $r_2 = r_3 = 0$ (that is, when the lateral pile resistance is less than 3% of the axial resistance), $\Delta'_x = -54.90$ kips, or the lateral displacement of the foundation decreases six times. Thus, a completely misleading and much more alarming picture of the lateral foundation displacement is obtained if the lateral pile resistance is ignored, even when it is quite small.

c. Although the horizontal displacements of the foundation, Δ'_x, are always negative (that is, are always directed to the left), the vertical displacements

Δ'_v and the rotational movements α vary in direction, depending on the pile-soil conditions. The combination of these displacements sometimes results in most baffling values of the pile loads.

Thus, one would naturally expect that the greatest axial load is carried by pile 1. This is so in the medium and weak soils, but in the very weak soil, especially with the less rigid timber piles, pile 3 becomes the one which is loaded most. This condition is particularly pronounced in case of piles completely devoid of lateral resistance (case 1). Then the axial load on pile 3 is $P = 59.8$ kips, whereas on pile 1 it is only 23.5 kips. This striking result is accentuated by the negative axial load on the adjacent pile 4 and the positive load of 12.7 kips on pile 5.

In spite of this apparent irregularity of the pile loads in different groups the loads in the same group vary linearly. Thus, the loads on pile 2 are the mean between the load values on piles 1 and 3.

d. The greatest axial load when piles are endowed with lateral resistances in medium, weak, and even, in some cases, of very weak ground varies between 35 kips and 39 kips, and it mostly acts on pile 1. This means that, when the lateral pile resistances are ignored (case 1), the greatest axial pile load is exaggerated by approximately 60%, and often a wrong pile is indicated as carrying the maximum load.

e. The constancy of the maximum axial load for nearly all pile-soil combinations is also surprising. Apart from cases 1, 2, and 4, involving very weak soil and flexible piles, the greatest variation of this load amounts to only 10%.

In comparing the more exact solutions based on all three pile ratios with the approximate ones involving the same values of r_1 and omitting the other ratios, the results on the whole are quite favorable to the approximate method. Both with respect to the deformations and to the pile loads, the two corresponding solutions in most cases are related in a definite manner, which may be predicted beforehand. In a few special cases this regularity is broken, evidently because of the peculiarities of the combinations of the pile ratios. These deviations are not serious, however, and the difference in the results by the two methods is nearly always quite small.

f. When the pile ratios r_2 and r_3 are ignored, the horizontal displacement of the foundation is usually increased by an amount close to 10%, making the approximate method that much on the safe side. In one case of very weak soil (case 8) the approximate method is about 28% on the safe side. On the other hand, in the medium soils, and partly in the weak soils, the approximate method somewhat underestimates the horizontal displacements, which under these conditions are all comparatively small.

g. The values of the lateral pile loads, in nearly all cases, are found by the approximate method to be slightly higher than the ones found by the more exact method. In the two cases in which this relation is reversed, the difference between the two methods is very small, and may possibly be attributed to the errors of computation. The amount by which the lateral pile load found by the approximate method exceeds the value found by the exact method is usually close to 10%, although in cases 8, 9, and 10 this excess varies between 15% and 30%. Except for cases 7 and 13, involving rigid piles in medium ground, all lateral pile loads are quite small.

h. The greatest axial loads are usually slightly greater when found by the approximate method, the difference varying between 0% and 10%. The reverse is true in five cases, in four of which the approximate method is on the unsafe side by from 2% to 4%, and in one case (case 9), by 10%.

From these remarks it may be seen that, on the whole, the approximate method compares favorably with the exact method within the limits of the example considered—that is, for the assumed arrangement of a five-pile bent, carrying the specified forces, under virtually all possible conditions of deformability of soil and piles. Even the extreme and rather rare discrepancies of 10% on the unsafe side and of 28% on the safe side should not be considered as excessive in a subject as uncertain as that of foundation problems. It is desirable, however, to verify this conclusion on the examples of some other pile arrangements and some other combinations of the external forces X, Y, and M.

CONCLUSIONS

The approximate method of analysis of pile loads presented in this paper has the following advantages:

1. It is reasonably simple and not too laborious.
2. It is free from the two common weaknesses of other current methods of analysis: Failure to investigate the horizontal foundation displacement, which is as important as the vertical displacement, and failure to make allowance for the lateral pile resistance, which leads to an enormous exaggeration of the lateral displacement accompanied sometimes by a drastic rearrangement of the pile loads, thus leading to an entirely erroneous picture of the mechanical behavior of the foundation.
3. The only physical property of piles and soil involved in the method is the ratio r_1, which can be determined simply as the ratio of the allowable lateral and axial pile loads, which data must be known to the designer, no matter what method of analysis he is prepared to use.
4. The error involved in disregarding the two minor pile ratios, r_2 and r_3, present in the more exact method, appears to be not excessive.

ACKNOWLEDGMENT

The writer wishes to express thanks to his two former students, V. L. Mosher and J. D. Anderson, for some of the preliminary calculations involved in this research.

DISCUSSION

Jacob Karol,[9] M. ASCE.—In his desire to make the method of analysis simpler to use, Professor Hrennikoff has introduced reduced foundation constants into the basic equations of equilibrium from which the pile loads are more easily determined, and has developed a criterion, $\Delta'_x \leqq P_a$ (Eq. 25), which presumably indicates the adequacy of the design. Considering the cases in Table 3 for which r_2 and r_3 are zero (pin-ended piles), the criterion indicates that the even-numbered cases, 2*a* to 12*a*, are weaker than their respective odd-numbered counterparts, 3*a* to 13*a*. It will subsequently be shown that, considering displacements only, the opposite is true. Cases 2*a* and 8*a* must be deemed inadequate because the allowable axial pile load of 40 kips has been exceeded.

The actual displacements of the foundation are an inverse function of n. Thus, if the piles were infinitely rigid axially (a case approximated by very short piles driven to rock), the lateral displacement of the foundation even in case 1 would be zero, since, by Eq. 11*a*, $\Delta_x = \Delta'_x/n$. Furthermore, considering the example in Fig. 1, if it is assumed that the axial displacement of the pile is $-3/80$ in., or one tenth of the original value, it follows that the lateral displacement of the foundation will be reduced to one tenth of 0.928 in., or 0.093 in. Most designers would probably decide that the foundation was adequate for both load and lateral displacement, even though the latter was 2.6 times as great as the vertical displacement. The examples illustrate the important fact mentioned by the author, but insufficiently emphasized, that the pile loads are functions of the "relative" values of the pile constants (pile ratios), whereas the foundation displacements are functions of the "absolute" values of these constants. Since the adequacy of a foundation is determined by both the pile loads and the foundation displacements, the designer must establish the absolute values as accurately as possible for any particular design.

Dividing through the reduced foundation movements in Table 3 by the values of n from Table 2, the actual foundation movements are obtained and are shown in Table 4. As a matter of interest, the movement of the top of the retaining wall at the inner face is also given, assuming that the wall rotates as a rigid body. The elastic deformation of the stem would have to be added to the lateral movement shown to obtain the final displacement of the top of the wall.

If a solution of the pile loads is obtained by the conventional method for vertical pile groups using Eq. 16, and a part of the transverse shear is assumed to be carried by the horizontal components of the axial loads in the batter piles, the resulting axial loads are practically the same as those shown for cases 7*a* and 13*a*. The unbalanced transverse shear, as indicated by these cases, could be assumed to be carried by the vertical piles only. Therefore, the conventional method of analysis, with this slight modification, may be considered satisfactory for "medium" soils. In addition, the lateral movement of the foundation is given by the expression:

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$$\Delta_x = \frac{Q}{t_3} \quad \text{(32)}$$

in which Q is the unbalanced transverse shear carried by one vertical pile.

The error in the interpretation of the results of Table 3 is due to the faulty definition of the value of r_1 as the ratio of the allowable lateral load to the allowable axial load on the pile, and to the use of this value in developing the criterion. This definition arbitrarily assumes that the allowable transverse and vertical displacements of the pile head are the same. For example, for timber piles in medium soil, this results in a value of Q_a for case 6*a* of 1.18 kips and for case 7*a* of 10.4 kips. There is no reason why the value of Q_a should not be the same in the two cases, since the pile and the soil are the same. The discrepancy indicates that in general the allowable lateral displacement of the pile will not be equal to the allowable vertical displacement. Hence the value of r_1 as originally defined in Eq. 8*a*, should be used in the analysis, and the criteria in Eqs. 23 should be used to determine the adequacy of the foundation. Despite these criticisms, Professor Hrennikoff has made a notable contribution to the analysis of pile foundations.

TABLE 4.—MOVEMENTS OF WALL IN FIG. 13 UNDER VARIOUS COMBINATIONS OF PILE RATIOS

Case No.	n	FOUNDATION MOVEMENTS			MOVEMENTS AT TOP OF WALL	
		Δ_x (in.)	Δ_y (in.)	α (radians)	Δ_x (in.)	Δ_y (in.)
2*a*	265	−0.870	−0.090	+0.00156	−0.308	−0.076
3*a*	60	−2.164	−0.027	+0.00196	−1.458	−0.009
4*a*	530	−0.307	−0.016	+0.00040	−0.164	−0.012
5*a*	60	−0.650	+0.301	+0.00238	+0.207	+0.322
6*a*	530	−0.104	+0.028	−0.00018	−0.170	+0.026
7*a*	60	−0.211	+0.393	−0.00360	−1.507	+0.361
8*a*	1,000	−0.173	−0.011	+0.00024	−0.086	−0.009
9*a*	280	−0.298	+0.030	+0.00006	−0.277	+0.030
10*a*	2,000	−0.049	+0.003	+0.00001	−0.044	+0.003
11*a*	280	−0.086	+0.076	−0.00066	−0.324	+0.070
12*a*	2,000	−0.014	+0.010	−0.00009	−0.045	+0.009
13*a*	280	−0.034	+0.085	−0.00078	−0.317	+0.078

J. OWEN LAKE,[10] ASSOC. M. ASCE.—Because of the variable physical and mechanical properties of soils the actual forces induced in piled foundations cannot be predicted with any considerable degree of accuracy no matter how mathematically refined the analysis; yet, until a rigorous solution—even though based inevitably on certain simplifying assumptions—has been made, the degree of inaccuracy inherent in empirical expressions or formulas, based on assumptions which are obviously not realized in practice, can only be surmised. Therefore, Mr. Hrennikoff's analysis of the effect of fixity at the heads of piles embedded in footings supported on nonparallel piles is of considerable value. However, the particular example presented by the author exaggerates the importance of neglecting the lateral resistance of the piles, for not only are the piles spaced at exceptionally close centers but the passive resistance of the soil in front of the retaining wall is neglected.

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[10] Engr., The British Steel Piling Co., Ltd., London, England.

Under the heading "Discussion of the Solution Results," paragraph *b*, the author states that the effect of the lateral pile resistances on the carrying capacity of the bent is small but that their effect on the lateral rigidity of the foundation is tremendous; for the particular example presented in the paper the lateral displacement is shown to be reduced six times by a lateral pile resistance which is less than 3% of the axial resistance. In view of this condition it must be observed that most foundations will have a deep capping to the piles capable of providing a resistance somewhat greater than the aforementioned 3%.

It should be observed that under certain conditions it is inadmissible to allow for any lateral resistance of the piles—for instance, where piles supporting relieving platforms to sheet pile bulkheads are located within the zone of expanding soil immediately behind the wall.

Under the heading, "Existing Methods of Analysis," the author states that the method developed by Swedish engineers,[3] for vertical piles loaded laterally, does not give a method for determining the unknown depth to the point of fixity. Another paper, however, by Donald Hamish Little,[11] describes an approximate graphical method, and an analytical solution of the problem is presented by the writer[12] in the written discussion of Mr. Little's paper.

George S. Murphy,[13] Assoc. M. ASCE.—A retaining wall supported wholly on vertical piles is subjected to translation and forward rotation about the toe. With batter piles present, however, the wall is subject to translation and backward rotation about the heel. This latter condition is brought about by the upward push of the heads of the batter piles, resulting from their rotation about an instantaneous center as the piles bend.

With the usual 30° batter, a forward translation of the pile head will produce an upward movement 0.577 times the horizontal movement. Since the height of the wall is usually about 2.2 times the base, the wall will tilt backward an amount equal to 2.2 times the upward movement of the toe. It follows that, with a forward translation, x (Fig. 14), the backward tilt of the top of the wall will be $2.2 \times 0.577\, x = 1.27\, x$, or about 25% more than the forward movement of the base.

The problem of relating these relative displacements to the pressures developed may be viewed in three ways: First, if the entire earth wedge behind the wall is considered to move forward with the displacement of the base of the wall, but compressed by the backward tilt, then the relative backward movement of the top of the wall against the earth is 5 $x/4$ rather than $x/4$.

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[11] "Some Dolphin Designs," by Donald Hamish Little, *Journal*, Inst. of Civ. Engrs., November, 1946, p. 48.

[12] Discussion by J. Owen Lake of "Some Dolphin Designs," by Donald Hamish Little, in Supplement, *ibid.*, October, 1947, p. 453.

[13] Toledo, Ohio.

The other extreme would be to assume that the plastic flow of the earth mass occurs only where the wall permits, so that only an active pressure is exerted on the wall except for the small section at the top where the absolute movement of the wall is backward, with a maximum of $x/4$ at the apex.

A third action is conceivable by assuming a cylindrical surface of shear failure from the heel of the wall to the surface of the fill, with the axis of rotation at the wall top (see Fig. 15). This would impose no more than active pressure on the vertical wall, but such an assumption has doubtful validity, especially since the axis of rotation also has a vertical translation.

It seems likely either that the true condition is intermediate between the first two, or that it is a mean of all three, with the several components having variable weights for different types of soil, but that, in any case, some passive earth pressure is developed by the wall when batter piles are used.

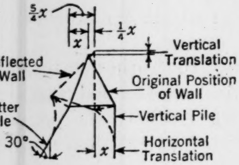

FIG. 14

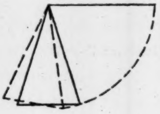

FIG. 15

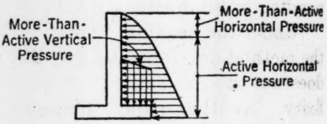
More-Than-Active Vertical Pressure
More-Than-Active Horizontal Pressure
Active Horizontal Pressure

FIG. 16

When the horizontal translation is sufficiently great, uplift is developed in the most forward vertical pile. In this case, even if the pressure, in excess of all the active horizontal pressure that can be developed, is neglected (as the author has done), some passive pressure must be developed on the upper surface of the base.

Taking into account the passive pressure developed against the back of the vertical wall, all the vertical piles may be hung from the base, as a result of which some greater-than-active pressure (Fig. 16) would develop along the entire top surface at the base. The only relief from this condition is that derived from the axial compression of the batter piles; but, in the case here considered, the depression so caused is less than the rise caused by the forward translation of the batter pile head.

In summary, it would appear that the author has not considered the development of horizontal and vertical passive earth pressures in the case of walls supported partly by batter piles, in so far as such pressures affect the distribution of pile loads and the stresses in the wall.

Such pressures are generated, at any rate, in the period during which the piles are developing deflections corresponding to the maximum earth pressures. Possibly, after the lapse of sufficient time, some of the passive resistance may leach away through the continued operation of plastic adjustments of the internal structure of the soil (such as by permanent consolidation), with the result that intensities equivalent to active pressures are approached at all points; nevertheless, it is reasonable to assume that at least some of the pressure intensities in excess of the active pressure will remain as elastic compression of the soil.

A. Hrennikoff,[14] Assoc. M. ASCE.—The misunderstanding that led to the criticism voiced by Mr. Karol is regrettable. The writer's sole reason for computing the approximate numerical values of the pile stiffness factors n, t_3, t_a, and m_a, for different piles and soils, was to establish the range of the pile ratios for the purpose of demonstrating that the error involved in the writer's approximate theory (based on a single pile ratio r_1) is small when compared to the more exact theory, under virtually all possible pile-soil conditions (see under the heading, "Approximate Formulas for the Pile Constants"). Once the pile ratios are established there is no need to go back to the stiffness factors, since it is the ratios and not the stiffness factors which determine the distribution of loads among the piles of a given bent.

Accordingly, in his example, the writer makes use of the computed pile ratios, but not of the stiffness factors. In the paragraph immediately preceding Table 3 it is stated that the allowable axial pile load in all the cases listed in Table 3 is assumed to be the same—namely, 40 kips. From the viewpoint expounded by the writer, this assumption denotes that, in all cases, the piles are equally deformable axially—that is, they possess equal axial stiffness factors n. Their principal pile ratios $r_1 = \frac{t_3}{n}$, therefore, are directly proportional to the lateral stiffness factors t_3; consequently, case 2a, which has a smaller r_1 than case 3a, involves a lower lateral stiffness and is more dangerous to the stability of the foundation than is case 3a, as is indicated by Table 3.

In comparing cases 2a and 3a, Mr. Karol assumes that they both involve the same lateral stiffness factors t_3, and that a smaller value of r_1 in case 2a is caused by a greater axial stiffness factor n. By these assumptions he concludes correctly that case 2a is the safest; but, under the conditions prescribed by Mr. Karol, it would be incorrect to assign the same values of the allowable axial load to both cases, as is done and explicitly stated by the writer. Table 4 and its subsequent interpretation are affected by the same misunderstanding.

In connection with Mr. Lake's discussion, the writer wishes to point out that no progress, can be made, in the analysis of complex engineering problems, by considering all the factors in their entirety. The proper approach is to separate the individual factors and to account for them one by one. The writer's problem was limited solely to the analysis of pile behavior in a foundation under a known set of loads, and it did not extend to a consideration of different factors affecting these loads.

Mr. Lake is quite correct in stating that the variable passive pressure in front of the pile cap would contribute substantially to the safety of the piles. The writer admittedly did not consider this factor. Moreover, even if he had known how to allow for this variation of the passive pressure, he would have excluded it intentionally from his example in order not to obscure the picture of pile response to the known loads by an extraneous factor associated with variation of one of these loads. In an actual design, by all means, the passive soil resistance must be allowed for in the best manner known; but this has no bearing on the theory of pile analysis presented. Likewise, the writer's statement (borne out by the results of the example discussed) to the effect that the lateral pile resistance, even when very small, has a tremendous effect on the lateral stiffness of the bent, is not invalidated by the fact that a certain variable factor (the increased passive pressure on the cap) may also contribute in a substantial manner to the stiffness of the bent.

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[14] Prof., Civ. Eng., Univ. of British Columbia, Vancouver, B. C., Canada.

Incidentally, if the passive resistance is expressed as a known linear function of the horizontal cap displacement Δ_x, the solution of Eqs. 1 and the computation of the pile loads and displacements are not complicated in any way, as compared with computations in the example presented. Even if the passive pressure on the cap were expressible as a known nonlinear function of Δ_x, Eqs. 1 could still be solved without much extra trouble by a few successive approximations.

Although the effect of the passive earth pressure on the resistance of the foundation to lateral loads is significant, it must not be overrated, or there would be no need for either the lateral pile resistance or the batter piles in pile foundations.

One must agree with Mr. Lake that there are cases in which it is not safe to rely even on a small lateral pile resistance; but one must not ignore the necessity of determining the lateral displacement of the bent. The writer's method is convenient for computing that displacement.

The paper by Donald Hamish Little,[11] and Mr. Lake's discussion of it,[12] deal with the design of "dolphin" piles, whose long projection into open water distinguishes them sharply from foundation piles. In the former the lateral deflection is an advantage and the bending stress is a matter of primary concern, whereas in the latter the deflection is a major fault, and the bending stress, as a rule, is of minor significance.

Mr. Lake made the following assumptions in connection with his method:

1. The pile is acted upon by the passive pressure in front and by the active pressure at the back, determined by the classical Rankine theory of pressure exerted by cohesionless materials on walls (observe that it is the walls and not the piles).
2. The side friction on the pile is allowed for by judgment in a manner not explained. No distinction seems to be made between round and square piles.
3. The earth pressure of full passive and active intensity is assumed to act for a limited depth along the pile, a length necessary to give the pile complete fixation, with no earth pressure extending farther down.

All these assumptions seem to be rather arbitrary, and, although it is possible that the effect of errors resulting from them may be not so great in application to the long free-standing "dolphin" piles, the adaptation of the method to the foundation piles embedded in earth does not appear likely to produce satisfactory results. Furthermore, the method would involve a greater number of constants describing each particular pile condition than the single constant used by the writer—the principal pile ratio, r_1.

In his discussion restricted to consideration of the example of the retaining wall analyzed by the writer, Mr. Murphy examines qualitatively the effects of the displacement of the base slab of the wall on the variation in values of the active and passive pressures exerted by the earth on the structure. Although in substantial agreement with Mr. Murphy's conclusions, the writer still feels justified in ignoring, in his example, the variations in the earth pressure for the reasons explained in connection with Mr. Lake's discussion.

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WIND STRESSES IN SEMI-RIGID CONNECTIONS OF STEEL FRAMEWORK

By Basil Sourochnikoff[1]

With Discussion by Messrs. C. Batho, George S. Murphy, and Basil Sourochnikoff.

Synopsis

Much has been written on wind stresses in general and also on stresses in semi-rigid framework. Some disagreement remains, however, on how to combine the stresses due to gravity loads with those due to wind in the connections of semi-rigid framework, particularly when the connections are stressed beyond the yield point in bending. As the semi-rigid connections are often allowed to work beyond the yield point, this question is of importance. This paper presents a theoretical discussion of stresses under these conditions, and derives general conclusions applicable to rectangular framework.

Introduction

Connections of windbracing beams in steel framework are often encountered in practice, as such beams are used when X-bracing cannot be used because of clearances, when the stiffening effect of walls or partitions is considered to be insufficient, and when, at the same time, the horizontal forces are of sufficient magnitude to warrant taking special provisions to resist their action. Where wind connections are used in the bents of simple framework the girders are usually connected to the columns by top and bottom angles or tees so as to resist within certain limits the change of angle between the girders and the columns. Therefore, the connections are of the semi-rigid type and the study of stresses in them necessitates an analysis of semi-rigid framework.

Study of the stresses in semi-rigid framework has been receiving attention for many years, and many methods of analysis, both exact and approximate, have been developed. However, there are few data on the manner of combining wind stresses with gravity stresses, and there is considerable disagreement among designers on this point.

In its 1946 code[2] the American Institute of Steel Construction (AISC) recommends that wind connections be adequate to resist moments induced by

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Note.—Published in February, 1949, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Structural Designer, Eng. Dept., E. I. du Pont de Nemours & Co., Wilmington, Del.

[2] "Code of Standard Practice for Steel Buildings and Bridges," A.I.S.C., 1946.

gravity loads and wind loads at the increased wind stresses permitted therefor, or that the connections carrying wind moments be so designed that larger moments induced by the gravity loads under the actual conditions of restraint will be relieved by deformation of the connecting material without overstress of the welds. Many designers, however, use an approximate method in which the stresses due to gravity loads are entirely neglected and the connections are designed for wind moments alone. Other designers take the opposing view and recommend that the connections be designed for the algebraical sum of stresses caused by gravity loads and wind loads computed independently of each other with the assumption in each case that the connections act elastically.

Although the last two views seem to contradict each other, each one may hold true for a certain degree of flexibility of the connections. When the connections are very flexible—that is, if they can undergo large inelastic deformations without failure—it may be permissible to neglect the effect of gravity loads. On the other hand, if the breaking point is reached at a comparatively small deformation beyond the elastic limit, both the gravity loads and the wind loads must be considered.

The purpose of this paper is to analyze the manner in which gravity stresses are combined with wind stresses after the elastic limit has been reached.

SUMMARY OF DATA ON THE DESIGN OF SEMI-RIGID FRAMEWORK

Available data on the design of semi-rigid frames can be divided into two sections for examination: (*a*) The properties of semi-rigid connections, and (*b*) the analysis and design of semi-rigid framework.

Properties of Semi-Rigid Connections.—The deformation of a semi-rigid connection is too complex to be determined mathematically. In fact, it depends on the deflection of the top and bottom angles or tees, the deflection of the column flange to which it may be connected, and the deformation and possible slip of the shear rivets. The last factor cannot be represented by a simple mathematical expression. Therefore, the experimental determination of stress-strain curves of the type given by J. Charles Rathbun,[3] M. ASCE, and J. L. Brandes, Jun. ASCE, and R. M. Mains,[4] Assoc. M. ASCE, is of value.

The stress-strain curves are plotted with the moment M acting on the connection as ordinate and the rotation α of the connection as abscissa. These curves have a generally parabolic shape (Fig. 1), the rotation ceasing to be proportional to the moment at rather low values of the moment, so that for most of the length of the curve the slope, $1/Z$, is not constant. However, up to a certain value of the moment the change in Z is small and the connection within these limits may be considered to be elastic. As a rule, the curves do not show a definite yield point comparable to the one that appears on the stress-strain curves of steel tested in tension, and the variation of Z is progressive. When the connection is loaded to a point such as C (Fig. 1) and then unloaded, the representative point travels along a line CC′, which is practically straight. Some irregularities occur in this line, however, especially in the vicinity of point C′, where the moment is zero, and are probably due to the slip that occurs in the rivets when the load is reversed. All unloading lines of a connection are closely parallel to each other, and the value of Z of the unloading lines depends but little on the point C reached during loading. Apparently, not many tests have been made on the behavior of a connection under the reversal of moment, or on the fatigue strength of a connection. Professor Rathbun[3] gives stress-strain curves of two specimens under reversal of moment. These curves show that after the connection is unloaded along a line such as CC′ (Fig. 1) and the moment is reversed, the stress-strain curve under negative moment is somewhat similar to the original loading curve; that is, the slope of the curve decreases gradually with the increase of the negative moment. After reaching a point such as C″ on this curve, if the connection is again unloaded—that is, if the negative moment is reduced—the new unloading curve C″ C‴ is practically a straight line and is closely parallel to the unloading line CC′. If the load is reversed, the corresponding curve C‴ C⁗ is closely similar to loading curve OC.

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[3] "Elastic Properties of Riveted Connections," by J. Charles Rathbun, *Transactions*, ASCE, Vol. 101, 1936, p. 524.

[4] "Report of Tests of Welded Top-Plate and Seat Building Connections," by J. L. Brandes and R. M. Mains, *Welding Journal*, March, 1944, p. 146-s.

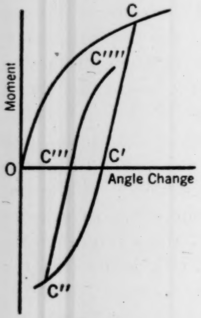

Fig. 1

Analysis and Design of Semi-Rigid Framework.—The various methods of analysis and design may be classified into two types depending on whether or not the rotation of the connections is assumed to be proportional to the moment.

In the first case, in which the rotation of the connections is assumed to be proportional to the moment, the structure as a whole, including connections, is considered to be elastic. A connection is thought of as a locally weakened section between the end of the beam and the face of the column, the effect of which is the inverse of the effect produced by end haunches or added cover plates. The structure as a whole is therefore a rigid frame comprising a number of locally weakened sections, and the ordinary methods of analysis of rigid frames, such as the slope-deflection method or the moment-distribution method, are applicable. However, the stiffness and distribution factors must be adjusted to provide for the local weakening of sections at connections.[3,5,6,7]

In the second case, since the stress-strain curves have no simple mathematical expression, the structure must be analyzed either by successive approximations or by graphical methods, as indicated in a British report.[8]

ANALYSIS OF GRAVITY AND WIND STRESSES IN CONNECTIONS WORKING BEYOND THE PROPORTIONAL LIMIT

The types of design methods previously mentioned are rather complicated and time consuming in their application, and they assume that the properties

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[5] "Analysis of Building Frames with Semi-Rigid Connections," by Bruce Johnston and Edward H. Mount, *Transactions*, ASCE, Vol. 107, 1942, p. 993.

[6] "The Analysis of Rigid Frames," by A. Amirikian, U. S. Govt. Printing Office, Washington, D. C., 1942, p. 338.

[7] "Theory of Modern Steel Structures," by L. E. Grinter, The Macmillan Co., New York, N. Y., Vol. 2, 1936.

[8] "Second and Final Reports," Steel Structures Research Committee, Dept. of Scientific and Industrial Research of Great Britain, H. M. Stationery Office, London, England, 1934.

of the connections are well known. These properties, however, are known only in the case of a few typical connections, and even for these connections they depend on the history of the connection—that is, on the previous loading the connection has sustained and on the permanent set it may have taken. Therefore, too much refinement in the analysis is not warranted.

It is claimed[4,5,9] that, as a result of partial continuity, a saving of from 15% to 20% of steel can be made by using semi-rigid connections in place of standard connections. In practice, however, the semi-rigid connections are used mostly to resist side loads only, the possible saving of steel being of secondary importance. It seems, therefore, that it would be useful to have a design method that would permit checking a connection for overstress under the combination of gravity and side loads without going through the process of exact determination of the stresses caused by gravity loads. The usual design method, which neglects entirely the stresses from gravity loads, is of this type. Its drawback, however, is that it may be open to question as to just why the frame should be considered to be rigid for side loads and, at the same time, pin connected for gravity loads.

In the following analysis, in which an attempt is made to determine the limits within which the usual design method is justifiable, it is assumed that:

1. The stress-strain diagram is of the general type shown in Fig. 1, although the exact shape and the numerical relationship between the angle of rotation and the corresponding moment may not be known for all points of the curve.
2. The connection can undergo a considerable deformation without failure. The connections described by Professor Rathbun[3] and by Mr. Brandes and Professor Mains[4] are of this type. However, if the connection is made more rigid by welding, by using splice plates instead of top angles, or by other methods, the frame will be classified as a rigid frame rather than a semi-rigid one, and the analysis will not apply.

It is assumed first that the moments in the connections due to the combination of gravity and side loads are small enough so that the rotations can be considered to be proportional to the moments. In this case, the structure as a whole is elastic and the moments can be obtained by one of the standard methods of analysis of rigid frames, taking into account the local reduction of the rigidity of the beams at the connections. Under these conditions, obviously, the stress from the side loads may be computed independently and added algebraically to the stress caused by the gravity loads to obtain the total stress. If the total stress exceeds the proportional limit, the rigid frame method is not applicable; but the frame is not necessarily unsafe, as a certain amount of nonelastic deformation is acceptable in semi-rigid frames. In order to visualize what happens in semi-rigid frames in which the connections are stressed beyond the proportional limit, two examples are analyzed: First, a preliminary example of a symmetrical rectangular frame under symmetrical gravity loads and a side load; and, second, a general case of a semi-rigid frame under gravity loads and side loads. It is assumed in both cases that the connections are stressed above

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[9] "Design Economy by Connection Restraint," by Bruce Johnston and Robert A. Hechtman, *Engineering News-Record*, October 10, 1940, p. 74.

the proportional limit under gravity loads alone, and that the side load enhances this condition. The same analysis holds, however, if the connections are stressed below the proportional limit under gravity loads and the proportional limit is exceeded only when the side load is applied. In both cases, it is assumed also that the side load may act in either direction; that is, it may be reversed after being applied in a given direction, then removed and reapplied in either direction.

Symmetrical Rectangular Frame Under Symmetrical Vertical Load and a Side Load.—In a rectangular bent ABB′A′ under symmetrical vertical load (Fig. 2(*a*)), because of the partial restraint at the connections A and B there will be a restraining moment M_0 in these connections. Therefore, there will also be equal and opposite shearing forces V_H in the columns. The state of the connections may be represented by the point C_0 on the stress-strain diagram (Fig. 3). This point, and M_0, can be determined either by successive approximations or graphically as described elsewhere.[8]

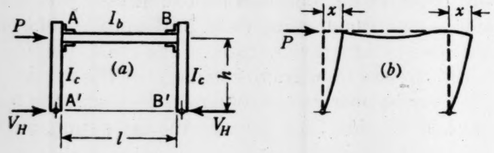


FIG. 2

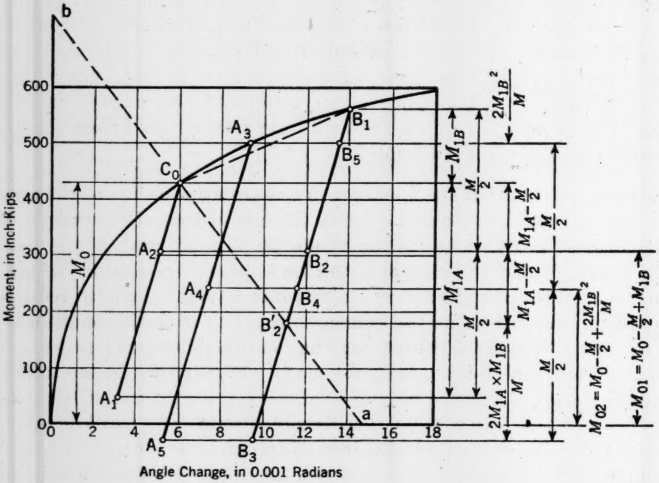


FIG. 3

It is now assumed that a side load P is applied at point A. The angle at point A will open by a quantity α_1, and the moment will decrease by M_{1A} and become $M_0 - M_{1A}$. The point representing the connection at A will move to A_1 on the return branch of the stress-strain curve. As this line is practically straight,

$$\alpha_1 = Z\, M_{1A} \qquad \text{(1}a\text{)}$$

Similarly, the angle at point B will close by a quantity α_2, and the moment will increase by M_{1B} and become $M_0 + M_{1B}$. The point representing the connection at B will move to B_1 on the original stress-strain curve. It may be assumed that the section C_0B_1 of the curve is practically straight, so that

$$\alpha_2 = Z_1\, M_{1B} \qquad \text{(1}b\text{)}$$

The angle changes α_1 and α_2 also can be determined from the deflection of the structure due to moments M_{1A} and M_{1B} by ordinary formulas of the strength of materials. Thus,

$$\alpha_1 = \frac{x}{h} - \left(\frac{l}{3\,E\,I_b} + \frac{h}{3\,E\,I_c}\right) M_{1A} + \frac{l}{6\,E\,I_b}\, M_{1B} \qquad \text{(2}a\text{)}$$

and

$$\alpha_2 = \frac{x}{h} - \left(\frac{l}{3\,E\,I_b} + \frac{h}{3\,E\,I_c}\right) M_{1B} + \frac{l}{6\,E\,I_b}\, M_{1A} \qquad \text{(2}b\text{)}$$

Eliminating α_1 and α_2 from Eqs. 1 and 2,

$$\frac{M_{1A}}{M_{1B}} = \frac{\dfrac{h}{3\,E\,I_c} + \dfrac{l}{2\,E\,I_b} + Z_1}{\dfrac{h}{3\,E\,I_c} + \dfrac{l}{2\,E\,I_b} + Z} \qquad \text{(3)}$$

Because of the side loads the shear at point A′ will decrease by M_{1A}/h and the shear at point B′ will increase by M_{1B}/h. The total variation is equal to the side load P, so that

$$P = \frac{M_{1A} + M_{1B}}{h} \qquad \text{(4)}$$

From Eqs. 3 and 4 it is possible to derive

$$M_{1A} = \frac{P\,h}{1 + M_{1B}/M_{1A}} \qquad \text{(5}a\text{)}$$

and

$$M_{1B} = \frac{P\,h}{1 + M_{1A}/M_{1B}} \qquad \text{(5}b\text{)}$$

Inasmuch as $Z_1 > Z$, it follows that $M_{1A}/M_{1B} > 1$ and $M_{1B} < 1\ M_{1A}$. This means that the side load mostly decreases the moment at point A whereas the increase of moment at point B is comparatively small. It should be noted that if the connections at point A and point B were rigid—that is, in the case of a rigid frame—the decrease of the moment at A would be equal to the increase of moment at B and would have the value $P\,h/2$. This shows that the increase of moment in a connection caused by the side load is smaller in a semi-rigid frame than in a rigid frame of the same dimensions and subject to the same load.

It is next assumed that the side load is removed. The frame will sway back, but will not reach the original position because it has undergone a plastic deformation when the side load was first applied. The point representing the connection at point B will travel from B_1 to B_2 on the return branch of the stress-strain curve (Fig. 3); the point representing the connection at point A will travel from A_1 to A_2. Both connections act elastically. The total variation of moment is

$$M = P\,h \quad \text{(6)}$$

and is divided equally between the two connections. It is seen that the moments restraining the beam at A and B become smaller than they were before application and removal of the side load, their value becoming

$$M_{01} = M_0 - M/2 + M_{1B} \quad \text{(7)}$$

This means that if restraining moments are depended on to reduce the stress in the middle of the beam, as is done in the rigid frames, it should be remembered that the restraining moments are reduced, and, consequently, the factor of safety at the midspan of the beam is also reduced, because of the application and subsequent removal of a side load. If the side load is again applied in the same direction, the connections will again act elastically, and the points reached on the stress-strain curve will again be A_1 and B_1.

On the contrary, if from the unloaded position the side load is applied in the direction opposite to the first load—that is, in the direction BA—the moment in the connection at point A will increase and the moment in the connection at point B will decrease. The point representing the connection at A will move from A_2 to C_0 along the elastic line, then to A_3 on the plastic line. The point representing the connection at B will move from B_2 to B_3 on the elastic line, reaching point B'_2 when point C_0 is reached by the point representing the connection at A. When points C_0 and B'_2 are reached, the total change of moment in both connections is $2\,(M_{1A} - M/2)$ and there remains a moment of

$$M - 2\,M_{1A} + M = 2\,(M - M_{1A}) = 2\,M_{1B} \quad \text{(8)}$$

which will be taken up by the movement of the representative points from C_0 to A_3 and from B'_2 to B_3, respectively. As the connection at point A acts plastically from point C_0 on, this remaining moment will be distributed between the connections at point A and at point B in the ratio M_{1B}/M_{1A}, so that the increment M_{2A} of moment in the connection at point A for the movement from C_0 to A_3, and the decrement M_{2B} of the moment in the connection at point B for the movement from B'_2 to B_3 will be determined by

$$M_{2A} + M_{2B} = 2\,M_{1B} \quad \text{(9a)}$$

and

$$M_{2A}/M_{2B} = M_{1B}/M_{1A} \quad \text{(9b)}$$

Thus,

$$M_{2A} = \frac{2\,M_{1B}}{1 + M_{1A}/M_{1B}} = \frac{2\,M^2_{1B}}{M} \quad \text{(10a)}$$

and

$$M_{2B} = \frac{2\,M_{1B}}{1 + M_{1B}/M_{1A}} = \frac{2\,M_{1A}\,M_{1B}}{M} \qquad (10b)$$

Eqs. 10 determine the location of points A_3 and B_3, and therefore the moments in the connections at this time.

If the side load is now removed, the frame will sway back in the direction AB, the moment in the connection at point A will decrease, and the representative point will move from A_3 to A_4. Also, the moment in the connection at B will increase and the representative point will move from B_3 to B_4. Both connections behave elastically, so that the variation of moment is $M/2$ in each connection. It will be seen that the moment restraining the beam at points A and B is decreased still further and becomes

$$M_{02} = M_0 + \frac{2\,M^2_{1B}}{M} - \frac{M}{2} \qquad (11)$$

From points A_4 and B_4 the side load can again be applied in either direction AB or direction BA. Both connections will work elastically, the representative points moving from A_4 to A_3 and from B_4 to B_3 for the load in the direction BA, or from A_4 to A_5 and from B_4 to B_5 for the load in the direction AB. No additional plastic deformation will occur during these and subsequent loadings.

From the preceding it may be deduced that:

a. The maximum increase of moment in the connections due to side load is M_{1B} determined by Eqs. 3 and 5 and is smaller than the increase of $M/2$ which would occur if the frame were perfectly elastic and if the moment due to side load were algebraically added to the moment due to gravity load considered independently.

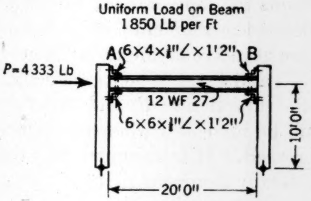


FIG. 4

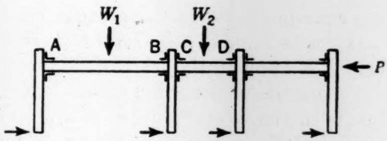


FIG. 5

b. The moments restraining the beam at A and B decrease because of the application, reversal, and removal of the side load, the value of these reduced moments being given by Eq. 11.

c. After the first application, reversal, and removal of the side load, the connections act elastically, and the increment or decrement of moment in each connection is $M/2$, counted from the mean value of moment given by Eq. 11, rather than from the original restraining moment caused by the gravity loads only.

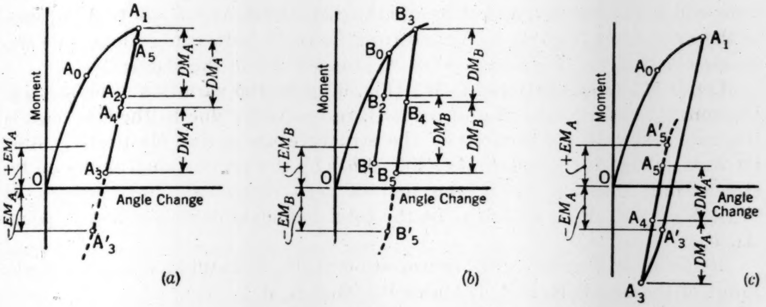
Numerical Example.—The frame to be analyzed is shown in Fig. 4. To simplify computations, the columns are assumed to be perfectly rigid. The stress-strain diagram of the connection is shown in Fig. 3, which is taken from Professor Rathbun's[3] specimen 10. The following computations are self-explanatory:

$$\begin{aligned} \text{Fixed-end moment} &= 1{,}850 \times 20^2 \times 12/12 = 740 \text{ in-kips} \\ \text{Free-end rotation} &= w\,l^3/24\,E\,I \\ &= (1{,}850 \times 20^3 \times 12^2)/(24 \times 30 \times 10^6 \times 204.1) \\ &= 0.0145 \text{ radian} \end{aligned}$$

In Fig. 3 beam line, ab, determines the actual end moment $M_0 = 435$ in-kips. Assume that the side load is such that point B₁ is reached by the point representing the connection at B; that is, the total moment in the connection at point B is 565 in-kips. Then $Z_1 = \dfrac{0.014 - 0.006}{565 - 435} = 6.15 \times 10^{-5}$ radians per in-kip. From the return line of the stress-strain diagram $Z = 0.755 \times 10^{-5}$ radians per in-kip, and from the properties of the beam $\dfrac{l}{2\,E\,I} = \dfrac{20 \times 12}{2 \times 30 \times 10^6 \times 204.1}$ $= 1.96 \times 10^{-5}$ radians. The ratio of wind moments is $M_B/M_A = (0.755 + 1.96)/(6.15 + 1.96) = 0.333$. For the assumed $M_B = 130$ in-kips, it is found that $M_A = 390$ in-kips and $M = 520$ in-kips. Note that if the connections were assumed to be elastic, $M_{1B} = M_{1A} = 520/2 = 260$ in-kips. The slope of the line C_0A_2 is slightly different from the slope of C_0B_1. Assume that the two slopes are equal so that Eq. 11 may be used. The final end restraint moment is then $435 - 520/2 + 2 \times 130^2/520 = 240$ in-kips instead of the original 435 in-kips, a reduction to 55% of its original value.

General Case of Semi-Rigid Framework Under Vertical Loads and Side Loads.—Under gravity loads, the connections in a semi-rigid framework as a rule will exert restraining moments on the beams. When side loads are applied, these moments are changed. In some of the connections the moments are increased because of the side loads, and in other connections the moments are reduced. For example, in the frame shown in Fig. 5 there are negative moments in the connections at A, B, C, and D due to vertical loads W_1 and W_2. When a horizontal load P is applied, the frame tends to sway in the direction of P, and therefore the moments in the connections at A and C will be increased whereas the moments in the connections at B and D are decreased. It is assumed that the curves of Figs. 6(*a*) and 6(*b*) show the stress-strain diagrams of two typical connections at A and B, and that the connection at A is typical of all the connections in which the moment is increased because of a side load acting in a given direction, whereas the connection at B is typical of all the connections in which the moment is decreased because of this load. Suppose that points A₀ (Fig. 6(*a*)) and B₀ (Fig. 6(*b*)) are reached on these diagrams under vertical load only. When the side load is applied, the moment in the connection at point A is increased, and the new point reached will be A₁, located on the original stress-strain curve. At the same time the moment in the connection at point B is decreased and the new point reached will be B₁, located on the return branch of the diagram.

Now assume that the side load is removed. The moment in the connection at point A is decreased and the new point reached on the diagram is A₂, on the return branch from A_1. Similarly, the moment in the connection at point B is increased and the new point reached on the diagram is B_2. During unloading, the rigidity of the connection at point B is practically the same as during loading, but the rigidity of the connection at point A is greater during unloading than during loading. Therefore, the variation of moment during unloading is not the same as during loading, and the state of the connections after the side load is removed is not the same as it was before the side load was applied. This new state is represented by points A_2 and B_2 instead of by points A_0 and B_0. It is seen, incidentally, that the restraining moment at the ends of the beams becomes smaller after the side load has been applied and subsequently removed.



(a) (b) (c)

Fig. 6

By applying the side load again in the same direction as the original load, the moment in the connection at point A increases and the representative point moves from A_2 to A_1. On the other side, the moment in the connection at point B is decreased and the point moves from B_2 to B_1. The structure will then be in the same state of loading, stress, and deformation as when the side load was first applied. There is no additional plastic deformation, and the load can again be removed and applied without further plastic deformation.

Suppose now that to the horizontally unloaded structure there is applied a side load of the same intensity as the original side load, but of opposite direction. The moment in the connection at point A will decrease, and the representative point will move from A_2 to A_3. The moment in the connection at point B will increase and the point will move from B_2 toward B_0, possibly going beyond B_0 to B_3 on the plastic line. The structure may therefore undergo an additional plastic deformation. When the structure is unloaded, the representative points will move, respectively, to A_4 and B_4. During this unloading the rigidity of the connections is not the same as during loading and therefore there will remain a residual strain, the point A_4 being below the point A_2. It is seen again that the restraining moment against gravity loads is further reduced, and that, as before, the structure can again be loaded in the same directions and unloaded without any new plastic deformation, the representative points traveling, respectively, between A_4 and A_3, and between B_4 and B_3. If, however, from the last unloaded state the structure is loaded in the same direction as the original load—that is, in the direction opposite to the load just discussed —the moment in the connection at point A will increase and the representative point will move from A_4 to A_5. The moment in the connection at point B will decrease and the representative point will move from B_4 to B_5. Both connections will act elastically, and therefore the distance A_4A_5 is equal to A_2A_1, since in the movement from A_2 to A_1 and from B_2 to B_1 both connections were also acting elastically. Therefore, point A_5 is below A_1 on the elastic line.

Let $D\,M_A$ (Fig. 6(*a*)) and $D\,M_B$ (Fig. 6(*b*)) be the variation of moments in the connections at point A and point B, respectively, due to the side load, in the case when all connections of the structure are acting elastically. Also, let $E\,M_A$ (Fig. 6(*a*)) and $E\,M_B$ (Fig. 6(*b*)) be the proportional limits on the stress-strain diagram below the line of zero moments. Furthermore, let M_{A1}, M_{B1}, $\cdots$, M_{An}, and M_{Bn} be the total moments corresponding to points A_1, B_1, $\cdots$, A_n, and B_n.

Referring to Fig. 6(*b*), the lowermost point, B_5, will still be within the elastic range of the line B_3B_5 if B_5 is above B'_5; that is, if

$$M_{B5} > -\,E\,M_B \qquad (12a)$$

As

$$M_{B5} = M_{B3} - 2\,D\,M_B \qquad (12b)$$

this condition becomes

$$M_{B3} - 2\,D\,M_B > -\,E\,M_B \qquad (12c)$$

or

$$D\,M_B < \tfrac{1}{2}\,(M_{B3} + E\,M_B) \qquad (12d)$$

Point B_3 was assumed to be beyond the proportional limit; that is,

$$M_{B3} > E\,M_B \qquad (13a)$$

therefore, point B_5 will be within the elastic range of the line B_3B_5 if

$$D\,M_B < \tfrac{1}{2}\,(E\,M_B + E\,M_B) \qquad (13b)$$

or

$$D\,M_B < E\,M_B \qquad (13c)$$

In Fig. 6(*a*), the lowermost point, A_3, will still be within the elastic range of the line A_1A_3 if A_3 is above A'_3; that is, if

$$M_{A3} > -\,E\,M_A \qquad (14a)$$

or

$$D\,M_A < \tfrac{1}{2}\,(M_{A5} + E\,M_A) \qquad (14b)$$

It was assumed that M_{A1} is greater than $E\,M_A$, but it is not known if M_{A5} is greater than $E\,M_A$ or not. Therefore, the conclusion arrived at for the connection at point B does not necessarily hold for the connection at point A.

Assume now that A_3 is outside the elastic range, as shown in Fig. 6(*c*). Then A_5 will be within the elastic range of the line A_3A_5 if

$$M_{A3} + 2\,D\,M_A < E\,M_A \qquad (15a)$$

As M_{A3} is assumed to be smaller than $-E\,M_A$, the condition holds if

$$-E\,M_A + 2\,D\,M_A < E\,M_A \quad \text{(15}b\text{)}$$

that is,

$$2\,D\,M_A < 2\,E\,M_A \quad \text{(15}c\text{)}$$

or

$$D\,M_A < E\,M_A \quad \text{(15}d\text{)}$$

Therefore, after the first horizontal load is applied, reversed, and removed, the connections will work within the elastic range for all subsequent applications of load of a magnitude not exceeding the original load provided that

$$D\,M_A < E\,M_A \quad \text{(16}a\text{)}$$

and

$$D\,M_B < E\,M_B \quad \text{(16}b\text{)}$$

In some cases, the connections may work within the elastic range even if these conditions are not satisfied.

In the preceding it was assumed that points A_1 and B_3 were reached without failure of the connections. This assumption is justified only in case the connections can undergo large deformations before failure. It is seen that $M_{A1} - M_{A0}$ and $M_{B3} - M_{B0}$ are smaller than $D\,M_A$ and $D\,M_B$, respectively, and that therefore the maximum increment in moment in the connection is smaller than the increment computed on the assumption that the connections are elastic. Likewise, it is seen that A_4 is below A_0 and B_4 is below B_0 and, therefore, that the wind load moments tend to reduce the restraining moment at the ends of the beams.

CONCLUSIONS

Further experimental determination of the behavior of semi-rigid connections under reversing moments will be helpful in establishing a correct design method. Assuming, however, that the connections can safely sustain repeated reversing stresses not exceeding a certain percentage of the proportional limit of the connections hardened by the initial strain, the following conclusions may be drawn:

(1) When connections are stressed beyond the proportional limit, the distribution of wind moments is not the same as it would be if the connections were elastic. The increments of moments above those due to gravity loads are smaller, whereas the decrements are larger.

(2) The moments restraining the ends of beams against gravity loads are reduced after wind loads have been applied and removed.

(3) In checking the strength of connections it is desirable to analyze moments caused by gravity loads as well as those caused by wind load. However, if the connections are such that they can undergo large inelastic deformations without failure, it will be found that, if the connections are adequate to resist wind stress moments alone at design stresses computed on the assumption that the connections are elastic, they are also adequate for the combination of gravity loads and wind loads.

DISCUSSION

C. Batho.[10]—The beam-line method for solving problems on frameworks having rigid, elastic, or nonelastic connections, which Mr. Sourochnikoff makes some use of in his paper, was first given in three papers in the Reports of the Steel Structures Research Committee (SSRC).[11,12,13] Since then it has been greatly extended, and, in particular, it has been applied to problems involving sway.

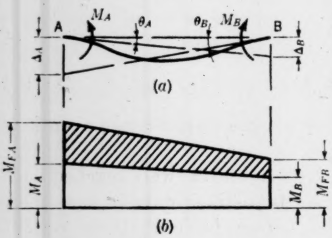


Fig. 7

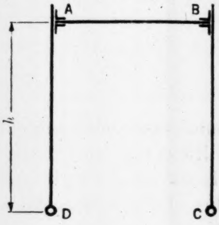


Fig. 8

The writer proposes to confine this discussion to a consideration of its application to a rectangular portal frame hinged at the feet, having nonelastic connections between the beam and the stanchions and subjected to vertical and side loads, since this is the main subject of Mr. Sourochnikoff's paper.

Consider a beam AB (Fig. 7(*a*)) loaded in any manner and partly restrained at the ends by moments M_A and M_B, and let θ_A and θ_B be the slopes at the ends of the beam. If M_{FA} and M_{FB} are the moments necessary to reduce the slopes at the ends to zero, the beam may be regarded as having been initially restrained in this manner and the moments gradually lowered to M_A and M_B. The drop from B to the tangent to the beam at A(Δ_B) is therefore given by $\frac{A\,\bar{x}_B}{E\,I}$, in which A is the area of the shaded part of the diagram (Fig. 7(*b*)), $\bar{x}_B$ is the distance of its centroid from B, and I is the relevant second moment of area of the beam. Thus,

$$\theta_A = \frac{A\,\bar{x}_B}{E\,I\,l} \dots\dots\dots\dots\dots\dots\dots\dots (17a)$$

and

$$\theta_B = \frac{A\,\bar{x}_A}{E\,I\,l} \dots\dots\dots\dots\dots\dots\dots\dots (17b)$$

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[10] Prof., Dept. of Civ. Eng., Univ. of Birmingham, Birmingham, England.

[11] "Investigations on Beam and Stanchion Connections," by C. Batho and H. C. Rowan, 2d Rept., Steel Structures Research Committee, Dept. of Scientific and Industrial Research of Great Britain, H. M. Stationery Office, London, England, 1934, p. 94.

[12] "The Analysis of the Moments in the Members of a Frame Having Rigid or Semi-Rigid Connections, Under Vertical Loads," by C. Batho and H. C. Rowan, *ibid.*, p. 177.

[13] "The Analysis and Design of Beams Under Given End Restraints," by C. Batho, Final Rept., Steel Structures Research Committee, Dept. of Scientific and Industrial Research of Great Britain, H. M. Stationery Office, London, England, 1936, p. 364.

in which l is the length of the beam. This leads to the equations

$$6\,E\,K\,\theta_A = 2\,(M_{FA} - M_A) + (M_{FB} - M_B) \ldots\ldots\ldots (18a)$$

and

$$6\,E\,K\,\theta_B = 2\,(M_{FB} - M_B) + (M_{FA} - M_A) \ldots\ldots\ldots (18b)$$

in which K is I/l, the stiffness of the beam. By adding and subtracting Eqs. 18, two further equations may be obtained:

$$\frac{(M_A + M_B)}{2} = \frac{(M_{FA} + M_{FB})}{2} - 2\,E\,K\,\frac{(\theta_A + \theta_B)}{2} \ldots\ldots (19a)$$

and

$$(M_A - M_B) = (M_{FA} - M_{FB}) - 6\,E\,K\,(\theta_A - \theta_B) \ldots\ldots (19b)$$

Eq. 19*a* is the equation of a straight line connecting the mean of the bending moments at the ends and the mean of the slopes, and is called the mean beam line. Eq. 19*b* is the equation of another straight line connecting the difference of the moments at the ends and the difference of the slopes, and is called the difference beam line (see, subsequently, Fig. 10). They were first given in the Final Report of the SSRC.[13] If the restraining moments are equal and the loading symmetrical, they reduce to the one equation:

$$M = M_F - 2\,E\,K\,\theta \ldots\ldots\ldots\ldots\ldots\ldots (20)$$

which is the equation of the original beam line introduced to interpret the results of experiments on beam and column connections.[11] It is this beam line that Mr. Sourochnikoff has used in his example (see under the heading, "Analysis of Gravity and Wind Stresses in Connections Working Beyond the Proportional Limit: Symmetrical Rectangular Frame Under Symmetrical Vertical Load and a Side Load—Numerical Example").

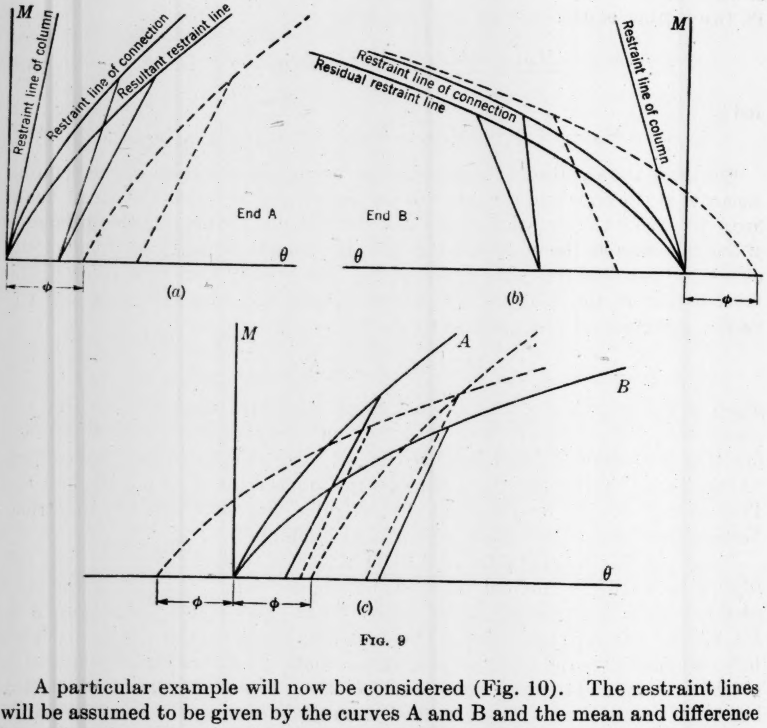
Now consider the portal frame ABCD hinged at the base (Fig. 8). Let the relation between the moment, M, and the angular deformation, θ, of the connection (which is supposed to be known from experiment) be as shown in Fig. 9(*a*) for end A of the beam. The unloading line is assumed to be straight. If the columns were rigid this would give the relation between the slope at end A of the beam, θ_A, and the restraining moment M_A. It will be called the restraint line of the connection. The attached member is elastic, however, and since it is pinned at the foot, the restraint line for this member will be

$$M = 3\,E\,K_{SA}\,\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots (21)$$

in which K_{SA} is the stiffness of the column. The resultant restraint line is obtained by adding together the abscissas of these restraint lines.

The restraint line for end B (Fig. 9(*b*)) is found in a similar manner. For the purpose of the following graphical constructions this line is reversed and drawn from the same origin as the restraint line for A (Fig. 9(*c*)).

If sidesway occurs, the horizontal movements at ends A and B must be equal and in the same direction, and, as the members AD and BC are of the same length, the angles of sway at A and B must be equal. Further, since the vertical members are pinned at ends C and D, sway does not produce any deformation of these members. Therefore the effect of sway may be represented by shifting the restraint line horizontally through a distance equal to the angle of sway (ϕ). For a sway to the right both restraint lines are shifted to the right, but, when the restraint line at end B is reversed, as in Fig. 9(c), this will appear as a shift to the left.


M
Restraint line of column
Restraint line of connection
Resultant restraint line
End A
θ
φ
Restraint line of connection
Residual restraint line
Restraint line of column
M
End B
θ
φ
M
A
B
θ
φ
φ

(a) (b) (c)

Fig. 9

A particular example will now be considered (Fig. 10). The restraint lines will be assumed to be given by the curves A and B and the mean and difference beam lines for a given vertical loading of the beam to be as shown in the figure.

It is now necessary, as in the slope-deflection method and moment-distribution method, to determine the sway equation. If there is a horizontal force P acting at A it must be balanced by the horizontal forces produced by the moments at A and B. Therefore, with the convention of signs for the moment used in deriving the beam line, $M_A - M_B = P\,h$ if the load is to the left. If it is to the right, the right-hand side of the equation becomes $-\,P\,h$, and if there is no horizontal load it is zero.

It will be seen from the foregoing that the conditions to be fulfilled are:

(a) The shift of the restraint line for end A must be equal and opposite to that for end B;

(b) The sway equation must be satisfied;

(*c*) $M_A - M_B$ and $\theta_A - \theta_B$ must be in accordance with the difference beam line; and

(*d*) The mean of the moments M_A and M_B must lie on the mean beam line.

From condition (*c*) it follows that the relative position of the points on the ($M\ \theta$)-diagram (Fig. 10(*a*)) giving the moments at A and B must be the same as given by the difference beam line. For example, if there is a horizontal load P to the left, the relevant points on the difference beam-line diagram are O and T. It follows that if a straight line is drawn between the points representing M_A and M_B in Fig. 10(*a*) it must be parallel and equal in length to the radius vector $\overline{OT}$ of the difference beam line. Further, from condition (*d*) it must be bisected by the mean beam line.

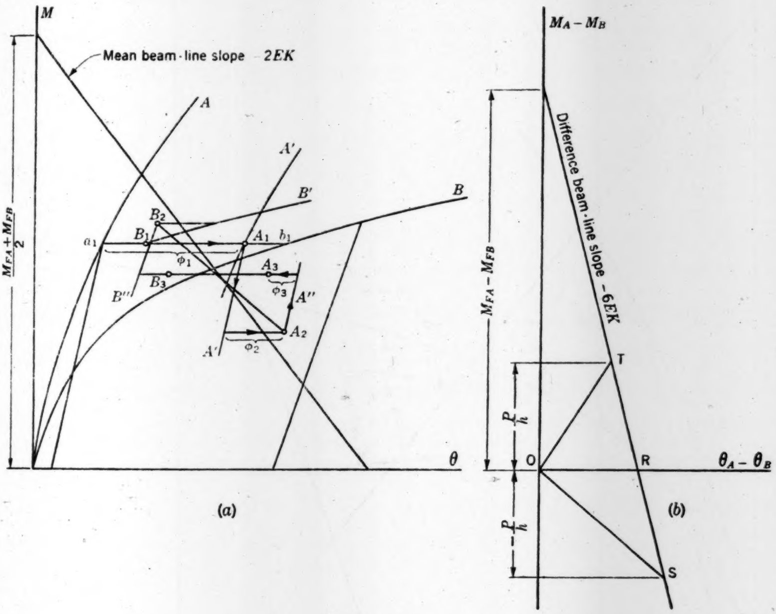

M
Mean beam·line slope – 2EK
A
A′
B′
B
B_2
a_1
B_1
A_1
b_1
φ_1
A_3
B_3
φ_3
A″
B″
A_2
A′
φ_2
$\frac{M_{FA}+M_{FB}}{2}$
θ
(a)
$M_A - M_B$
Difference beam·line slope −6EK
$M_{FA} - M_{FB}$
T
$\frac{P}{h}$
O
R
$\theta_A - \theta_B$
$-\frac{P}{h}$
S
(b)

Fig. 10

It is important to keep in mind that the end of the radius vector on the beam line is related to restraint line A and the end at the origin to restraint line B. The behavior of the frame under cycles of loading will now be considered.

1. *No Horizontal Load.*—Moment M_A must be equal to M_B. Determine the horizontal line $a_1\,b_1$ on the ($M\ \theta$)-diagram bounded by the two restraint lines and bisected by the mean beam line. The relevant radius vector of the difference beam line is $\overline{OR}$. Set off half of this on each side of the mean beam line, obtaining points A_1B_1. These give the moments and the slopes at ends

A and B of the beam. The horizontal distance from A_1 to the restraint line A is the angle of sway, ϕ_1, at A. Condition (*a*) is fulfilled since this is equal to the distance from B_1 to the restraint line B. The restraint lines now shift to A′B′.

2. *Horizontal Load to the Right.*—The quantity $M_A - M_B$ is now $-Ph$ and the relevant radius vector of the difference beam line is $\overline{OS}$. Under this loading the moment M_A decreases; therefore, the unloading part of the restraint line A′ must be used. Determine the line A_2B_2 equal in length and parallel to OS, bisected by the mean beam line and having its ends equidistant horizontally from restraint lines A′ and B′. (This is best done by drawing the difference beam-line diagram on tracing paper and superimposing it on the $(M\ \theta)$-diagram.) Lines A_2 and B_2 give the moments and slopes at the ends of the beam, and the additional sway produced by the horizontal load is ϕ_2 as shown in the figure. The restraint lines are now A″B″. Only the unloading part of these lines is shown in the figure.

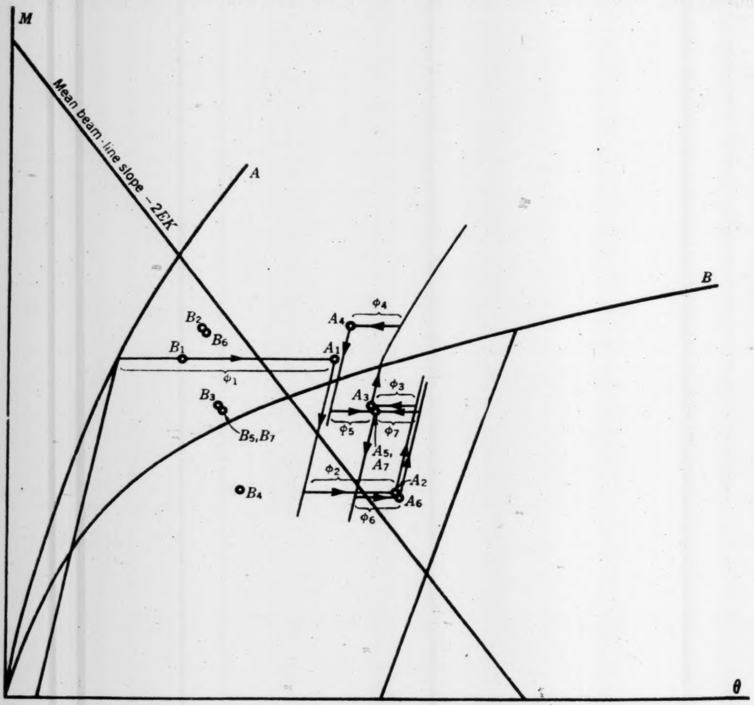


FIG. 11

3. *Horizontal Load Removed.*—Moment M_A is once more equal to M_B. Determine the horizontal line bounded by the two unloading restraint lines and

bisected by the mean beam line and proceed as for the case of no horizontal load (step 1). Lines A_3 and B_3 give the moments and slopes at the ends of the beam. The reduction in moment due to the application and removal of the side load is evident from the figure.

4. *Horizontal Load to the Left.*—

5. *Horizontal Load Removed.*—The method of procedure has now been explained and it is necessary to give only the results. These are shown in Fig. 11. One cycle of loading has now been completed and it will be seen that steps 4 and 5 result in a further slight reduction of the residual moments when the side load is removed.

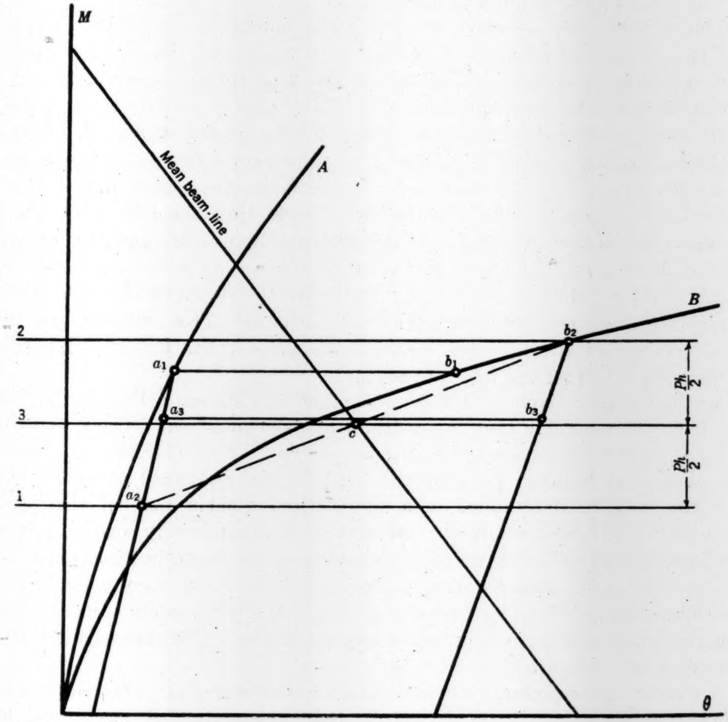


Fig. 12

Fig. 11 also shows the effect of succeeding cycles of loading. In order to avoid confusion the sways are only indicated for point A and some of the construction lines are omitted.

It will be seen that after the first cycle there is a slight change in the moments produced by the side load, but no further reduction in the moments when the load is removed. The structure behaves elastically when the relevant restraint lines are the straight unloading lines.

The horizontal movements of the point of application, A, of the side load may be found by multiplying ϕ by the height of the frame, h.

If there are no vertical loads on the beam, or if the side load is large compared with the vertical loads, some of the moments become negative, and it is necessary to know the restraint lines for the connections under reversed moments.

Simplified Procedure.—The foregoing is the complete solution of the problem. If only the moments are required, a simpler procedure, suggested to the writer by A. J. Francis, may be adopted.

This procedure is shown in Fig. 12. The points $a_1 b_1$ are obtained as in step 1, and the unloading line for end A, not allowing for sway, is drawn from a_1. Now draw two parallel lines 1 and 2 on tracing paper making the vertical distance between them equal to $P h$. Line 1 is to be used in conjunction with restraint line A, and line 2 with restraint line B. If the horizontal load is to the right, line 2 must be above line 1. Also, draw a third line midway between these lines. Superimpose these lines on the $(M\ \theta)$-diagram, keeping them parallel to the θ-axis and adjust them until the points a_2 c and b_2 are collinear. The ordinates of a_2 and b_2 give M_A and M_B. When the load is removed the residual moments are given by a_3 and b_3. The abscissas of the a-points and b-points do not give the slopes at the ends of the beam since sway has not been taken into account. The ordinates give the correct values of the moments because the sway shifts the restraint lines horizontally by equal and opposite amounts.

In the example given by Mr. Sourochnikoff and shown in Fig. 3, it will be seen from the foregoing that the restraint line shown is really two coincident restraint lines superimposed, and the beam line ab becomes the mean beam line when sway occurs. By means of the foregoing simplified procedure, the moments may be determined in a simpler and quicker manner than that employed by Mr. Sourochnikoff. His results are substantially correct, but the line between corresponding points such as A_4 and B_4 should be bisected by the beam line ab. Mr. Sourochnikoff begins by taking an arbitrary point B_1 and determines the side load corresponding to it. If the moments corresponding to a given side load were required, it would presumably be necessary to use a process of trial and error.

By means of the complete method given at the beginning of this discussion, which requires only a little more time than the simplified procedure, the sway—which may be of some importance—may also be determined.

It may be added that the reduction in the restraining moments at the ends of the beam produced by the application and removal of a side load was noted in the Final Report of the SSRC.[14]

"[14] The Effect of Wind Loads on Frames with Semi-Rigid Connections," J. F. Baker and E. Leader Williams, Final Rept., Steel Structures Research Committee, Dept. of Scientific and Industrial Research of Great Britain, H. M. Stationery Office, London, England 1936, p. 434.

George S. Murphy,[15] Assoc. M. ASCE.—The determination of the actual strength of semi-rigid connections depends (as the author states) on the availability of experimentally determined empirical curves, one for each joint combination, assuming identical quality of workmanship amd materials in every case. Therefore, the problem is to discover what limiting assumptions will keep one on the safe side of uncertainty.

Seldom does ordinary design procedure go beyond computations for the tension, bearing, and shear in the rivets or welds, while assuming the base material of the members and the connection pieces to be fully nondeformable—despite a full awareness, by the designer, of the flexibility and plasticity of the parts of a standard connection. The frame (excluding wind bents) is designed assuming zero end moments (that is, assuming that there is no restraint against the pulling away of the top of a connection); and yet, subsequently, the superimposed concrete floor is designed as if the framing connections were fully continuous. Running cracks in the concrete floor, along the steel girders to which the steel floor beams are framed, are not unknown occurrences. They bear witness to the ill effects of this contrasting approach to the design of different parts of the same structure; and any effort to correct this approach should be encouraged.

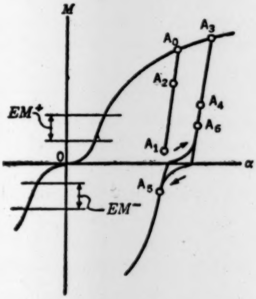


Fig. 13.—Stress Diagram for Connection at A (Fig. 2), Showing Possible Effect on Curve from Looseness at Time of Reversal of Moment

The general type of stress diagram for connections (Figs. 3 and 6), in which moments are plotted against angular deflections, might be questioned in the area of positive or negative increase from zero moment, as demonstrated by Fig. 13. An analogous illustration would be the free play in the steering column of a car or boat when the direction of the turning force is reversed. A break in the elastic curve, such as this, would have a considerable effect in altering the distribution of moments in the structure where not all the joints would be subjected to a reversal of moment. Stress states A6 and A4 would then not be identical.

In defining the conditions for keeping a connection within the elastic range, the author's association of algebraic and absolute numbers may lead to confusion. This danger is particularly apparent when one considers that a given connection may have different elastic limits ($E\,M$, Fig. 13) for different directions of rotation. Eq. 12a would be more mathematically precise in the form:

$$M_{B5} > -|E\,M_B|_{\mathbf{neg.}} \qquad (22)$$

In other words, M_{B5} is algebraically greater than the negative of the absolute value of the elastic limit of the joint when rotated in the negative direction.

Conclusions (1) and (2) are undeniably correct. Conclusion (3), however, is a qualitative conclusion within more or less indeterminate limits; and caution is required to avoid its improper application to specific cases. Judgment is required in estimating the capacity inherent in the connection of undergoing "large" inelastic deformations without failure, and in estimating the magnitude of the inelastic deformations it will be required to undergo. Furthermore, the statement that "adequacy" of the connection for combined gravity and wind loads will be found if "adequacy" to resist wind moments alone is found, when the latter are computed assuming elasticity of the connections, is contingent, not only upon sufficient deformability of the connection, but also upon a sufficiently large ratio of the wind moments to those caused by the gravity loads. Otherwise, in windless climes, no connections would be required. This is recognized by the design specification that, in any case, the connection be "adequate" for the gravity loads.

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[15] With Samborn, Steketee, Associates, Toledo, Ohio.

BASIL SOUROCHNIKOFF.[16]—In his interesting and instructive discussion, Mr. Batho agrees in substance with all the conclusions in the paper. Considerable research on the subject was done by Mr. Batho and his associates, which was not available when the paper was being prepared. Mr. Batho's treatment of the "beam line" is probably simpler than the writer's treatment, but this fact is mainly of academic interest as practical designers will seldom find enough time to make a complete analysis of the frame. The important conclusion is that in most cases gravity moments may be neglected and connections can be analyzed for wind moments alone by the usual methods of analysis of elastic frames with reduced moment of inertia at the ends of beams.

The writer does not agree with the statement by Mr. Murphy that it may be concluded from the paper that "* * * in windless climes, no connections would be required." Since moment connections alone are considered, the conclusion should be that no moment connections are required when wind is not considered, which, of course, is obvious. The writer agrees with Mr. Murphy that irregularities in the unloading line in the vicinity of zero moment may cause considerable error in the determination of moments. These irregularities, however, occur in a limited number of connections only, and such connections should preferably be avoided.

In any case, more experimental determination of properties of connections is necessary before a simple and reliable design method can be established.

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MODERN CONCEPTS APPLIED TO CONCRETE AGGREGATE

By R. F. Blanks,[1] M. ASCE

With Discussion by Messrs. Herbert K. Cook, Thomas B. Kennedy, and Bryant Mather; and R. F. Blanks.

Synopsis

Aggregates influence the durability of concrete through numerous physical and chemical properties, most important of which are the size, continuity, and abundance of internal voids; the bonding characteristics; and the alkali reactivity. In some instances, other properties, particularly swelling characteristics associated with water absorption, may cause distress or actual deterioration of concrete.

Testing methods, as currently applied, yield statistical but not specific correlations with serviceability of aggregates; they do not reveal the fundamental qualities on which acceptance or rejection of individual aggregates should depend. The three most important properties are not determined even qualitatively. The Bureau of Reclamation (USBR) requires petrographic examination of all aggregates to be used in construction on its projects, particularly to detect potential alkali reactivity or the need for supplementary testing of materials of marginal quality.

There is great need for research to develop new testing techniques, so that reorientation of acceptance specifications can be consummated. Research on the problem of alkali reactivity, by USBR, has developed a chemical test to detect potential reactivity of aggregates; and the mechanism of this type of deterioration has been proved. Studies of capillary phenomena in rocks and concrete have revealed a diagnostic relation between pore characteristics and freezing and thawing durability. Other properties of aggregates, particularly bonding characteristics and volume change with wetting and drying, should be investigated fully.

Improvements in testing of aggregates must accompany improvements in the quality of cement, and in the techniques of mixing, placing, and curing the concrete. Only in this way will the full potentialities of concrete be realized.

Note.—Published in April, 1949, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

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INTRODUCTION

The serviceability of concrete depends on many factors, among which are the quality of the aggregate and cement and the methods of fabrication, placement, and curing. With respect to durability, present standard tests fail to evaluate the quality of aggregate properly. Inadequacy of currently applied methods of selecting materials is demonstrated by repeated cases of poor service of concrete resulting from some characteristic of the aggregate; also, although less commonly, sound aggregates are rejected as a result of accepted tests, thus entailing unnecessary expense for additional exploration or the use of more costly materials.

The primary difficulty with present aggregate testing lies in the concepts that strong and durable rocks necessarily contribute to the strength and the durability of concrete, and that weak rocks contribute directly to unsoundness of concrete. Standard tests are designed to determine the quality of aggregate as rock—not as a component of concrete—whereas experience shows that good rocks do not always produce superior concrete, and weak rocks do not always cause deterioration.

Revised concepts of aggregate testing by the USBR regard the aggregates as components of concrete. Testing techniques must be designed that will determine adequately the effect of an aggregate on the properties of concrete, if satisfactory concrete is to be produced consistently.

EFFECT OF ROCK PROPERTIES ON CONCRETE

The properties of aggregates include (*a*) those of the individual particles, and (*b*) those characterizing the entire assemblage. The latter comprise gradation, moisture content, and bulk weight. They are easily determined and interpreted and will not be considered here. The properties of aggregates related to the individual particles, their significance, and methods of measuring them will be discussed.

The individual rock particles affect quality of concrete through either physical or chemical processes. The physical properties of aggregates are related particularly to the mechanical and structural characteristics of the aggregate particles, and to a lesser extent to unusual characteristics of component minerals—such as ability to swell and to contract with wetting and drying. Chemical properties relate more specifically to the stability of the constituent minerals and rock substances while they are enclosed by Portland cement. Consequently, the chemical and physical properties of aggregate particles depend on petrographic characteristics. However, rocks of a given type do not always possess the same properties, for their properties can be changed by weathering or solution by rain or ground water, or by fracturing and jointing related to the geologic history of the region. Thus, basalts can be hard or soft; and rhyolites can be deleteriously reactive with high-alkali cements, or they may be innocuous if the reactive glass has been crystallized into stable minerals by metamorphic processes. Therefore, the quality of aggregates must be evaluated by measurement of certain chemical and physical properties which are significant to concrete.

Size, Abundance, and Continuity of Pores.—The voids or spaces within solid materials exist in several forms. They may be isolated more or less by intervening impermeable material as, for example, gas bubbles (vesicles) in glassy lavas like obsidian. They may be ramifying passageways, or approximately plane fissures, or fractures. The voids in rocks may be empty or partly or completely filled with chemically precipitated minerals such as quartz, or insoluble residues such as clay.

The size, abundance, and continuity of the voids influence the physical and chemical stability of the rock; they are the most important physical properties of aggregate particles. These characteristics of pores affect the strength of any material; they control water absorption and permeability; and they determine the quality of bond with cement. Consequently, they control durability under freezing and thawing conditions and strongly influence the rate of chemical alteration.

The durability of solid materials during freezing and thawing is controlled by the freedom with which they are permeated by water. Obviously, materials which are impermeable are not disrupted by freezing. However, highly porous, absorptive rocks likewise will not be disrupted if freezing is progressive and proceeds slowly enough to permit escape of water in excess of that necessary to fill the void spaces. Susceptible to breakdown with freezing and thawing are rocks having those intermediate degrees of permeability which allow water to enter the material but not to drain freely. Evidence in the literature (1)[2] and experimentation by the USBR indicate that, under natural conditions of freezing, voids less than 0.005 mm in diameter, and particularly those less than 0.004 mm in diameter, will drain effectively only at hydrostatic pressures that exceed the tensile strengths of some rocks and concrete.

Important to the durability of concrete is the relative size of continuous voids in the aggregate particles and of those in the hydrated cement paste. Unless the concrete remains under hydrostatic head, the water content of the aggregate will be controlled by this relative pore size. Large voids in the aggregate will be drained of water, even if initially filled, through progressive withdrawal by capillarity into the hydrating cement paste. Conversely, voids in the aggregate smaller than those in the cement paste will remain filled or will tend to become filled by extraction of water from the capillaries of the cement paste. Of course, the relative size of pores in the cement paste might vary considerably with continued hydration or alteration and dissolution.

This interchange of water between aggregate particles and cement paste may affect, significantly, the freezing and thawing durability of concrete, because the saturated particles will be more susceptible to breakdown than would unsaturated ones. Furthermore, large voids in unsaturated aggregates will accommodate the water migrating through the concrete during progressive freezing, whereas voids filled with water or voids of very small size or limited continuity might block the penetrable capillaries through which water is advancing during freezing, consequently causing rupture at or near the aggregate-to-paste contact (2).

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[2] Numerals in parentheses, thus: (1), refer to corresponding items in the Bibliography (see Appendix)

The relation of size, abundance, and continuity of voids to bond of the aggregate particle with cement will be discussed subsequently.

Bonding Character.—The tenacity of the bond between aggregate and cement paste is extremely important to the quality of concrete. The properties of this aggregate-cement bond depend on surface texture and chemical stability of the aggregate. Chemical reactions that dissolve or decompose (weaken) the aggregate obviously reduce strength of bond, but reactions that form new compounds of considerable strength and elasticity may weld the aggregate to cement. Reactions of the former type are exemplified particularly by the reaction between aggregate and cement alkalies. Although they undoubtedly exist, reactions of the latter type have not been identified.

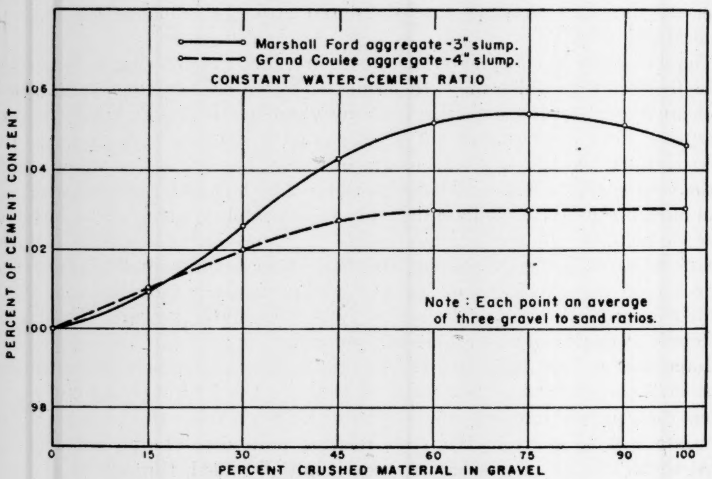

FIG. 1

Two aspects of surface texture exert an influence on bond—the surface roughness and the pore characteristics of the zone underlying the surface of the particle. Salients and depressions on the particles, particularly when the sides of these roughnesses are almost perpendicular to the general surface, assist the adherence of the paste to the aggregate. Undulatory roughness is less helpful and may even be harmful to bond as the mortar changes in volume. Rugosity is probably less significant than physical penetration of cement into the aggregate.

Particle Shape.—Flat or elongated particles in aggregate tend to decrease workability and thus to require the use of more sand, cement, and water (3). Moreover, they tend to reduce bulk weight and decrease compressive strength. During the placement of the concrete, if flat particles become oriented with their large faces in a horizontal position, bled water will be entrapped under them, a condition which prevents the development of good bond.

Sharp, angular rock fragments may require increased mortar to produce adequate workability and thus may significantly increase cement and water content (Figs. 1 and 2) if a constant water-cement ratio is to be maintained.

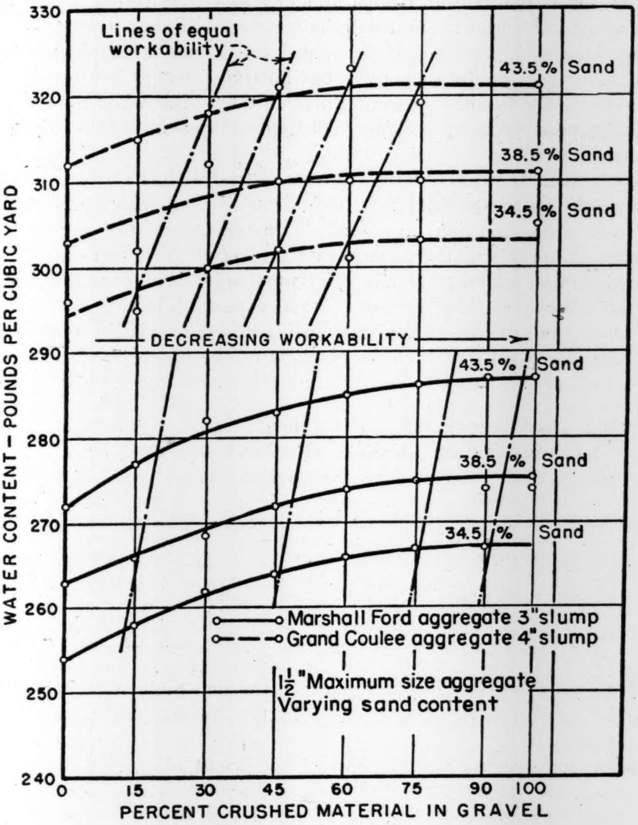

FIG. 2

Thermal Expansivity.—Thermal expansivity of aggregates affect concrete in two ways:

1. Through the influence of the average coefficient of expansion on the expansivity of the concrete as a whole; and
2. Through the action of individual particles whose coefficients differ widely from the coefficient of the cement paste.

Rarely can the effect of the expansivity of individual components of the aggregate on the concrete be measured. Failure of concrete has been attributed to thermal properties of the aggregate (4); but, in general, the effects of expansivity are masked because they merely augment, significantly or insignificantly, other deteriorative actions. It is becoming continually more evident that failure of concrete commonly results from a combination of processes which, acting individually in comparable degree, would be ineffectual.

Aggregate particles with unusually low expansivities are more objectionable than are those with unusually high expansivities, since few rocks and minerals possess higher coefficients than hydrated Portland cement. Most destructive to concrete are the tensile strains produced in the mortar when the temperature is declining, possibly being accompanied by development of tensile fractures in the cement paste.

The surprisingly rapid deterioration associated with the use of the so-called "sand-gravel" aggregate from the Republican River in Kansas (5) appears to result from a combination of several characteristics of the aggregate. The aggregates are only slightly or moderately reactive with high-alkali cements, but they contain high proportions of individual crystals of orthoclase and microcline feldspars whose coefficient of thermal expansion is very low (0.0×10^{-6} to 0.6×10^{-6} per degree Fahrenheit) (6)(7) in one crystallographic direction. These minerals constitute high proportions of the coarse sand, ranging from 34% to 45% of the No. 4 to No. 8 fraction in the samples studied (Table 1).

TABLE 1.—Petrographic Analysis of Sand, Republican River, Cloud County, Kansas (Percentages by Weight, No. 4 to No. 8 Fraction)

Item	Rock types	No. 6609A	No. 6609B	No. 6609C
1	Orthoclase and microcline	45.0	38.0	34.2
2	Granites	8.0	5.7	6.9
3	Quartz	45.0	52.5	55.6
4	Basic volcanic rocks	0.5	1.6	0.8
5	Siliceous limestones	0.5	....	0.8
6	Chalcedony	1.0	2.2	1.7

In such proportion, the great difference in thermal coefficients of particle and cement will create considerable stress in the cement paste with decline in temperature.

Among common rocks, the thermal coefficient of expansion ranges from about 0.5×10^{-6} to 9.0×10^{-6} per degree Fahrenheit (8). Rocks that commonly possess exceptionally low coefficients (1.0×10^{-6} or less) include granite, limestone, and marble. The thermal coefficient of hydrated neat Portland cement usually is between 5.9×10^{-6} and 9.0×10^{-6} per degree Fahrenheit (8). The freezing and thawing durability of concrete is not recognizably affected by moderate variations in expansivity of the aggregates in the presence of other influencing factors (Table 2).

Properties Affecting Volume Change.—Two sets of physical properties adversely influence volume change of concrete during wetting and drying—those which increase expansion, and those which increase or decrease shrinkage. Concrete may be distressed by local expansions caused by excessive expansion of aggregate particles as they become wet; with drying, these same particles will contribute to excessive shrinkage. In addition, particles that are especially compressible increase drying shrinkage, because they fail to restrain the normal drying shrinkage of the cement paste (9).

Clay minerals of the montmorillonite type (bentonitic) expand and contract significantly with wetting and drying. The expansion is accompanied by potential pressures which occasionally are several thousand pounds per square inch (10). An altered dolerite, apparently sound according to standard tests, was found to contain about 30% of interstitial clay (the mineral nontronite). Tests indicated that the rock expanded and contracted 0.06% linearly with wetting and drying, and after repeated cycles an increment of length remained. Concrete beams containing the dolerite as aggregate lost more than 68% of their original dynamic modulus of elasticity in twenty cycles of freezing and thawing. Some limestones, although sound according to standard tests, contribute to deterioration of concrete when their interstitial clay swells as it absorbs water from the concrete. Some sandstones expand as much as 0.08% linearly with wetting (11), probably because of interstitial clay.

TABLE 2.—EFFECTS OF THE THERMAL EXPANSIVITY OF AGGREGATE ON THE DURABILITY OF CONCRETE MAY BE MASKED BY OTHER FACTORS

Concrete	Type of aggregate (crushed) in the concrete	Thermal coefficient of the aggregate,[a] α	CYCLES OF FREEZING AND THAWING OF THE CONCRETE	
			To 25% loss by weight	To 0.3% linear expansion
A	Diorite No. 1	4.7	13	6
	Quartz diorite	4.4	25	19
	Limestone A	2.3	44	41
	Diorite No. 2	3.8	59	45
	Limestone B	1.8	84	88
B	Limestone C	4.4	18	10
	Graywacke	5.1	105	65
	Dolomite	4.5	535	540
C	Limestone C	4.4	40	50
	Graywacke	5.1	148	134
	Dolomite	4.5	351	351
D	Quartz	5.7	40	31

[a] According to the relation $\alpha = \frac{\Delta L}{\Delta T} \times \frac{10^6}{L}$, in which L is the length of the rock specimen, ΔL is the change in length, and ΔT is the change in thermometer temperature, expressed in degrees Fahrenheit. The expectable coefficient of expansion for hydrated, neat Portland cement ranges from 5.9×10^{-6} to 9.0×10^{-6} per degree Fahrenheit (8).

Strength and Elasticity.—Most rocks possess strengths and elasticities considerably greater than those of concrete. The range in crushing strength of rocks generally acceptable as concrete aggregate is from 10,500 lb per sq in. to 28,700 lb per sq in. (6); yet in individual cases, these rocks are incapable of producing concrete with adequate strength and durability.

Conversely, weakness of rocks may not be reflected proportionately in the strength of the concrete. For example, a very soft, porous, pumice that could be easily crushed in the fingers and which included numerous particles that were so porous as to float on water possessed a "structural strength" in mortar equivalent to approximately 78% that of the sand used in construction of Grand

Coulee Dam in Washington, one of the best aggregates tested in the USBR laboratories. After 28 days, the average compressive strength of standard 2-in. by 4-in. mortar cylinders containing fine pumice was 3,680 lb per sq in.—equal to that obtained on mortars containing some natural sands.

Other Properties.—Specific gravity of aggregates has a marked effect on the unit weight of concrete and for that reason specific gravity of aggregates sometimes is of importance in the design of gravity structures or of structural members that should be unusually low in weight. The specific gravity of aggregates must be known for purposes of mix design, but the estimation of the soundness of aggregates from specific gravity is a tenuous extrapolation. It is true that apparent specific gravity will decrease as porosity increases, but porosity can be determined from this measurement only if the absolute specific gravity is known also. Even if the absolute and apparent specific gravities are known, only porosity, but not the size, number, or continuity of pores, will be calculable. Experience indicates that tests of specific rock types or aggregates of similar lithology will yield a correlation between specific gravity and durability; but the same correlation cannot be carried over to aggregates of other lithology. Particles with very low specific gravity commonly move to the surface and thus cause poor finish and reduced abrasion resistance.

Specific heat and heat conductivity of aggregates strongly affect the rate of temperature change in concrete and the rate at which heat is transmitted through concrete (12). These factors are important in the preparation of design and specifications for massive structures, but are of limited significance for small structural members. To the extent that the temperature of concrete is changed more rapidly or by a greater amount, then the thermal expansivities of the aggregates exert a stronger influence on the concrete.

The properties of aggregates are changed by the presence of coating materials. These materials commonly change adversely the physical or chemical properties of the aggregate, but the change is caused by the same properties that affect quality of aggregate particles as a whole. In the experience of the Denver (Colo.) laboratories of the USBR, coatings are composed of silt, clay, gypsum, impure carbonates of lime and magnesia, iron oxides, opal, manganiferous substances, soluble phosphates, or mixtures of these substances.

Chemical Properties of Aggregates.—The traditional view regarded aggregates as inert fillers which passively contributed weight and bulk to the concrete. Actually, few, if any, aggregates are inert while enclosed by cement. Any type of chemical reactivity will cause a series of complex, mutually dependent changes in the concrete, but four general types of chemical activity appear to be identifiable:

First, the aggregates may contain substances that are soluble in the aqueous solution permeating the concrete. Leaching of these substances will increase porosity of the concrete and decrease its strength. The dissolved substances can form efflorescence on and near the surface of the concrete, and thus cause scaling.

Second, soluble substances or products of decomposition of aggregate particles may modify normal hydration of the cement. Thus, organic substances

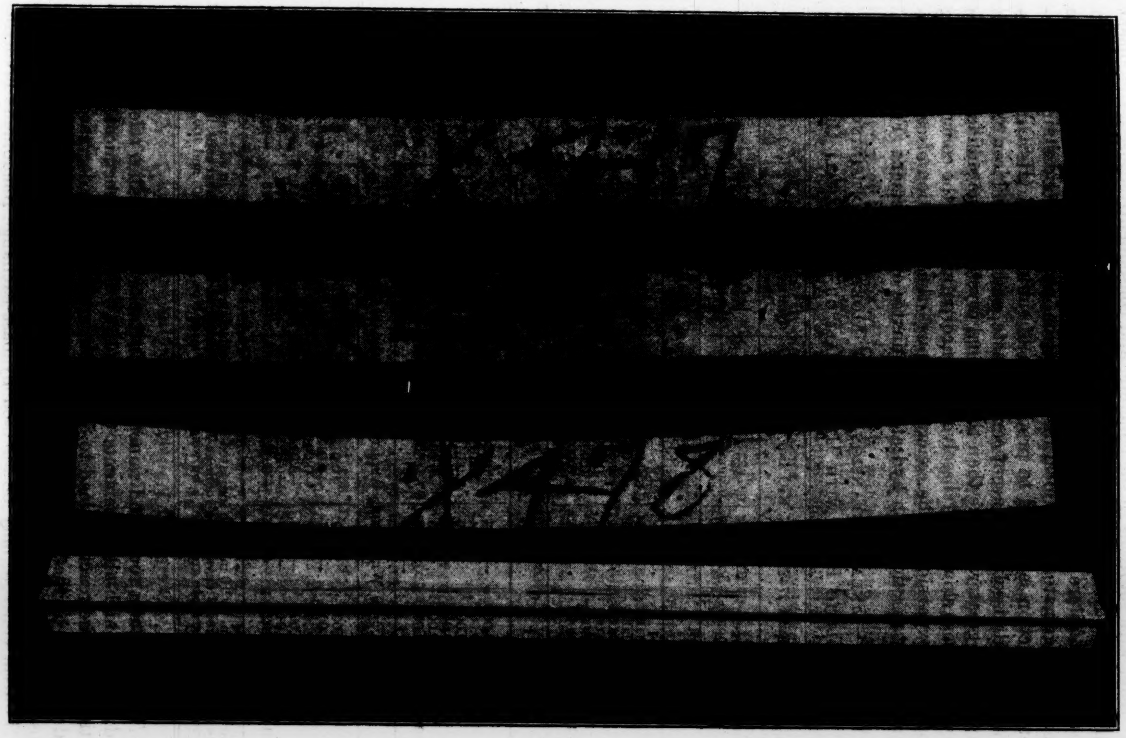

FIG. 3

may react with solutions in the concrete, and consequently cause development of abnormal hydration products. Strength and durability are reduced by such changes, and failure of the concrete can result. Oxidation of pyrite or marcasite (FeS_2) in aggregate causes local disintegration, pop outs, and staining of concrete; this action is related to production of sulfuric acid (H_2SO_4), which decomposes and dissolves adjacent cement paste, and hydrated iron oxides (which may occupy greater volume than the original crystals of the sulfide). Sulfates may decompose slowly and dissolve after incorporation in the concrete, causing progressive expansion of the concrete, through the formation of calcium sulfoaluminate by reaction with hydrated aluminates of the cement (Fig. 3).

TABLE 3.—LINEAR EXPANSION (%) OF MORTARS CONTAINING ZEOLITES OR ZEOLITIC ROCK WITH HIGH-ALKALI OR LOW-ALKALI CEMENT[a]

Replacement in crushed quartz (%)	CEMENT 2742; 0.12% K_2O AND 1.30% Na_2O				CEMENT 2735; 0.14% K_2O AND 0.04% Na_2O			
	Age, 1 month	Age, 3 months	Age, 6 months	Age, 1 year	Age, 1 month	Age, 3 months	Age, 6 months	Age, 1 year
(a) ANALCITE; GOLDEN, COLO.								
5	0.004	0.015	0.017	0.033	−0.002	0.001	−0.002	0.007
10	0.003	0.014	0.015	0.031	−0.005	0.000	−0.003	0.006
20	0.002	0.014	0.016	0.031	−0.004	0.000	−0.002	0.005
50	0.001	0.010	0.013	0.028	−0.001	0.001	−0.003	0.009
75	0.002	0.007	0.008	0.022	−0.006	−0.001	−0.003	0.010
100	0.000	0.005	0.003	0.017	−0.004	0.000	−0.002	0.010
(b) STILBITE; NOVA SCOTIA								
5	0.006	0.014	0.020	0.028	−0.001	0.003	0.003	0.004
10	0.007	0.014	0.019	0.026	−0.003	0.001	0.002	0.004
20	0.006	0.016	0.016	0.019	−0.001	0.002	0.003	0.006
50	0.005	0.013	0.013	0.017	....	....	....	....
75	0.001	0.005	0.006	0.009	....	....	....	....
100	0.000	0.003	0.004	0.008	−0.002	0.007	0.002	0.004
(c) WEATHERED ANORTHOSITE; SOLEDAD CANYON, CALIFORNIA								
5	0.007	0.009	0.015	0.027	−0.003	−0.007	−0.007	−0.004
10	0.006	0.007	0.013	0.029	−0.002	−0.005	−0.006	−0.001
20	0.003	−0.002	−0.001	0.009	−0.003	−0.007	−0.011	−0.004
50	−0.006	−0.013	−0.014	−0.006	−0.004	−0.011	−0.014	−0.007
75	−0.013	−0.019	−0.023	−0.015	−0.006	−0.013	−0.017	−0.009
100	−0.020	−0.028	−0.030	−0.026	−0.007	−0.015	−0.020	−0.007

[a] Bars, 1 in. by 1 in. by 10 in., with indicated materials replacing crushed quartz in all size fractions. Concrete mix, 1:2, cured 7 days in fog room at 70° F; then stored at 100° F in sealed cans, in the presence of excess water. Aggregate graded 20% by weight each of sizes Nos. 8, 16, 30, 50, and 100, no pan.

Third, certain rocks and minerals are attacked by alkalies released during hydration of high-alkali cements (13)(14)(15). Physical-chemical phenomena initiated by the reactions lead to expansion, cracking, and decreased elasticity and strength of the concrete. These phenomena will be discussed in detail subsequently.

Fourth, concrete may be affected by base-exchange phenomena involving components of the aggregate and the caustic solution within the cement paste. This process usually involves exchange of alkalies in the aggregate for the calcium contained in the solution. The released alkalies attack aggregate particles susceptible to cement-aggregate reaction, or they may form deposits wherever the water is evaporated from the concrete. A somewhat altered granitic rock (granodiorite) used as aggregate in construction of a thin dam in southern California apparently has participated in base-exchange reactions of this kind, because efflorescent deposits of impure alkali carbonates occur on the downstream face of the structure and scaling of the concrete has taken place. A zeolite similar to laumontite (or leonhardite) $(Ca, Na_2) Al_2Si_4O_{12} \cdot 4H_2O$, occurring in the rock called anorthosite is associated with failure of cast stone and stucco in southern California. G. F. Loughlin (16) attributed the failure to volume change of the zeolite with change in water content. It is possible, however, that base-exchange reactions contributed to the deterioration. Chemical studies of the zeolite containing anorthosite indicate that the rock has a significant proportion of replaceable bases (1.4% by weight), mainly calcium, sodium, and magnesium.

Although zeolites may contribute to deterioration of concrete through base exchange, they do not cause significant expansion of moist-stored mortars when present alone or mixed with nondeleterious aggregate material (Table 3). In theory at least, zeolites with exchangeable sodium or potassium could contribute to alkali-aggregate reaction in concrete or mortar containing reactive aggregate by release of the alkalies upon adsorption of calcium released by hydration of the cement. However, contractions as great as 0.030% were measured after 6 months on the bars containing the weathered anorthosite (with included laumontite) possessing significant base-exchange capacity.

Tests of Concrete Aggregate for Quality

For many years, engineering laboratories have applied a series of standardized tests to determine the quality of aggregates proposed for use in construction. The tests usually applied include the determination of: (*a*) Specific gravity and absorption; (*b*) soundness, usually with sodium or magnesium sulfate; (*c*) abrasion resistance; (*d*) organic impurities in sand; (*e*) quantities of coal or lignite, clay lumps, and shale; and (*f*) properties of standard mortars to find the structural strength of fine aggregate.

The results of these tests correlate statistically with the service of concrete containing the aggregates, but cannot be depended on to diagnose the quality of a particular aggregate. For this reason, engineers have come to rely on tests and examination of concrete wherever possible. If an aggregate has been used in construction, service histories will be available; or, if laboratory facilities are at hand, the aggregates may be incorporated in concrete or mortar specimens. Service histories of concrete containing the proposed aggregate and subject to conditions similar to those anticipated at the project under investigation are most diagnostic. On the other hand, service histories can be dangerously misleading if data regarding the cement, mix design, or placement and curing methods are incomplete, or if the conditions of service are not similar.

Tests of concrete or mortar containing proposed aggregates will indicate the quality of the material if the mixes are prepared and testing procedures are designed to simulate anticipated conditions on the job. Influence of the aggregate on durability, strength, elasticity, permeability, expansivity, thermal properties, drying shrinkage, or bulk weight can be determined in the laboratory. However, the time necessary for adequate tests of concrete frequently precludes application of such methods to studies of concrete aggregate. During a 6-month period in 1946, the USBR laboratories in Denver were called on to test more than 200 different concrete aggregates. Neither personnel nor facilities are available in any laboratory for tests of concrete on a comparable scale. Consequently, for most purposes, the concrete engineer must select aggregate by methods other than those which involve testing of concrete.

The standard tests usually applied to unconfined aggregates do not measure directly the properties of aggregates that affect the quality of concrete. Properties as fundamental as size, continuity, and abundance of pores; bonding characteristics; and thermal expansivity are not measured even qualitatively.

Specific Gravity and Absorption Determinations.—Specific gravity and absorption commonly correlate with freezing and thawing durability if lithologically similar materials are tested, but no fundamental relation to durability can be adduced. Pumice commonly used as lightweight aggregate might have a bulk specific gravity (American Society for Testing Materials (ASTM) designation: E12) of 1.0 and a 24-hour absorption of 24%, and yet produce a concrete which is sound in freezing and thawing. Determinations of specific gravity and absorption by standard methods (ASTM designations: C127–42 and C128–42) do not distinguish those aggregates whose specific gravity is low and whose absorption is high because of the presence of a few unsound particles, from those which contain an abundance of particles whose specific gravity is but slightly low and absorption but slightly high. For this reason, absorption and specific gravity of lithologically complex aggregates cannot be related quantitatively to soundness. Statistical correlations indicate that rigorous application of specified limits for specific gravity and absorption does not assure elimination of unsound aggregates and may cause rejection of some satisfactory aggregates.

Accelerated Soundness Tests.—Accelerated soundness tests involving use of sodium or magnesium sulfate (ASTM designation: C88–41T) are commonly used to estimate susceptibility of the aggregate to breakdown under freezing and thawing conditions because the sulfate soundness tests are more rapid and involve less equipment than do actual freezing and thawing of aggregates. Breakdown is believed to result from hydration and consequent expansion of crystals of Na_2SO_4 or $MgSO_4 \cdot H_2O$, following reimmersion of the samples after the drying period (17). This mechanism is certainly different from the processes by which freezing and thawing of water cause breakdown. Consequently, the results of the sulfate soundness tests do not parallel the results of freezing and thawing soundness tests. Moreover, as pointed out previously, the durability of aggregate does not always correlate with the durability of concrete. In addition to these fundamental deficiencies of the test, extraneous variations may vitiate results. Inconsistencies in the sulfate soundness tests arise from: (*a*) Differences in the types of containers used; (*b*) differences in techniques of drainage following the immersion period; (*c*) inadequate temperature control; (*d*) variations in length of the drying period; (*e*) differences in techniques of screening the samples; and (*f*) differences in grades of salt and concentration of the solutions used in the test. . Frequently, aggregates that fail the test prove to be satisfactory, and those that pass actually might contribute to the breakdown of concrete during freezing and thawing.

Abrasion Test.—The Los Angeles abrasion machine (ASTM designation: C131-39) largely has replaced the Deval machine (ASTM designation: D2-33) in tests of concrete aggregate. Extensive tests have established a general relation between strength of concrete and the percentage of wear in the Los Angeles abrasion test (18). The Los Angeles abrasion test yields information of value on the degree to which aggregates containing inordinate quantities of soft particles can be processed effectively. Specifically, aggregates that lose excessively in the first one hundred revolutions, but that lose less than 40% in five hundred cycles, commonly can be made suitable for use by scrubbing or special processing.

Breakdown of the sample is produced mainly by impact of the cast-iron or steel spheres (19), and hence is not predominantly an abrasion loss. The test reveals the presence of soft, friable, or exceptionally brittle particles; but limited loss in the test does not assure high quality of the aggregate. A crushed, milky quartz used as a control aggregate shows only 33.8% wear in the five hundred revolutions of the abrasion test; yet it consistently fails to produce durable concrete (Table 2); the low "structural strength" of the highly quartzose Ottawa sand is well known.

Test of Organic Impurities in Sand.—Organic materials in aggregates are partly dissolved by the solution of sodium hydroxide used in the standard test for organic impurities in sand (ASTM designation: C40-33). However, dark coloration of the solution merely indicates the need for further testing (particularly in mortar) rather than the rejection of the aggregate. Some organic substances are innocuous, yet produce a strongly colored solution. Organic substances occur as woody particles, coatings on aggregate particles, or as waxy or petroliferous materials disseminated through rock particles.

Determination of Content of Unsound Particles.—The standard tests to determine content of coal or lignite (ASTM designation: C123-39), and shale (ASTM designation: D546-41) involve gravity separation of the sample in a heavy liquid. Presumably, the coal or lignite, and the shale, float on the surface, but other particles of low specific gravity also will separate in the same manner. Clay lumps are identified in the standard test (ASTM designation: C142-39) as those particles that can be broken up into fine material with the fingers.

These three tests have been replaced in the USBR laboratories by petrographic examination, thus eliminating erroneous classification of lightweight or soft particles.

Petrographic Examination.—For several years petrographic examinations have been made of all aggregates proposed for use in concrete construction on projects of the USBR; but, with passing time and increasing knowledge of the significance of the petrographic properties of aggregates, the results of these examinations have become more significant and increasing reliance has been placed on them. The techniques and methods of petrographic examination of coarse aggregates have progressed from mere visual examination to detailed analysis and description of complex aggregates (8)(20).

Petrographic examination is used to supplement the results of the standard tests or to judge the need for special tests. Suspected chemical instability or physical unsoundness can be checked by tests of mortar or concrete. Detailed knowledge concerning the lithologic composition of aggregates commonly will permit precise determination of the cause for observed deficiencies, so that ameliorative measures can be undertaken. Petrographic examination is the only method heretofore available by which alkali-reactive aggregates can be identified, quickly and reliably.

RESEARCH ON CONCRETE AGGREGATES

Deficiencies of the standard tests have led to attempts at development of additional methods by which the physical and chemical properties of aggregates could be determined. Because of their great significance, work was conducted first on alkali reactivity and on the characteristics of pores which occur in solid materials. These researches are being continued, but results to date (1949) will be summarized.

Research on Alkali-Aggregate Reaction.—Soon after the experiences of T. E. Stanton, M. ASCE, in California (13) were made available, the USBR became intensely interested in the problem of alkali-aggregate reaction because of experiences at Parker Dam on the Colorado River in California and Arizona (14)(20)(21). Adequate facilities and specialized personnel were procured in 1941 to undertake a thorough investigation of the types of aggregate material capable of participating in this reaction, the mechanism producing the expansion and decline in strength of the concrete, and possible ameliorative methods where reactive aggregates are encountered. (The problem of cement-aggregate reaction represents an interesting example of the manner in which engineering problems can be attacked by scientific methods. The research herein summarized, and reported elsewhere (22)(23) in 1948, was obtained by highly trained and specialized scientific personnel. Integration of research and coordination of science and engineering have yielded results of immediate practical value.)

It has been well established that the deterioration manifests itself as decline in strength of the concrete and as expansion. Outwardly, the concrete exhibits typical patterns of cracking, which may be accompanied by exudation of siliceous gels at the surface. These criteria furnish the engineer with methods for preliminary diagnosis of the disease, but complete diagnosis often requires thorough laboratory study. Even under favorable circumstances it is not always possible to obtain an unambiguous diagnosis if the deterioration has been extremely severe, because secondary effects may mask the primary cause.

Numerous examples of concrete structures affected by alkali-aggregate reaction are now recognized, and many of these are related to the aggregates of several specific river systems and drainage areas. The map (Fig. 4) shows only the locations of structures that are definitely known to be affected by alkali-aggregate reaction (22*a*). West of the states bordering the Mississippi River there are several areas wherein no affected structures are indicated, notably

Texas, Oklahoma, and New Mexico; most of Nevada and Utah; and Montana, North Dakota, and South Dakota. However, this condition does not necessarily imply that deleterious aggregates are absent in these areas. It merely indicates that no affected structures in these areas have come to public attention. Acid to intermediate volcanic rocks are present in some of these areas and, where significant quantities of these rocks have been used as aggregate in combination with high-alkali cements, deterioration caused by alkali-aggregate reaction is certain to occur. East of the Mississippi River, volcanic rocks are comparatively rare at the surface, but opaline or chalcedonic cherts are widespread and are known to have caused deterioration of concrete through alkali-aggregate reaction in Georgia and Alabama. Although only a few spots appear on the map, again it should be recalled that this must not be taken to imply that alkali-aggregate reaction can be disregarded in this large area.

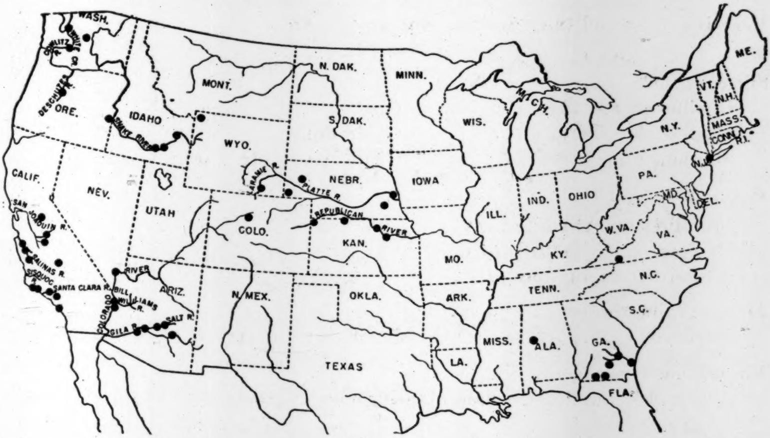
WASH.
COWLITZ
ORE.
DESCHUTES
IDAHO
SNAKE RIVER
MONT.
WYO.
N. DAK.
S. DAK.
MINN.
WIS.
MICH.
ME.
VT.
N.H.
MASS.
CONN.
R.I.
N.Y.
N.J.
PA.
MD.
DEL.
CALIF.
NEV.
UTAH
LARAMIE R.
NEBR.
PLATTE R.
IOWA
REPUBLICAN RIVER
COLO.
KAN.
ILL.
IND.
OHIO
W.VA.
VA.
SAN JOAQUIN R.
SALINAS R.
SISQUOC
SANTA CLARA R.
RIVER
BILL WILLIAMS R.
COLORADO
ARIZ.
GILA R.
SALT R.
N. MEX.
OKLA.
MO.
ARK.
KY.
TENN.
N.C.
S.C.
TEXAS
MISS.
ALA.
GA.
LA.
FLA.

FIG. 4.—LOCATION OF STRUCTURES THAT HAVE BEEN AFFECTED BY ALKALI-AGGREGATE REACTION

Some of the earliest laboratory investigations conducted by the USBR were directed toward the identification of rocks and minerals that react deleteriously and their comparative reactivities. Although previous work (13)(14)(21)(24)(25) had indicated that opal, a phyllite, and certain intermediate to acidic volcanic rocks were susceptible to attack, the degree of reactivity of most rocks and minerals was unknown. Therefore, samples of many common rocks and minerals were obtained from various sources and, after crushing and grading, were fabricated into mortar bars with low-alkali and high-alkali cements. Each sample of test aggregate was examined by USBR petrographers to determine its mineralogical and textural properties.

As a result of these studies, it was demonstrated that all silicate and silica minerals react with alkalies of Portland cement; but the greater number of minerals and rocks react insignificantly, without causing deleterious expansion or other distress accompanying cement-aggregate reactions. The rocks and minerals that contribute to deterioration of concrete in this way can be detected by petrographic examination. The reactive minerals, and their chemical and physical characteristics, are (22*b*):

Reactive minerals	Chemical composition	Physical character
Opal	$SiO_2 \cdot nH_2O$	Amorphous
Chalcedony	SiO_2	Cryptocrystalline fibrous
Tridymite (26)	SiO_2	Crystalline

Reactive rocks and their reactive components are:

Reactive rocks	Reactive component
Siliceous rocks—	
Opaline cherts	Opal
Chalcedonic cherts	Chalcedony
Siliceous limestones	Chalcedony, opal, or a combination of both
Volcanic rocks—	
Rhyolites and rhyolite tuffs	....
Dacites and dacite tuffs	Volcanic glass, devitrified glass and tridymite
Andesites and andesite tuffs	
Metamorphic rocks—	
Phyllites	Hydromica or very fine-grained quartz

Miscellaneous rocks—

Any rocks containing veinlets, inclusions, or grains of the foregoing rocks or minerals.

To be sure, aggregate material may be deleterious for other reasons. In addition to the aforementioned reactive rocks, for example, artificial glasses, such as pyrex glass, are known to be deleteriously reactive. Whereas the volcanic types listed are known to be deleteriously reactive, basalts and at least certain phonolites are known to be innocuous, and data regarding trachytes and latites are lacking.

Reliance on the technical skill and experience of petrographers to interpret the potential reactivity of aggregates has several obvious disadvantages. To overcome these difficulties, experimentation was directed toward the development of a chemical test capable of predicting the potential reactivity in quantitative terms. Several varieties of physical and chemical tests have been advocated during the past several years (24)(27)(28)(29) but none of these yields information diagnostically related to the fundamental causes of expansion, decline in strength, etc. Moreover, all are impractical for one or more reasons such as:

1. They require too much time;
2. They depend on properties that cannot be measured quantitatively; or
3. They cannot be applied simultaneously to all the different rock constituents of a complex aggregate.

As early as 1934, R. E. Stevens (30) demonstrated that many common minerals are slightly soluble in water and that slurries of finely ground minerals quickly become alkaline. Some minerals yield very mildly alkaline solutions whereas others produce moderately strong alkalinity. Attempts to correlate these types of data on aggregates with their deleteriousness, as indicated by mortar bar expansions, produced unsatisfactory results and these experiments were abandoned. However, the studies aided in formulation of later concepts.

Basically, testing a concrete aggregate for reactivity is simple. It should consist merely of crushing the aggregate to an appropriate size, adding some specific amount of alkaline solution, and measuring the effects of the interaction. Unfortunately, early tests of this type gave results that correlated rather poorly with the service histories and mortar bar data on the aggregates. Opal will usually give a diagnostic test by almost any procedure, as it did in most leach tests (27)(28)(29), but the problem of differentiating between the less reactive and the innocuous materials was not resolved by these early methods, which were based on determination of the quantity of silica dissolved by the caustic solution. The experiments demonstrated that some chemically innocuous aggregates (such as basalt) released more silica into solution than did certain moderately deleterious aggregates.

During tests of materials in different size fractions, it was discovered that the innocuous aggregates released from two to four times more silica into a strongly alkaline solution from pulverized material than from somewhat coarser sizes. This result indicates a normal relationship between the surface area and the quantity of silica released into solution. However, release of silica did not occur in this straightforward way when deleterious aggregates were tested. In these experiments, the reduction of active alkali in the solution as well as the amounts of silica taken into solution from the aggregate were determined analytically. The measurements indicated that the potency of the caustic solution was reduced more rapidly than silica was dissolved from deleterious materials, and thus demonstrated that the earlier leach tests were not diagnostic of deleteriousness because the amount of silica held in solution is not proportional to the amount of reaction. A successful method of testing requires measurement of two factors—the effect of the solution on the aggregate and the effect of the aggregate on the solution.

Correlation of the chemical data with mortar bar expansion is presently obtained by comparing the quantity of silica released from an aggregate with the reduction in the alkalinity of a 1N sodium hydroxide solution during the reaction. For the test, specially constructed stainless steel vessels containing the aggregate material and caustic solution are immersed for 24 hours in a liquid bath controlled at 80° C ± 1°. The test requires rigorous control and precise analytical methods, to yield reliable results. Diagnostic correlations have been obtained (Fig. 5) for more than seventy rocks, minerals, sands, and gravels on which mortar bar data are also available. In Fig. 5, only a few of the tested materials are indicated merely to demonstrate the expectable range of analytical results and the separation of innocuous from deleterious aggregates.

The chemical data obtained during the development of an aggregate test for reactivity, combined with important information obtained from microscopic investigations and other experiments, form the basis for a general concept of the mechanism by which alkali-aggregate reaction causes a breakdown of concrete. The process of deterioration may be summarized as follows: During the mixing period and the subsequent several hours, a large part of the alkalies of the Portland cement dissolve in the liquid phase. As hydration progresses, the calcium aluminates and the silicates of the paste extract water, thus causing the liquid to become even more strongly caustic. The caustic solutions then attack susceptible particles of the aggregate and produce alkalic silica gels. These gels have a high affinity for water. As they imbibe water, they swell and produce hydrostatic pressures within the confining concrete.

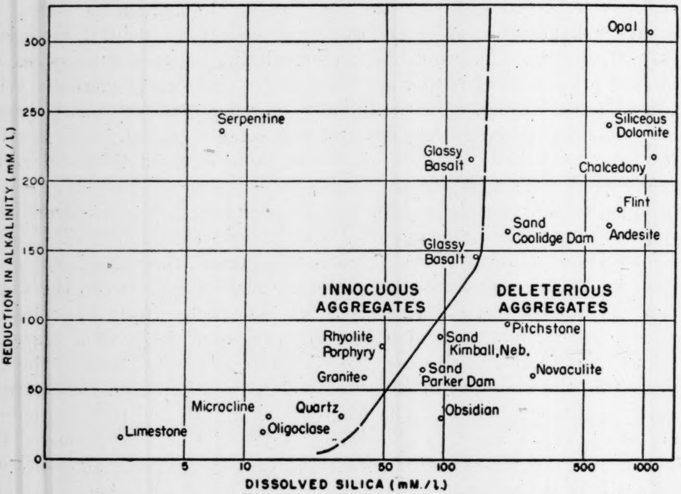

FIG. 5.—CHEMICAL INTERACTION OF CONCRETE AGGREGATE AND SODIUM HYDROXIDE (NaOH) SOLUTION

Osmotic pressures in excess of 550 lb per sq in. have been measured (22c) in the laboratory (Fig. 6), and similar pressures have been computed in accordance with recognized laws of physical chemistry. Thus, pressures that exceed the tensile strength of concrete can be expected from the formation and the hydration of the gel. As a consequence of rupture of the paste, microfractures develop and the pressures are momentarily relieved by the intrusion of the gels into the microfractures. These gelatinous materials may ultimately exude at the surface of moist-stored concrete affected by this malady. The microfractures account qualitatively for both the increase in volume of the concrete and the decline in strength; a quantitative demonstration of this fact would be rather difficult to obtain.

Capillarity and Its Relation to Durability of Solid Materials.—Formation of ice within solid materials results in breakdown through the operation of one or more of three separate processes:

a. The development of excessive hydrostatic pressures within materials of low permeability during progressive freezing;

b. Formation of ice lenses; or

c. The excessive thermal expansion of ice filling voids while temperatures are rising, in the range below the freezing point.

The effects of hydrostatic pressure during progressive freezing cannot be doubted, but this mechanism alone probably will not explain all the observable variation in disintegration. Stephen Taber (31) has proved the development of ice lenses in soils of low permeability. Both of these mechanisms require that pores within the materials be minute, for neither is operative if the pores are predominantly large. The effect of excessive thermal expansivity of ice crystals is probably less significant than the other two mechanisms; yet the thermal properties of ice may cause distress of rock or concrete, especially if the temperature changes below the freezing point are large. The thermal coefficients of linear expansion of rocks and hydrated Portland cement rarely exceed 9.0×10^{-6} per degree Fahrenheit (8), but in the range from 32° F to −4° F, the coefficient of linear expansion of ice is 28×10^{-6} per degree Fahrenheit (32).

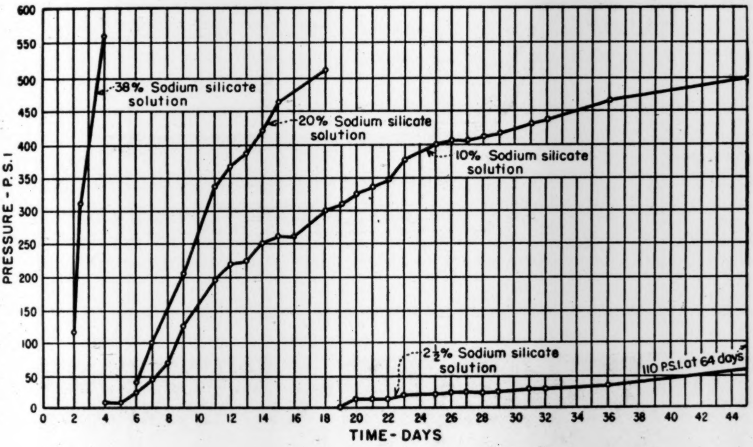

FIG. 6

The extent of breakdown in solid materials due to hydrostatic pressure, on the one hand, or to lens formation, on the other, is unknown. Breakdown of materials of low permeability through the development of high hydrostatic pressures is undoubtedly widespread, and plausibly will account for observed distress of concrete (2) and disintegration of rock in many situations (11). Mr. Taber developed the concept of lens formation specifically for soils, but A. F. Gill and J. F. J. Thomas (33) applied it to studies of masonry, bricks, and concrete. Subsequently, in 1944, A. R. Collins (34) utilized it to explain the disruption of porous concrete subjected to freezing and thawing. Mr. Taber (35) found the potentialities for stress development in natural materials to be relatively small, since he inhibited the formation of ice lenses by surface loads

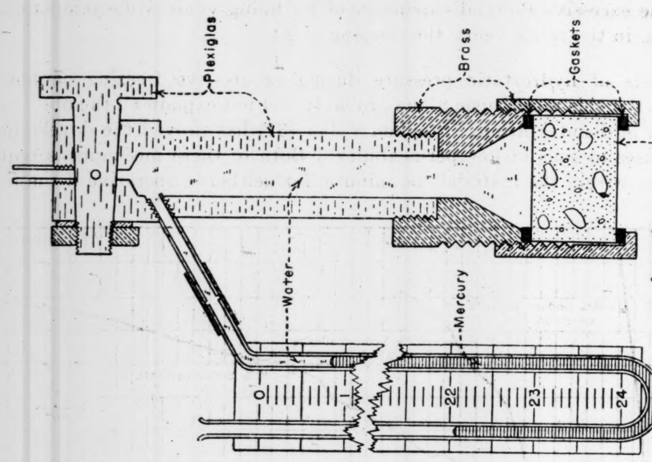
Plexiglas
Brass
Gaskets
Water
Mercury
Concrete or rock specimen
0
1
22
23
24

FIG. 8

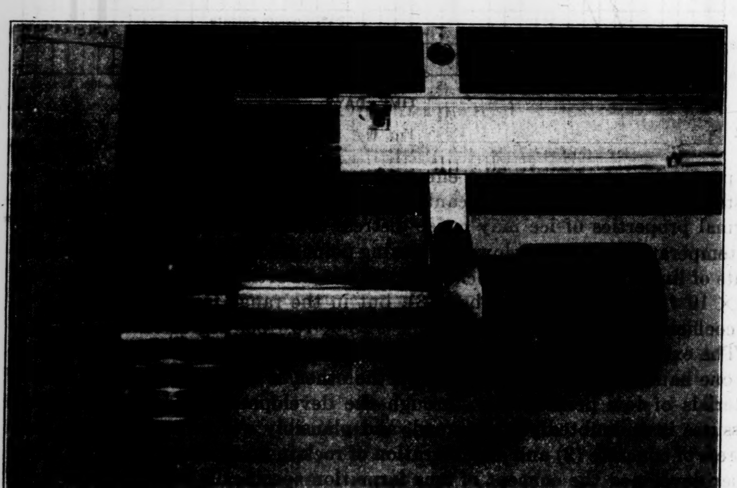

FIG. 7

equivalent to 15 kg per sq cm (213 lb per sq in.), although he believed that tensile stresses as high as 1,350 kg per sq cm (approximately 19,000 lb per sq in.) are possible in ideal systems. Nevertheless, Mr. Collins considered that the stress accompanying lens formation was the result of crystal growth and that it was limited in magnitude only by the melting pressure of ice at any given temperature (namely, 30,000 lb per sq in. at −8° F).

Experiments in the USBR laboratories, conducted under conditions that preclude the development of the high hydrostatic pressures associated with low permeability, indicate that lens formation can cause disruption of inherently weak rocks of low permeability or of rocks containing fractures. Lens formation also occurs in unset (36*a*) or very porous concrete (34), and porous bricks (33), but it would appear that this mechanism is operative only where tensile strength is small.

Because freezing and thawing durability is related to the size, continuity, and abundance of pores, an index of durability for any particular material can be obtained by measuring the rate of evaporation from saturated specimens and the force associated with capillary movement of water into the material. By modifying apparatus used in the Department of Scientific and Industrial Research, London (37), England, an instrument (Fig. 7 and 8) was designed in the USBR laboratories to measure these properties of rock, concrete, and mortar.

The apparatus is composed of a clamp in which the specimen is sealed, a water reservoir, and a mercury manometer. The specimens tested in the apparatus are cylinders 1.5 in. in diameter and up to 1 in. thick. The specimen is clamped into the apparatus and sealed by ring gaskets in such a way that air or water cannot pass around it. The cell above the specimen and all space above the mercury in the manometer are filled with distilled water. Air is eliminated from the specimen (by saturation under vacuum) and from the reservoir prior to the initiation of the test.

After the cock at the top of the reservoir has been closed, water that evaporates from the surface of the specimen will be replaced by water from the reservoir because of capillary attraction. However, the system is open to atmospheric pressure at the other side of the manometer, so that the forces exerted by the capillary openings can be measured provided they are at less than atmospheric pressure. The rate of evaporation is controlled by the relative area of continuous capillaries exposed at the lower surface. The pressure in the reservoir will continue to decline either until it equals the vapor pressure of water at the temperature of operation, or until the air is able to force the water from the largest pores and thus to prevent further increase of the pressure deficiency within the reservoir.

It is clear that materials containing abundant, continuous pores of small size will be indicated by the rapid increase and the ultimately high magnitude of pressure deficiency. Materials characterized by a few large pores will be indicated by slow increase in pressure deficiency, and this deficiency will still be comparatively small when air begins to pass through the specimen into the reservoir. All combinations of rate of rise and ultimate magnitude of pressure deficiency are possible for different materials. In every case, the ultimate pressure deficiency is controlled by the minimum diameter of the largest continuous channel way in the specimen. Results of measurements on rocks and concretes are indicated in Figs. 9 and 10; supplementary data are compiled in Tables 4 and 5.

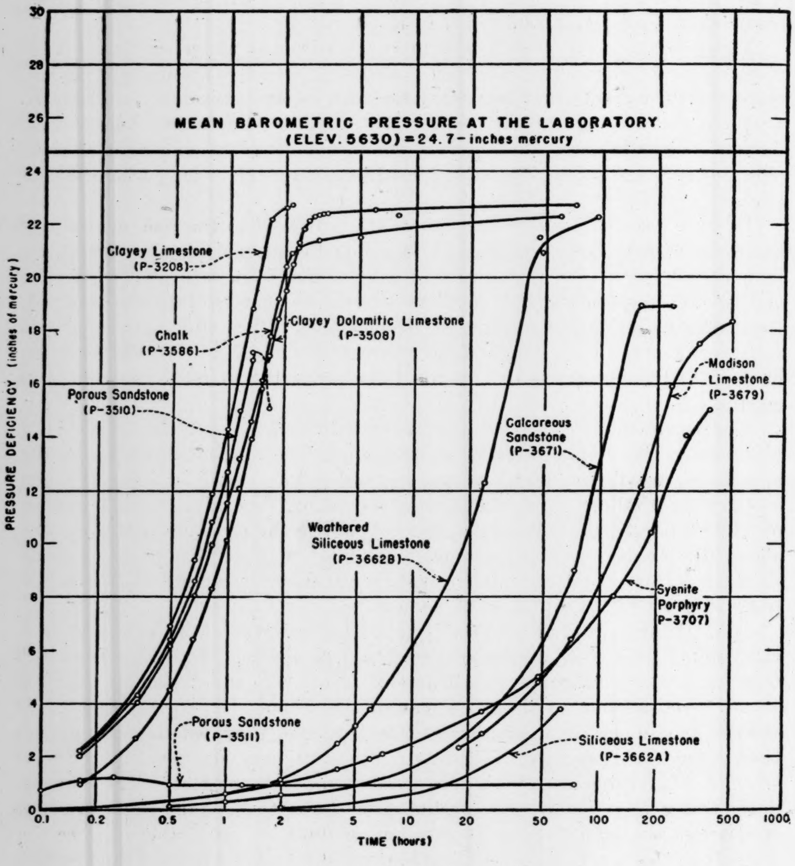

FIG. 9

The rocks that failed during the freezing and thawing test (Nos. P-3208, P-3586A, P-3508, and P-3662B, Table 4) caused rapid increases in pressure deficiency, which ultimately exceeded 22 in. of mercury. The weathered siliceous limestone (P-3362B) is represented by a curve (see Fig. 9) below and to the right of curves representing the other unsound rocks. This rock disintegrated more slowly and less completely than the other unsound rocks. Apparently, three of the sound rocks (P-3662A, P-3679, and P-3707) would sustain high pressure deficiencies if the test were continued long enough. Evidently, their soundness in freezing and thawing is controlled largely by their inability to become critically saturated while immersed in water.

The two sandstones (P-3510 and P-3511A, Table 4) are friable and weak (having lost 95% or more in the Los Angeles abrasion test), yet they lost very small amounts of material in the freezing and thawing test. Their soundness is evidently related to the large pores which can drain effectively during progressive freezing. One of these sandstones (P-3510) sustained a pressure deficiency of 17.1 in. of mercury, before air was forced by atmospheric pressure through the specimen. The rapid rise in pressure deficiency represents the high effective porosity. The sound calcareous sandstone (P-3671) apparently is characterized by relatively few pores, yet the pores are comparatively large, the ultimate pressure deficiency being 18.9 in. of mercury.

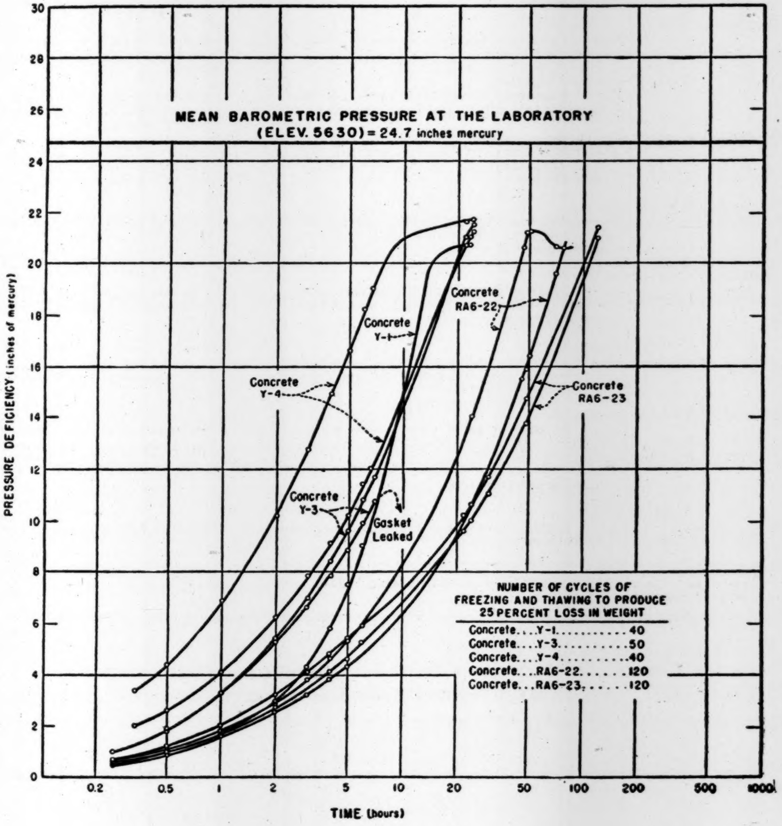

Fig. 10

TABLE 4.—OBSERVATIONS ON ROCKS USED IN THE MEASUREMENT OF CAPILLARITY

No.	Type	Description	%[a]	Results of soundness tests[b]
P-3208	Clayey Limestone:	Firm, fine grained, finely porous, highly argillaceous	5.2	Specimens cracked at four cycles; broke down to small pieces at six cycles
P-3586A	Chalk:	Firm, exceedingly fine grained, finely porous; contains 5% clay	16.4	Specimens broke down to small chips and sludge during four cycles
P-3508	Clayey Dolomitic Limestone:	Firm, exceedingly fine grained, finely porous; contains 13% of silt, clay, and fine sand	....	Specimens broke down to small pieces and sludge during twenty cycles
P-3662B	Weathered Siliceous Limestone:	Moderately hard, brittle, fine grained, red; opal, chalcedony, clay, and sand constitute 42% of rock	6.6	Specimen split in two, and one part broke down during fifteen cycles into several small pieces
P-3662A	Siliceous Limestone:	Moderately hard, brittle, fine grained, yellowish buff; opal, chalcedony, clay, and sand constitute 46% of rock	2.7	Specimen lost 1.4% by spalling during fifteen cycles
P-3510	Porous Sandstone:	Friable, porous, fine grained, quartzose sandstone; lost 96.7% in five hundred cycles of Los Angeles abrasion test	3.6	Specimens lost 0.1% to 0.7% by granulation during fifteen cycles
P-3511A	Porous Sandstone:	Friable, porous, medium grained, quartzose sandstone; lost 95.0% in five hundred cycles of Los Angeles abrasion test	7.6	Specimens lost 0.2% to 2.0% by granulation during fifteen cycles
P-3679	Madison Limestone:	Hard, massive, slightly porous, fine grained oolitic	0.2	Specimens lost 0.2% by spalling at edges in fifteen cycles
P-3707	Syenite Porphyry:	Hard, porous, fine grained; contains large phenocrysts (crystals)	2.0	No loss in fifteen cycles
P-3671	Calcareous Sandstone:	Hard, slightly porous, fine grained, cemented by calcium carbonate	1.7	No loss in fifteen cycles

[a] 24-hour absorption. [b] Results of freezing and thawing in 24-hour cycles; freezing in cold room at −10° F, thawing in tap water.

TABLE 5.—MIX PROPERTIES OF CONCRETES USED IN THE MEASUREMENTS OF CAPILLARITY

Concrete[a]	Aggregate[b]	Cement	MIX, IN PARTS BY WEIGHT[c]				Slump (inches)	UNIT WEIGHT	WATER CONTENT	Cement $\left(\frac{bbl}{yd^3}\right)$	Cycles[d] (%)
			w/c	C	S	G		Pounds per cubic yard			
Y-1	P	5784	0.76	1.00	3.85	3.85	2¾	144.5	313	1.10	40
Y-3	L	5784	0.61	1.00	2.65	2.65	5	147.0	350	1.53	50
Y-4	L	4701	0.61	1.00	2.65	2.65	1	147.0	350	1.53	40
RA6-22	P	5784	0.55	1.00	2.65	2.65	6½	145.7	315	1.52	120
RA6-23	P	5784	0.55	1.00	2.65	2.65	6½	145.7	315	1.52	120

[a] Cylinders, 3 in. by 6 in. [b] Maximum size of aggregate, ¾ in. Definitions: P = Platte River aggregate, Denver, Colo.; and L = crushed limestone. [c] Definitions: w/c = water-cement ratio; C = cement; S = sand; and G = gravel. [d] Cycles of freezing and thawing to 25% loss of weight. Specimens were cured 14 days in a fog room at 70° F. Then they were stored at 50% relative humidity at 70° F. The freezing and thawing soundness test was conducted after 180 days; and the capillarity experiment was run at an age of 2 years for concretes Y-1, Y-3, and Y-4 and at an age of 3 years for the remaining two concretes.

Calculations, based on the relation of surface tension of water to size of capillary and height of rise, indicate that the pressure deficiency of the three rocks (P-3208, P-3586A, and P-3508, Table 4), which sustained a pressure deficiency of 22.7 in. of mercury, is equivalent to that which could be created by a system of pores 0.0037 mm in diameter. The pressure deficiencies sustained by the sandstones (P-3671, P-3510, and P-3511A) are equivalent to pores 0.0044 mm, 0.0050 mm, and 0.0846 mm in diameter, respectively.

Concrete specimens were tested similarly in the apparatus (Fig. 10 and Table 5), and three (Y-1, Y-3, and Y-4) were found to be unsound, equivalent specimens having lost 25% by weight at forty cycles, fifty cycles, and forty cycles, respectively. The remaining two concretes (RA6-22 and RA6-23, Table 5) are satisfactory in quality, since equivalent specimens lost 25% by weight only after one hundred and twenty cycles of freezing and thawing. The rates of rise and the magnitudes of pressure deficiency are comparable to those measured during testing of the rock specimens. The pore characteristics of the rocks are therefore comparable to those of the concrete. Difficulty was experienced in testing concrete Y-1, Table 5, for in three attempts gasket leaks occurred, and reliable measurements could not be obtained for one specimen of the two prepared. The slow initial rise in pressure deficiency may be caused by the precipitation of soluble salts within the pores of the specimen during the wetting and the drying entailed by repeated testing.

As for the unsound rocks, poor durability of the concrete is indicated by rapid rise to high pressure deficiency (20.7 in. of mercury or greater). Sound concretes are indicated by much slower increases in pressure deficiency but the values ultimately attained are high (20.6 in. of mercury or greater). Some variation in results is related to position in the original concrete from which the tested specimen was cut, as is implied by the separation of curves representing specimens from different ends of the same 3-in. by 6-in. cylinder (Fig. 10). The rate and the magnitude of rise are indifferent to the kind of aggregate and cement, apparently being related only to durability.

The ultimate values of pressure deficiency (20.6 in. to 21.7 in. of mercury) measured on the concrete correspond to pores 0.0045 mm to 0.0040 mm in diameter. Thus, the controlling pores in the cement paste are larger than the controlling pores in the four unsound rocks tested. Capillary movement of water from the cement paste into the aggregate particles of weathered or argillaceous limestones is therefore possible. Consequently, these particles can attain and retain a high degree of saturation while enclosed by cement paste. The pertinence of this conclusion is indicated in the section of this paper concerning size, abundance, and continuity of pores in aggregate.

The apparatus used in these experiments can be modified so as to measure the abundance of voids of different sizes. However, techniques required for these measurements are time consuming, and will be unnecessary if experience continues to indicate that durability of rock and concrete can be predicted adequately from a function of effective porosity and pore size.

CONCLUSIONS

Many physical and chemical properties of aggregates affect quality of concrete. Most important are: (*a*) The size, continuity, and abundance of internal voids; (*b*) the bonding characteristics; and (*c*) the alkali reactivity. Other properties (especially volume change with wetting and drying) occasionally are critically important. The standard tests generally applied to determine acceptability of aggregates are inadequate. They yield data only statistically, not specifically related to the fundamental quality of aggregates. Deficiencies of these tests are mitigated if they are supplemented by an examination by petrographers experienced in problems of concrete. However, a reorientation of testing techniques is needed urgently.

Research has developed a chemical test which quickly and reliably identifies the aggregates causing distress of concrete containing high-alkali cement. A promising technique for quickly evaluating freezing and thawing durability of aggregates and concrete and for exploring quantitatively the character of internal voids has been devised. Research on the properties that determine bond between aggregate and cement should be promoted with vigor. More must be learned about the occasionally significant properties, such as thermal expansivity and the ability to shrink and expand with drying and wetting. Efficient and practicable tests of the fundamental properties of aggregates should be developed; yet, inevitably, economy will not permit application of complex tests to all aggregates proposed for use. Thus, the petrographic examination may serve to establish the testing program necessary for each aggregate, with a view to measuring properties that cannot be estimated with sufficient accuracy.

Improvements in the testing of aggregates should proceed hand in hand with research to improve cement and techniques of mixing, placing, and curing. Only thus can the full potentialities of concrete be realized.

ACKNOWLEDGMENTS

The research on alkali reactivity and on the relation of capillarity to freezing and thawing durability, described in this paper, was conducted by the Petrographic Laboratory, Geology Section, Research and Geology Division of the Bureau of Reclamation in Denver. Research activities of the laboratory were performed by and under the direction of Duncan McConnell, former head of the laboratory; Richard C. Mielenz, present head of the laboratory; William Y. Holland, petrographer; Kenneth T. Greene, chemist-petrographer; and N. Cyril Schieltz, X-ray crystallographer. The data presented in this paper were compiled by Messrs. McConnell and Mielenz.

APPENDIX. BIBLIOGRAPHY ON CONCRETE AGGREGATES

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(1) **"The Weathering of Natural Building Stones,"** by R. J. Schaffer, *Special Report No. 18*, Dept. of Scientific and Industrial Research, Building Research, London, 1932, p. 1.

(2) **"A Working Hypothesis for Further Studies of Frost Resistance of Concrete,"** by T. C. Powers, *Proceedings*, A.C.I., Vol. 41, 1945, p. 245.

(3) **"Concrete Manual,"** Bureau of Reclamation, U. S. Dept. of the Interior, Denver, Colo., 1942, p. 1.

(4) **"A Concrete Failure Attributed to Aggregate of Low Thermal Coefficient,"** by J. C. Pearson, *Proceedings*, A.C.I., Vol. 38, 1941, p. 29.

(5) **"Effect of Various Coarse Aggregates Upon the Cement-Aggregate Reaction,"** by C. H. Scholer and W. E. Gibson, *ibid.*, Vol. 44, 1948, p. 1009.

(6) **"Handbook of Physical Constants,"** *Special Paper No. 36*, Geological Soc. of America, 1942, p. 31.

(7) **"Factors Affecting the Thermal Expansion of Concrete Aggregate Materials,"** by W. H. Parsons and W. H. Johnson, *Proceedings*, A.C.I., Vol. 40, 1944, p. 457.

(8) **"Petrography of Concrete Aggregate,"** by Roger Rhoades and R. C. Mielenz, *ibid.*, Vol. 42, 1946, p. 581.

(9) **"Drying Shrinkage of Concrete As Affected by Many Factors,"** by R. W. Carlson, *Proceedings*, A.S.T.M., Vol. 38, 1938, p. 430.

(10) **"Hydration-Pressure Relationships in Clays and Heaving Shales,"** by H. H. Powers, B. L. Towle, and J. B. Plaza, *Oil and Gas Journal*, Vol. 41, 1942, p. 215.

(11) **"Experiments on the Freezing of Certain Building Materials,"** by W. N. Thomas, *Technical Paper No. 17*, Dept. of Scientific and Industrial Research, Building Research, London, 1938, p. 1.

(12) **"Thermal Properties of Concrete,"** *Bulletin No. 1*, Boulder Canyon Project Final Reports, Pt. VII, Cement and Concrete Investigations, Bureau of Reclamation, U. S. Dept. of the Interior, Denver, Colo., 1940, p. 1.

(13) **"Influence of Cement and Aggregate on Concrete Expansion,"** by T. E. Stanton, *Engineering News-Record*, February 1, 1940, p. 40.

(14) **"Concrete Deterioration at Parker Dam,"** by R. F. Blanks, *ibid.*, March 27, 1941, p. 46.

(15) **"Notes on the 'Effect of Alkalies in Portland Cement on the Durability of Concrete,' "** *Proceedings*, A.S.T.M., Vol. 43, 1943, p. 199.

(16) **"An Interesting Case of Dangerous Aggregate,"** by J. C. Pearson and G. F. Loughlin, *Proceedings*, A.C.I., Vol. 19, 1923, p. 142.

(17) **"Studies of the Accelerated Soundness Tests,"** by L. V. Garrity and H. F. Kriege, *Proceedings*, Highway Research Board, National Research Council, Vol. 15, 1935, p. 237.

(18) **"The Relation Between Los Angeles Abrasion Test Results and the Service Records of Coarse Aggregates,"** by D. O. Woolf, *ibid.*, Vol. 17, 1937, p. 350.

(19) **"Looking Inside the Abrasion Machine,"** by W. S. Cook, L. C. Gilbert, and H. F. Kriege, *Pit and Quarry*, May, 1941, p. 43.

(20) **"Petrographic Examination of Concrete Aggregates,"** by Richard C. Mielenz, *Bulletin*, Geological Soc. of America, Vol. 57, 1946, p. 309.

(21) **"Cracking in Concrete Due to Expansive Reaction Between Aggregate and High-Alkali Cement As Evidenced in Parker Dam,"** by H. S. Meissner, *Proceedings*, A.C.I., Vol. 37, 1941, p. 549.

(22) **"Cement-Aggregate Reaction in Concrete,"** by Duncan McConnell, Richard C. Mielenz, William Y. Holland, and Kenneth T. Greene, *Proceedings,* A.C.I., Vol. 44, 1948, pp. 93–128. (*a*) p. 95, Fig. 2. (*b*) p. 110, Table 3. (*c*) p. 124, Fig. 20.

(23) **"Chemical Test for Reactivity of Aggregates with Cement Alkalies; Chemical Processes in Cement-Aggregate Reaction,"** by Richard C. Mielenz, Kenneth T. Greene, and Elton J. Benton, *ibid.,* pp. 193–221.

(24) **"California Experience with the Expansion of Concrete Through Reactions Between Cement and Aggregate,"** by T. E. Stanton, O. J. Porter, S. Meder, and A. Nicol, *ibid.,* Vol. 38, 1942, p. 209.

(25) **"Investigation of Causes of Delayed Expansion of Concrete in Buck Hydroelectric Plant,"** by H. A. Kammer and R. W. Carlson, *ibid.,* Vol. 37, 1941, p. 665.

(26) **"Notes on Effect of Alkalies in Portland Cement on the Durability of Concrete,"** by F. B. Hornibrook, H. Insley, and L. Schuman, *Proceedings,* A.S.T.M., Vol. 43, 1943, p. 218.

(27) **"Studies to Develop an Accelerated Test Procedure for the Detection of Adversely Reactive Cement-Aggregate Combinations,"** by T. E. Stanton, *ibid.,* p. 875.

(28) **"Alkali Etching Tests on Concrete Aggregates,"** by Willard H. Parsons and Herbert Insley, *Proceedings,* A.C.I., Vol. 40, 1944, p. 229.

(29) **"Reactivity of Aggregate Constituents in Alkaline Solutions,"** L. Bean and J. J. Tregoning, *ibid.,* Vol. 41, 1944, p. 37.

(30) **"Studies on the Alkalinity of Some Silicate Minerals,"** by R. E. Stevens, *Professional Paper 185-A,* Geological Survey, U. S. Dept. of the Interior, 1934, p. 1.

(31) **"Frost Heaving,"** by Stephen Taber, *Journal of Geology,* Vol. 37, 1929, p. 428.

(32) **"Handbook of Chemistry and Physics,"** edited by C. D. Hodgman, Chemical Rubber Pub. Co., Cleveland, Ohio, 29th Ed., 1945, p. 1.

(33) **"Some Observations Regarding Frost Action,"** by A. F. Gill and J. F. J. Thomas, *Proceedings,* Highway Research Board, National Research Council, Vol. 19, 1939, p. 281.

(34) **"The Destruction of Concrete by Frost,"** by A. R. Collins, *Journal,* Inst. C. E., Vol. 23, 1944, p. 29.

(35) **"The Mechanism of Frost Heaving,"** by Stephen Taber, *Journal of Geology,* Vol. 38, 1930, p. 303.

(36) **"A Working Hypothesis for Further Studies of Frost Resistance of Concrete,"** by T. C. Powers, *Proceedings,* A.C.I., Vol. 41, 1945, pp. 245–272. (*a*) p. 272–9.

(37) **"Report of the Building Research Board for the Year 1931,"** H. M. Stationery Office, London, 1932, p. 21.

(38) **"Deterioration of Concrete Dams Due to Alkali-Aggregate Reaction,"** by R. F. Blanks and H. S. Meissner, *Transactions,* ASCE, Vol. 111, 1946, p. 743.

(39) **"Automatic Accelerated Freezing-and-Thawing Apparatus for Concrete,"** by Charles E. Wuerpel and Herbert K. Cook, *Proceedings,* A.S.T.M., Vol. 45, 1945, p. 813. (*a*) p. 823.

(40) **"Automatic Accelerated Freezing-and-Thawing Apparatus for Testing the Durability of Concrete and Concrete Materials,"** by Herbert K. Cook, *Bulletin No. 30,* Waterways Experiment Station, Vicksburg, Miss., 1947, p. 19.

(41) **"Report on Significance of Tests of Concrete and Concrete Aggregates,"** A.S.T.M., 1943, p. 92.

(42) **"Physical and Chemical Tests of Mineral Aggregates and Their Significance,"** by Harold S. Sweet, *Special Technical Publication No. 83,* A.S.T.M., 1948, p. 49. (*a*) p. 69.

(43) **"The U. S. Corps of Engineers' Approach to More Durable Concrete,"** by Jesse R. Gilbert, *The Crushed Stone Journal,* June, 1949, p. 3. (*a*) p. 8.

(44) **"The Testing of Aggregates in Air-Entrained Concrete,"** by Bryant Mather, *Bulletin No. 30,* Waterways Experiment Station, Vicksburg, Miss., 1947, p. 41.

(45) **"A Study of the Influence of Thermal Properties on the Durability of Concrete,"** by Albert Weiner, *Proceedings,* A.C.I., Vol. 43, 1947, p. 997.

(46) **"Petrographic Identification of Reactive Constituents in Concrete Aggregate,"** by Bryant Mather, *Proceedings,* A.S.T.M., Vol. 48, 1948, p. 1120. (*a*) p. 1122.

(47) **"A Wetting-and-Drying Test for Predicting Cement-Aggregate Reaction,"** by C. H. Scholer, *ibid.,* Vol. 49, 1949 (1949 Preprint No. 46, June, 1949).

DISCUSSION

Herbert K. Cook,[3] **Thomas B. Kennedy,**[4] **and Bryant Mather.**[5]—The paper by Mr. Blanks, describing the extremely interesting studies of concrete aggregate which have been made in recent years by the Bureau of Reclamation, United States Department of the Interior, is a valuable contribution to concrete research and will be studied by workers in this field for many years. The emphasis that he has placed on the importance of studying aggregate materials as components of concrete rather than as isolated unconfined rock and mineral particles is perhaps the most important concept developed from recent research on aggregates. The importance of this approach has been recognized by the Concrete Research Division of the Waterways Experiment Station (Clinton, Miss.), and its work likewise has been directed toward developing suitable procedures for evaluating materials in accordance therewith. The automatic accelerated freezing-and-thawing apparatus for concrete which was described in 1945 (39)(40) was recommended as a method of testing aggregate, among other reasons, "because it permits materials for concrete to be tested in concrete" (39*a*). It is suggested, however, that in emphasizing this point and stating that "* * * experience shows that good rocks do not always produce superior concrete, and weak rocks do not always cause deterioration," Mr. Blanks may have implied, more strongly than the data permit, that the reverse is frequently true. The reviews of the significance of the standard tests of aggregates (41)(42) suggest that, in the absence of information such as revealed by the more recently developed procedures discussed by Mr. Blanks, the standard tests will generally provide a satisfactory method of separating good aggregates from bad aggregates. Also, as stated by Harold S. Sweet, Jun. ASCE (42*a*): "* * * it is pertinent that these classical techniques be recognized in that they may serve as guides for further aggregate research."

In his discussion of the effects of thermal expansivity of aggregates on concrete, Mr. Blanks has suggested that great differences in the thermal coefficient between aggregate particles and cement paste may cause failure in concrete. Studies conducted at the Concrete Research Division have suggested that such differences, when considered as occurring between coarse aggregate particles and the mortar matrix of the concrete, may provide a more useful index than is provided by considerations of aggregate-cement paste relationships. A large number of studies, involving determinations of the linear coefficient of thermal expansion of rocks used as coarse aggregates and of cement mortars containing different natural and manufactured sands as fine aggregate, and accelerated freezing and thawing tests of concretes composed of such coarse aggregate-mortar combinations, have provided the basis for statistical analysis which has yielded a strong inverse correlation between thermal coefficient differences and resistance of concrete to accelerated freezing and thawing. The results of this study have not been published in detail but are mentioned briefly in the papers by J. R. Gilbert (43) and Bryant Mather (44). Mr. Gilbert has stated (43*a*).

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> "It must be concluded, therefore, that a large difference between these coefficients [of expansion of the coarse aggregates and mortars] should be avoided since the stresses set up at the interfaces of coarse aggregate particles and mortar can be sufficient to cause loss of bond and cracking from the interface which will eventually destroy watertightness in hydraulic structures."

Mr. Blanks has stated that specific heat and heat conductivity of aggregates strongly affect the rate of temperature change in concrete and the rate at which heat is transmitted through concrete. He then states that these factors are of limited significance for small structural members. In an investigation conducted at the Concrete Research Division (45) it was found that differences in thermal diffusivity between coarse aggregate and mortar appeared to be responsible for differences in resistance of concretes to accelerated freezing and thawing.

Mr. Blanks' discussion of alkali-aggregate reaction is extremely interesting and valuable particularly in view of the fact that his laboratory has, perhaps, done more work in this field than any other. His statement (under the heading, "Research on Concrete Aggregates: Research on Alkali-Aggregate Reaction"):

> "Acid to intermediate volcanic rocks are present in some of these areas and, where significant quantities of these rocks have been used as aggregate in combination with high-alkali cements, deterioration caused by alkali-aggregate reaction is certain to occur."

—appears, however, to be somewhat more positive and inclusive than the data warrant. It is believed that more than an editorial modification would be involved if this statement were changed so that it would read

> "Acid to intermediate volcanic rocks, containing significant quantities of siliceous volcanic glass, are present in some of these areas and, where significant quantities of these rocks have been used as aggregate in combination with high-alkali cements, deterioration caused by alkali-aggregate reaction may be expected to occur."

The sentence should also be qualified by a further statement, such as the following which is taken from a previous paper (46)(46*a*):

> "The possibility that the expected reaction will not occur or that it may be forestalled—as a result of proper attention to the chemical composition of the cement, the use of an inhibitor, the fact that the deleterious constituent is present in an amount much greater than or much less than the 'pessimum' amount, or the favorable location and exposure of the structure, should always be borne in mind * * * However, in the present state of our knowledge these are mere possibilities—none can be counted on to give a safe guarantee of freedom from trouble if reactive materials are in the aggregate."

What is sorely needed by the materials-testing profession is a series of accelerated performance tests to evaluate the behavior of proposed aggregates in concrete. One such test is the accelerated freezing-and-thawing test previously referred to in this discussion. Another is a nonfreezing weathering test, by which the influences of temperature and moisture-content changes could be studied. A number of combinations of influences such as alternate wetting and drying, alternate heating and cooling, and more complex conditions (such as heating-wetting with cooling-drying and heating-drying with cooling-wetting) appear to play an important part in the deterioration of concrete. Aggregates that will produce concrete of satisfactory durability when exposed to a nonfreezing natural environment, such as is found in the southern United States, may not produce satisfactory concrete if exposed to freezing and thawing.

C. H. Scholer, M. ASCE (47), has been pioneering in the development of nonfreezing testing techniques for concrete. The Concrete Research Division and the division concrete laboratories of the Corps of Engineers have developed equipment which shows great promise in testing the aggregates, or cement, in concrete, under the influences of heating-wetting with cooling-drying and of heating-drying with cooling-wetting.

The device described by Mr. Blanks for measuring the rate of evaporation from saturated specimens and the force associated with capillary movement of water into the material has promise of becoming a valuable tool in helping to find the reasons for the lack of durability exhibited by some materials. In its present state of development the device appears to have the limitation of being used with only a relatively small specimen (1.5 in. in diameter and 1 in. thick). Although the results obtained from such tests may be valid when the specimen is representative of a material of constant pore characteristics, the interpretation of the data obtained by testing specimens from a heterogeneous gravel deposit or quarry, or from concrete specimens containing such materials, rapidly becomes more complex; and the usefulness of the data in terms of establishing an index of durability for such materials becomes a matter of the adequacy of the sampling and the number of tests made. The question is also raised as to the validity of application of the data obtained on a specimen of the size used in the test to mass concrete containing aggregate 6 in. in size.

The writers agree with the statements made by Mr. Blanks in the "Conclusions" that, at the present state of knowledge of the evaluation of concrete aggregates, the petrographic examination may serve to establish the testing program necessary for each aggregate. The petrographic examination, for example, would be valuable in determining the number of pressure-deficiency tests which might be required in the investigations of the heterogeneous materials referred to in the preceding paragraph.

The general acceptance of air entrainment in concrete may eventually lead to a general overhauling of current concepts of aggregate suitability and method of aggregate testing; also, the increasing use of manufactured sand with crushed stone as aggregate for concrete may necessitate modifications of present-day concepts of what are suitable gradations and test limitations.

R. F. Blanks,[6] M. ASCE.—It is gratifying to note the care with which Messrs. Cook, Kennedy, and Mather have studied the paper, and the cogency of their discussion. As stated in the paper, tests of concrete containing the proposed aggregate, including the analysis of service records, are the most diagnostic methods for determining aggregate quality, as long as the tests simulate, and conditions of previous service are comparable to, the conditions anticipated for the new work. However, there is a "rub": The kind and intensity of physical and chemical phenomena of importance to concrete can be incorporated readily into laboratory procedures, but accelerated tests cannot simulate closely the natural conditions of exposure in frequency or duration of the phenomena. The value of standardized laboratory tests of concrete and mortar has been demonstrated for the investigation of resistance to freezing and thawing, wetting and drying with heating and cooling, and alkali-aggregate reaction. However, it must be remembered that facilities for the performance of these tests are not available in most testing laboratories. Moreover, even accelerated tests may require several weeks or months to yield definitive results. There is real need for the development of tests that can be performed in a few hours or days with a minimum of equipment and personnel. Clearly, development and application of such tests must be predicated upon a thorough, quantitative understanding of the interrelationship of the properties of aggregates and the serviceability of concrete. This understanding can come only through research.

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In commenting on the writer's statement (under the heading, "Introduction")—"* * * experience shows that good rocks do not always produce superior concrete, and weak rocks do not always cause deterioration"—the discussers state that the writer may have implied rather strongly that the reverse is frequently true. The statement does not imply or suggest that weak rocks usually produce superior concrete nor that good rocks usually cause deterioration. The truth of the statement as published seems adequately demonstrated by the voluminous work on alkali-aggregate reaction, the deleteriousness of many aggregates in Kansas and Nebraska which characteristically pass the "classical" tests, and the preoccupation of the discussers with the influence of thermal properties of aggregate on the durability of concrete.

The discussion by Messrs. Cook, Kennedy, and Mather of the relative thermal expansivity of coarse aggregate and mortar and its influence on the resistance of concrete to accelerated freezing and thawing is interesting. It is unfortunate that the investigations to which they refer, mentioned briefly by Jesse R. Gilbert (43) and Mr. Mather (44), are not generally available for study. As has been emphasized by many writers, analysis of stresses (including a study of the resultant distress in concrete related to differential thermal volume change of the ingredients) is exceedingly difficult. However, it would appear that the role of thermal expansivity in the durability of concrete and mortar must include the relationship, first, of the fine and coarse aggregate to the cement paste and, secondly, of the aggregate to the mortar or concrete as a whole. In the first category must be analyzed the difference in expansivity of particles in the aggregate from that in the cement paste, the effects of restraint of volume change of the cement paste, the effects of plastic deformation of the paste, and other related factors. The second category includes the relationship of the expansivity of coarse sand to that of mortar and of the expansivity of coarse sand and coarse aggregate to that of concrete.

Inasmuch as there is no real physical separation in a well-graded aggregate between a coarse aggregate and a fine aggregate, the method of comparing the properties of a coarse aggregate with those of the mortar appears to be without fundamental basis. A more real comparison would be between the properties of a given particle of coarse aggregate and the properties of the concrete as a whole. The stresses associated with thermal relationships in the second category can extend for distances through the concrete greater than the radius of the particle producing tension in the mass.

In this connection, the reader is referred to the treatment of thermal expansivity in Table 2 of the paper and in the text, under the heading, "Effect of Rock Properties on Concrete: Thermal Expansivity."

The discussers have concluded that the differences between diffusivity of coarse aggregate and mortar appear to account for the differences in the resistance of concretes to accelerated freezing and thawing. If the observation is correct, the fact would seem more to militate against the validity of the accelerated test than to demonstrate the significance of diffusivity in the resistance of concrete to natural freezing and thawing. Differences in diffusivity of substances indicate differences in rate of temperature change that would occur under similar conditions of exposure, but the magnitude of the difference in temperature established between two phases of a mass, such as between a particle of coarse aggregate and the enclosing concrete, is also dependent on the rate and magnitude of temperature change in the mass as a whole. In an accelerated freezing and thawing test, the rate of cooling of the interior of a 3-in. by 6-in. cylinder of concrete is about 80° F per hr for the first 30 min following initiation of the freezing half cycle; and the rate of cooling of the exterior will be much greater. Under natural conditions of exposure to insolation, the rate of temperature rise of near-surface concrete will be less than one fourth of this value, and cooling of the near-surface concrete rarely will exceed one half of this value. Moreover, large particles of coarse aggregate always will be separated from the surface of the mass by mortar and concrete, so that the coarse aggregate particle will necessarily exchange heat with only the concrete. Consequently, the temperature of the aggregate is not independent of that of the concrete, as would be more nearly true if the aggregate received heat directly from the air or solar radiation.

In addition, the concrete of "small structural members" (to which reference is made in the paper under the heading, "Effect of Rock Properties on Concrete: Other Properties") will contain aggregate less than 1.5 in. in size, so that the distance from the center of the coarse aggregate to its periphery—the distance in which the thermal gradient will be established—will not exceed $\frac{3}{4}$ in. Furthermore, the only serious departure in diffusivity of common aggregate-forming rocks and minerals from that of mortar and concrete occurs when particles of quartz constitute the entirety of an aggregate particle. The diffusivity of quartz is slightly more than twice that of concrete containing quartzite, limestone, or dolomite as coarse aggregate, approximately three times that of concrete containing granite as coarse aggregate, and about four times that of concrete containing rhyolite or basalt as coarse aggregate.

As an example to determine the maximum possible effect of differences in diffusivity of aggregate and concrete on the resistance of concrete to freezing and thawing, consider a concrete in which a piece of quartz, 1.5 in. in diameter, is exposed at the surface, the temperature at the surface being instantaneously decreased from 120° F to 70° F. After 30 min, the temperature of the quartz at a depth of 0.75 in. will be approximately 77° F, whereas that of concrete containing basalt as the remainder of the coarse aggregate will be 81° F at the same depth. Taking these temperatures as the average for the entire depth of 1.5 in. and assuming coefficients of expansion of 5.7×10^{-6} and 4.6×10^{-6} per degree Fahrenheit for the quartz and concrete, respectively, calculation will show a difference of 66×10^{-6} in. per in. in contraction of the two materials, of which only 23×10^{-6} in. per in. relates specifically to the difference in diffusivity. This total strain is equivalent to a compressive stress of 330 lb per sq in. in the concrete adjacent to the quartz, of which only 115 lb per sq in. results from the difference in diffusivity alone. Comparable tension could be computed with equivalent rise in temperature, but a comparable increase in temperature is impossible under natural conditions. Similarly, tension in the concrete could be determined by assuming that the concrete contained a more siliceous coarse aggregate, and, consequently, possessed a coefficient of thermal expansion greater than that of quartz. However, the siliceous aggregate would cause an increase also in diffusivity of the concrete, with a consequent decrease in the difference in temperature developed between the quartz and the concrete. A similar calculation could be made for mass concrete, assuming a 6-in. piece of quartz at the surface. In this situation, compressive stress in the concrete adjacent to the quartz would approximate 500 lb per sq in.

It will be seen that these stresses are much less than those which could be calculated with similar temperature change where the concrete possesses a thermal coefficient of expansion of 4.6×10^{-6} or greater, and the aggregate possesses a thermal coefficient of 1.0×10^{-6} or less (such as might be exhibited by some limestones, granites, or marbles).

Messrs. Cook, Kennedy, and Mather have discussed at length, and suggested qualification of, a statement in the paper regarding the occurrence of alkali-aggregate reaction where acid to intermediate vocanic rocks are used with high-alkali cements. There is no doubt concerning the prevention or inhibition of alkali-aggregate reaction by several conditions, external or internal to the concrete, and these conditions have been discussed at length elsewhere (22)(24)(27). Their objections will be mitigated if the word "almost" is inserted in the sentence under discussion between the words "is" and "certain."

The discussers have pointed out several factors, in the testing of aggregate and concrete, which, in its present state of development, delimit the use of the device for measuring capillarity of solid materials. No attempt has been made to introduce the method as a part of standardized testing procedure. On the contrary, however, the device does yield valuable information concerning the size, continuity, and distribution of voids in materials, and has contributed significantly to an understanding of the factors controlling resistance to freezing and thawing.

The work of the Concrete Research Division of the Waterways Experiment Station at Clinton, Miss., has been watched with interest and benefit, and publication of details of their investigations is awaited. Only through penetrating and coordinated research on all aspects of concrete manufacture can its deficiencies be remedied.

AMERICAN SOCIETY OF CIVIL ENGINEERS
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TRANSACTIONS

Paper No. 2404

AIR-INLET VALVES FOR STEEL PIPE LINES

By John Parmakian[1]

With Discussion by Mr. W. E. Evans.

Synopsis

An analytical basis is presented for determining the size and location of air-inlet valves in steel pipe lines in order to prevent their collapse due to external pressure. The allowable internal pressure for steel pipe shells is first computed from formulas published elsewhere for the collapse of cylindrical shells under external pressure. The conditions under which the hydraulic gradient may fall below atmospheric pressure at some points in the pipe line are then investigated and the size and location of air-inlet valves required to prevent the collapse of the pipe shell are determined on the basis of adiabatic expansion of air. Various charts are presented to facilitate the required computations.

Under certain conditions of operation water may be removed from some sections of a steel pipe line faster than it can be supplied from the reservoir. Such conditions may occur as a result of gate operation, the failure of the pipe line at a low point, or other causes. If the pipe shell is thin, air must be admitted into the pipe to prevent an excessive reduction of the internal pressure and the subsequent collapse of the pipe throughout a considerable length because of the external atmospheric pressure. The strength of the pipe shell under this differential pressure will depend on the length-to-diameter ratio, the thickness-to-diameter ratio, and the modulus of elasticity of the steel. The presence of stiffener rings may exert a considerable influence on the strength of the pipe shell. However, if the distance between stiffener rings is great, they will have no appreciable effect on the section of the pipe shell midway between them and the collapsing pressure will be the same as that of a pipe shell of infinite length. The collapsing pressure of steel pipe shells with and without stiffener rings has been studied and experimentally determined by various investigators and the resulting formulas are given elsewhere.[2]

To obtain the collapsing pressure of steel pipes more conveniently a chart has been prepared (Fig. 1) based on the following instability formulas:[2]

Note.—Published in June, 1949, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Engr., Branch of Design and Constr., Bureau of Reclamation, Denver, Colo.

[2] "Collapse by Instability of Thin Cylindrical Shells Under External Pressure," by Dwight F. Windenburg and Charles Trilling, *Transactions*, A.S.M.E., 1934, p. 819.

For cylindrical shells without stiffener rings,

$$p' = 50.2 \times 10^6 \left(\frac{t}{D}\right)^3 \dots\dots\dots\dots\dots\dots \text{(1)}$$

and, for cylindrical shells with stiffener rings,

$$p' = 73.4 \times 10^6 \frac{(t/D)^{5/2}}{L/D} \dots\dots\dots\dots\dots\dots \text{(2)}$$

In Eqs. 1 and 2, p' is the collapsing pressure for the pipe shell, in pounds per square inch; t is the thickness of the pipe shell, in inches; D is the inside diameter of the pipe, in inches; and L is the spacing of the stiffener rings, in inches.

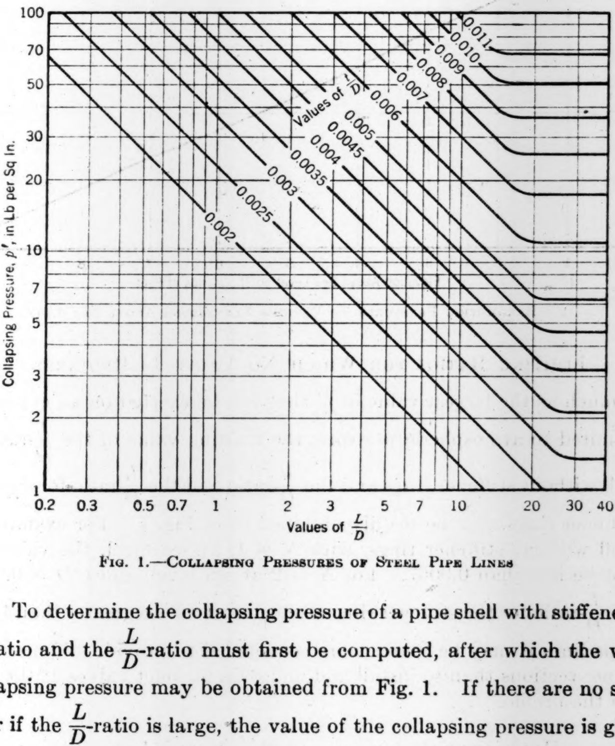

FIG. 1.—COLLAPSING PRESSURES OF STEEL PIPE LINES

To determine the collapsing pressure of a pipe shell with stiffener rings, the $\frac{t}{D}$-ratio and the $\frac{L}{D}$-ratio must first be computed, after which the value of the collapsing pressure may be obtained from Fig. 1. If there are no stiffener rings, or if the $\frac{L}{D}$-ratio is large, the value of the collapsing pressure is given by the horizontal lines on the right-hand side of the chart.

Application of a suitable factor of safety, N, to p' will give p_2, the allowable pressure on the inside of the pipe. Thus,

$$\frac{p'}{N} = p_1 - p_2 \dots\dots\dots\dots\dots\dots \text{(3a)}$$

or

$$p_2 = p_1 - \frac{p'}{N} \qquad (3b)$$

in which p_1, the atmospheric pressure on the outside of the pipe, may be obtained from Fig. 2. Selection of a value of N depends on the judgment of the designer, who must also consider the probable frequency of the assumed operation which causes a reduction of the internal pressure.

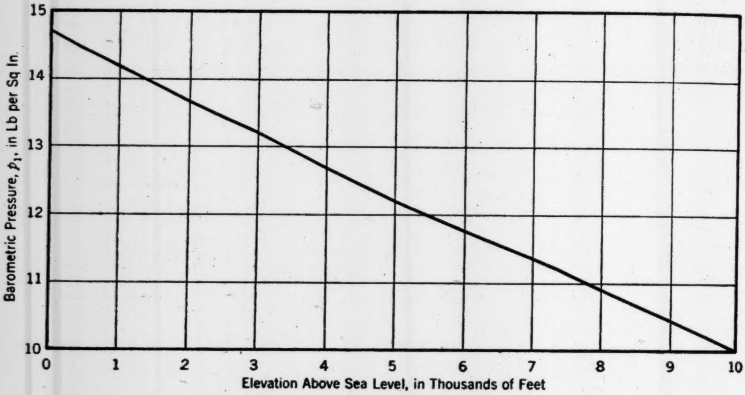

FIG. 2.—BAROMETRIC PRESSURES AT VARIOUS ELEVATIONS ABOVE SEA LEVEL

LIMITING RATIOS FOR WHICH NO VALVE IS REQUIRED

Inasmuch as the largest value of p' that can be exerted on an exposed pipe line is limited to atmospheric pressure, the limiting value of the $\frac{t}{D}$-ratio for a pipe shell without stiffener rings and the $\frac{t}{D}$-ratio and the $\frac{L}{D}$-ratio for a pipe shell with stiffener rings may be readily obtained from Fig. 1. For example, for a pipe shell without stiffener rings, with $N = 1$, at sea level, the value of t/D must not be less than 0.0067. For $N = 2$ at sea level, and $t/D = 0.003$, the required $\frac{L}{D}$-ratio for the stiffener ring spacing must not be greater than 1.3. On some pipe lines it may be found more economical to provide stiffener rings at the thinner sections than to install and maintain air-inlet valves at the various crests in the profile.

HYDRAULIC CONDITIONS AT THE VALVE LOCATION

Under normal conditions of operation the flow in the pipe line is essentially uniform and the hydraulic gradient is above atmospheric pressure throughout the entire length. If the flow is increased abnormally, as a result of a rupture in the line at a low point, or any other cause, the hydraulic gradient may drop considerably below atmospheric pressure at the crests in the pipe line profile.

External atmospheric pressure may then cause collapse of a considerable length of the pipe shell, although this collapse may be prevented by installation of an air-inlet valve of ample size to maintain the pressure inside the pipe at any desired allowable pressure, p_2. The hydraulic flow conditions that will exist inside the pipe at the air-inlet valve location with the valve open may then be determined from the following considerations: The minimum allowable internal pressure at the air-inlet valve is p_2; therefore, the pressure head at the air-inlet valve is $2.31(p_2 - p_1)$. The water velocities in the pipe line leaving and approaching the air-inlet valve location, designated as V_1 and V_2, respectively, may then be computed by means of the Bernoulli energy equation. If $V_1 > V_2$, or V_1 and V_2 are in opposite directions, an air-inlet valve is required to supply enough air to the pipe interior to occupy the space created by the breaking of the water column; that is,

$$Q = W v_2 = \frac{\pi D^2}{4 \times 144} \Delta V \qquad (4)$$

in which Q is the quantity of air, in cubic feet per second, that must be supplied to the pipe interior at the pressure p_2; W is the weight of air flowing through the air-inlet valve, in pounds per second; v_2 is the specific volume of air, in cubic feet per pound, at pressure p_2; and

$$\Delta V = V_1 - V_2 \qquad (5a)$$

It should be noted that, if V_2 is opposite in direction from V_1,

$$\Delta V = V_1 - (-V_2) = V_1 + V_2 \qquad (5b)$$

FLOW OF AIR THROUGH INLET VALVES

The air drawn through the inlet valve will undergo a volumetric change as it passes from atmospheric pressure, p_1, to the lower pressure, p_2, inside the pipe. Ordinarily, this takes place so rapidly that there is little interchange of heat between the water and the air. Therefore, the air flow through the valve may be considered as an adiabatic expansion; that is,

$$144\, p_1 v_1^{\gamma} = 144\, p_2 v_2^{\gamma} \qquad (6)$$

in which v_1 is the specific volume of air, in cubic feet per pound, at pressure p_1; and γ is the ratio, for air, of the specific heat at constant pressure to the specific heat at constant volume.

The weight of air per second flowing through an air-inlet valve is then given by the expressions:[3] For $p_2 > 0.53\, p_1$,

$$W = C \left(\frac{\pi d^2}{4 \times 144} \right) \frac{1}{v_2} \sqrt{2 g\, (144\, p_1)\, v_1 \left(\frac{\gamma}{\gamma - 1} \right) \left[1 - \left(\frac{p_2}{p_1} \right)^{(\gamma - 1)/\gamma} \right]} \qquad (7a)$$

and, for $p_2 \leqq 0.53\, p_1$,

$$W = C \left(\frac{\pi d^2}{4 \times 144} \right) \left(\frac{2}{\gamma + 1} \right)^{1/(\gamma - 1)} \sqrt{2 g \left(\frac{\gamma}{\gamma + 1} \right) \frac{144\, p_1}{v_1}} \qquad (7b)$$

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[3] "Mechanical Engineers Handbook," by Lionel S. Marks, McGraw-Hill Book Co., Inc., New York, N. Y., 1941.

in which C is the coefficient of discharge for the air-inlet valve; d is the inside diameter of the valve, in inches; and g is the gravitational acceleration. Substituting Eqs. 4 and 6 in Eqs. 7, the ratio of the air-inlet valve diameter to the pipe diameter is, for $p_2 > 0.53\ p_1$,

$$\frac{d}{D} = \sqrt{\frac{\Delta V}{C}} \left\{ 2\,g\,(144\,p_1)\,v_1 \left(\frac{\gamma}{\gamma - 1}\right)\left[1 - \left(\frac{p_2}{p_1}\right)^{(\gamma-1)/\gamma}\right]\right\}^{-\frac{1}{4}} \quad \ldots.(8a)$$

and, for $p_2 \leqq 0.53\ p_1$,

$$\frac{d}{D} = \sqrt{\frac{\Delta V}{C}\left(\frac{p_2}{p_1}\right)^{1/(2\gamma)}} \left\{ 2\,g\,(144\,p_1)\,v_1 \left(\frac{\gamma}{\gamma + 1}\right)\left(\frac{2}{\gamma + 1}\right)^{2/(\gamma-1)}\right\}^{-\frac{1}{4}} \quad \ldots(8b)$$

Eqs. 8 may be further simplified by using an average value for the term $[2\,g\,(144\,p_1)\,v_1]^{-\frac{1}{4}}$, as its numerical value is nearly constant regardless of altitude, air temperature, or humidity. The assumed average conditions will be

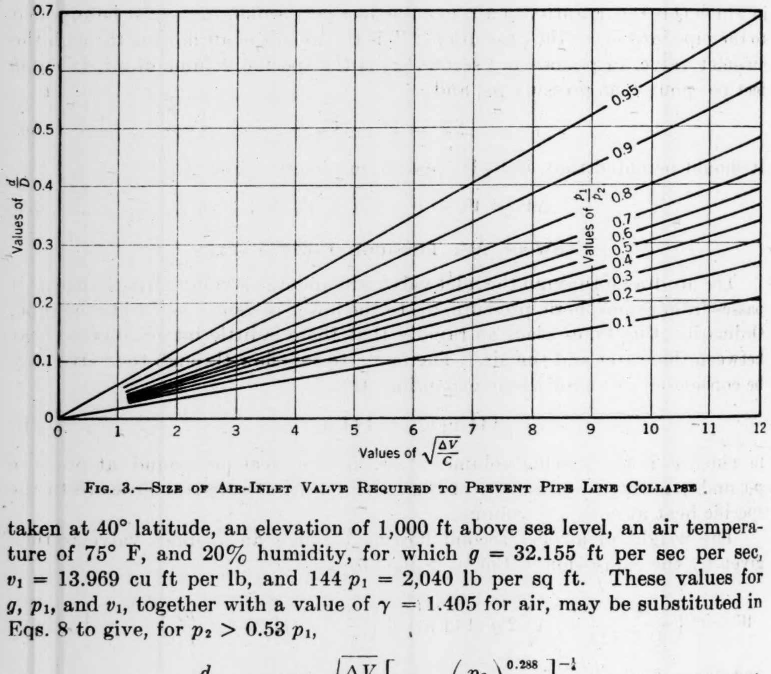

FIG. 3.—SIZE OF AIR-INLET VALVE REQUIRED TO PREVENT PIPE LINE COLLAPSE

taken at 40° latitude, an elevation of 1,000 ft above sea level, an air temperature of 75° F, and 20% humidity, for which $g = 32.155$ ft per sec per sec, $v_1 = 13.969$ cu ft per lb, and $144\ p_1 = 2{,}040$ lb per sq ft. These values for g, p_1, and v_1, together with a value of $\gamma = 1.405$ for air, may be substituted in Eqs. 8 to give, for $p_2 > 0.53\ p_1$,

$$\frac{d}{D} = 0.0199\sqrt{\frac{\Delta V}{C}}\left[1 - \left(\frac{p_2}{p_1}\right)^{0.288}\right]^{-\frac{1}{4}} \quad \ldots\ldots\ldots\ldots\ldots(9a)$$

and, for $p_2 \leqq 0.53\ p_1$,

$$\frac{d}{D} = 0.0391\sqrt{\frac{\Delta V}{C}\left(\frac{p_2}{p_1}\right)^{0.356}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(9b)$$

Determination of the ratio $\frac{d}{D}$ from Eqs. 9*a* and 9*b* is facilitated by a chart (Fig. 3), which is entered with the calculated values of p_2/p_1 and $\sqrt{\Delta V/C}$.

If the value of C, the coefficient of discharge of the air-inlet valve and connecting pipe, is known, its value should be used in Eqs. 9. A value ranging from 0.5 to 0.6 is believed to be reasonable for the ordinary design of air-inlet valves.

NUMERICAL EXAMPLE

To illustrate the procedure to be followed in determining the size of an air-inlet valve required in an actual pipe line installation, consider the steel siphon whose profile is shown in Fig. 4. In the vicinity of the crest at point B the pipe shell is $\frac{1}{4}$ in. thick and there are stiffener rings spaced 30 ft apart. It is desired to calculate the size of the air-inlet valve at point B if a complete rupture of the pipe is assumed to occur at point C. The required factor of safety for this assumed condition of operation is taken as 2, and the coefficient of discharge of the air-inlet valve is taken as 0.5.

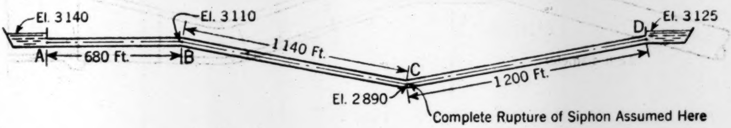

FIG. 4.—PROFILE OF SIPHON INSTALLATION

Value of p_2/p_1.—Given: $t = 0.25$ in., $D = 78$ in., $t/D = 0.0032$, $L/D = 4.62$, and $N = 2$. From Fig. 1, $p' = 9.2$ lb per sq in.; and, from Fig. 2, $p_1 = 13.1$ lb per sq in. By Eq. 3*b*, $p_2 = 8.5$ lb per sq in., and $p_2/p_1 = 0.65$.

Value of ΔV.—Using the Bernoulli equation for the pipe line from point A to point B, it is found that $V_2 = 27.0$ ft per sec.

Similarly, using the Bernoulli equation for the pipe line from point B to point C it is found that $V_1 = 54.7$ ft per sec. Thus, by Eqs. 5, $\Delta V = V_1 - V_2 = 27.7$ ft per sec.

Size of the Air-Inlet Valve.—Using the assumed value of C and the value of ΔV just found, $\sqrt{\Delta V/C} = 7.44$. From Fig. 3, $d/D = 0.26$. Therefore, the diameter of the air-inlet valve required at point C is $0.26 \times 78 = 20.3$ in.

ACKNOWLEDGMENTS

The writer wishes to acknowledge the assistance of W. E. Evans and C. C. Crawford of the Bureau of Reclamation in the preparation of this paper.

DISCUSSION

W. E. Evans.[4]—The size of air-inlet valve required to maintain a certain internal pressure in a pipe line during transient flow conditions can be determined easily by the method proposed in this paper. Undoubtedly, there are many conditions that may exist in a pipe line to which these data would apply, and it is one of these special cases that will be described herein.

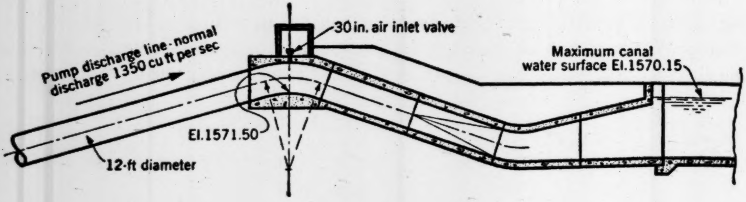
30 in. air inlet valve
Pump discharge line - normal discharge 1350 cu ft per sec
Maximum canal water surface El.1570.15
El.1571.50
12-ft diameter

Fig. 5

At the discharge outlet of the pumping plant pipe lines at Grand Coulee Dam in Washington there is a siphon as shown in Fig. 5. The problem of breaking the siphon action for return flow through the discharge line following power failure at the pumps arose. This problem was not associated with the collapse of the pipe line but rather with the loss of water from the canal by siphon action maintaining return flow through the pump.

The siphon head available was limited and depended upon the canal water surface elevation, relative to the siphon invert elevation. This relationship determined the internal pressure that had to be obtained and maintained in the discharge line to break the siphon action.

In addition to this internal pressure, the rate at which air must be supplied had to be determined. This aspect of the problem was handled by recourse to the pump characteristics of the model pumps for Grand Coulee.[5] From these characteristics the steady-state reverse discharge at normal head was obtained as approximately 0.5 times normal pump discharge. Transient values of reverse discharge were higher than this steady-state value but were considered as too short-lived to be a basis for selecting the size of air valve needed.

Then, knowing the internal pressure necessary to prevent siphon action and expressing the steady-state reverse discharge in terms of velocity, the curves shown in Fig. 3 were used to obtain the size of the air value required. It was concluded that an air valve 30 in. in diameter, or its equivalent, was needed. This air valve would supply air at a rate equal to the steady-state reverse discharge through the pumps with the internal pressure in the pipe line at the valve adequate to break the siphon action.

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[4] Engr., Branch of Design and Constr., Bureau of Reclamation, Denver, Colo.

[5] "Centrifugal-Pump Performance as Affected by Design Features," by R. T. Knapp, *Transactions*, A.S.M.E., Vol. 63, 1941, p. 251.

AMERICAN SOCIETY OF CIVIL ENGINEERS
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Paper No. 2405

PHOTOGRAMMETRY, POSTWAR

BY LEON T. ELIEL[1]

WITH DISCUSSION BY MR. COURTLANDT EATON.

SYNOPSIS

Photogrammetry developed at a normal, cautious pace in the decade before World War II and had only achieved a certain minor acceptance among engineers in general. The war gave such a spectacular demonstration of the many applications of aerial photography and photogrammetry that engineers everywhere have grown to recognize these new tools as something very practical and ready for immediate use in their problems. This paper concerns the status of the art of aerial photography and photogrammetry as influenced by wartime experience.

By 1941 the aerial cameras used during the war had already been developed to a large extent. The only thing that the war did was to take some of these cameras and apply them in a new way. Photogrammetric methods also had been producing contour maps successfully in the United States for a period of more than ten years. Although acceptance was far from universal, many engineers had used the photogrammetric contour map for topography in dam design, reservoir studies, highway location, geological problems, pipe line location, transmission line location, city planning, real estate developments, and a great number of other problems.

The Tennessee Valley Authority (TVA) was the first agency of the federal government to foster photogrammetric topography on an extensive scale, beginning in 1937. The Geological Survey (United States Department of the Interior) was using the photogrammetric method for producing a substantial part of its total topographic program by 1941. The Coast and Geodetic Survey (United States Department of Commerce), the Forest Service (United States Department of Agriculture), the Engineer Department (United States Army), the Soil Conservation Service (United States Department of Agriculture), and the Hydrographic Office (United States Navy) were all making extensive use of the new methods.

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[1] Vice-Pres., Fairchild Aerial Surveys, Inc., Los Angeles, Calif.

Considerable experimentation had been made in color prior to World War II, and color had developed to a point where very satisfactory work was being done from the air, although the control of tone and contrast has proved difficult.

The main developments that occurred in aerial photography and photogrammetry with the start of World War II were as follows:

a. The grouping together of three cameras in the so-called "Tri-Metrogon" system for taking photographs simultaneously from horizon to horizon. This method was largely developed by the U. S. Army for purposes of making aeronautical charts; and at one time it was the purpose of the Army to map the entire world, to which it had access, by this method.

b. The recognition by the Army of the necessity for better performance in photographic airplanes. The airplanes were not fast enough, they did not have ceiling enough, and they were defenseless in the face of antiaircraft and fighter attacks. Minton Kaye, Colonel, U. S. Army Air Force, fathered the development of the "Tri-Metrogon" system, and was responsible for the assignment of fast, high-performance, military aircraft for aerial photography. He created squadrons of P-38s, capable of flying fast enough to elude enemy fighters, and high enough to elude antiaircraft fire. Toward the end of the war cameras were being carried by V-17s, B-24s, and B-29s. Some of these photo ships carried a formidable array of variegated cameras.

At Wright Field in Dayton, Ohio, George W. Goddard, Colonel, U. S, Army, was making great progress in camera design. He had prepared 40-in. telephoto lenses and developed the 9-in. by 18-in. reconnaissance camera, which made great strides in color photography, in addition to being responsible for the development of the very effective "Sonne" strip camera. The Navy, under L. A. Pope, R. H. Quackenbush, Jr., and J. H. McElroy, Captains, U. S. Navy, had readied itself for the particular aspects of photography suitable to it purposes.

There was an extensive domestic mapping program consisting of topographic sheets of coastal areas and also a substantial program of aeronautical charts and mosaics covering hundreds of thousands of square miles of North and South America. Extensive areas in many foreign countries had already been mapped by one method or another prior to hostilities, and these maps from many sources were recompiled into standard military maps.

"Tri-Metrogon" aerial photography was the most economical way of obtaining fair information of vast territories. It has much to commend it for general utility, for small-scale maps, and for charts.

For more detailed purposes considerable high-altitude, vertical photography was also done, at altitudes as great as 35,000 ft above the ground. At these altitudes pictures were procured at scales ranging from 1:50,000 to 1:70,000, and these were compiled into mosaic maps and used for topographical maps and other purposes.

The "Sonne" strip camera, an outgrowth of the photo-finish camera of the race track, was one of the few really new developments of the war. This camera has a roll of film moving continously across the focal plane at the same angular rate the airplane is moving over the ground. In conjunction with the camera is a view finder, with "travelers." These are moving marks, which the operator can control so that they move at the same rate of speed as the image traverses the view finder. By so controlling the speed of the travelers, the speed of the film is regulated to move properly. The picture is taken through a narrow slit, and by this combination sharp pictures are obtainable from very low altitudes in a fast-moving airplane. This picture may be one continuous strip several hundred feet long. The "Sonne" pictures were extremely useful in taking inventory of troop movements along roads, and in photographing beaches at close range before landings.

In war there are two very satisfactory ways of obtaining aerial photography. The first of these is to fly high and fast, to be out of the range of antiaircraft fire, and high enough so that the danger of being attacked by fighters is minimized. The second satisfactory method is to come in low and fast, at altitudes of only a few hundred feet—perhaps at 100 ft or 200 ft. It is in the latter type of photography that the "Sonne" camera proved so effective. No other camera, moving at 300 miles per hr or more, at an elevation of 100 ft, would obtain anything except a magnificent blur.

The engineering profession emerged from the war with all these old and new tools to choose from. The writer has many biases in favor of the systems he knows best and believes in most thoroughly. Many of the arguments in this paper are expressions of personal opinion on subjects that are controversial, and it is hoped that possibly other people qualified to discuss the subjects will take issue with them, thus provoking constructive discussion.

The "Tri-Metrogon" system has only a limited application to civil demands in the fields of either aerial photography or photogrammetry. It may be used advantageously where working weather is at a great premium; and therefore it is desirable to photograph as much terrain as possible in the rare instances when suitable weather is available. Thus, for example, in Colombia, South America, where something less than an average of 2 min (photographic) per day has been experienced, there is justification for using "Tri-Metrogon" photography, because in the 60 min per month that can be expected on an average, much more ground can be photographed by "Tri-Metrogon," even though this ground is not nearly as well photographed and as usefully photographed as it could be by vertical methods. The "Sonne" strip camera is finding commercial application in conjunction with new developments such as the airborne magnetometer. It is also being used for taking large-scale inventory maps of railways and highways.

Aerial photography and photogrammetry are presented to the engineer today in much the same garb which dressed them in 1941, but with a much better press. In other words, more people are favorably impressed with aerial photography and photogrammetry, and the engineering profession in general is ready now (1949) to give it full acceptance.

A review of the types of project undertaken today by aerial photography and photogrammetry is probably the best way to crystallize engineering thinking on how these tools may best be applied to the requirements of modern engineering. For example, aerial photography first became important in the southwest in about 1938, when the Soil Conservation Service, the Forest

Service, and the Bureau of Indian Affairs (United States Department of the Interior), between them, completed aerial maps and mosaics of practically the entire eastern half of the State of Arizona, and the western half of the State of New Mexico. About this same time the Soil Conservation Service mapped the "dust bowl" area, comprising nearly 70,000 sq miles in Colorado, Kansas, New Mexico, and Texas.

In its earliest days photogrammetry was used to map the workings around the Boulder Dam site in Arizona and Nevada (1930); and later, when the gates of the dam were closed and the reservoir started filling with water, a contour map of the entire area of the reservoir was made. This map of the empty reservoir was designed as a base from which studies could be made to determine the rate at which the reservoir was being filled with silt.

Other dam sites of the Colorado River system have been explored extensively by photogrammetric methods. This work has been sponsored by the Bureau of Reclamation (United States Department of Interior). A contour map was made, for example, of the Bridge Canyon Dam site (about 10 miles above the headwaters of Lake Mead), at a scale of 1 in. = 500 ft, with 10-ft contours.

A very extensive survey was completed along the San Juan River in 1947, extending from approximately the confluence of the San Juan with its parent stream, the Colorado, up to the four corners where the states of Colorado, Utah, Arizona, and New Mexico come together. The topographical map to be prepared of this proposed reservoir is at a scale of 1 in. = 1,000 ft, with 10-ft and 20-ft contours.

This San Juan project is perhaps one of the most difficult topographical surveys ever undertaken. In fact, topographical surveys of any part of the Colorado chasm are very difficult, because the canyons themselves are almost impassable, and access to the canyons for establishing control is extremely difficult. Control of the San Juan job was largely done by jeep, pack horse, and on foot.

For years the Salt River Valley Water Users' Association of Phoenix, Ariz., has approached many of its problems from the starting point of aerial photography. Many of the tributary streams of the Salt River Valley system have been mapped from that valley up toward their sources. The entire irrigated area was mapped many years ago and has been used continuously for a variety of purposes. These maps have proved valuable to the Salt River Valley Water Users' Association as a complete inventory of physical conditions on the ground with which it is so vitally concerned; and in several instances they have given valuable record information which was used in litigation to prove conditions as they actually existed.

Over the years, the Bureau of Reclamation has had many photographic maps made of the unruly Colorado River. Areas subject to flood were studied from photographs made early in the project history. The delta of the Colorado, lying partly in the United States and partly in Mexico, has been photographed at least twice (the first time in 1922) to study the vagaries of this menacing river, which in flood thrashes itself from old channels to new channels, and back again. The aerial photographic maps showed this delta in minutest detail, so that every channel, both new and old, was apparent, and the work of planning the control of the river was thus greatly expedited. Attempts have been made to hack ground survey lines through this brush-choked delta region, a prohibitively tedious and expensive task, yielding, at best, a bleak skeleton of data compared to the full body of exquisite photographic detail.

The mining areas of the southwest have utilized photogrammetry extensively. Many of the mines have had vast tracts photographed for geological exploration, whereas others have had detailed contour maps made of their operating properties. A photogrammetric map of a mining area offers two general types of valuable information. In the first place, the contact prints can be viewed stereoscopically and mosaics can be assembled, which together give a continuity and detail for delineating geology unobtainable by any other method. Secondly, the contour map that may be compiled from the same photographs gives an accurate base map on which roads can be located and construction projects of all kinds designed.

A paper such as this can never escape some discussion of relative costs. In the past it has been possible to give only the most generalized kind of cost information on aerial surveys, and that situation still persists. The aerial survey is not of a sufficiently definite nature so that its cost can be given, such as the costs of commodities or of tons of steel, or of bags of cement. The question—How much will an aerial survey cost—is comparable to asking: How much does a bridge cost to build? The estimator may even be told how long a bridge was to be, but he would still have difficulty in estimating its cost without knowing a great many other facts. The problem of supplying information about the cost of an aerial survey, and a topographical map based on photogrammetric methods, is just as complex (see Table 1). The following factors all affect such cost:

1. The location of the area;
2. The size of the area;
3. The weather conditions;
4. The specifications of the final map;
5. The accessibility of the area from the standpoint of control;
6. The existence or nonexistence of usable control already in the area;
7. The character of terrain from the standpoint of control;
8. The type of vegetation, from the viewpoint of both interference with control operations and interference with the ultimate precision of the contour map;
9. The maximum differences in elevation, which are an index of the number of contours that are to be drawn and drafted;
10. The number of roads, dams, railroads, and buildings, to be shown on the map;
11. The possible need of a map made on spherical coordinates;
12. The form in which the map is to be delivered—as lithographed sheets, tracings, or other form;
13. The number of geographical names to be shown;

14. The extent of field editing required: How critical is the map from the standpoint of showing every small building, and how serious would it be if a haystack were mistaken for a building and shown on the map?;
15. The season of the year the work must be done;
16. The time to be allowed the contractor; and
17. The required scale, the contour interval, and the accuracy.

Some data relating to costs of contour maps for highways, and for city planning and land subdivision, are offered in Table 1. In addition, the following general data are offered as applying to reservoir mapping:

Item	Description	Unit cost
1	Thirty-seven reservoirs, some at scales of 1:1,200 with 5-ft contours and some at scales of 1:12,000 with 10-ft contours; dollars per reservoir	$4,000
2	Two reservoirs at 1:6,000 with 10-ft contours; dollars per reservoir	6,500
3	Sundry reservoirs aggregating 75 sq miles, some at scales of 1:10,000 and some at scales of 1:24,000—all at 5-ft contours; dollars per acre	1.02
4	Sundry reservoirs, one group at 5-ft contours aggregating 40 sq miles and another group at 20-ft contours aggregating 24 sq miles—all at scales of 1:6,000; dollars per acre	0.90
5	Sundry reservoirs aggregating 31 sq miles at a scale of 1:4,800 with 5-ft contours; dollars per acre	0.98

TABLE 1.—COST OF CONTOUR MAPS BY PHOTOGRAMMETRY, AT 1947 PRICES

Item	Contour interval (ft)	(a) HIGHWAYS				(b) CITY PLANNING AND SUBDIVISION		
		Scale 1 in. equals (ft)	Dimensions, in miles[a]	Dollars per acre	Dollars per sq mile[a]	Scale 1 in. equals (ft)	Area, in acres	Dollars per acre
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
1	5	100	0.38 by 20	3.34	(870)[b]	100	5,760	1.00
2	5	100	....	6.00	....	100	400	7.25
3	5	200	(300 ft wide)	15.40	(570)[b]	100	1,500	4.76
4	5	200	1 by 8	1.38	885	200	9,600	1.72
5	5	200	1.14 by 20.5	0.98	629	200	450	4.90
6	5	200	....	0.85	545	....	....	....
7	5	200	(6 sq miles)	1.82	....	....	....	....
8	5	200	(49 sq miles)	1.02	....	....	....	....
9	10	....	....	....	....	200	9,600	0.77
10	10	....	....	....	....	300	1,400	0.93
11	20	....	....	....	....	....	....	....

[a] Except quantities in parentheses. [b] Dollars per linear mile.

In considering the cost of topographic maps by photogrammetric methods, one very essential differentiation must be kept in mind between this method of map making and the older and more conventional methods. With the photogrammetric method, the horizontal scale is usually not the determining factor in cost; the vertical accuracy is the all-important element. Flying height is largely a question of vertical accuracy of the contours. Various methods of photogrammetry permit flying the airplane at certain altitudes for certain results, and therein lies the efficiency of the method. The cost may vary manyfold depending on the method and the experience at the disposal of the map maker.

Contour maps made by photogrammetric methods within a certain range can be enlarged and reduced to almost any horizontal scale; but the minute that 5-ft contours are specified, and it is required that 90% of the points checked on the contours shall be accurate to half a contour interval, then it is immediately required that flying altitude (with the Fairchild system, for example) be 2,500 times 2.5 ft, or 6,250 ft aboveground. Thus, in all large-scale contour maps made by photogrammetric methods, the contour interval is a very important factor. For example, if a 5-ft contour is required, the map could be produced very efficiently at 1 in. = 200 ft, or 1 in. = 400 ft. Any one of several other scales could be selected without affecting the cost very greatly. However, if the contour interval were changed from 5 ft to 10 ft, although the scale remained the same, many phases of the operation would be reduced to one fourth—there would be only one fourth the number of control points to establish, and only one fourth the number of pictures from which the map had to be drawn. Also, there would be one half the amount of flying. Thus, the contour interval and vertical accuracy are all-important factors in the cost of topographical mapping by photogrammetric methods.

This same thought can be expressed in another manner. For example, if a 10-ft contour map is being prepared at a horizontal scale of 1 in. = 1,000 ft, and the specification requires that 90% of the points tested on the contours shall be within a half interval of their true elevation, the resultant map will cost nearly twice as much as if only 75% of the points tested has to be within a half contour interval of the true elevation. The flight can be made twice as high if only 75% of the points must check in vertical accuracy.

DISCUSSION

Courtlandt Eaton,[2] M. ASCE.—The long strides that have been made in aerial photography in recent years are brought out in this informative paper. Of particular interest to the writer are the applications of aerial photography to the production of contour maps. About 1930, the Fairchild Aerial Surveys under contract with the Los Angeles County (California) Flood Control District made two aerial surveys in particularly difficult terrain—one, a contour map of the main San Gabriel River Canyon from its mouth, including the east and west forks; and the other, a contour map of a quarry site in the lower canyon about 600 ft high with a practically vertical face.

From the San Gabriel Canyon survey some fourteen possible dam sites were located, from which, by field reconnaissance and office studies, the sites that finally resulted in San Gabriel rock-fill dams Nos. 1 and 2 were selected. Much of the territory was passable only on horseback and on foot. Time was important. The contour maps were prepared in a fraction of the time necessary for field party surveys. Cost data are not available, but the cost was less than was possible by ordinary field methods. The contract called for a high degree of accuracy and field survey spot check areas indicated that aerial contours were more accurate than field surveys. This accuracy was due to the facilities, under aerial methods, for following up small creeks and draws, which in the aggregate add up to substantial areas.

The quarry survey was equally satisfactory as developed by spot checks. Because the vertical face made a field survey extremely difficult, hazardous, and costly, it would have involved delays that were inadvisable.

Aside from costs and time savings, there are many advantages in aerial surveys well known to engineers who have used them. One advantage is that aerial surveys can decide questions of accuracy of original ground surveys arising between the contracting parties in the case of large excavations such as dams. Each party may claim the superiority of his survey. With a photographic record such claims either can be reconciled or will not be made.

Another advantage is that aerial photographs bring out details not visible by ground inspections—for instance, rights of way where photographs give evidences of old watercourses through improved properties. The writer remembers an aerial survey of lands now occupied by the Santa Anita (Calif.) race track which showed evidences of an early race track said to be one built by Lucky Baldwin and later changed in direction for better sun conditions.

Since there are also many other advantages in aerial surveys, topographic surveys by present archaic and laborious methods probably will become obsolete for major areas.

The writer's only "criticism" of the paper is that it is too short. In his closure, it is hoped that Mr. Eliel will elaborate on this most interesting subject.

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[2] Denver, Colo.

AMERICAN SOCIETY OF CIVIL ENGINEERS
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TRANSACTIONS

Paper No. 2406

DEVELOPMENT OF CBR FLEXIBLE PAVEMENT DESIGN METHOD FOR AIRFIELDS

A SYMPOSIUM

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Note.—Published in January, 1949, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

FOREWORD

At the beginning of the construction program for World War II, the Corps of Engineers, United States Army, after a survey of available flexible pavement design methods, adopted the empirical method in use by the California Division of State Highways, known as the California bearing ratio (CBR) method. This method uses the CBR of the soil, which is an index of strength obtained by a simple shear test, and a set of design curves for determining the total required thickness of base and pavement for a given wheel load.

When the method was adopted, design curves were available for wheel loads as great as approximately 12,000 lb. Since the situation was urgent, these curves were immediately extrapolated to wheel loads of 60,000 lb, and a research program was initiated to provide information for verifying or revising the curves. Furthermore, since the CBR test had not been used widely outside California, a project to study it was included in the program. Many difficulties were encountered in the adaptation of the CBR method to the design of airfield pavements for heavy wheel loads and the research program was extended and revised accordingly.

Various phases of the program were completed during World War II and reports on these phases, which are mentioned in the several papers, were published. Other phases of the program were completed on which formal reports were not prepared. The results of both the published and unpublished investigations have provided design criteria which appear in the *Engineering Manual for War Department Construction.*[1]

The purpose of this Symposium is to present the work that has been accomplished by the Corps of Engineers in connection with the adaptation of the CBR method of design from highway construction to airport construction. In particular, it is desired to paint a true picture of the CBR test and its value as a design tool for flexible pavements. Paper No. 1 gives the background of the adoption of the method, a summary of the study that has gone into the improvement of the method, and the future of the method. Paper No. 2 outlines the development of the original method for highway design. In paper No. 3, the procedures by which the design curves were extrapolated from 12,000 lb to 60,000 lb are presented. Paper No. 4 gives the results of the study conducted on the CBR tests. Papers Nos. 5, 6, and 7 describe, in considerable detail, the three largest accelerated traffic tests that were conducted to obtain data for the design curves and paper No. 8 describes a number of smaller accelerated traffic tests. Paper No. 9 shows the correlation of the CBR curves with the data that have been obtained from the accelerated traffic tests and from studies of actual airfield pavements. Paper No. 10 presents a procedure whereby the design curves for single wheels can be translated into design curves for multiple wheel assemblies of practically any combination. Paper No. 11 is a discussion of the utility of the method. The good features and the shortcomings of the method are indicated and the general applicability is given.

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[1] "Airfield Pavement Design," *Engineering Manual for War Department Construction*, Office of Chf. of Engrs., U. S. Army, Washington, D. C., May, 1947, Pt. XII, Chapter 2.

NOTATION

The letter symbols in this Symposium conform essentially with "Soil Mechanics Nomenclature," ASCE *Manual of Engineering Practice No. 22*. Discussers are requested to use the same notation and to select new symbols accordingly.

d = distance between the inside edges of the tracks made by dual wheels of a test plane;
G = specific gravity;
I_P = plasticity index;
k = modulus of soil reaction;
N = a load concentration factor;
s = spacing center to center of dual wheels; and
w = water content:
  w_L = liquid limit;
  w_P = plastic limit.

ACKNOWLEDGMENTS

The CBR test and the CBR method of design for highways were originally developed by the California Division of Highways under the general supervision of T. E. Stanton, M. ASCE, material and research engineer, and under the immediate direction of O. J. Porter, M. ASCE, senior physical testing engineer.

At the beginning of World War II this method was selected by the Corps of Engineers for use in the design of flexible pavements. The responsibility for the development of design criteria for airfield construction was delegated by the Corps of Engineers to the Military Construction Branch, Engineering and Development Division (subsequently Airfield Branch, Engineering Division), Office, Chief of Engineers, under the direction of James H. Stratton, followed by L. C. Urquhart, both Members, ASCE, Henry C. Wolfe, and D, S. Burns. In addition to Gayle McFadden and Thomas B. Pringle, Members, ASCE, chief of the Airfield Branch and chief of the Runway Section, respectively, the early development work was under the responsible direction of Reuben M. Haines, Assoc. M. ASCE, until ill health forced him to resign his position. Consultants to the Corps of Engineers assisted materially in the adaptation of the California method to airfield pavement design for heavy wheel loads. At the time the CBR method was adopted these consultants were A. Casagrande, M. ASCE, T. A. Middlebrooks, Assoc. M. ASCE, and Mr. Porter. Later a formal Board of Consultants was established. Included on this board were the three consultants previously named and H. M. Westergaard and J. L. Land, Members, ASCE, P. C. Rutledge, Assoc. M. ASCE, and F. C. Lang (deceased).

Laboratory, field, and office studies required to verify and modify the California method were primarily the responsibility of the Soils Division, Waterways Experiment Station, Mississippi River Commission. This work was directed by W. J. Turnbull, W. H. Jervis, and W. K. Boyd, Members, ASCE. The studies included a comprehensive investigation of the method of test and application of test results, consolidation of accelerated traffic test section studies, tests on existing airfields, special studies, and the development of CBR design criteria for multiple wheel loads.

DEVELOPMENT AND SCOPE OF INVESTIGATION

BY GAYLE MCFADDEN[2] AND THOMAS B. PRINGLE,[3]
MEMBERS, ASCE

During the latter part of November, 1940, the responsibility for the design and the construction of military airfields was assigned to the Corps of Engineers, United States Army. Later the same function pertaining to civil aeronautics airfields, of military occupancy, was delegated to this same organization.

It soon became apparent, for a variety of reasons, that a simple, practical, as well as a uniform method of design would be necessary. For example, the prior experience of the majority of the Corps' engineers had been generally confined to river and harbor, flood control, and seacoast fortification. This explains the necessity for the elimination of a diversity of designs based on judgment of personnel not fully conversant with the requirements of airplane traffic. Other reasons for seeking a simple, practical, and uniform method of design were: (*a*) To obviate the use of untried methods; (*b*) to insure adequately designed pavements; (*c*) to provide a method not subject to variation occasioned by arbitrary cost differentials of local and competitive materials; (*d*) to avoid reductions of pavement thickness in order to balance cost; and (*e*) to establish a procedure for further development of design methods through tests, investigations, and study of the actual behavior of pavements.

The Westergaard method of design of rigid type (concrete) pavements seemed fairly universal in its application and was quickly adopted by the Engineer Department as a standard of design for that type of pavement.

The selection of a design method for flexible pavements presented a more difficult problem. There were several methods in vogue that had been developed primarily for highway design. All these were studied and all had considerable merit; but most had some apparent fallacy that seemed to defeat the aforestated objectives.

In most of the methods in current use at that time, the basic design assumption was the bearing capacity of the subgrade. The question was raised as to whether the procedures for determining this value, as related to flexible pavements, had been adequately determined.

Since it was desirable to select a method that could be adapted with the least delay, and since the selection of one in common usage seemed the most expeditious manner, personnel of the Corps of Engineers performed several special field investigations (Langley Field (Virginia), Bradley Army Airfield (Connecticut), and a Virginia Highway Department test section) to try and develop the procedures for determining the subgrade bearing capacity values in order that they might be used in the theoretical formulas of the several methods.

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The results of these special field investigations indicated that:

1. The length of time required to develop a satisfactory test procedure would preclude its use in the war emergency program then being faced;
2. In the field plate-bearing test, the proper deflection to determine the "bearing capacity" depends on the basic assumptions in the formula and varies according to combinations of many factors; and
3. In most cases the results of the plate-bearing test would not be applicable to the soil-moisture conditions expected ultimately to develop below a pavement, and it would be difficult to develop a satisfactory method for adjusting the test results to the various moisture conditions.

Some methods contemplated using plate-bearing tests on pavement surfaces for the design of flexible pavements. Field tests made on pavements and base courses led to the conclusion that the same factors which must be considered in using this test for subgrades must also be considered for pavements. In addition, the compressibility of the pavement and that of the base material entered into the problem. These conclusions were considered applicable primarily to flexible pavements because, in this case, the stress distribution from load to pavement, base and subgrade, results in shear failure; and it is practically impossible to differentiate between the deflection under plate-bearing tests due to consolidation and plastic deformation. These conclusions naturally do not hold true in designing rigid type pavements since the modulus of soil reaction k of the subgrade is essentially a measure of consolidation, and service behavior has demonstrated that a plate-bearing test is satisfactory for determining its value. The logical inference follows that a shear test is proper in the design of flexible type pavements.

Further studies by the Corps of Engineers indicated that the stress conditions and the behavior of subgrades and the base courses under repetitive wheel loads were extremely complex and could not be determined by a single test.

From these studies a definite conclusion was reached that some method other than the application of a theoretical formula would have to be used, at least during this period of pressing need. Therefore, adaptation of an empirical method that had been proved for highway loading appeared to be the only solution. This decision narrowed the field; and, after some months of investigation of suggested methods, the principles used by the California Highway Department in designing flexible pavements for highway loading, known as the CBR method, were adopted tentatively.

The controlling reasons for the adoption were many. Among these reasons were: (1) The CBR method had been correlated to the service behavior of flexible pavements and construction methods and successfully used by the State of California for a number of years; (2) it could be more quickly adapted to airfield pavement design for immediate use than any other method; (3) it was thought to be as reasonable and as sound as any of the other methods investigated; (4) two other states were known to have methods of a similar nature that had been successful; (5) the subgrade modulus could be tested with simple portable equipment either in the laboratory or in the field; and (6) testing could be done on samples of soil in the condition representative of the foundation-moisture state under most pavements.

The selection of the method was only the first step in its assimilation by the Corps. A lecture course was organized and conducted in the United States Engineer Office, Sacramento, Calif., on the "California Method of Determining the Relative Bearing Value of Soils and Its Application to the Design of Highways and Runways." The scope of the course was limited to the design of flexible type pavements. Its purpose was to provide an intensive short course to acquaint personnel of the Corps of Engineers with the CBR method. The course was attended by representatives of the then eleven regional offices of the Corps, known as Division offices; representatives of the Office of the Chief of Engineers; and representatives of what was later to become the Flexible Pavement Laboratory. The representatives were in turn to instruct the personnel in the subordinate offices of the Divisions (known as District offices) in order that application of the procedure could be made rapidly available.

The adoption of the method did not stop with the familiarization course given the personnel of the Corps of Engineers. An exhaustive study was initiated and has become a continuing program in an attempt to obtain and to maintain the maximum degree of beneficial use from the method.

Since the CBR method is empirical and its design curves are used in connection with a modulus of shearing resistance of soils determined by a special CBR test, it was considered necessary to investigate this test to develop a correlation with field performance for airfield construction similar to that already accomplished by the California Division of Highways for road work. This study was authorized by the Office of the Chief of Engineers, in September, 1942, for accomplishment by the Waterways Experiment Station, Vicksburg, Miss.

The scope of the study was expanded as it became obvious that there were further fields to explore. Interim reports were published from time to time for dissemination to the field offices of the Corps and to provide information in the preparation of the *Engineering Manual for War Department Construction*. After approximately 3 years of study, the investigation culminated in a comprehensive final report.[4]

When the CBR method was first adopted, the design curves for total pavement thickness requirements had only been correlated with field performance sufficiently to accommodate wheel loads normally found in highway usage. Airplane wheel loads generally began where highway loads ended. Therefore, it was necessary to provide an extension of these curves. The best talent available, including men familiar with the development of the original curves, was set to work to evolve a tentative extrapolation to fulfil an immediate requirement. These extrapolations were published as "tentative" design curves pending their validation.

Work was begun immediately to provide the necessary correlation with field performance under military airplane loads to validate the extrapolations.

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[4] "The California Bearing Ratio Test As Applied to the Design of Flexible Pavements for Airports," *Technical Memorandum No. 213-1*, U. S. Waterways Experiment Station, Vicksburg, Miss., 1 July 1945.

This operation was accomplished by determining the CBR of the subgrade and base and performing accelerated traffic tests (*a*) on existing pavements whose construction history was known and (*b*) on specially constructed full-scale test sections where all construction items were carefully controlled. In all tests, airplane traffic was simulated as closely as possible with regard to load and tire characteristics. In the specially built sections, the various items were intentionally overdesigned and underdesigned to bracket the results and to give the widest range of information. Post-traffic examination of surface and subsurface conditions was made and recorded.

From the plot of the test results, the design curves were modified where necessary and this modification and other pertinent design information developed from the tests were immediately made available to the designers of the Corps of Engineers and incorporated into the *Engineering Manual for War Department Construction.*

One or two traffic tests were not considered satisfactory to develop the required information. Not less than four were run on existing pavements at Corpus Christi, Tex.; Dothan, Ala.; Fargo, N. Dak.; and Lewistown, Mont. Not less than six specially built test sections were subject to traffic—among which were those at Langley Field, Virginia; Stockton Army Airfield, California (Sections Nos. 1 and 2); Eglin Field, Florida; Barksdale Field, Louisiana; and Marietta Aircraft Assembly Plant, Georgia. The wheel loads ranged from 15,000 lb to 200,000 lb. The traffic was generally continued until the behavior of the pavement indicated failure, incipient failure, or satisfactoriness.

A detailed account of the studies and tests that have gone into the improvement of the CBR method and the CBR test, to determine the modulus of shearing resistance of soils, is contained elsewhere in this Symposium.

It is recognized that the CBR method may not supply the ultimate answer in flexible pavement design. Pavement failures have occurred in cases where the design was apparently adequate according to the CBR method; however, these instances are in minority. The enthusiastic reports from engineers in the theaters of operation, where speed and simplification of design were mandatory, the ease with which the method was assimilated in construction in the continental limits of the United States, and the growing demand from the engineering profession in the United States and other countries for further information on the developments made by the Corps of Engineers indicate that the CBR method has a definite place in flexible pavement design for some time to come. It was considered the most desirable available method at the time of adoption, and the studies and the testing that have gone into its improvement, together with growing uniformity in laboratory test procedures, have maintained it as the most desirable method to date.

However, the Corps of Engineers is not inseparably "wedded" to the CBR method. Continuing studies have as their purpose the development of basic understanding of the stresses and the strains that occur in subgrades, bases, and wearing courses under wheel loads, and the development of test procedures to measure behavior characteristics. The results of these studies, together with data being obtained on the effect of moisture changes and other weather features, may permit the development of a more rational method of design. However, results to date (1948) indicate that such a method may not be available for quite some time. When a more rational method is developed, by the Corps of Engineers or others, the superiority of which can be proved and which contains required features of simplicity of testing and application, there is no doubt that its adoption by the Corps is a foregone conclusion. Until this advance occurs, the Corps of Engineers is continuing to use and to develop the CBR method.

DEVELOPMENT OF THE ORIGINAL METHOD FOR HIGHWAY DESIGN

BY O. J. PORTER,[5] M. ASCE

During 1928 and 1929, the California Division of Highways made an intensive investigation of pavement failures throughout the state. Areas that had failed were investigated to determine the local drainage conditions and other factors affecting the stability of the pavement. Holes were drilled through pavements and undisturbed samples were obtained which would furnish information as to density and moisture content of the materials underlying the pavement surface. These investigations pointed to three principal types of pavement failure: (1) By lateral displacement of the subgrade material as the result of the pavement having absorbed water and softened after construction; (2) by differential settlement of materials underlying the pavement; and (3) by excessive deflection of the materials underlying the pavement, resulting from repetitions of load.

Types (1) and (2) of failure could often be traced directly to poor compaction during construction. Although poor drainage conditions contributed to the rate of increase of moisture beneath the pavement, the ultimate moisture conditions were similar regardless of drainage conditions. Examination of pavements which did not fail indicated an increase of moisture after construction, but the extent of the moisture increase was limited by the degree of compaction of the material. Pavement conditions were found to be more dependent on compaction than on other factors unless, of course, the material showed excessive swelling in the presence of water. Usually type (3) of failure was brought about by insufficient thickness of pavement and by the existence of a base course over soils inherently weak in shear strength—because of either the nature of the soil or poor compaction. Investigation also showed that identification of the soil type did not provide sufficient assurance that a soil was, or was not, a satisfactory subgrade material, since the same soil would behave differently under different conditions of moisture and density.

A method of test was sought, therefore, which would be satisfactory for establishing the density that should be used in the construction of subgrades and the shear strength that was required. The experience gained from the study of failures also was helpful in determining the thickness of pavement and base course that would be required over the various soils, taking into account the condition in which it was placed.

To predict the probable behavior of materials underlying the pavement, it was desired to have a simple and quick test that would evaluate the quality of subgrade, subbase, and base course materials. For this purpose, static field load tests were first tried. These tests were influenced to a large extent by the elastic and the plastic deformation of the material beneath the test load. The extent of each of these types of deformation varies greatly with the moisture and the density conditions in the soil beneath the test area. It was practically impossible to moisten the soil properly in the field to the depth affected by the load test; and, as a consequence, the results of such tests might represent a test condition not related to the ultimate condition and the action of the subgrade during service.

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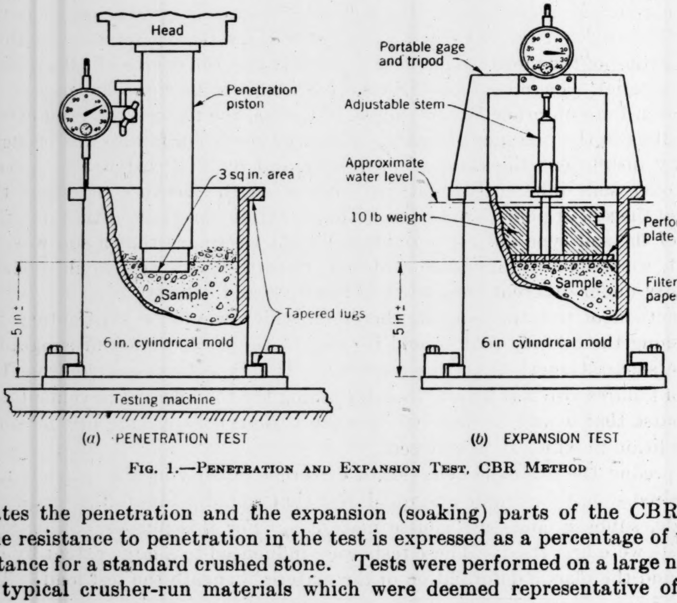
[5] Cons. Engr., Sacramento, Calif.; Consultant to the Office, Chf. of Engrs., Dept. of the Army, Washington, D. C.

A bearing ratio test was devised in 1929 in an attempt to eliminate some of the objections to field loading tests and to provide a quick method for comparing local base and subbase materials available for reinforcing the subgrade. This test has been called the CBR test. An empirical relationship was subsequently established between the test value and the suitability of base and subgrade materials. Samples of the material to be investigated were first thoroughly consolidated to the density obtainable under good construction methods or to approximately the density ultimately produced by traffic in good subgrade materials. This procedure eliminated, to a very large degree, most of the consolidation deformation that often influences static load tests made in the field. The compacted specimen was next soaked for 4 days under a surcharge representing the weight of the pavement, to permit the specimen to swell and absorb moisture, with consequent loss of strength. The CBR test was then made on the specimen to determine the resistance to lateral displacement, thus measuring the combined influence of cohesion and internal friction. Fig. 1 illustrates the penetration and the expansion (soaking) parts of the CBR tests. The resistance to penetration in the test is expressed as a percentage of the resistance for a standard crushed stone. Tests were performed on a large number of typical crusher-run materials which were deemed representative of high-quality base course materials. The average of these results was taken as a CBR of 100% and results of tests on other materials were expressed as a per-



(a) PENETRATION TEST (b) EXPANSION TEST

FIG. 1.—PENETRATION AND EXPANSION TEST, CBR METHOD

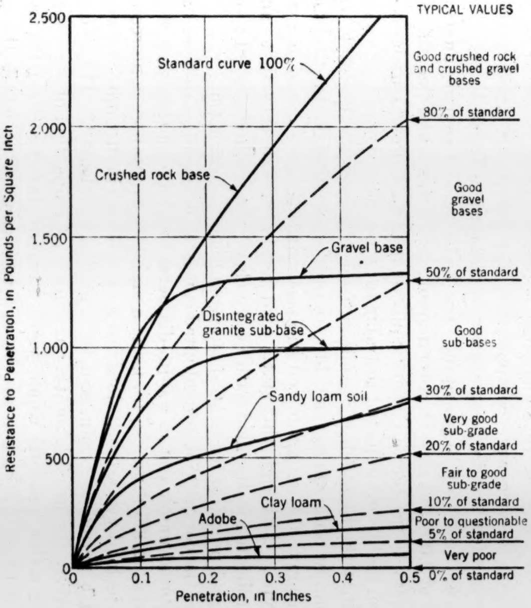


FIG. 2.—RESISTANCE TO PENETRATION, COMPACTED AND SOAKED SPECIMENS

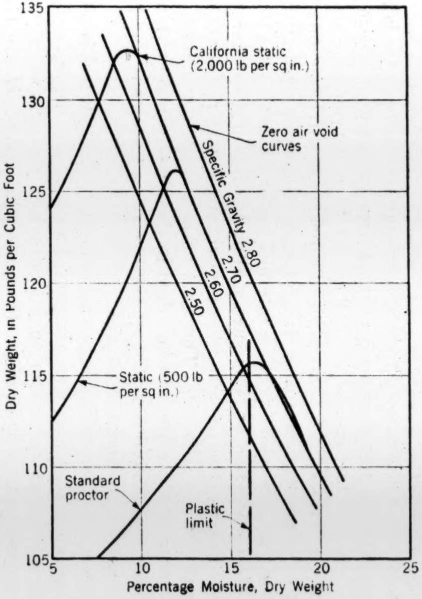


FIG. 3.—MOISTURE-DENSITY RELATIONSHIP, CLAY-LOAM SOIL

centage of the 100% value. Typical values for a number of soil and base course materials are shown in Fig. 2.

As the increments of penetration are large with respect to the loaded area, the results are influenced only to a minor degree by elastic deformation. In view of the fact that it is not practical to test large specimens in the laboratory and to penetrate the test specimens over an area comparable in size to field loading conditions, the results are influenced to a considerable degree by cohesion and interlocking of particles. Therefore, a surcharge on the area adjacent to the penetration piston is necessary to develop friction in clean, round-grained sand in which particle interlocking and cohesion are low. The results of the penetration tests on such materials without a surcharge should be interpreted as representing conditions in which the material is not confined by the weight of a pavement. The tests with a surcharge indicate the increased stability of the material when used in the lower portion of foundation courses where the confining weight of the overlying pavement and the base course increase the frictional resistance.

It has been noted in a preceding paragraph that the CBR test is performed on a soil specimen that has been compacted to a density equivalent to that obtained in good construction or to that obtained by satisfactory subgrade soils after consolidation by traffic. During studies made by the California Division of Highways in 1928 and 1929, subgrade materials which had proved satisfactory in service were compacted in the laboratory under static loads ranging from a few hundred pounds per square inch to 10,000 lb per sq in. and, in some cases, to 20,000 lb per sq in. On the basis of these preliminary tests, a static load of 2,000 lb per sq in. was adopted after it was found that this pressure was necessary to produce a density equivalent to that present under old highways which had been subjected to traffic for a period of years. It was evident not only that the high density resulting from this load was desirable, but also that it could be obtained in the field with proper moisture control and thorough rolling whenever the subgrade and subbase materials were not of an extremely adverse type. Although this static load results in a density that cannot be readily duplicated in the field with heavy clays, the method proved to be both desirable and practical for control of embankment construction.

Briefly, the compaction test adopted consisted of placing a sufficient amount of the material under test in a 6-in.-diameter mold and compacting it under a static load of 2,000 lb per sq in. using a standard laboratory compression machine. The load was applied slowly at a rate of 0.05 in. per min and was maintained for a period of 1 min, and then gradually released. The density was determined by the usual laboratory methods. Compaction of several samples at varying water contents would yield different densities, and the optimum moisture content was taken as the moisture content at which the greatest density was obtained. For field control, where a compression machine was not available, an impact method of compaction using twenty blows of a 10-lb tamper with an 18-in. drop on each 1-in. layer was adopted. Thus, a sufficiently accurate duplication was obtained for field use of optimum moisture and density as determined by the static test.

Fig. 3 shows the relationship between the static and the standard Proctor compaction tests obtained on a clay-loam soil which is classified as A4 by the Public Roads Administration (PRA) classification or as CL by the Casagrande classification.[6,7] (Other characteristics of the soil were: CBR, 8.0%; expansion, 5.3%; liquid limit (w_L), 23%; plastic limit (w_P), 16%; and plasticity index (I_P), 70%. Its component parts were: Sand, 48%; silt, 26%; and clay, 26%.)

Expansion during the soaking period indicates the tendency of the materials to swell when they are allowed access to water. The uplift, unless absolutely uniform, may roughen and crack nonrigid pavements. Because this volume change is seldom uniform throughout the length of a roadway or runway, the surface drainage may also be disrupted. Subgrades which have proved satisfactory usually show an expansion of less than 3% of the initial height of the specimen during the 4-day soaking period whereas good base and subgrade materials show less than 1%. Many of the poorer clay and adobe soils often show an expansion ranging from 7% to 20%. Studies of soil expansion have indicated that each material has a state of equilibrium regarding density and moisture content for each condition of confinement and that the moisture capacity of a soil varies inversely with the superimposed load. As an example, tests on an adobe that was soaked until all expansion ceased are demonstrated by Fig. 4.

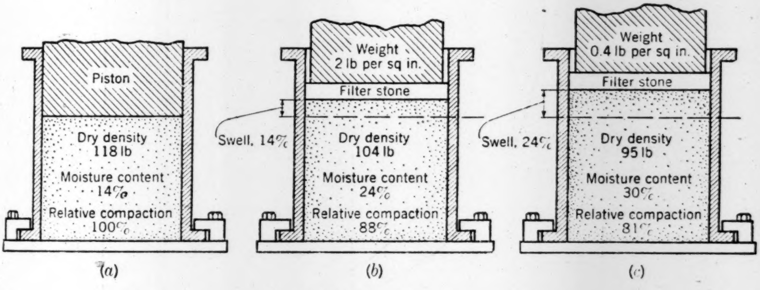

Piston
Dry density 118 lb
Moisture content 14%
Relative compaction 100%
Weight 2 lb per sq in.
Filter stone
Swell, 14%
Dry density 104 lb
Moisture content 24%
Relative compaction 88%
Weight 0.4 lb per sq in.
Filter stone
Swell, 24%
Dry density 95 lb
Moisture content 30%
Relative compaction 81%
(a)
(b)
(c)

Fig. 4.—Expansion of Adobe Soil When Surcharged, with Two Pavement and Subbase Loads

Fig. 4(*a*) represents the sample before soaking, when the soil has been compacted to its maximum subgrade density. Figs. 4(*b*) and 4(*c*) represent the sample soaked until expansion ceased. In Fig. 4(*b*), the surcharge (2 lb per sq in.) is assumed equal to the weight of a 6-in. concrete pavement and a 1-ft gravel base, whereas the surcharge in Fig. 4(*c*) (0.4 lb per sq in.) is equal to the weight of a 5-in. concrete pavement only. Fig. 4(*b*) indicates that the material will swell 14% with an increase in moisture content from 14% to 24% under 2 lb per sq in. whereas under 0.4 lb per sq in. the swell will be 24% with an increase in moisture content from 14% to 30%.

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[6] "Classification and Identification of Soils," by Arthur Casagrande, *Transactions*, ASCE, Vol. 113, 1948, p. 910, Table 3.

[7] *Ibid.*, pp. 912–915, Table 4.

Studies also indicated that the expansion varied with the quantity of air contained in the material. At a given density, minimum expansion occurs when the voids are approximately full of water. It is desirable, therefore, that expansive types of soils be compacted at a density near the no-swell state and at a moisture content that will approach minimum air voids for this density. The maximum density at which the material will not expand an excessive amount is also desirable to secure the highest possible stability.

Investigations made from 1928 to 1942, on both adequate pavements and flexible pavements that failed, furnished considerable empirical data for correlation of the CBR requirements with service behavior. From these data, curves were formulated for determining the thickness of pavement and base course required to carry traffic adequately over compacted subbase and subgrade materials.[8] Curves A and B, Fig. 5, show the minimum thickness of reinforcement used in 1942 for light and medium heavy traffic on the California highway system. Curve A was developed on the basis of current experience and is believed to be adequate generally for average traffic conditions. Curve B represents values developed from the original survey of the Materials and Research Department of the California Division of Highways. It demonstrates that foundation conditions under the pavement and the surfacing may be considered satisfactory at the time of the survey. These latter values are con-

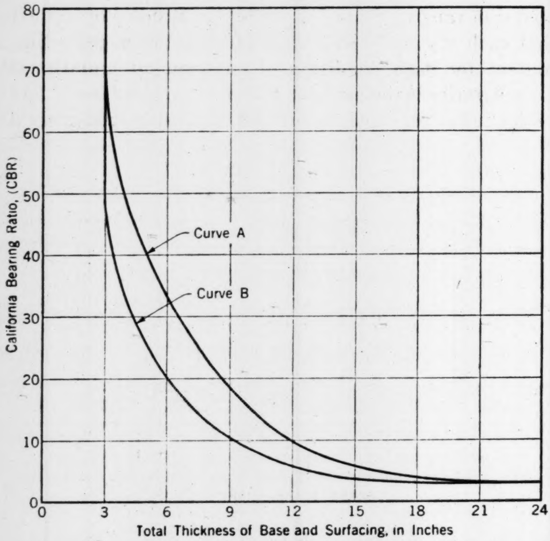

FIG. 5.—TOTAL THICKNESS OF BASE AND SURFACING IN RELATION TO CBR VALUES

sidered a minimum requirement on roads with light traffic. With the help of these curves pavements are designed by securing samples of the proposed materials, and by preparing and testing them in accordance with the procedures described in the preceding paragraphs. The bearing ratio obtained from the test is then projected horizontally from the left-hand side of Fig. 5 until it intersects the design curve applicable to the project. The thickness of pavement and base required above the soil tested is then read from the bottom scale directly below the point of intersection. The same procedure may be repeated for one or more successively stronger base materials, thus developing a pavement design having a layered base construction. The minimum thickness of surfacing required is 3 in., and no surfacing is laid directly on a base with a bearing ratio of less than 50% (see Fig. 5).

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[8] "Designing Foundation Courses for Highway Pavements and Surfaces," by Fred J. Grumm, *California Highways and Public Works*, March, 1942, p. 6.

ADAPTATION TO THE DESIGN OF AIRFIELD PAVEMENTS

By T. A. Middlebrooks,[9] Assoc. M. ASCE, and G. E. Bertram[10]

In the development of the design method based on the use of the CBR test, the California Division of Highways tested adequate sections of flexible highway pavements and sections that failed and correlated the service behavior of the pavement with its bearing ratio. These data were consolidated into a design chart as explained in paper No. 2. This design chart shows the total thickness of base and surfacing required for a flexible highway pavement on a subgrade with a given CBR value. On this chart (see Fig. 5), as stated by Mr. Porter, curve A represents a total thickness of pavement and base course above the subgrade adequate for average highway traffic conditions, and curve B indicates the total thickness considered to be the minimum requirement for light highway traffic.[11]

The empirical nature of the tests made it possible to adapt the CBR equipment and procedures with only minor changes; however, to make use of the design chart, it was essential to evaluate curves A and B in terms of airplane traffic. As drawn on the highway chart, the lines represented a large number of repetitions of highway traffic moving in narrow lanes. In converting these data to equivalent airplane wheel loads, the relationships between the factors—

a. Relative weights of loaded vehicles;
b. Characteristics of airplane and automobile tires; and
c. Lateral distribution of traffic

—were considered. As it was believed that curve A, Fig. 5, was the most reliable, it was used as a basis for the conversion. Although the highway curves were originally drawn for lighter wheel loads, it was known from service behavior of the pavements that 9,000-lb truck wheel loads were supported without distress throughout the life of the pavement. Highway loadings were carried on tires with a deformation of less than 10% whereas data for airplane tires indicated a general deformation of about 35% which resulted in a larger area of contact for any given load. Highway traffic was considered as moving essentially in the same narrow lane whereas the limited information available indicated that runway traffic was fairly well spread over the width of the slab with about 50% of the traffic concentrated in the center third of the pavement. On the basis of these factors, recognizing that the limited data would necessarily need to be supplemented by further tests and observations, it was decided that curve A, Fig. 5, could be assumed to represent a 12,000-lb airplane wheel load. Curve B was judged on the same basis to represent a 7,000-lb wheel load since this was the approximate wheel loading of training planes and represented the lightest traffic requirement for airfields.

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[10] Prin. Engr., Soils Branch, Eng. Div. (Civil Works), Office, Chf. of Engrs., Dept. of the Army, Washington, D. C.

[11] "Designing Foundation Courses for Highway Pavements and Surfaces," by Fred J. Grumm, *California Highways and Public Works*, November, 1941, p. 4.

Following the establishment of values for the basic curves, wheel load graphs for heavier airplane loadings were determined by extrapolation. The program of static load bearing tests conducted by the Corps of Engineers had shown that the deformation, under wheel load, of an adequately designed flexible pavement is made up of three factors—settlement of the subgrade, compaction of the base and the surface, and elastic deformation. Shear deformation was eliminated because, in a satisfactory pavement, the shearing stress does not exceed the shearing strength. Service behavior records of adequate pavement had indicated that it was necessary for elastic deformation to govern over an extensive period of use. Accordingly, the Office of the Chief of Engineers decided to develop empirical curves by extrapolating the original data on the basis of the elastic theory. Since all bearing tests (even those conducted on confined specimens such as the CBR) are essentially shear tests and since shear deformation must be eliminated in a satisfactory pavement, shear stresses were used as a guide in making the extrapolation.

A review of airplane tire data indicated that a uniform tire pressure of 60 lb per sq in. could be selected for the entire group of planes in use. As B-17's and B-24's were the prevalent types, wheel loads of 25,000 lb, 40,000 lb, and 70,000 lb were selected to cover the range of heavy aircraft loads. Contact areas were computed from wheel loads and tire pressures. Circular areas were used for ease of computation and also because the difference in shear stresses in the base course and subgrade did not vary materially for elliptical and circular areas. Shear stresses were computed as shown in Fig. 6 by the use of the stress tables.[13] The thicknesses of base course and pavement corresponding to bearing ratios of 3%, 5%, 7%, and 10% were located on the stress curve

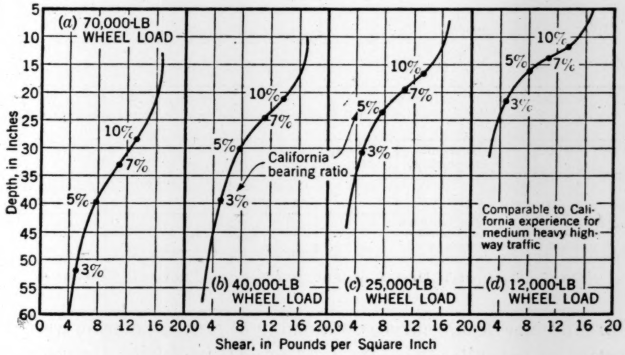

FIG. 6.—EXTRAPOLATION OF HIGHWAY PAVEMENT THICKNESSES BY THE ELASTIC THEORY (TIRE PRESSURE, 60 LB PER SQ IN.)

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[13] "Application of Elastic Theory and Plasticity to Foundation Problems," by Leo Jürgenson, *Journal*, Boston Soc. of Civ. Engrs., July, 1934, p. 242.

for the 12,000-lb load curve and the stresses corresponding to these thicknesses were noted. On the basis that these stresses should not be exceeded for other wheel loads to retain a uniform standard of design, the stress values were located on the curves for the 25,000-lb, 40,000-lb, and 70,000-lb wheel loads (Figs. 6(*c*), 6(*b*), and 6(*a*)). (The values indicated by the curve in Fig. 6(*d*), for the 12,000-lb wheel load, were comparable to the California experience for medium heavy highway traffic.) The thickness corresponding to these stresses was transferred to the graph of thickness versus bearing ratio, and curves similar to those shown in Fig. 7 were drawn.

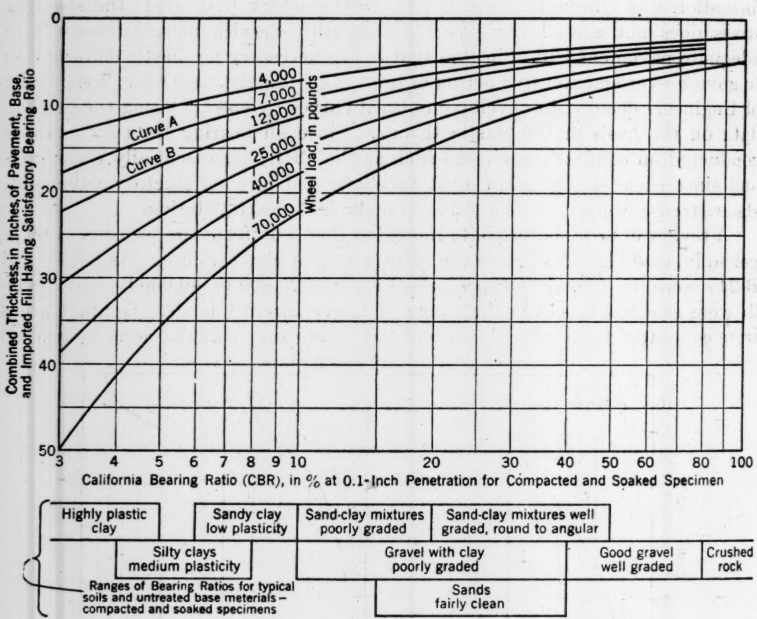

Fig. 7.—Tentative Design of Foundations for Flexible Pavements

The preliminary design curves obtained by this extrapolation were presented to a meeting of consultants in Washington, D. C., which included engineers from the Office of the Chief of Engineers, Mr. Porter, who developed the California method for highway pavements, and Arthur Casagrande, M. ASCE. The consultants had each made independent calculations to extrapolate the basic curves. Those of Mr. Porter were based on an allowable deformation for the various loads, whereas those of Professor Casagrande were based on the relationships between the relative size of the loaded areas. The three sets of computations were in substantial agreement when the data were reviewed by the group. It was decided that the average thicknesses shown by the three extrapolations were reasonable for the low CBR values; however, the majority of the members agreed that the less conservative values should be chosen for the higher CBR values. Based on the aforementioned extrapolations and conference decisions, tentative design curves were drawn, as shown in Fig. 7, which expressed the best judgment of the engineers from the Office of the Chief of Engineers, and of the consultants.

At this same meeting an initial program of field and laboratory testing was drawn up to check the tentative design curves immediately, before they were used for design of airfield pavements. The tests are described in subsequent papers; however, it is not amiss to state that, in general, the results of the studies have substantiated the tentative design curves and no appreciable changes in the design curves have been necessitated.

The ranges for the soil types indicated at the bottom of Fig. 7 are approximate. It is intended that a given design shall be based on actual test results.[13] Thicknesses derived from Fig. 7 are for average conditions based on a tire pressure of 60 lb per sq in., and they should be increased or decreased as much as 20%, depending on tire pressure, thickness and type of pavement and base, characteristics of imported fill, ground-water and drainage conditions, possible effects of frost action, and frequency of loading. In determining the required thickness of pavement and base for the total thicknesses of less than 6 in., the minimum CBR will be used instead of the CBR at 0.1-in. penetration.

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[13] "The Preparation of Subgrades," by O. J. Porter, *Proceedings*, Highway Research Board, National Research Council, December, 1938, Pt. II, p. 324.

ACCEPTED PROCEDURE FOR THE CBR TEST

By William H. Jervis,[14] M. ASCE, and Joseph B. Eustis,[15] Assoc. M. ASCE

From time to time after the adoption of the California method of design for flexible pavements, the Corps of Engineers found it advisable to modify the method of preparing remolded specimens, so that the CBR test would give results more nearly applicable to field conditions found throughout the United States and in overseas military areas. These modifications were based on the results of a comprehensive laboratory study performed by the Waterways Experiment Station.

The purpose of this paper is to present the pertinent results from the aforementioned investigation and to describe in detail recommended procedures for the preparation of remolded and undisturbed specimens and for the penetration part of the test. However, to give the reader a better understanding of the development of the CBR test by the Corps of Engineers, a brief discussion of the original California method of preparing test specimens and a review of the control tests used by the California Division of Highways and the Corps of Engineers, together with a review of the initial modifications to the California method of preparing test specimens made by the Corps of Engineers, will be presented first.

California Method of Preparing Remolded Specimens for Design Test

For the purpose of design, the California State Highway Department developed a mechanical laboratory procedure of preparing the soil for test, as follows:

1. The moisture-density relation for the soil is determined by compacting specimens at several moisture contents in a 6-in.-diameter mold with a static load of 2,000 lb per sq in.
2. For the CBR test, the soil is remolded and compacted at optimum moisture as determined in step 1, under a static load of 2,000 lb per sq in.
3. The test specimen is soaked from the top and bottom for a period of 4 days. During the soaking period the top of the specimen is confined with a surcharge weight of $12\frac{1}{4}$ lb (equivalent to the weight of about 5 in. of pavement).

Information gained from a study of the California procedure indicates that, if because of special conditions it is anticipated that the usual field compaction will not be obtained or that the laboratory method will not duplicate field compaction, a special laboratory procedure is used for preparing the soil samples that will produce a density equivalent to that expected in the field. If the soil is not to be compacted, penetration tests are conducted on undisturbed soaked soils.

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[15] Cons. Engr., Eustis Eng. Co., Vicksburg, Miss.; formerly Engr., Asst. Chf., Embankment & Foundation Branch, Soils Div., U. S. Waterways Experiment Station, Vicksburg, Miss.

CALIFORNIA COMPACTION TESTS

For compaction tests, California used the static load method as a standard procedure, both in the laboratory and in the field. In this standard test the moisture-density relation was determined by compacting specimens with a static load of 2,000 lb per sq in. in the CBR mold. However, a dynamic compaction method was used as an alternate field procedure for control on a large percentage of construction jobs, where quick control was required, because of variation in the material.

COMPACTION CONTROL TEST SPECIFIED BY THE CORPS OF ENGINEERS

The California static compaction test was not considered desirable for construction control because the field laboratories did not have access to the compression machines needed for the tests. The California dynamic compaction test also required special equipment which would have to be constructed. Time was not available to construct equipment and train personnel in the use of this new equipment. The laboratories did have equipment for conducting the compaction test[16] developed by the American Association of State Highway Officials (AASHO); therefore, a modification of this test was developed for construction control. This modification consisted of increasing the weight of the hammer from 5 lb to 10 lb, the height of drop from 12 in. to 18 in., and the number of layers in which the soil was placed in the mold from 3 to 5, and required changing the hammer used in the AASHO compaction test but did not change the molds or other apparatus. A comparison of compactive efforts from limited data available indicated that the density results obtained by this modification of the AASHO compaction test closely approximated those obtained by the California dynamic compaction tests.

EARLY MODIFICATIONS TO CALIFORNIA METHOD OF PREPARING REMOLDED SPECIMENS FOR DESIGN TEST

The modified compaction test first appeared as a construction control in the June, 1942, publication of the *Engineering Manual for War Department Construction*.[17] It has since become known as the modified AASHO compaction test. A partly modified California procedure for preparing test specimens was included in this manual. The procedure included the use of the modified AASHO compaction test as a control, and required that CBR test specimens be prepared under a static load of 2,000 lb per sq in. at an optimum water content predetermined by the modified AASHO compaction test.[17]

After a short period of compacting the samples for CBR design tests in accordance with this procedure, it was evident that:

1. The densities of the samples being prepared for the CBR test were at considerable variance with those obtained by field equipment. This variation was particularly true for sandy materials of low plasticity, for which field

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[16] "Standard Specifications for Highway Materials," AASHO, Washington, D. C., 1947, test No. T99-38.

[17] "Airfield Pavement Design," *Engineering Manual for War Department Construction*, Office of Chf. of Engrs., U. S. Army, Washington, D. C., June, 1942, Pt. II, Chapter XX.

densities were greater than could be obtained by the static laboratory compaction method.

2. This differential in density between the laboratory and the field on soils of low plasticity gave CBR values which were less than could be expected in the field. Therefore, unless the laboratory density on soils of low plasticity was improved, the CBR values for plastic soils and soils of low plasticity would not be comparable.

In view of the shortcomings of the partly modified California method of preparing test specimens, it was considered necessary to modify the method further so that, when penetrated, the specimens would more nearly reflect the relative stability of the soil as compacted during construction and as later affected by moisture changes. Two methods of compacting the soil for CBR design tests were therefore considered:

a. A static load method using a variable load to produce the density desired; and

b. An impact method similar to that adopted for field compaction control tests (modified AASHO test).

After preliminary study, method *b* was selected to compact the samples for the CBR design test, since (1) the method was similar to that adopted for compaction control tests; (2) the equipment for the dynamic method was available; and (3) static loads greatly in excess of 2,000 lb per sq in. were required to compact sandy samples to the density required. As previously stated, laboratories were not supplied with static load equipment of the high capacity required; and, because of the war, such equipment could not be obtained.

The use of the impact method for the preparation of specimens for design tests was adopted as a second modification by the Corps of Engineers in September, 1942. It was assumed at the time that the main factor in the preparation of the specimens for design tests was to produce the required density. Also, at that time, the use of a surcharge during penetration was introduced into the test procedures to make the test more applicable for use with cohesionless materials. During the test the penetration surcharge was specified to be equal to the anticipated overburden in the field. For cohesionless soils it was specified that a drainage period of 15 min should be allowed for removing any free water remaining on the surface of the specimen after soaking, thus preventing any surface disturbance or softening that would result in low CBR values.

The California Division of Highways, as well as several other state highway departments and agencies, found through experience and field observations that subgrade and base course soils (except clean sands) under impervious pavements usually increased in water content by capillarity and condensation of moisture, regardless of the ground-water elevation, and in some cases became saturated. Although it was recognized that this condition would not occur below all pavements, it was impossible to predict the ultimate moisture accurately. Therefore, the Corps of Engineers, to be conservative, specified a design based on moisture conditions comparable to those obtained in soaked laboratory specimens which were considered to approach the field saturated condition. For this reason it was stipulated that the CBR design penetration test should be performed on soaked specimens. However, provisions were made to reduce the total pavement thickness 20% for ideal subgrade conditions and for subgrades with low moisture.

COMPREHENSIVE LABORATORY STUDIES ON THE CBR TEST

Further modification and refinement in the procedure for the preparation of test specimens and in the method of conducting the CBR test appeared desirable, because of variations in test results obtained between different laboratories on the same material and between successive tests on the same material in one laboratory. Therefore, a program of investigation was formulated in November, 1942, to be conducted at the Waterways Experiment Station. The program included laboratory tests on twenty soils, to determine the effect of different variables on the CBR, such as the method of compaction, water content, density, time of soaking, soaking surcharge, drainage time after soaking, penetration surcharge, rate of penetration, and effect of oversize particles. The principal conclusions and recommendations obtained from the laboratory study are given in the following paragraphs. Details of the study have been published by the Waterways Experiment Station.[4]

The laboratory investigation showed that variations in CBR test results are largely explained by the method of preparing the test specimens. The variations are systematic, however, and are caused primarily by the effects of three variables—molding water content, density, and method of compaction. The variations are probably valid qualitatively; but they may not be strictly valid quantitatively, partly because of the confining effect of the 6-in.-diameter CBR mold. Consistent laboratory results can be obtained only when the variables listed are given full consideration. Unconfined compression tests and triaxial shear tests indicated that the shearing resistances at low strains show trends similar to those found in the CBR test for comparable test specimens. Inasmuch as the molding water content appears to be a prime factor in controlling the structural properties of all except free-draining soils, it follows that, in remolded soil, duplicate laboratory specimens cannot be prepared unless molding water contents are duplicated, even if the water contents and densities obtained subsequent to molding are duplicated. In other words, if a soil is molded at a given water content and then this water content is allowed to increase or decrease by a given amount, another identical soil specimen cannot be reproduced unless the entire cycle is reproduced, starting at the same molding water content.[4] It was felt that dynamic compaction rather than static laboratory compaction produces a soil structure which more nearly simulates that obtained in the prototype by the usual construction methods. This premise was based principally on logic rather than on actual compaction data, since little information was available as to the soil structure produced by field compaction equipment. Subsequently, some field work was accomplished which indicated that, in general, the structure produced by field compaction is more nearly duplicated by laboratory dynamic compaction than by static compaction, as evidenced by unconfined compression tests and by triaxial shear tests.

Conclusions.—As a result of the laboratory studies, the following conclusions with respect to the preparation of remolded specimens appear warranted:

1. The maximum dry densities obtained by the modified AASHO dynamic compaction test do not agree with those obtained by the California static method with 2,000 lb per sq in. The variation may be as much as 20 lb per cu ft, depending on the type of soil.

2. Under laboratory dynamic (dropping hammer) compaction, practically all soils develop compaction curves with definite optimum water contents. The compaction curves for soils of low plasticity developed by the static method of compaction often do not show a definite optimum water content.

3. Material should not be used more than once in establishing the water content-density relationship or physical characteristics of the original soil. All control compaction for CBR tests should be performed in the 6-in.-diameter CBR mold with the mold placed on a concrete floor or pedestal for firm support.

4. Small changes in density greatly affect the CBR, especially at high densities in the order of magnitude of modified AASHO or higher. The effect of density is most pronounced for cohesionless soils and soils of low plasticity.

5. Small changes in molding water content greatly affect the CBR of unsoaked laboratory samples, except clean sands and gravels. At a constant density in the unsoaked condition, the higher the molding water content the lower the CBR.

6. The CBR for all soils compacted statically, and soaked, showed an increase with increase in molding water content at a constant density.

7. The CBR for impervious, high-swelling soils compacted dynamically and soaked showed an increase with increase in molding water content for a constant density.

8. The CBR for specimens of soils of low plasticity with little or no swell, when compacted dynamically and soaked, showed an appreciable decrease with a slight increase in molding water content for a constant density This result is especially significant, since it is generally believed that the shearing resistance is not sensitive to the molding moisture within normal laboratory control for soaked specimens compacted to a given density.

9. Specimens compacted statically, on the dry side of optimum, swell more when soaked than specimens compacted dynamically to the same density at the same moisture.

10. Laboratory soaked specimens of soils with low plasticity compacted at optimum moisture content or compacted at drier than optimum moisture content by the dynamic method have greater CBR values than corresponding specimens compacted by the static method. On one sand-clay tested, the CBR for dynamically compacted soaked specimens was approximately two and one-half times greater than statically compacted soaked specimens at equal densities and at approximately modified AASHO optimum.

11. With one exception, the same trends described for the CBR test in conclusions 6 through 10 also occur in unconfined compression tests and quick triaxial tests when maximum strength and stresses at low strain are considered. The exception is the test involving maximum strength values, with soaked

specimens compacted dynamically, which showed trends similar to those obtained for soaked specimens compacted statically. This agreement is considered to be qualitative only, thus indicating the effect of mold size in CBR tests at high density.

12. The CBR specimen height should not be less than 4½ in. for all soils.

13. The most practical method for soaking test specimens is by submergence (soaking from top and bottom) for 4 days. Less time is required for some cohesionless soils.

14. In general, the effect of varying the soaking surcharge on the CBR increases with increase in plasticity of the soil. The soaking surcharge greatly affects the CBR of soils with from medium to high plasticity; it moderately affects that of soils of low plasticity; and it has practically no effect on the CBR of cohesionless soils.

15. A satisfactory drainage time for all soils is 15 min.

16. The increase in CBR values for some soils above standard AASHO density may be partly caused by the confining effect of the 6-in.-diameter mold.

Conclusions with respect to the penetration part of the test are as follows:

(1) The penetration surcharge greatly affects the CBR of cohesionless soil, moderately affects that of soils of low plasticity, and has practically no effect on that of soils of from medium to high plasticity.

(2) A penetration rate of 0.05 in. per min will give satisfactory results for all soils.

(3) For consistency and more uniform results, all stress-penetration curves that are concaved upward should be corrected by adjusting the zero point of the curve to establish a new origin.[4]

(4) A closely-controlled constant-strain type of loading apparatus should be used.

(5) Test results on samples containing stones are erratic and modification to the test procedure is needed for these soils. Until such a procedure is developed, several tests should be performed to obtain average representative results.

RECOMMENDATIONS

Specimen Preparation Procedure.—The equipment required for preparing and testing remolded specimens and undisturbed samples, or for performing field, in-place, CBR tests, is described in detail elsewhere.[18] All soils prepared for the design test should have the same density and moisture conditions as expected in the field. Whenever the modified AASHO method of control compaction is specified, the compaction tests should be performed in the 6-in.-diameter CBR mold using 55 blows per layer. The mold should be placed on a concrete floor or a pedestal and material should not be reused. The following procedures for various types of soil are applicable to specifications of the Corps of Engineers. Two methods of tests are described.

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[18] "The California Bearing Ratio Test As Applied to the Design of Flexible Pavements for Airports," *Technical Memorandum No. 213–1*, U. S. Waterways Experiment Station, Vicksburg, Miss., 1 July 1945, p. 30, Figs. 4, 5, and 6, and p. 40, Fig. 9.

Method 1.—When no unusual construction or weather conditions exist, method 1 will furnish satisfactory CBR information on plastic soils exhibiting little or no swell. The recommended step-by-step procedure for preparing remolded specimens for test is as follows (the procedure being extended to cover soaking, where testing under this condition is required):

(*a*) All material larger than ¾ in. should be removed and replaced with an equal proportion of material between 0.18 in. (No. 4 sieve) and ¾ in. in size.

(*b*) Control compaction tests should be conducted with a sufficient number of test specimens to establish, definitely, the optimum water content for 100% of modified AASHO density. Four or five specimens should be compacted with water contents within ± 2% of optimum water content, so that the optimum condition can be established rigidly. The fall of the hammer must be controlled carefully and the blows must be distributed uniformly over the specimen. This procedure establishes the moisture content at which specimens for CBR tests should be molded.

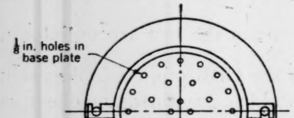
1/8 in. holes in base plate
(a) PLAN
1/16 in. chamfer on end of studs
15/16 in.
1 5/16 in.
6 1/2 in. diameter
6 in. diameter
1 in.
1/4 in.
2 1/2 in.
3/8 in. studs opposite
3/4 in.
7 in.
5 15/16 in. diameter
3 1/2 in.
Disc
2 in.
1/16 in.
3/8 in.
6 7/32 in. diameter
8 3/8 in. diameter
9 3/8 in. diameter
(b) ELEVATION

FIG. 8.—VIEW OF STEEL CYLINDER WITH COLLAR

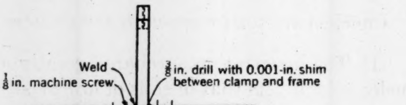
Weld
1/8 in. machine screw
3/8 in. drill with 0.001-in. shim between clamp and frame
1/4 in.
1/4 in.
1/2 in.
2 1/2 in.
1/2 in.
(a) PLAN
6 1/2 in.
Clamp
Dial
1/4 in.
1/2 in.
3/4 in.
4 3/4 in.
3/8 in.
3/8 in.
3/8 in.
3 in.
1/2 in.
1/2 in.
1/16 in.
1/16 in.
3/4 in.
6 in.
7 1/2 in.
3 in.
(b) ELEVATION
(c) SIDE VIEW

FIG. 9.—VIEW OF STEEL TRIPOD FOR DETERMINING EXPANSION

(*c*) For the CBR tests the mold should be fitted with an extension collar and base plate (Fig. 8).[19] The mold should be clamped with the fitted extension collar to the base plate and the spacer disk should be inserted over the base plate. A 6-in.-diameter coarse filter paper or wire mesh should be placed on top of the disk.

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[19] "The California Bearing Ratio Test As Applied to the Design of Flexible Pavements for Airports," *Technical Memorandum No. 213-1*, U. S. Waterways Experiment Station, Vicksburg, Miss., 1 July 1945, p. 30 Fig. 5.

(*d*) When results are required for a soil at 95% modified AASHO density, three specimens should be compacted at the optimum water content for 100% of modified AASHO compaction, using a different number of blows for each specimen—namely, 55, 25, and 10 blows per layer. The maximum allowable variation in the molding water content should not be more than ±0.5%. Any specimens not falling within this range should be discarded and a new specimen compacted that does meet this requirement. If specifications call for other than 95% AASHO density or other than 100% AASHO moisture, revisions must be made in this procedure to obtain specimens at the required density.

(*e*) The collar should be removed, the specimen trimmed, a screen or 6-in.-diameter coarse filter paper placed over the top of the specimen, and a perforated base plate clamped to the top of the test mold.

(*f*) The test mold should be inverted, the base plate and the spacer disk removed, and the density of the specimen determined.

(*g*) The adjustable stem and plate should be placed on the surface of the specimen and an annular weight applied to produce an intensity of loading equal to the weight of the base material and the pavement within ±5 lb—but the weight shall not be less than 10 lb.

(*h*) The mold and the weights should be immersed in water to allow free access of the water to the top and the bottom of the test specimen. Initial measurements should be taken for swell, using the dial gage and tripod (Fig. 9).[19] Specimens must be allowed to soak for 4 days. A shorter period of time may be permissible for more pervious materials. Final swell measurements should be taken at the end of the soaking period and the swell computed in percentage of initial specimen height.

(*i*) The specimen must be taken out of water and free surface water removed, taking care not to disturb the surface of the specimen. The specimen must be permitted to drain downward for 15 min. When removing surface water from impervious samples, it is necessary to tilt the samples. When this is done, the weight should be held firmly in place. The perforated plate and surcharge weights should then be removed and the specimen weighed. The specimen is now considered ready for the penetration test.

(*j*) The results of tests on all specimens should be plotted to show the relation between density and CBR. The use of this plot can be seen from Fig. 10.[20]

To obtain this plot, specimens were prepared with modified AASHO effort to determine the compaction curve for this effort (shown with points indicated as circles). This series of tests determines the moisture content at which CBR tests should be made (which is optimum for the modified AASHO effort). Two more specimens are then prepared for test at the optimum water content for modified effort using lower efforts of 26 blows and 10 blows per layer.

Three tests have now been made at the optimum moisture for modified AASHO effort, with this effort and two lower efforts. When these specimens have been soaked and penetrated they yield a curve, such as shown in Fig. 10, of CBR versus dry density at constant molding moisture content. As will be shown later under method 2, this represents one curve (8% moisture) in the center plot in Fig. 11. This curve intersects the 95% density intercept at a CBR of 45% which is the value which should be used in design.

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[20] "The California Bearing Ratio Test As Applied to the Design of Flexible Pavements for Airports," *Technical Memorandum No. 213-1*, U. S. Waterways Experiment Station, Vicksburg, Miss., 1 July 1945, p. 38, Fig. 7.

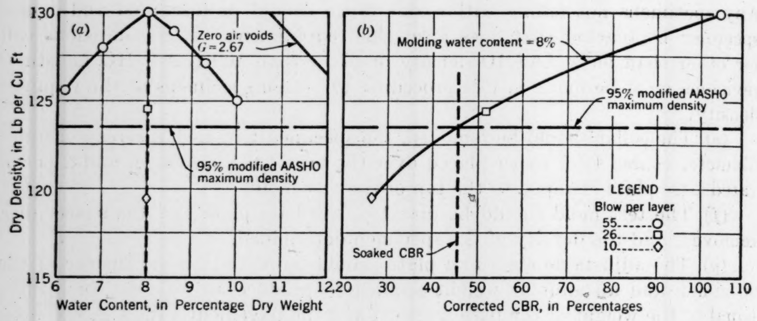

FIG. 10.—METHOD 1; LABORATORY TEST PROCEDURE RECOMMENDED WHEN LACK OF TIME OR EQUIPMENT PREVENTS A MORE THOROUGH INVESTIGATION

It is emphasized that method 1 will give satisfactory results in the laboratory only when the molding water content is controlled closely within the tolerance specified in recommendation (*d*). Since the tests are performed for one molding water content, no indication (either quantitative or qualitative) is given as to how this soil group will behave if placed in the field at any water content other than that for which the laboratory tests are performed.

Method 2.—Where unusual construction and weather conditions are likely to occur and where soils are encountered that are very critical with changes in moisture, method 2, as described herewith, is recommended for plastic soils exhibiting little or no swell, as it furnishes more complete information. CBR test results are affected by the density and molding water content of the soil specimens. If detailed information is desired regarding the effect of molding water content and density on CBR over a wide range of values, it is recommended that the variations be determined for typical soils encountered on the job. A series of specimens should be prepared and tested, as follows:

(*a*) All specimens should be prepared in a manner similar to that outlined under method 1, except that each specimen used in the development of the 55-blow compaction curves should be penetrated. In addition, the complete compaction curves for the 25-blow and 10-blow per layer compactive efforts should be developed and each test specimen compacted should be penetrated. If both soaked and unsoaked data are desired, the specimen may be molded with alternate samples being soaked before penetration. All compaction is performed in the 6-in.-diameter CBR mold, using the 10-lb hammer dropped 18 in. on each layer. It may be necessary to include an effort greater than the 55-blow effort, in the event greater compaction in the field is possible, as a result of initial rolling or traffic with very heavy wheel loads.

(*b*) The data from method 2(*a*) should be plotted as shown in Fig. 11.[21] As in method 1, the material was a clay-sand with a liquid limit $w_L = 17$ and a plasticity index $I_P = 4$. It is assumed that specified field control required a minimum dry density of the 95% modified AASHO maximum and a water content range of from 6% to 8%. All specimens were packed in five layers in the CBR mold using the 10-lb hammer and an 18-in. drop, after which they were soaked for 4 days in the CBR mold and then penetrated. As in method 1, a surcharge of 10 lb was applied for the soaking and the penetration tests. The tests plotted in Fig. 11(*b*) revealed no swelling during the time that the specimen was being soaked. The shaded area in this illustration is the region that will yield a CBR range of from 46% to 105%.

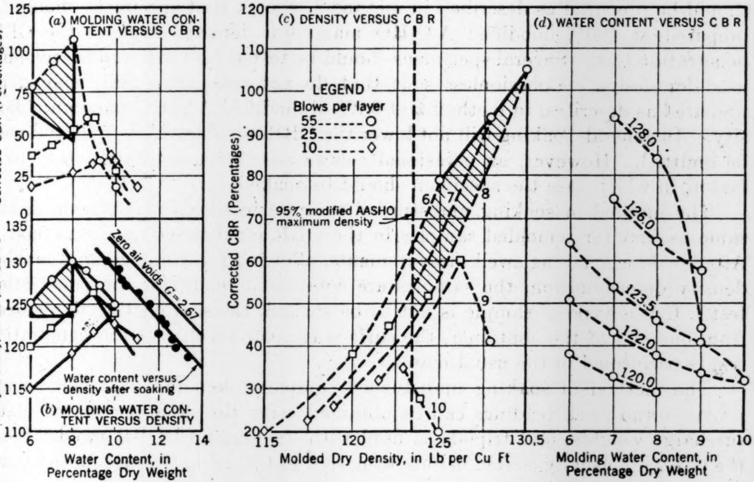

FIG. 11.—METHOD 2; LABORATORY TEST PROCEDURE RECOMMENDED WHEN A THOROUGH INVESTIGATION IS DESIRED

Method 2 is valuable to obtain test results on soils that are greatly affected by small changes in density and molding water content; it supplies a picture of the CBR characteristics, within the range of the field control expected, which will be useful in establishing the limiting CBR values. The test results as obtained by method 2 should be used in connection with the design curves, with the full understanding that the variations obtained may be valid only qualitatively.

Comment.—The procedure for preparing specimens of all swelling soils is the same as that for soils of low plasticity, except that the specimens should be prepared at a water content and a density controlled by swelling tests. The swelling tests are performed on the samples used in developing the compaction curves. Method 2 is the most desirable in this case, in that full cognizance can be taken of the swelling effects for three compactive efforts. On the basis of the density and the moisture conditions to be expected in the field, the proper CBR value can be selected from the family of curves. If method 1 is used, the moisture content at which swelling ceases is noted. This cessation usually occurs at optimum or on the wet side of optimum. Specimens for the CBR test are then prepared at the moisture and the density indicated. If the moisture control at the point of swell is appreciably on the wet side of modified AASHO optimum a compactive effort (less than modified) should be chosen for which the selected moisture is approximately optimum.

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[21] "The California Bearing Ratio Test As Applied to the Design of Flexible Pavements for Airports," *Technical Memorandum No. 213-1*, U. S. Waterways Experiment Station, Vicksburg, Miss., 1 July 1945, p. 39, Fig. 8.

Cohesionless soils (I_P less than 2) that will readily compact under rollers or traffic, to the maximum density specified by the modified AASHO method, should be prepared as described in method 1, except that specimens should be prepared at 100% modified AASHO maximum density only for the CBR penetration test. Several specimens should be tested for CBR and the average used for design. Cohesionless soils that do not compact readily should be prepared as described in method 1 at 95% of modified AASHO maximum density. In general, soaking will not lower the CBR of cohesionless soils and may be omitted. However, on occasional cohesionless soils, saturation may be a factor, in which case the specimen should be soaked.

The method of soaking undisturbed samples in molds or cylinders is the same as that for remolded samples in the CBR mold described in method 1. After soaking, making swell measurements, allowing for drainage, and making density determination, the samples are considered ready for the penetration test. If this type of sample is not to be soaked, the seal coating is removed from one end of the container, the surface is made level, and the penetration test is performed in the usual manner.

The method of soaking undisturbed samples taken in boxes can be improvised and swell readings can be obtained using the same perforated plate, surcharge weights, and tripods as used with the standard CBR mold. Since the boxes are usually coated before being sent to the field, the swell in the boxes themselves is negligible. In the event soaking is not required, the seal coating can be removed from one end, the surface leveled with a thin cover of sand if necessary, and the penetration test conducted with the standard testing equipment in the usual manner. The fact that the box construction must be rigid should be stressed.

Penetration Test Procedure.—The laboratory step-by-step procedure for this phase of the design test is as follows, and should be used on undisturbed or remolded samples after the testing surface has been prepared:

(*a*) A penetration surcharge should be applied on all soils sufficient to produce an intensity of loading equal to the weight of the base material and the pavement (within ±5 lb)—if a pavement is to be constructed which will overlie the soil in the prototype represented by the sample, except that the weight shall not be less than 10 lb. The weight applied must be estimated; and, if it does not produce the intensity present in the final design, the test should be repeated. If the sample has been previously soaked, the penetration surcharge should be equal to the soaking surcharge which, in turn, should have been estimated and governed by the conditions described in method 1(*g*). On soils in a condition for which it is expected that a low CBR value will be obtained, it is advisable to apply the penetration piston and the penetration surcharge weights in either one of two ways, to prevent upheaval of the soil into the hole of the surcharge weights. In the first method, one 5-lb annular disk surcharge should be applied to the soil surface, the penetration piston then seated with a 10-lb load, and finally the remainder of the surcharge applied by the use of slotted 5-lb surcharge weights. In the alternate method, a special locking and alinement device as shown in Fig. 12 can be used.[22] (The ¼-in. pin at the top is so fitted that the penetration piston is flush with the bottom of the surcharge weight.)

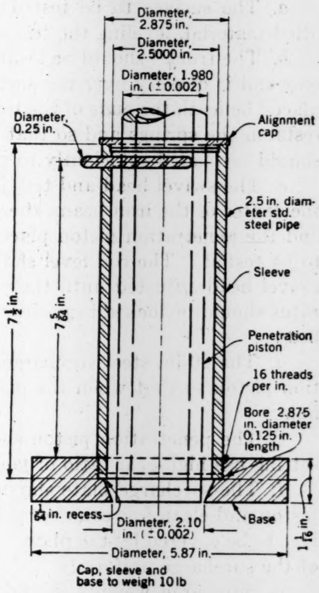
Diameter, 2.875 in.
Diameter, 2.5000 in.
Diameter, 1.980 in. (± 0.002)
Diameter, 0.25 in.
Alignment cap
2.5 in. diameter std. steel pipe
Sleeve
Penetration piston
16 threads per in.
Bore 2.875 in. diameter 0.125 in. length
7½ in.
7 5/64 in.
1/64 in. recess
Diameter, 2.10 in. (± 0.002)
Base
1 1/16 in.
Diameter, 5.87 in.
Cap, sleeve and base to weigh 10 lb

FIG. 12.—LOCKING AND ALINEMENT DEVICE FOR PENETRATING SURCHARGE WEIGHTS AND PISTON

(*b*) The penetration piston should be seated with a 10-lb load and the dial gage set at zero. The purpose of a 10-lb load before starting the penetration test is to insure satisfactory seating of the piston, and it should be considered as the zero load when determining stress-strain relations.

(*c*) Load on the penetration piston should be applied so that the rate of penetration is approximately 0.05 in. per min. Load readings should be obtained at deformations of 0.025 in., 0.050 in., 0.075 in., 0.1 in., 0.2 in., 0.3 in., 0.4 in., and 0.5 in. In using manually operated loading devices, it may be necessary to take more load readings as an assistance in controlling the rate of penetration.

(*d*) The moisture content should be determined in the upper inch and, in the case of the laboratory tests, for the entire depth of the sample.

(*e*) The penetration load in pounds per square inch should be computed and the stress-penetration curve drawn. To obtain true penetration loads from the test data, the zero point of the curve should be adjusted to correct for the initial concave-upward shape, if present.

(*f*) The corrected load value should be determined at 0.1-in. and 0.2-in. penetration. Next, the corrected CBR should be determined for 0.1-in. and 0.2-in. penetration by dividing the load at 0.1 in. and 0.2 in. by the standard loads of 1,000 lb per sq in. and 1,500 lb per sq in., respectively. Each ratio should be multiplied by 100 to convert to percentages.

(*g*) The CBR usually selected is at 0.1 in. penetration. If the CBR at 0.2-in. penetration is greater than that at 0.1-in. penetration, the test should be rerun. If check tests give similar results, the CBR at 0.2-in penetration should be used.

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[22] "The California Bearing Ratio Test As Applied to the Design of Flexible Pavements for Airports," *Technical Memorandum No. 213-1*, U. S. Waterways Experiment Station, Vicksburg, Miss., 1 July 1945, p. 31, Fig. 6.

The step-by-step procedure recommended in the conduct of field in-place CBR tests is as follows:

a. The surface to be tested should be prepared by removing loose and dried material, leveling the test area as perfectly as practicable.

b. The truck[23] should be located so that the center of the iron beam on the rear end is directly over the surface to be tested. The track jacks should be placed beneath the ends of the iron beam and the truck lifted so that no weight rests on the springs and so that the truck is approximately level. The truck should be loaded sufficiently to give a counterreaction of about 7,000 lb.

c. The swivel head and test jack should be installed at the underside and the center of the iron beam, the proving ring connected to the end of the jack, and the penetration piston placed to from within 1 in. to 2 in. of the surface to be tested. The rod level should be fastened to the pipe extension and the swivel head adjusted until the penetration piston is plumb. Then the apparatus should be locked in position by tightening the clamping nut in the swivel head.

d. The 10-in. steel surcharge plate should be placed beneath the penetration piston so that when the piston is lowered it will pass through the center hole.

e. The penetration piston should be seated under a 10-lb load (for rapid setting use high-gear ratio of jack).

f. The surcharge plate should be raised while the seating load is on the piston and clean fine sand spread to a depth of from ¼ in. to ⅜ in. over the surface to be covered by the plate. This procedure serves to distribute the weight of the surcharge uniformly.

g. Surcharge weights should be applied to the steel plate equivalent to the load intensity of the material and to the pavement which will overlie the subgrade or the base, except that the minimum weight applied should be made up of the 10-lb circular steel plate plus one 20-lb surcharge weight. This minimum weight creates an intensity of loading equal to that created by the 10-lb weight used in the 6-in.-diameter CBR mold in the laboratory.

h. The penetration dial clamp should be attached to the piston so that the dial rests on the dial support.

i. The dial gages should be set at zero.

j. Load should be applied to the penetration piston so that the rate of penetration is approximately 0.05 in. per min. By using the low-gear ratio of the jack during the test, a uniform rate of penetration can be maintained by the operator. The deflection of the proving ring should be recorded at penetration depths of 0.025 in., 0.05 in., 0.075 in., 0.1 in., 0.15 in., 0.2 in., 0.3 in., 0.4 in., and 0.5 in. The CBR is then computed as in the laboratory test.

k. At the completion of the test, a sample should be obtained at the point of penetration for water content determination. A sample should also be obtained from about 4 in. to 6 in. away from the point of penetration, for density determination.

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[23] "Development of California Bearing Ratio Apparatus," by W. J. Turnbull, *Civil Engineering*, May, 1945, p. 234.

TEST SECTION NO. 1, STOCKTON FIELD, CALIFORNIA

BY O. J. PORTER,[5] M. ASCE

One of the major field tests in the research program instigated to check the tentative design curves was the accelerated traffic test conducted at Stockton Field in 1942. Authorized by the Office of the Chief of Engineers, the work was done by personnel of the California Division of Highways and of the Sacramento District of the Corps of Engineers, under the supervision of the writer.

DESCRIPTION OF PROJECT

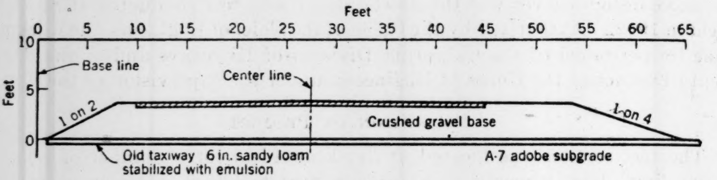
The test section was located at Stockton Field for a number of reasons. In the first place, runways and taxiways that had been constructed in 1936 over the adobe subgrade (CL-OL in the Airfield Classification,[7] A-6 and A-7 in the PRA classification[6]) had failed during the winter of 1940–1941 under the operation of AT6A training planes having wheel loads of only 2,000 lb. This performance indicated that the subgrade was of about as poor a quality as could be expected at any proposed airport site; hence, tests at this site would be valuable in checking the tentative design curves in the low CBR range.

In the second place, the moisture content of the subgrade and the subsoil had become uniformly equalized under an old paved taxiway that had not been subjected to traffic and therefore had not failed. This taxiway provided a satisfactory area for the test section and also for determining the magnitude of the pavement deflection caused by the action of the AT6A planes (2,000-lb wheel load) that caused failure of the adjacent runways.

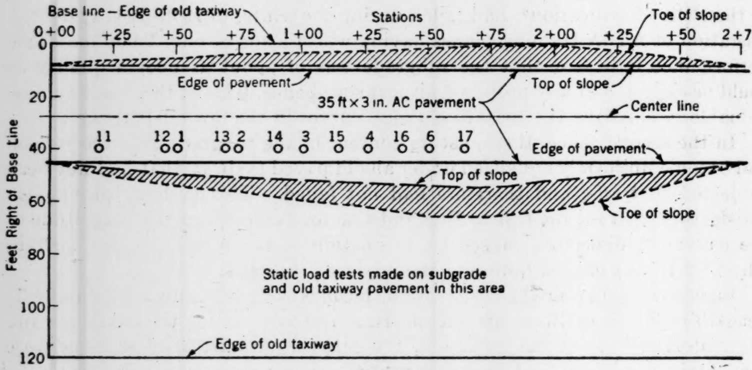
Before constructing the test section, borings were made to a depth of 8 ft beneath both the taxiway and the old runway to determine the character and the uniformity of the subgrade and the subsoil. Undisturbed samples were also obtained under the selected area to determine the moisture content and the bearing ratio of the subgrade. These explorations indicated that the old runways and taxiways consisted of about 6 in. of sandy loam soil, the top 3 in. or 4 in. being stabilized and then sealed with emulsified asphalt. Both the untreated sandy loam and the stabilized material appeared well compacted. The stabilized material also appeared relatively dry and very stable, although somewhat brittle. The subgrade and the subsoil to a depth of 3 ft consisted of black adobe of CL-OL classification (A-7 in the PRA classification). This material contained approximately 30% moisture, and laboratory tests on undisturbed samples gave CBR values ranging from 4% to 8% at 0.1-in. deflection and averaging approximately 5%. Remolded samples compacted to the field density at the natural water content gave CBR values of 2% when confined by a surcharge of 0.45 lb per sq in. during the soaking period and of 8% when confined by a surcharge of 2 lb per sq in. (the approximate pressure beneath 2 ft of pavement and base) during the soaking period.

The subsoil below the black adobe to a depth of 8 ft was found to be a relatively compact gray clay of CL-OL classification (A-7 in the PRA classification) containing from 25% to 30% moisture. In general, the soil conditions were similar beneath the runways that failed and the taxiway area selected for the runway test section, which did not fail.

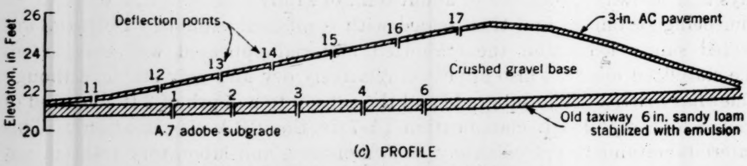
The runway test section (Fig. 13) was constructed on top of the old taxiway after measurements had been made of the deflection produced by both static and dynamic wheel loads of the AT6A plane. Some of the deflection points were left in place to permit measurement of the deflection of the bottom of the base beneath the completed test section. The old pavement was scarified lightly and then rolled before placing new material to provide a bond and to eliminate any possibility of the base slipping on the hard smooth surface of the old taxiway. With the exception of the thickness of sandy loam on the old taxiway, the entire base course consisted of a well-graded mixture of sand and gravel, and of crushed gravel. Whenever the total thickness of pavement


Feet
0
5
10
15
20
25
30
35
40
45
50
55
60
65
Feet
10
5
0
Base line
Center line
1 on 2
1 on 4
Crushed gravel base
Old taxiway 6 in. sandy loam stabilized with emulsion
A-7 adobe subgrade

(a) TYPICAL SECTION


Base line – Edge of old taxiway
Stations
Toe of slope
0+00
+25
+50
+75
1+00
+25
+50
+75
2+00
+25
+50
2+75
0
20
40
60
80
100
120
Feet Right of Base Line
Edge of pavement
Top of slope
35 ft × 3 in. AC pavement
Center line
11
12
1
13
2
14
3
15
4
16
6
17
Edge of pavement
Top of slope
Toe of slope
Static load tests made on subgrade and old taxiway pavement in this area
Edge of old taxiway

(b) PLAN


26
24
22
20
Elevation, in Feet
Deflection points
3-in. AC pavement
11
12
13
14
15
16
17
1
2
3
4
6
Crushed gravel base
Old taxiway 6 in. sandy loam stabilized with emulsion
A-7 adobe subgrade

(c) PROFILE

FIG. 13.—STOCKTON RUNWAY TEST SECTION

and base is referred to in this paper, it includes the 6-in. thickness of sandy loam on the old taxiway.

The base material was obtained from a commercial plant at Tracy, Calif., and the aggregate gradation was closely controlled to insure uniformity throughout the length and the depth of the base on the test section. This material is approximately comparable in quality to many of the pit-run and bank-run gravel bases readily available in many areas of the western states. After soaking, this material had a CBR of 85%. The base was thoroughly compacted in layers from 4 in. to 5 in. thick, placed at optimum moisture to a relative compaction of from 87% to 100% of the maximum density obtained in the static compaction test of the California Division of Highways. Although each layer was rolled and compacted with a 12-ton roller, a small degree of consolidation occurred during the traffic tests as the result of the repeated action of the heavy wheel loads. Where the thickness of the base course ranged from 3 ft to 4 ft, settlements from ½ in. to ¾ in. occurred during the traffic test without causing failure of the pavement. This action is consistent with highway experience where repetitions of dynamic wheel loads produce a greater density than is obtainable during construction. Even when the material is placed in thin layers, the moisture content is carefully controlled, and heavy rollers are used.

Asphaltic concrete pavement 3 in. in compacted thickness was placed over the entire test section. The pavement mixture contained approximately 5% material passing a 200-mesh sieve and 5.4% of 70 penetration asphalt (by weight) to provide a highly flexible pavement without inducing instability in the mixture. The pavement remained stable and showed no signs of either flushing or displacing during the traffic tests. It also proved very flexible and withstood the repetitions of the wheel loads, satisfactorily and without cracking, where the thickness of the base was sufficient to eliminate plastic flow of the subgrade. In areas having insufficient thickness, substantial grooving developed before serious cracking and initial failure occurred.

TESTING PROGRAM

The objective of the test section was to check or to modify the tentative design curves by the actual service behavior of a test pavement constructed on a subgrade with a low CBR value. Information was desired for wheel loads ranging from a few thousand pounds to approximately 53,300 lb. Because of lack of time, it was impossible to make complete traffic tests with large numbers of repetitions of wheel loads over the full range investigated; hence, it was desired that indirect observations be used to the fullest possible extent to predict the behavior of the test section. Therefore, direct traffic tests were made under wheel loads of 25,000 lb and 40,000 lb, and the deflections of the pavement and of the subgrade were measured under both static and moving wheel loads of 2,000 lb, 5,000 lb, 10,000 lb, 25,000 lb, 40,000 lb, and 53,300 lb. Static bearing tests on circular steel plates were also made to provide a basis for possible correlation with measurements under pneumatic wheel loads used in the traffic tests and in the deflection observations.

Pavement Deflection.—Deflection measurements under both static and moving wheel loads had been used rather extensively by the California Division of

Highways in its study of the performance of flexible pavements. Special electrical equipment developed by that division permitted the measurement of very small deflections under rapidly moving wheel loads. Highway experience indicated that the deflection of flexible pavements under the maximum highway wheel load of 10,000 lb should not exceed 0.02 in. or 0.03 in. in cases where a large number of repetitions (a million or more) is anticipated. Because of the lesser number of repetitions anticipated on airfield pavements, deflections of approximately 0.05 in. were judged permissible provided that the pavement is good and does not become brittle. Also, the allowable deflections under the heavier wheel loads should be slightly larger because of the larger contact areas of the heavier loads which produce a flatter curvature of the pavement.

Gages for the measurement of deflection (vertical movement) were placed at five points on the surface of the subgrade and at seven points on the surface of the pavement (for positions, see Fig. 13(*c*)). The gages were installed at locations selected to cover the range between 12 in. and 48 in., the total thickness of pavement and base, and were placed 5 ft inside the right edge of the pavement. The deflection measurements for each wheel load were first made at the point having the least thickness and then progressed up the ramp to the point of maximum thickness. This order was followed to reduce the effect of load repetition to a minimum. The deflection tests were made with the following wheel loads in the order listed:

Wheel loads, in pounds	Equipment used
2,000	AT6A airplane
5,000	Dump truck
10,000	RU LeTourneau Carryall
25,000	RU LeTourneau Carryall
25,000	Bomber B-24-D
40,700	RU LeTourneau Carryall
53,300	RU LeTourneau Carryall

The deflection of the pavement under dynamic wheel loads was measured at a speed of about 10 miles per hr. For each wheel load dynamic tests were made prior to the static tests. In the static tests, the total deflection was recorded after the wheel load had remained over the deflection point until movement had apparently ceased. The time varied from 3 min to 7 min.

A summary of deflection measurements under the foregoing wheel loads is presented in Table 1. Points of interest relative to each of the wheel loads are reviewed subsequently.

2,000-Lb Wheel Load.—Deflections were observed at five locations on the old taxiway pavement (6 in. thick), under a wheel load of 2,000 lb. The average results of the five readings were as follows:

Loading condition	Deflection, in inches
Dynamic	0.079
Static	0.092

All these areas contained cracks; hence, an extra test was made in a location free of cracks. This test resulted in a dynamic deflection of 0.034 in. and a static deflection of 0.048 in., which are both less than half the values observed for the cracked areas. Fine hair cracks developed while the plane was taxiing over the test area. This behavior clearly indicated that the pavement was not capable of withstanding deflections of approximately 0.05 in. for even a few repetitions. Additional repetitions would have been required, of course, to increase the number and the size of the cracks and to cause complete failure.

TABLE 1.—PAVEMENT AND SUBGRADE DEFLECTION, IN INCHES, STOCKTON (CALIF.) TEST SECTION NO. 1

Pressure[a]	Wheel load, in kips[b]	3-In. AC Pavement							Old Taxiway				
		Deflection Points; see Figs. 13(b) and 13(c):											
		11	12	13	14	15	16	17	1	2	3	4	6
Thickness[c]		12	18	24	30	36	42	48	20	26	32	38	45
60	5.0D.....	0.068	0.045	0.030	0.033	....	....	0.023	0.018	0.015	0.012	....	0.005
	5.0S......	0.096	0.060	0.042	0.045	....	....	0.048	0.035	0.019	0.015	....	0.009
58	10.0D....	0.094	0.088	0.074	0.068	0.062	0.050	0.046	0.056	0.036	0.025	0.018	0.013
	10.0S.....	0.162	0.137	0.082	0.074	0.064	0.056	0.059	0.083	0.046	0.034	0.024	0.018
58	25.0D....	0.246	0.210	0.160	0.157	0.134	0.110	0.095	0.143	0.119	0.097	0.075	0.046
	25.0S.....	0.485	0.345	0.205	0.184	0.160	0.129	0.114	0.240	0.156	0.130	0.094	0.062
	25.0D[d]...	....	....	....	....	....	....	....	....	0.093	0.073	....	....
	25.0S[d]....	....	....	....	....	....	....	....	....	0.114	0.086	....	....
58	40.0D....	0.550	0.350	0.187	0.180	0.160	0.128	0.110	....	0.148	0.129	0.095	0.067
	40.0S.....	....[e]	....[e]	0.252	0.237	0.190	0.151	0.146	....	0.206	0.167	0.120	0.086
58	53.3D....	....[e]	....[e]	0.261	0.217	0.173	0.164	0.138	....[e]	0.216	0.158	0.110	0.086
	53.3S.....	....[e]	....[e]	....[e]	0.272	0.226	0.210	0.176	....[e]	0.269	0.223	0.162	0.122

[a] Contact pressure, in pounds per square inch. [b] 1 kip = 1,000 lb or 1 "kilo-pound"; D = dynamic and S = static. [c] Thicknesses of pavement and base in inches. [d] B-24-D bomber. [e] Deflection was too great to be measured properly with electric gages.

5,000-Lb Wheel Load.—Deflections under the 5,000-lb wheel load are given in Table 1. Bearing tests on a rigid plate resulted in pavement deflections from 110% to 150% of the static wheel load values. This relationship is just opposite to that found for the heavier wheel loads.

At this wheel load, tests were made to determine the effect of tire pressure on the deflection of the top of the pavement and on the deflection of the top of the subgrade. These tests indicated that the changes in air pressure in the tires of from 40 lb per sq in. to 80 lb per sq in. greatly influenced the deflection of the top of the pavement, but had little influence on the deflection of the top of the subgrade. Although the results were only sufficient to show a trend, it is logical to assume that the higher tire and contact pressures are of particular importance in selecting pavement and base materials for the upper part of a roadway or a runway.

10,000-Lb Wheel Load.—Bearing tests made on a rigid plate having a contact area comparable to that of the tire used for the 10,000-lb wheel load tests resulted in from about 20% to 25% less deflection than that obtained for the static wheel load. Referring to Table 1, it may be noted that a deflection of approximately 0.04 in. was obtained at the bottom of the base for a total thickness of 26 in. Highway experience in California indicates that a minimum of 24 in. of flexible pavement and base is desirable over this class of soil on roads that carry a large volume of heavy truck traffic (maximum wheel loads of approximately 9,000 lb). Deflection measurements have also indicated that for highway conditions the dynamic deflections should not exceed 0.03 in. or 0.04 in. and that even this amount is permissible only in the event the flexible pavement is good and not brittle. Observations made during seventy-six trips over the pavement required for this series of tests indicated that some plastic flow of the subgrade may have occurred during the time interval of the static load test for thicknesses less than 25 in. Thus, additional traffic of even a rolling load would probably have caused progressive plastic deformation resulting in grooving and ultimate failure after a sufficient number of load repetitions.

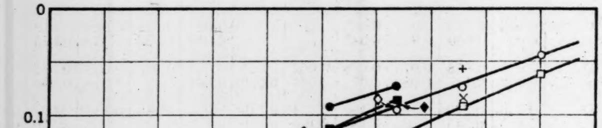


FIG. 14.—DEFLECTION OF THE BOTTOM OF THE BASE UNDER A LOAD OF 25,000 LB

25,000-Lb Wheel Load.—

LeTourneau Carryall.—Dynamic and static deflections measured under the 25,000-lb wheel load are included in Table 1. These data are also shown in Fig. 14, a typical plot of the deflection of the bottom of the base versus total thickness. Bearing tests with a footprint area and a contact pressure corresponding to that of the tire used for the 25,000-lb test resulted in deflections from 35% to 55% less than the values measured with the static wheel load. The deflection was checked at the 36-in. thickness after traffic was completed with the 40,000-lb wheel load to determine the increased deflection that resulted from the cracking and the weakening of the pavement and the base. The pavement at this point showed a 15% increase in deflection after the development of cracking. The deflection of the bottom of the base at the 38-in. thickness, where the cracking had not yet developed to an appreciable extent, was slightly less than readings obtained before traffic. The indications are that no plastic flow occured at a thickness of 38 in. Any plastic flow of the subgrade probably will cause progressive grooving, cracking, and failure with additional load repetitions. The rate of failure in relation to load repetition would undoubtedly depend to a large extent on the magnitude of the plastic deformation per repetition.

B-24-D Bomber.—A B-24-D bomber was made available for a few hours by the Fourth Air Force. In view of the limited time that the bomber was available, deflection tests were made only at the bottom of the base at two points. Dynamic and static deflections measured at those two points are included in Table 1 and Fig. 14. It will be noted that the deflections caused by the bomber were approximately 20% less than those for similar tests made with the carryall having a wheel load of 25,000 lb. An attempt was made to obtain a quick but accurate determination of the wheel load and footprint area for the bomber. Difficulty was encountered in making these measurements; therefore, the accuracy was not as great as desired. As closely as could be determined, the wheel load was 24,000 lb and the contact pressure was 60 lb per sq in. In addition to the deflection measurements for the B-24-D bomber that are included in Table 1, tests were made to determine the effect of the vibration of the motors. These tests (plotted in Fig. 14) indicated that vibrations increased the deflections only slightly over the values obtained for the static load with the motor dead. Although these results are 25% higher than the deflections measured under the rolling wheel load, it is probable that if the tests had been made on a lesser thickness of pavement the influence of the dynamic load would have been much greater.

40,000-Lb Wheel Load.—Values of deflection measurements for the 40,000-lb wheel load, given in Table 1, are maxima. The longitudinal distribution of deflections under the moving 40,000-lb wheel load is given in Fig. 15. The data for these curves were obtained from oscillograph films for the deflection of points 3 and 14, Fig. 13. The footprint was 20.25 in. wide by 31.25 in. long, or 568 sq in., and the average tire contact pressure during the tests was 72 lb per sq in. In tests with the 40,000-lb wheel load and the 53,000-lb wheel load, the 24-in. by 32-in. tire was overloaded and the air pressure (59 lb per sq in.) could not be dropped sufficiently to reduce the contact pressure to the desired range of from 60 lb per sq in. to 65 lb per sq in. Heavy bombers, however, usually carry higher air pressure and therefore the deflection curves probably approximate actual field service conditions. Bearing tests on rigid circular plates gave somewhat erratic results but in general indicated from 15% to 45% less deflection than was obtained under the static wheel load. A similar discrepancy has been found in other tests, and it appears that the shape and the rigidity of the bearing plate greatly influence the static load test results. Under actual tire loads the pavement can bend throughout the loaded area, whereas with the rigid plates essentially all the bending of the pavement and the base must occur outside the loaded area.

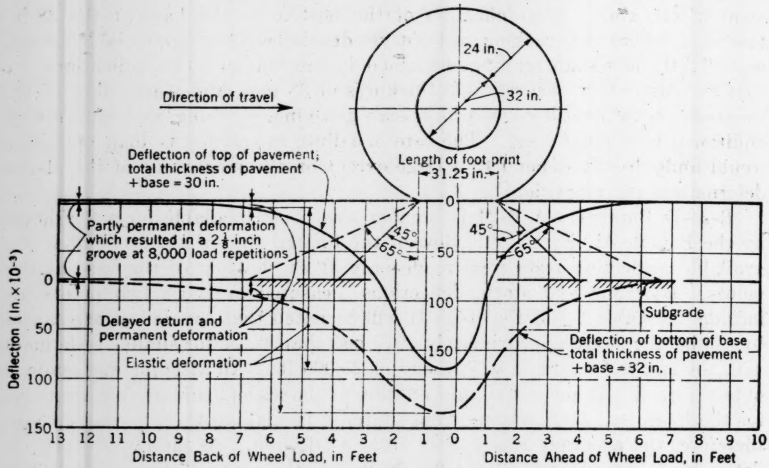


FIG. 15.—TYPICAL CURVES SHOWING THE LONGITUDINAL DISTRIBUTION OF PAVEMENT DEFLECTION; WHEEL LOAD OF 40,000 LB MOVING AT APPROXIMATELY 10 MILES PER HR

53,300-Lb Wheel Load.—Deflections measured under the 53,300-lb wheel load are included in Table 1. Bearing tests made on the pavement where the total thickness was 18 in. gave very erratic results. Disregarding these tests, the deflections at the other thicknesses varied from 10% to a very large indeterminate amount less than the values obtained for the static wheel load. Large deflections were obtained for the thinner pavement sections. Tests under this heavy load and also under the 40,000-lb load indicate large differences between the deflection at the top of the pavement and at the bottom of the base. The values measured at the top of the pavement were higher primarily because of elastic deflection and, to a lesser degree, because of the consolidation deformation of the base. Even if the thickness of base were entirely adequate to eliminate all plastic deformation of the subgrade, the flexible pavement must be of good quality and very flexible to withstand a large number of repetitions of deflections in the order of the minimum for this load, which was 0.138 in.

TRAFFIC TESTS

The left side of the test section was subjected to the traffic of a 25,000-lb wheel load for a maximum of 1,920 coverages after the static load bearing tests had been completed on top of the pavement. Similarly, the right half of the test section was subjected to 3,730 coverages of a 40,000-lb wheel load. For the first 1,600 repetitions, traffic with the 40,000-lb load was concentrated in one rut, but after 1,600 repetitions traffic was spread over an area about three times the width of the tires. Traffic with the 25,000-lb wheel load was spread over three tire widths for the entire period. Traffic lanes were developed near the edges of the test section for the outer wheels of the equipment and near the center of the section for the inner wheels of the equipment. Since the behavior of the outer lanes was influenced by the proximity of the wheels to the edge of the pavement, only the behavior of the inner lanes is evaluated in the traffic tests. As traffic progressed, measurements were made to determine the amount of grooving and the extent of cracking in the pavement.

Observations were recorded that show the extent of cracking of the pavement, rutting, grooving, and roughness. It is considered that the extent to which these detrimental behavior characteristics progressed from the thin end toward the thick end of the test section is an indication of the extent of shear deformation in the subgrade for the following reasons: The 3-in. asphaltic concrete was flexible and of good quality; therefore, no part of the detrimental behavior can be attributed to weakness of the asphaltic concrete. The base course was of good quality and well compacted; no evidence of shear deformation was found in the base course and only a nominal amount of consolidation occurred. Therefore, the cause of the detrimental behavior must have been in the subgrade. The adobe clay subgrade was practically saturated and was too impervious to drain, which prevented any appreciable compaction or consolidation occurring under traffic. The cracking, rutting, grooving, and roughness that occurred are attributed, therefore, to plastic flow or to shear deformation in the subgrade.

25,000-Lb Wheel Load.—Deep rutting and complete failure progressively developed during the 1,920 coverages of the 25,000-lb wheel load from the thin end of the pavement to the 16-in. thickness. Cracking and grooving of the pavement to the extent classified as a failure occurred to approximately the 20-in. thickness. Most of the 1,920 coverages were made in relatively warm weather. Because of this and the very flexible character of the pavement, cracks did not develop at greater thicknesses even though the surface grooved and roughened to the 30-in. thickness.

A 20-in. thickness of pavement and base was inadequate to protect the subgrade for 1,920 coverages. The data indicate that a 23-in. to 28-in. thickness would be required to prevent detrimental shear deformation in the subgrade. Since the subgrade had an average CBR of 5%, these values are in substantial agreement with the 26-in. thickness required by the design curves for a subgrade having a CBR of 5%.

40,000-Lb Wheel Load.—Failures developed progressively with traffic repetitions from the thin end to the thicker parts of the test section and ultimately to the point where the total thickness of pavement and base was 30 in. This thickness was not sufficient to prevent excessive shear deformation in the subgrade for 3,900 coverages of the 40,000-lb wheel load. Cracking also occurred to greater thicknesses of pavement and base, but generally disappeared under the action of traffic in warm weather.

A 30-in. thickness of pavement and base was not adequate to protect the subgrade from shear deformation. The data indicate that 32-in. to 36-in. thickness would be required to prevent excessive shear deformation in the subgrade. For a subgrade with a CBR of 5 (which is the average CBR of this subgrade), the design curves call for 32 in. of base and pavement for a 40,000-lb wheel load. The 32-in. to 36-in. range indicated by these tests is only slightly greater than the thickness required by the design curves.

Conclusions

It appears that the following conclusions can be drawn from the traffic and deflection data for soil, base, and pavement conditions comparable to those present on this test project:

(*a*) The results obtained from the traffic tests are indicative of the service behavior of runway pavements where traffic is not concentrated near the edge of the pavement.

(*b*) The thicknesses of pavement and base required according to the test are in reasonable agreement with those required by the CBR design curves.

(*c*) Some increase in the total thickness of pavement and base as shown by the design curves is indicated for taxiway operations where a large volume of traffic may be concentrated near the edges of the pavement.

(*d*) The base used in the test section was good and remained stable throughout the traffic tests.

SERVICE BEHAVIOR TESTS, BARKSDALE FIELD, SHREVEPORT, LA.

BY RALPH HANSEN,[24] ASSOC. M. ASCE

Another research program conducted to obtain information for the design curves was the flexible pavement investigation conducted at Barksdale Field, near Shreveport, where special test tracks were built to study the base requirements on a plastic clay subgrade. Barksdale Field was selected for the service behavior tests as it was believed that the subgrade at this site would be representative of the plastic clay subgrades which might be encountered in the construction of flexible pavements in the southern part of the United States. The specific purpose of the tests was to determine the following factors for flexible pavements using 20,000-lb and 50,000-lb wheel loads:

(1) The total thickness of pavement and base necessary to prevent detrimental shear deformation in the subgrade;

(2) The effect of the quality of base materials on the total specified thickness of pavement and base;

(3) The quality of base materials required directly under the pavement; and

(4) The adequacy of the compaction requirements as adopted by the Chief of Engineers[25] to prevent detrimental consolidation of the base materials and the subgrade.

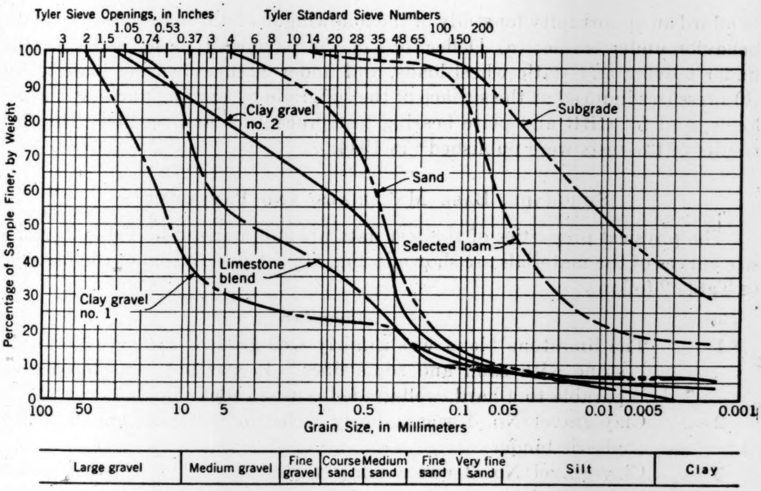


FIG. 16.—MECHANICAL ANALYSIS DIAGRAM FOR CONSTRUCTION MATERIALS (UNITED STATES BUREAU OF SOILS CLASSIFICATION)

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[24] Senior Engr., New England Div., Corps of Engrs., Boston, Mass.

[25] "Airfield Pavement Design: Flexible Pavements," *Engineering Manual for War Department Construction*, Office of the Chf. of Engrs., U. S. Army, Washington, D. C., May, 1947, Pt. XII, p. 1, paragraph 2–03.

Information was also desired pertaining to the action of the pavement, base materials, and subgrade under traffic conditions and to other data that might be useful in the development of design methods for flexible pavements. For these reasons, the service behavior tests were conducted in such a manner as to afford an opportunity for study of the following in addition to the study of the behavior under traffic: (*a*) Deflections of the pavement, base, and subgrade under moving and static wheel loads, and under circular bearing plate loads; (*b*) pressures induced at the surface of the subgrade by moving and static wheel loads; and (*c*) CBR and plate bearing test procedures and results. Detailed results of the tests were published[26] in 1944.

TABLE 2.—PHYSICAL PROPERTIES OF CONSTRUCTION MATERIALS IN THE SUBGRADE BASE AND PAVEMENT

Line	Material	ATTERBERG LIMITS[a]				CLASSIFICATION[b]		COMPACTION[c]		Los Angeles abrasion tests[d]
		Liquid limit, w_L	Plasticity index, I_P	Shrinkage limit	Linear shrinkage	Casagrande	PRA	Unit dry weight	Moisture content	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
1	Limestone blend	NP	NP	13.0	....	GW	A-3	145	4	46.7
2	Clay-gravel No. 1	27.7	15.7	16.8	7.4	GF	A-2	138	6	24.5
3	Clay-gravel No. 2	NP	NP	18.9	....	GF-SF	A-3	131	7	26.2
4	Sand	NP	NP	21.2	....	SW-SF	A-3	116	10	....
5	Selected loam	25.9	7.4	21.5	2.0	ML	A-4	118	13	....
6	Subgrade	45.1	27.5	11.7	16.1	CL-CH	A-7-6	114	14	....
7	Soil cement	....	....	....	....	....	....	112	14	....
8	Asphaltic concrete[e]	....	....	....	....	....	....	....	....	....

[a] In Cols. 2 and 3, lines 1, 3, and 4, NP denotes "nonplastic." [b] "Classification and Identification of Soils," by Arthur Casagrande, *Transactions*, ASCE, Vol. 113, 1948, p. 901. [c] Modified compaction tests adopted by the American Association of State Highway Officials: In Col. 8, the maximum unit dry weight expressed in pounds per cubic foot; and, in Col. 9, the optimum moisture content expressed as percentages of dry weight. [d] Percentage of wear. [e] For other characteristics, see text.

SUBGRADE, BASE MATERIALS, AND PAVEMENT

The physical properties of the test data are summarized in Table 2; grain-size curves of the materials are shown in Fig. 16. A description of the material in Table 2 follows:

Line	Description
1	The limestone blend was a fairly well-graded, soft, crushed limestone with sand and some fines. It was not considered comparable to a hard, well-graded crushed rock.
2	Clay-gravel No. 1 was a fairly well-graded sandy gravel with a plastic binder.
3	Clay-gravel No. 2 was a fairly well-graded gravelly sand with a silty binder.
4	This sand was fairly well graded and silty with a trace of clay. Lines 2, 3, and 4 were pit-run materials from local deposits.

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[26] "Report on Barksdale Field Service Behavior Tests," U. S. Engr. Office, Little Rock, Ark., October, 1944.

5 This material was a sandy clayey silt or a selected loam.

6 This material was a medium plastic clay.

7 The soil cement was a mixture of a selected raw soil loam the same as item 6, mixed with Portland cement. The tests revealed an average of 8.6% cement by volume. The cement content and the unit dry weight of the soil cement were lower than indicated as necessary by design tests.

8 The relative density of the asphaltic concrete was 98.7 and the field stability tests,[27] developed by Prévost Hubbard, Affiliate, ASCE, were performed under a load of 2,148 lb. The extraction test disclosed 6.4% bitumen and the penetration of the residue at 77°F (100 g, 5 sec) was 74.

The aggregate for the asphaltic concrete (line 8, Table 2) was the same as for the limestone blend (line 1), except that all the material was crushed to pass the ¾-in. screen, the sieve analysis for the mineral aggregate being

Screen	Percentage passing
¾ in.	100.0
½ in.	95.7
No. 4	61.8
No. 10	49.2
No. 40	28.3
No. 80	15.3
No. 200	9.1

Subgrade.—The subgrade was an alluvial deposit of reddish brown to black, medium plastic clay which was relatively uniform in structure and moisture content. The clay deposit extended to a depth of about 5 ft and was underlain by a deposit of sandy clayey silt which extended to an explored depth of 10 ft. During the traffic tests the water table elevations varied from about 1 ft to 4 ft below the surface of the subgrade. The CBR of the subgrade was between 5% and 7%.

Base Materials.—In this paper the term "upper base materials" has been used to indicate the limestone blend, clay-gravels Nos. 1 and 2, and sand (lines 1 to 4, Table 2).

Pavement.—The pavement consisted of a 3-in. wearing course of hot-mix asphaltic concrete which was excellent in quality and uniform throughout.

TEST PROGRAM

A test section designed in accordance with tentative design curves for 20,000-lb and 50,000-lb wheel loads was constructed on the medium plastic clay subgrade. Extensive tests on the subgrade and on the materials locally available for base course construction were conducted to determine the CBR, the compaction requirements, and the other properties prior to design and construction of the test pavements. The subgrade was stripped level and compacted, and a test section consisting of two test tracks, one for the 20,000-lb

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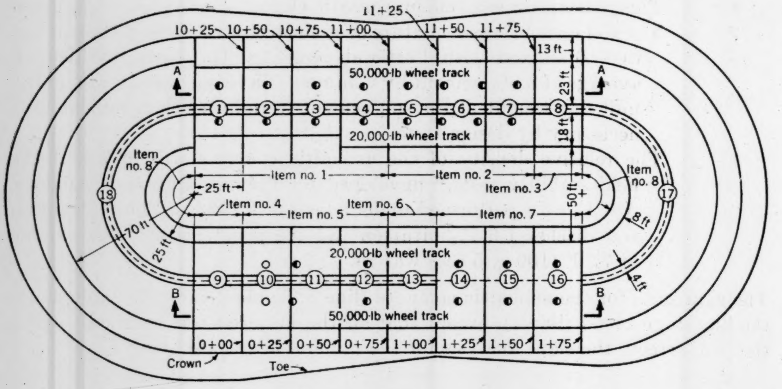
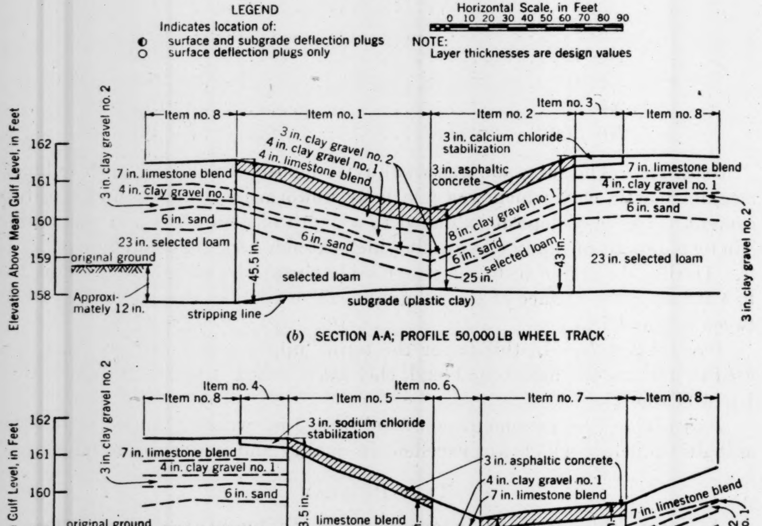
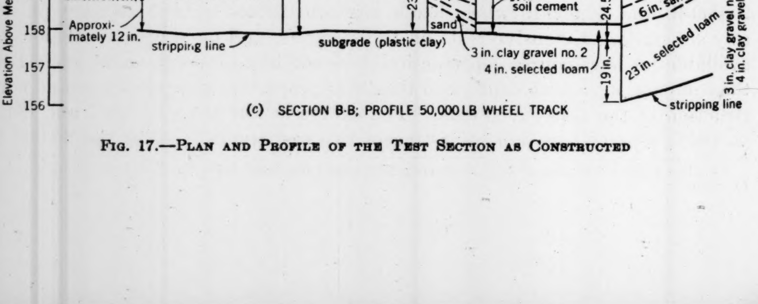
[27] "The Rational Design of Asphalt Paving Mixtures," *Research Series No. 1*, Asphalt Inst., October 15, 1935.


11+25
10+25
10+50
10+75
11+00
11+50
11+75
13 ft
23 ft
18 ft
50,000-lb wheel track
20,000-lb wheel track
A
A
1
2
3
4
5
6
7
8
Item no. 8
Item no. 1
Item no. 2
Item no. 3
Item no. 8
25 ft
18
17
50 ft
Item no. 4
Item no. 5
Item no. 6
Item no. 7
70 ft
25 ft
8 ft
4 ft
20,000-lb wheel track
9
10
11
12
13
14
15
16
B
B
50,000-lb wheel track
0+00
0+25
0+50
0+75
1+00
1+25
1+50
1+75
Crown
Toe

(*a*) PLAN


LEGEND
Indicates location of:
◐ surface and subgrade deflection plugs
○ surface deflection plugs only
Horizontal Scale, in Feet
0 10 20 30 40 50 60 70 80 90
NOTE:
Layer thicknesses are design values
Elevation Above Mean Gulf Level, in Feet
162
161
160
159
158
3 in. clay gravel no. 2
Item no. 8
Item no. 1
Item no. 2
Item no. 3
Item no. 8
3 in. clay gravel no. 2
4 in. clay gravel no. 1
4 in. limestone blend
3 in. calcium chloride stabilization
3 in. asphaltic concrete
7 in. limestone blend
4 in. clay gravel no. 1
6 in. sand
23 in. selected loam
original ground
Approximately 12 in.
45.5 in.
6 in. sand
selected loam
8 in. clay gravel no. 1
6 in. sand
selected loam
25 in.
43 in.
7 in. limestone blend
4 in. clay gravel no. 1
6 in. sand
23 in. selected loam
3 in. clay gravel no. 2
stripping line
subgrade (plastic clay)

(*b*) SECTION A-A; PROFILE 50,000 LB WHEEL TRACK

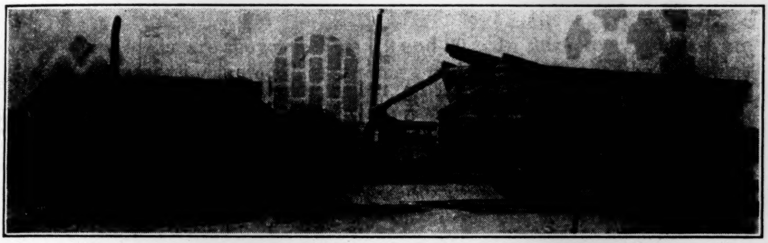
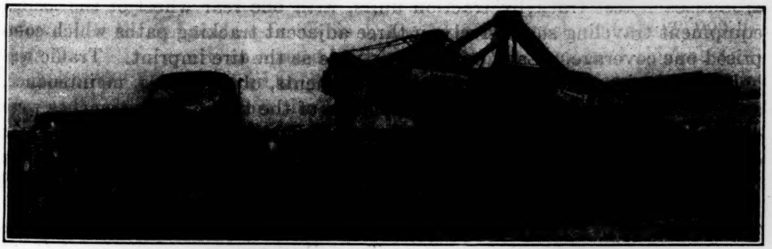

Elevation Above Mean Gulf Level, in Feet
162
161
160
159
158
157
156
3 in. clay gravel no. 2
Item no. 4
Item no. 6
Item no. 8
Item no. 5
Item no. 7
Item no. 8
3 in. sodium chloride stabilization
7 in. limestone blend
4 in. clay gravel no. 1
6 in. sand
original ground
23 in. selected loam
Approximately 12 in.
stripping line
43.5 in.
limestone blend
23 in.
sand
3 in. asphaltic concrete
4 in. clay gravel no. 1
7 in. limestone blend
17.5 ft
soil cement
24.5 in.
19 in.
subgrade (plastic clay)
3 in. clay gravel no. 2
4 in. selected loam
7 in. limestone blend
6 in. sand
23 in. selected loam
3 in. clay gravel no. 2
4 in. clay gravel no. 1
stripping line

(*c*) SECTION B-B; PROFILE 50,000 LB WHEEL TRACK

FIG. 17.—PLAN AND PROFILE OF THE TEST SECTION AS CONSTRUCTED

wheel load and the other for the 50,000-lb wheel load, was constructed. The test section was oval in shape with two parallel paved straightaways joined at the ends by semicircular unsurfaced "turnarounds." Inner and outer tracks were constructed for the 20,000-lb and the 50,000-lb wheel loads, respectively. A plan and typical profiles of the test section are shown in Fig. 17. Each test track comprised four different types of bases, as follows: (1) A layered base

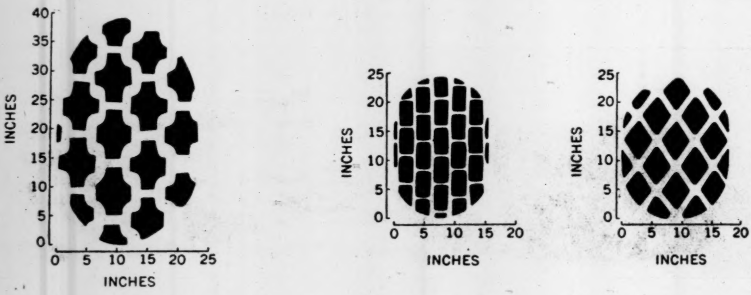
TABLE 3.—CHARACTERISTICS OF LOADING EQUIPMENT; SERVICE BEHAVIOR TEST SECTION

Coverage No.	Tracking wheel	Direction of travel	Tire size	Tire air[a]	Contact pressure[a]	Contact area[b]
(a) 50,000-LB WHEEL LOAD						
						
EUCLID FOUR-WHEEL RUBBER-TIRED TRACTOR AND LETOURNEAU MODEL R-30 "TOURNATRAILER"						
0-9	Left	Counterclockwise	30 by 40	60	76.5	653.5
10-1,248	Left	Counterclockwise	30 by 40	50	70.7	707.0
1,249-5,005	Right	Clockwise	30 by 40	50	80.4	621.6
(b) 20,000-LB WHEEL LOAD						
						
GAR WOOD MODEL 515, 14-YD SCRAPER AND INTERNATIONAL MODEL K8-F, SIX-WHEEL TRUCK TRACTOR						
0-9	Right	Counterclockwise	18 by 24	60	65.7	304.4
10-321	Right	Counterclockwise	18 by 24	50	63.6	314.2
322-2,621	Left	Clockwise	18 by 24	50	63.5	315.0
2,622-5,009	Left	Clockwise	18 by 24	42	56.6	353.1

[a] Pressure, in pounds per square inch. [b] Square inches.

construction consisting of several layers of different materials with a high CBR material directly under the pavement (item 1, Fig. 17); (2) a layered base construction similar to that in item 1 but with a lower CBR material directly under the pavement (item 2, Fig. 17); (3) a high CBR material for the full thickness of the base (item 5, Fig. 17); and (4) a soil cement base (item 7, Fig. 17). The construction of these items was similar for the 20,000-lb and the 50,000-lb wheel load tracks except that lesser thicknesses of base were used for the smaller wheel load. Variations in thickness of the items were obtained by sloping the surface of the test tracks as shown in the profiles, Figs. 17(*b*) and 17(*c*).

Loaded earth-moving equipment (Table 3) with tire contact areas approximately comparable to those of planes with similar wheel loads (Fig. 18) was used to produce the specified wheel loads. The loaded equipment was pulled around each test track by traction units, with one rear wheel of the loaded equipment traveling successively in three adjacent tracking paths which comprised one coverage. Each path was as wide as the tire imprint. Traffic was halted at various intervals for test measurements, observations, maintenance, and repairs to the tracks. After completion of the traffic tests, trenches were excavated in each test track to determine the lateral displacement (shear deformation) and the consolidation of the base and the subgrade. Extensive field and laboratory tests were made on the asphaltic concrete pavement, on all base materials, and on the subgrade to determine behavior under traffic.



(*a*) 0 to 5005 Coverages

(*b*) 0 to 2621 Covereges

(*c*) 2622 to 5009 Coverages

50,000 LB WHEEL LOAD

20,000 LB WHEEL LOAD

FIG. 18.—TYPICAL TIRE PRINTS OF LOADING WHEELS FOR COVERAGES IN TABLE 3

SUMMARY OF TEST RESULTS

In the analyses of the test results it was assumed that, if no shear deformation could be determined from the test data in the section during 5,000 coverages of a given load, the section would be satisfactory as an airfield pavement and would have adequate thickness of pavement and base to prevent detrimental shear deformation in the subgrade. Also, the base materials would have adequate stability to prevent detrimental shear deformation in the base. It was assumed, furthermore, that, during construction of an actual airfield pavement, if the soils are compacted to the same relative density as those in the test section after 5,000 coverages, excessive consolidation due to traffic compaction would not occur. Typical cross sections of the lateral displacement, and curvature of the 3-in. asphaltic concrete pavement resulting from repetitions of wheel loads, are shown in Figs. 19 and 20. In this paper, analyses are presented only for the layered bases (items 1 and 2, Fig. 17) and the limestone blend base (item 5) as these items particularly pertain to the development of design criteria for flexible pavements.

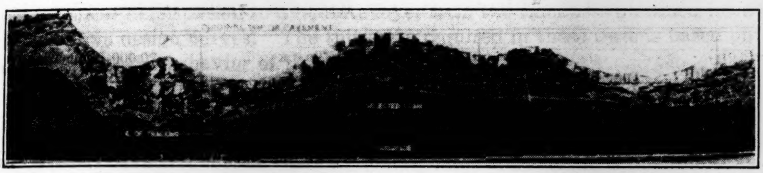


FIG. 19.—DISPLACEMENT AND CURVATURE AT STATION 11+45.7, FIG. 17(a) (ITEM 2, 20-KIP TRACK; THICKNESS OF PAVEMENT AND BASE, 22.5 IN.)

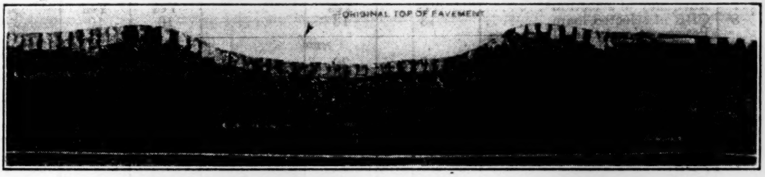


FIG. 20.—DISPLACEMENT AND CURVATURE AT STATION 11+01, FIG. 17(a) (ITEM 2, 50-KIP TRACK; THICKNESS OF PAVEMENT AND BASE, 25.5 IN.)

Thickness of Pavement and Base Required to Prevent Detrimental Shear Deformation in the Subgrade.—In Table 4, the 50,000-lb track and the 20,000-lb track, respectively, are compared with regard to the combined thickness of pavement and base required to prevent detrimental shear deformation in the subgrade as determined from the analysis of test data and as obtained by the tentative design curves of the California method.[28] It will be noted that when no shear deformation occurred in the base materials the thicknesses of pavement and base required to prevent detrimental shear deformation in the subgrade in the service behavior test are in substantial agreement with the thicknesses obtained from the tentative design curves (see items 1 and 5, 50,000-lb track, and item 5, 20,000-lb track). Where shear deformation occurred in the base materials, a greater thickness was required to prevent detrimental shear deformation in the subgrade. This fact is shown by comparing the thickness required in item 1, 50,000-lb track, with the thickness required in item 2, 50,000-lb track, and by comparing the thicknesses required in items 1 and 2, 20,000-lb track, with the thickness required in item 5, 20,000-lb track. The thicknesses required for items 1 and 2 for both the 20,000-lb and 50,000-lb tracks are in excess of those required for design when high-stability materials are used in the base. On this basis, the data for all items substantiate the validity of the tentative design curves.

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[28] "Airfield Pavement Design," *Engineering Manual for War Department Construction,* Office of Chf. of Engrs., U. S. Army, Washington, D. C., March, 1943, Pt. II, Chapter XX.

TABLE 4.—SUMMARY OF TEST RESULTS (ITEMS IN FIG. 17)

Line	Description	50,000-LB TRACK			20,000-LB TRACK		
		Item 1[a]	Item 2[b]	Item 5[c]	Item 1[a]	Item 2[b]	Item 5[c]
	(a) THICKNESS OF PAVEMENT AND BASE REQUIRED TO PREVENT DETRIMENTAL SHEAR DEFORMATION IN THE SUBGRADE						
1	Coverages completed.........	5,000	400	5,000	2,000	2,000	5,000
2	Location of deformation	12-ft area[d]	UBM[e]	None[f]	UBM[e]	UBM[e]	None[f]
3	CBR of subgrade	5 to 6	6	5 to 6	5 to 6	5 to 7	5
	Pavement and Base Thickness[g] (in Inches) by:						
4	Service behavior tests......	30 to 36	32 to 37	29 to 34	20 to 25	21 to 26	18 to 21
5	Tentative design curves....	27.5 to 31.0	27.5	27.5 to 31.0	19 to 21	17 to 21	21
	(b) REQUIREMENTS FOR LAYERED BASE DESIGN						
6	Coverages completed.........	5,000	400	....	2,000	2,000	....
7	Adequacy of selection[h].......	Yes	Yes	....	Yes	Yes	....
8	CBR of selected loam........	13	13	....	10	10	....
	Pavement and Base Thickness (in Inches) by:						
9	Service behavior tests......	20	20	....	14	14	....
10	Tentative design curves....	17	17	....	14	14	....
	(c) QUALITY OF BASE MATERIALS REQUIRED DIRECTLY UNDER THE PAVEMENT						
11	Thickness of layer[i]..........	4(7)[j]	8	24 to 43	2	5	11 to 29
	CBR:						
12	Before tracking...........	42(42)[j]	21	86 to 90	24	28	58 to 64
13	After tracking............	86(86)[j]	64	90 to 118	42	12	62 to 68
14	Adequacy..................	No(Yes)[j]	No	Yes	No	No	Yes

[a] Item 1 (see Fig. 17); layered base with a limestone blend (line 1, Table 2) directly beneath the pavement. [b] Item 2 (see Fig. 17); layered base with clay-gravel No. 1 (line 2, Table 2) directly beneath the pavement. [c] Item 5 (see Fig. 17); limestone blend (line 1, Table 2) for the entire depth of the base. [d] The upper base materials were just adequate in the 12-ft area between the points where the combined thickness of pavement and base was 41 in. and 43 in. They were inadequate in the remainder of item 1. [e] UBM is an abbreviation of "upper base materials." [f] No detrimental shear deformation in the base. [g] Total thickness of pavement and base required to prevent detrimental shear deformation in the subgrade. [h] Adequacy of the base above selected loam with respect to the thickness required to prevent detrimental shear deformation. [i] Thickness of layer, in inches. [j] Values in parentheses refer to the 12-ft area discussed in footnote [d].

Effect of Quality of Base Materials on the Total Required Thickness of Pavement and Base.—Referring again to Table 4(*a*), it will be noted that the required thickness of pavement and base for layered base construction (item 1, 50,000-lb track) is slightly greater than the thickness for the base with limestone blend for the full depth (item 5, 50,000-lb track). Although the thickness required in the service behavior test was slightly greater in item 1 than in item 5, it is believed that the difference can be considered as less than 5% when the effect of shear deformation in the base of item 1 on the total thickness is considered. Although the pavement surfaces were in better condition after traffic on the parts of the test section where the base was composed of limestone blend for the full depth, from the point of view of prevention of shear deformation in the subgrade the layered base performed essentially as well as did the full depth of high-quality base course material.

Requirements for Layered Base Design.—Table 4(*b*) shows a comparison of the thickness of pavement and base required in layered base construction to prevent shear deformation in a definite layer with the thickness obtained from the tentative design curves. The thickness required in these tests is based on an analysis of the behavior of the selected loam.

It will be noted in Table 4(*b*) and Fig. 17 that the thicknesses of pavement and base above the selected loam in items 1 and 2 were the same from the thin end of the items to the thick end. The results of the service behavior tests indicate that the thicknesses used were adequate; however, it is possible that lesser thicknesses might have been adequate to prevent detrimental shear deformation in the selected loam. Although a definite conclusion cannot be made regarding the validity of the design curves to determine the qualities and the thicknesses of each layer of base, the data do indicate that the tentative design curves are substantially valid.

TABLE 5.—COMPACTION REQUIREMENTS TO PREVENT EXCESSIVE CONSOLIDATION

Line	Description	ITEM 1				ITEM 2				ITEM 5			
		Upper base materials	Selected loam	Subgrade at:		Upper base materials	Selected loam	Subgrade at:		Limestone blend at:		Subgrade at:	
				Surface	1-ft depth			Surface	1-ft depth	Surface	1-ft depth	Surface	1-ft depth
	(*a*) 50,000-LB TRACK												
	Modified AASHO[a] Maximum Unit Dry Weight (%):												
1	Before tracking	97 to 101	89	88	86	93 to 100	92	87	88	96	94	89	91
2	After tracking	97 to 105	93	91	88	101 to 103	95	89	88	99	94	93	90
3	Depth of sampling, in inches[b]	....	20	41	53	....	20	42	54	..	15	41	53
	(*b*) 20,000-LB TRACK												
	Modified AASHO[a] Maximum Unit Dry Weight (%):												
4	Before tracking	83 to 99	91	91	88	93 to 101	91	84	88	97	88	89	87
5	After tracking	92 to 100	93	89	90	95 to 102	93	86	86	97	91	89	88
6	Depth of sampling, in inches[b]	....	14	28	40	....	14	27	39	..	15	26	38

[a] American Association of State Highway Officials. [b] Location below the surface of the pavement.

The Quality of Base Materials Required Directly Under the Pavement.—The data on the adequacy of the base materials directly below the 3-in. high-quality asphaltic concrete pavement from the standpoint of shear deformation for both the 50,000-lb and the 20,000-lb tracks are summarized in Table 4(*c*). At least 6 in. of material with a CBR of 80% or greater is required directly below a 3-in. asphaltic concrete pavement for a 50,000-lb wheel load. For the 20,000-lb wheel load, the data are not sufficient to determine the minimum thickness of layer required for material directly below a 3-in. asphaltic concrete pavement; however, the test results indicate that a material with a CBR of 60% for the full depth of base below the pavement is adequate, and may be more than adequate. If the pavement were thinner than 3 in., or not of good quality, it is possible that a material with a CBR greater than 60% would be required for a 20,000-lb wheel load.

Compaction Requirements to Prevent Excessive Consolidation.—The compaction attained during construction and the additional compaction resulting from traffic are shown in Table 5. A study of the data indicates that compaction to about 103% modified AASHO maximum unit dry weight was required in the upper base materials for the 50,000-lb track, whereas 100% probably would have been adequate for the 20,000-lb track. The data also indicate that compaction to 95% would have been adequate for the subgrade and the selected loam for either the 20,000-lb or the 50,000-lb wheel loads. The measurable effect of compaction from traffic extended to depths of from 27 in. to 40 in. from the surface of the pavement at the thick ends of the items in the 20,000-lb track and from 41 in. to 53 in. from the surface of the pavement at the thick ends of the items in the 50,000-lb track. The depth of traffic compaction was about 6 in. in the subgrade for the 20,000-lb track and about 12 in. in the subgrade for the 50,000-lb track at the thickness of pavement and base required to prevent detrimental shear deformation in the subgrade (29 in. to 37 in. for 50,000-lb track and 18 in. to 26 in. for 20,000-lb track).

CONCLUSIONS

General.—On the basis of the results obtained from the service behavior tests and of the assumptions made regarding the significance of the data with respect to the requirements for the quality of pavement desired, the following general conclusions are presented. These conclusions are applicable to the design of flexible type airfield pavements when the subgrade is a medium plastic clay.

1. In general, the service behavior tests at Barksdale Field have substantiated the CBR tentative design curves[28] when applied to flexible pavements constructed on medium plastic clay subgrades.

2. The stresses induced by a 50,000-lb wheel load require that 6 in. of material, with a CBR of 80% or more, be placed directly below a 3-in. high-quality asphaltic concrete pavement. The minimum thickness of layer required for material directly below the pavement was not determined for a 20,000-lb wheel load; however, the data indicate that for this wheel load a material with a CBR of 60% for the full depth of base below a 3-in. asphaltic concrete pavement was adequate and may be more than adequate. If the pavement were thinner, or not of good quality, it is possible that the CBR of the underlying base material should be greater than 60%.

3. The compaction requirements indicated as necessary to prevent excessive consolidation in the base materials and the subgrade under the traffic of 50,000-lb and 20,000-lb wheel loads are as follows:

(*a*) Base Materials.—For the 50,000-lb wheel load, the upper base materials (limestone blend, clay-gravel No. 1, clay-gravel No. 2, and sand for layered base construction or the 18 in. directly below the pavement for limestone blend for full thickness of base) should be compacted to a minimum of 103% of the modified AASHO maximum unit dry weight. For the 20,000-lb wheel load, the upper base materials (limestone blend, clay-gravel No. 1, clay-gravel No. 2, and sand for layered base construction or the 12 in. directly below the pavement for the limestone blend for full thickness of base) should be compacted to a minimum of 100% of the modified AASHO maximum unit dry weight. The lower base materials (below 12-in. depth) should be compacted to a minimum of 95% of the modified AASHO maximum unit dry weight.

(*b*) Subgrade.—For both the 20,000-lb and the 50,000-lb wheel loads, the plastic clay subgrade (if the stability can be increased by compaction) should be compacted to a minimum of 95% of the modified AASHO maximum unit dry weight. The depth of compaction should be 12 in. for a 50,000-lb wheel load and 6 in. for a 20,000-lb wheel load.

WHEEL LOAD TESTS, MARIETTA, GA.

By John M. Griffith,[29] Assoc. M. ASCE

When fully loaded, the B-29 Superfortress has a gross weight of approximately 120,000 lb, which is carried on dual wheel landing gear. When this plane was first introduced, the CBR method of design did not contain flexible pavement design criteria for dual wheel loads. Therefore, the Chief of Engineers, United States Army, was requested by Headquarters, Army Air Forces, to study the problem and to formulate design criteria.

The Office of the Chief of Engineers, through the Engineer Board, directed the Waterways Experiment Station to design, construct, and conduct tests at Marietta on a test section composed of flexible pavements on various types of bases. The primary purpose of the test section was to furnish data whereby design curves, similar to the tentative CBR design curves[1] could be developed for a dual wheel load of 60,000 lb. The application of the data to the development of design curves for the B-29 dual wheels is given in Symposium paper No. 10. The test section was also designed to furnish other subordinate information; but only a limited amount of data other than those required to accomplish the primary purpose is discussed in this paper because of necessary space limitations. The complete results of all tests conducted at Marietta are presented in a report of the Waterways Experiment Station.[30]

Fig. 21.—Boeing B-29 Superfortress (XB-118,335) Used for Pavement Behavior Tests

The test program at Marietta was conducted to determine the relationship between subgrade deflections and pressures under dual and single wheel loads so that thickness design curves similar to those used in the CBR method for single wheels could be prepared. In these tests, B-29 and B-24 planes were used to impose the desired dual and single wheel loads. Fig. 21 shows the B-29 plane used in the tests. Accelerated traffic testing was not contemplated in the event that these data proved adequate. The test section was designed, however, so that accelerated traffic tests could be conducted should they be considered desirable.

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[29] Engr., Chf. of Bituminous Section, Flexible Pavement Branch, Soils Div., Waterways Experiment Station, Corps of Engrs., Vicksburg, Miss.

[30] "Certain Requirements for Flexible Pavement Design for B-29 Planes," U. S. Waterways Experiment Station, Vicksburg, Miss., Appendix on "Flexible Pavement Tests, Marietta, Ga.," 1 August 1945.

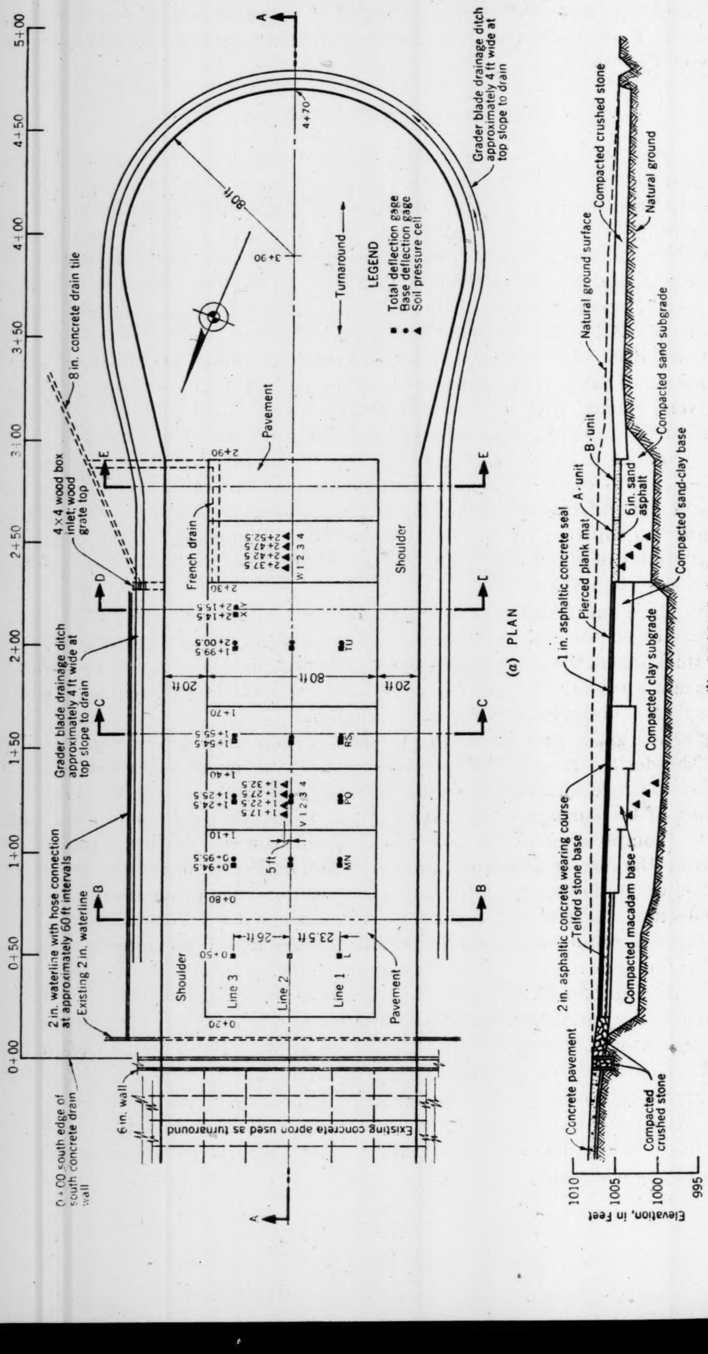
DESCRIPTION OF TEST SECTION

The test section was constructed as shown in Figs. 22 and 23 with ordinary construction equipment using standard construction procedures; however, close field control and inspection were maintained throughout construction to insure uniform conditions and close adherence to the desired design requirements. The in-place soil was a highly elastic micaceous silty clay that was considered unsuitable for the purpose of the test; therefore, an imported cohesive clay was placed as a subgrade beneath all the test section except the sand-asphalt item. The average CBR of the imported clay subgrade as constructed was about 4%. A clean well-drained sand was placed as the subgrade for the sand-asphalt item. The macadam bases were constructed of a hard, durable, crushed rock, in 4-in. layers, choked with limestone screenings, which produced a base with a CBR of 80% or more. The telford base was constructed of limestone rock averaging about 10 in. in height and not less than 3 in. in width. The stones were bedded into a sand cushion course over the clay subgrade. Voids between the stones were filled with crushed limestone spalls and screenings. The sand-clay base was constructed of a fairly well-graded sand-clay material having a plasticity index of about 5%. The average CBR of the sand-clay base as constructed was about 35%.

The telford and macadam items were surfaced with a 2-in. thickness of hot-mix asphaltic concrete of good quality. The sand-clay item was surfaced with a 1-in. thickness of the same asphaltic concrete and on this a steel pierced plank landing mat was laid. The sand-asphalt item was surfaced with 6 in. of hot-mix sand asphalt laid directly on the sand subgrade.

Deflection gages were located in the test section as shown in Fig. 22. To obtain the desired information, it was necessary to install two gages at each location; one gage furnished a measurement of the total vertical movement or deflection of the surface of the pavement, and the other deflection gage measured the compression in the base and wearing course. By subtracting the latter from the former deflection, the amount of deflection of the surface of the subgrade was determined. Only a single deflection gage to measure total surface deflection was installed in the telford item since it was considered that the single layer of telford stone would be incompressible. Pressure cells were installed at four elevations, both in the clay subgrade under the medium thickness of the macadam base item and in the sand subgrade of the sand-asphalt item.

Immediately on completion of construction and prior to any plane test runs, 20 coverages of a 20,000-lb wheel load were applied to all items except the telford. The objective of these initial coverages was to consolidate the test items to reduce the amount of consolidation appearing in the measured deflections. An attempt was made to consolidate the telford item with a 10,000-lb wheel load, but traffic was discontinued after the first coverage because of visibly large deflections.

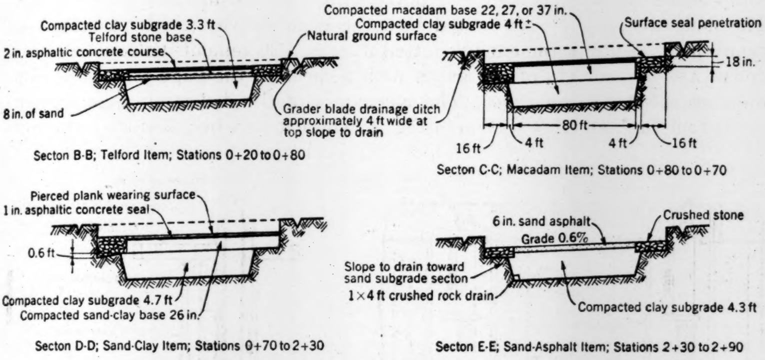

0+00 0+50 1+00 1+50 2+00 2+50 3+00 3+50 4+00 4+50 5+00
0+00 south edge of south concrete drain wall
2 in. waterline with hose connection at approximately 60 ft intervals
Existing 2 in. waterline
Grader blade drainage ditch approximately 4 ft wide at top slope to drain
4×4 wood box inlet; wood grate top
8 in. concrete drain tile
Grader blade drainage ditch approximately 4 ft wide at top slope to drain
6 in. wall
Existing concrete apron used as turnaround
Shoulder
Line 3
Line 2
Line 1
Pavement
French drain
Shoulder
Pavement
0+20 0+50 0+80 1+10 1+40 1+70 2+30 2+90
0+94.5 0+95.5 1+17.5 1+22.5 1+24.5 1+25.5 1+27.5 1+32.5 1+54.5 1+55.5 1+99.5 2+00.5 2+14.5 2+15.5 2+37.5 2+42.5 2+47.5 2+52.5
L MN PQ RS TU XY V 1 2 3 4 W 1 2 3 4
26 ft 23.5 ft 5 ft 20 ft 80 ft 20 ft
80 ft
3+90 4+70
Turnaround
LEGEND
■ Total deflection gage
● Base deflection gage
▲ Soil pressure cell
A A B B C C D D E E
1010 1005 1000 995
Elevation, in Feet
Concrete pavement
Compacted crushed stone
2 in. asphaltic concrete wearing course
Telford stone base
Compacted macadam base
1 in. asphaltic concrete seal
Pierced plank mat
Compacted clay subgrade
Compacted sand-clay base
A-unit
6 in. sand asphalt
B-unit
Compacted sand subgrade
Natural ground surface
Compacted crushed stone
Natural ground

(a) PLAN

(b) PROFILE A–A

FIG. 22.—FINAL PLAN AND PROFILE, TEST SECTION, MARIETTA, GA.

By reference to Fig. 22(*a*) it may be noted that there are three longitudinal lines of deflection gages, designated lines 1, 2, and 3. Transverse lines of gages are identified by letters. The three longitudinal lines of gages were installed to furnish one separate line of gages each for the single and dual wheel loads (lines 3 and 1, respectively) and a common line of gages for both loads (line 2).


Compacted clay subgrade 3.3 ft
Telford stone base
2 in. asphaltic concrete course
8 in. of sand
Natural ground surface
Grader blade drainage ditch approximately 4 ft wide at top slope to drain
Compacted macadam base 22, 27, or 37 in.
Compacted clay subgrade 4 ft ±
Surface seal penetration
18 in.
80 ft
16 ft
4 ft
4 ft
16 ft
Pierced plank wearing surface
1 in. asphaltic concrete seal
0.6 ft
Compacted clay subgrade 4.7 ft
Compacted sand-clay base 26 in.
6 in. sand asphalt
Grade 0.6%
Crushed stone
Slope to drain toward sand subgrade secton
1×4 ft crushed rock drain
Compacted clay subgrade 4.3 ft

Secton B-B; Telford Item; Stations 0+20 to 0+80

Secton C-C; Macadam Item; Stations 0+80 to 0+70

Secton D-D; Sand-Clay Item; Stations 0+70 to 2+30

Secton E-E; Sand-Asphalt Item; Stations 2+30 to 2+90

FIG. 23.—TYPICAL CROSS SECTIONS, TEST SECTION, MARIETTA, GA. (PLAN AND PROFILE IN FIG. 22)

On completion of the single wheel load tests, however, dual wheel tests were conducted over line 3 in addition to those on lines 1 and 2 for purposes of further correlation.

TESTS CONDUCTED

Deflection and pressure measurements were made under three loads for both the single wheel and the dual wheel. These loads were 20,000 lb, 25,000 lb, and 30,000 lb for the single wheel and 40,000 lb, 50,000 lb, and 60,000 lb for the dual wheel. It may be noted that the dual wheel loads were twice those on the single wheel; thus, the load on each tire of a dual assembly was the same as the single wheel loads. Deflections and pressures induced by both the single and dual wheels were measured. Initially, measurements were made for the following conditions: (*a*) Plane standing with motors dead, (*b*) plane standing with motors running about 1,000 rpm, (*c*) plane towed with motors dead, and (*d*) plane taxiing. After a part of the program was completed, it was determined that the deflections and pressures under some of the conditions were practically equal; therefore, certain conditions were deleted in the latter phases of the tests. Deflection data obtained for the single wheel loads of a B-24 plane for the foregoing conditions and similar data for the dual wheel loads of a B-29 plane, with one tire of the duals over the gage and with the center of the dual wheels over the gage, are shown in Table 6. Pressure data for these conditions are presented in Table 7 for the single (B-24) and dual (B-29) wheel load conditions. Each deflection or pressure measurement shown in Table 6 and 7 is the average of three or more individual tests.

INSTRUMENTATION

Deflection gages and pressure cells used in these tests were designed and constructed by personnel of the Waterways Experiment Station. General operational features of the deflection gage and the method of their installation in the pavement are illustrated in Fig. 24, a deflection gage sleeve being mounted in the pavement and base as shown. Where total surface deflection measurements were desired, a $\frac{3}{4}$-in. pipe reference was driven deeply into the subgrade and capped by a 2-in.-diameter steel gage foot. The pipe reference for base compression measurements was grouted into the base in the manner shown. The deflection gages were bolted into the gage sleeves which in turn were firmly fixed in the pavement by slotted flanges. Pavement deflections caused downward movements of the gages with respect to the gage foot, thus compressing the deflection gage. The mechanics of the deflection gage are illustrated in plan and cross section in Fig. 25 where it may be noted that compression of the gage causes movement of a tapered pin within the external shell of the gage. The pin shown (1-in. taper length) is interchangeable with a 0.3-in. taper length, the latter providing greater accuracy where deflections less than 0.3 in. were anticipated. Movement of the tapered pin causes bending in a cantilever beam which has two resistance strain gages mounted as shown. Bending of the cantilever thus causes a change in the resistance of the strain gages, cumulative for the two gages, which is measured in a Wheatstone bridge circuit. Prior to field use each gage was calibrated in the laboratory in a device in which the tapered pin was moved by measured increments and the change in resistance of the strain gages was determined.

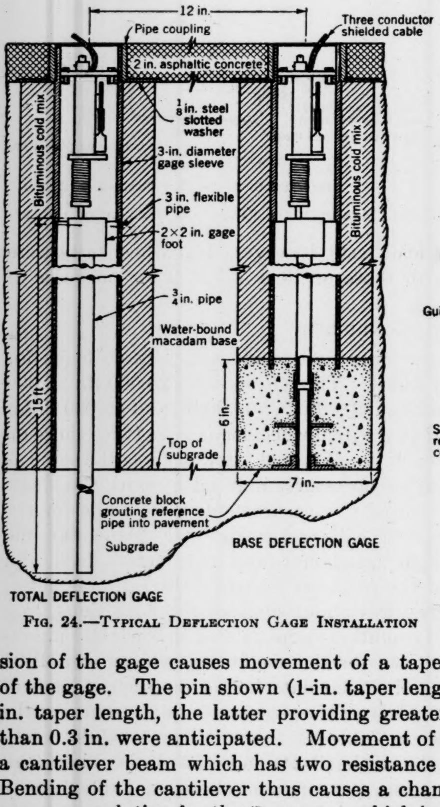

12 in.
Pipe coupling
Three conductor shielded cable
2 in. asphaltic concrete
$\frac{1}{8}$ in. steel slotted washer
3-in. diameter gage sleeve
3 in. flexible pipe
2×2 in. gage foot
$\frac{3}{4}$ in. pipe
Water-bound macadam base
Bituminous cold mix
Bituminous cold mix
15 ft
6 in.
Top of subgrade
7 in.
Concrete block grouting reference pipe into pavement
Subgrade
BASE DEFLECTION GAGE
TOTAL DEFLECTION GAGE

FIG. 24.—TYPICAL DEFLECTION GAGE INSTALLATION

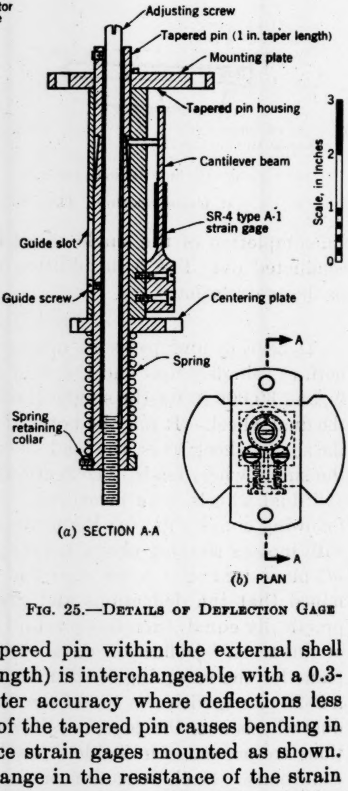

Adjusting screw
Tapered pin (1 in. taper length)
Mounting plate
Tapered pin housing
Cantilever beam
SR-4 type A-1 strain gage
Guide slot
Guide screw
Centering plate
Spring
Spring retaining collar
Scale, in Inches
0
1
2
3
A
A
(a) SECTION A-A
(b) PLAN

FIG. 25.—DETAILS OF DEFLECTION GAGE

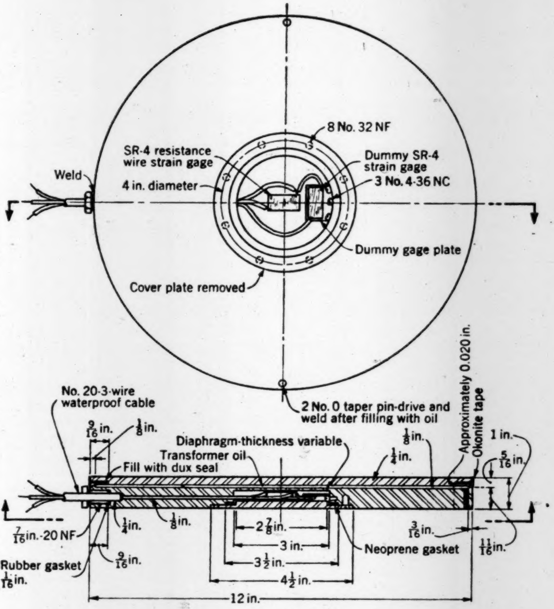
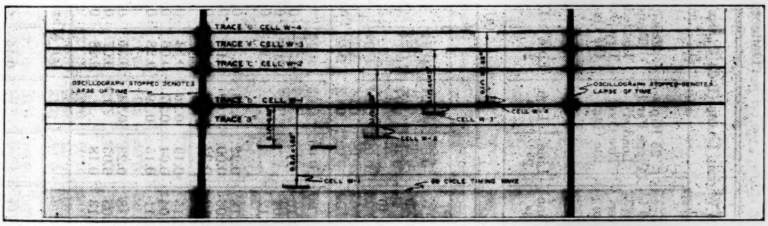
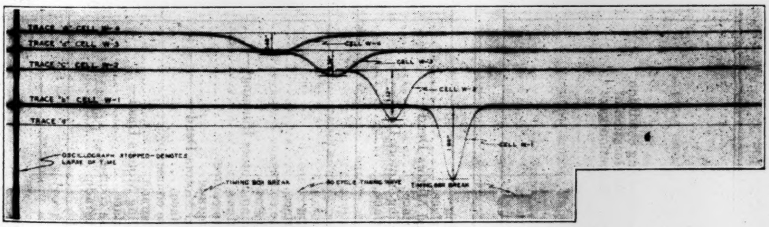

8 No. 32 NF
SR-4 resistance wire strain gage
Dummy SR-4 strain gage
Weld
4 in. diameter
3 No. 4-36 NC
Dummy gage plate
Cover plate removed
No. 20-3-wire waterproof cable
2 No. 0 taper pin-drive and weld after filling with oil
Approximately 0.020 in. Okonite tape
$\frac{9}{16}$ in.
$\frac{1}{8}$ in.
Diaphragm-thickness variable
Transformer oil
Fill with dux seal
$\frac{1}{8}$ in.
$\frac{1}{4}$ in.
1 in.
$\frac{5}{16}$ in.
$\frac{7}{16}$ in.-20 NF
$\frac{1}{4}$ in.
$\frac{1}{8}$ in.
$2\frac{7}{8}$ in.
3 in.
$3\frac{1}{2}$ in.
$4\frac{1}{2}$ in.
12 in.
$\frac{3}{16}$ in.
$\frac{11}{16}$ in.
Neoprene gasket
Rubber gasket $\frac{1}{16}$ in.
$\frac{9}{16}$ in.

Fig. 26.—Earth Pressure Cell



(a) CALIBRATION PRIOR TO TEST



(b) MOVING PLANE TEST OVER PRESSURE CELLS

Fig. 27.—Typical Moving Wheel Load Oscillogram

TABLE 6.—SUMMARY OF DEFLECTION DATA IN THE FLEXIBLE PAVEMENT BEHAVIOR TESTS, MARIETTA, GA.

(Deflections Are in Inches)

No.	Loading condition and location of deflection gage	SINGLE WHEEL LOADS (B-24 PLANE)						DUAL WHEEL LOADS (B-29 PLANE)											
		20 Kip (Plane Empty); Tire Over the Gage		25 Kip (Plane Partly Loaded); Tire Over the Gage		30 Kip (Plane Loaded); Tire Over the Gage		40 Kip (Plane Empty)					50 Kip (Plane Partly Loaded); Tire Over the Gage		60 Kip (Plane Loaded)				
								Tire Over the Gage			Center of Dual Wheel Over the Gage				Tire Over the Gage			Center of Dual Wheel Over the Gage	
		Line 2[a]	Line 3[a]	Line 2[a]	Line 3[a]	Line 2[a]	Line 3[a]	Line 1[a]	Line 2[a]	Line 3[a]	Line 1[a]	Line 2[a]	Line 2[a]	Line 3[a]	Line 1[a]	Line 2[a]	Line 3[a]	Line 1[a]	Line 2[a]
	(a) TELFORD ITEM																		
	Taxiing—																		
1	Surface	....	....	....	....	....	....	0.76	0.84	....	....	....	....	....	....	....	....	....	....
	Towed Motors Dead—																		
2	Surface	0.60	0.60	....	....	....	....	0.86	0.98	....	0.54	0.65	....	....	....	....	....	....	....
	Standing Motors Dead—																		
3	Surface	0.51	0.40	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....
	(b) THIN MACADAM ITEM																		
	Taxiing—																		
4	Surface	0.15	0.16	....	....	0.32	0.31	0.28	0.19	....	....	....	....	....	0.46	0.45	....	....	....
5	Base course	0.04	0.04	....	....	0.06	0.07	0.06	0.04	....	....	....	....	....	0.16	0.08	....	....	....
6	Subgrade	0.11	0.12	....	....	0.26	0.24	0.22	0.15	....	....	....	....	....	0.30	0.37	....	....	....
	Standing Motors Running[b]—																		
7	Surface	0.17	0.18	0.22	0.23	0.38	0.38	....	....	....	....	....	....	....	0.91	0.59	....	....	....
8	Base course	0.05	0.06	0.00	0.00	0.08	0.08	....	....	....	....	....	....	....	0.26	0.09	....	....	....
9	Subgrade	0.12	0.12	0.22	0.23	0.30	0.30	....	....	....	....	....	....	....	0.65	0.50	....	....	....
	Towed Motors Dead—																		
10	Surface	0.15	0.15	0.19	0.26	0.34	0.31	0.34	0.20	0.27	0.27	0.14	0.34	0.34	0.57	0.38	0.35	0.55	0.39
11	Base course	0.04	0.04	0.04	0.07	0.06	0.07	0.08	0.07	0.07	0.00	0.01	0.06	0.07	0.11	0.08	0.00	0.02	0.00
12	Subgrade	0.11	0.11	0.15	0.19	0.28	0.24	0.26	0.13	0.20	0.27	0.13	0.28	0.27	0.46	0.30	0.35	0.53	0.39
	Standing Motors Dead—																		
13	Surface	0.18	0.18	0.23	0.24	0.35	0.32	0.45	0.27	0.33	0.35	0.19	0.36	0.41	0.64	0.56	0.48	0.66	0.46
14	Base course	0.04	0.05	0.05	....[c]	0.07	0.07	0.09	0.08	0.08	0.00	0.00	0.05	0.08	0.15	0.13	0.05	0.00	0.01
15	Subgrade	0.14	0.13	0.18	....	0.28	0.25	0.36	0.19	0.25	0.35	0.19	0.31	0.33	0.49	0.43	0.43	0.66	0.45
	(c) MEDIUM MACADAM ITEM																		
	Taxiing—																		
16	Surface	0.10	0.11	....	....	0.15	0.19	0.17	0.11	....	....	....	....	....	0.41	0.25	....	....	....
17	Base course	....[c]	0.04	....	....	....[c]	0.05	0.04	0.00	....	....	....	....	....	0.07	0.00	....	....	....
18	Subgrade	....	0.07	....	....	....	0.14	0.13	0.11	....	....	....	....	....	0.34	0.25	....	....	....

	Standing Motors Running[b]—																		
19	Surface	0.10	0.11	0.15	0.15	0.20	0.23	....	....	....	....	....	....	....	0.62	0.31	....	....	....
20	Base course	....[c]	0.05	0.06	0.05	0.06	0.07	....	....	....	....	....	....	....	0.10	0.10	....	....	....
21	Subgrade	....	0.06	0.09	0.10	0.14	0.16	....	....	....	....	....	....	....	0.52	0.21	....	....	....
	Towed Motors Dead—																		
22	Surface	0.08	0.09	0.14	0.12	0.17	0.21	0.20	0.13	0.17	0.16	0.09	0.20	0.20	0.39	0.20	0.24	0.36	0.13
23	Base course	0.03	0.04	0.05	0.05	0.04	0.05	0.05	0.03	0.05	0.00	0.00	0.06	0.05	0.07	0.04	0.06	0.01	0.00
24	Subgrade	0.05	0.05	0.09	0.07	0.13	0.16	0.15	0.10	0.12	0.16	0.09	0.14	0.15	0.32	0.16	0.18	0.35	0.13
	Standing Motors Dead—																		
25	Surface	0.10	0.12	0.15	0.14	0.20	0.22	0.22	0.16	0.19	0.17	0.10	0.23	0.27	0.46	0.31	0.32	0.48	0.18
26	Base course	0.04	0.04	0.06	0.05	0.06	0.07	0.05	0.04	0.06	0.01	0.00	0.08	0.08	0.10	0.07	0.07	0.01	0.00
27	Subgrade	0.06	0.08	0.09	0.09	0.14	0.15	0.17	0.12	0.13	0.16	0.10	0.15	0.19	0.36	0.24	0.25	0.47	0.18
	(*d*) THICK MACADAM ITEM																		
	Taxiing—																		
28	Surface	0.07	0.06	....	....	0.10	0.08	0.07	0.07	....	....	....	....	....	0.22	0.11	....	....	....
29	Base course	....[c]	0.04	....	....	....[c]	0.04	0.03	0.03	....	....	....	....	....	0.09	0.06	....	....	....
30	Subgrade	....	0.02	....	....	....	0.04	0.04	0.04	....	....	....	....	....	0.13	0.05	....	....	....
	Standing Motors Running[b]—																		
31	Surface	0.07	0.07	0.10	0.08	0.11	0.09	....	....	....	....	....	....	....	0.29	0.13	....	....	....
32	Base course	....[c]	0.05	0.07	0.06	0.07	0.04	....	....	....	....	....	....	....	0.10	0.06	....	....	....
33	Subgrade	....	0.02	0.03	0.02	0.04	0.05	....	....	....	....	....	....	....	0.19	0.07	....	....	....
	Towed Motors Dead—																		
34	Surface	0.06	0.06	0.09	0.08	0.11	0.10	0.08	0.07	0.10	0.06	0.05	0.14	0.11	0.17	0.16	0.13	0.12	0.10
35	Base course	0.05	0.03	0.06	0.04	0.07	0.05	0.03	0.02	0.06	0.01	0.01	0.06	0.06	0.07	0.06	0.06	0.01	0.01
36	Subgrade	0.01	0.03	0.03	0.04	0.04	0.05	0.05	0.05	0.04	0.05	0.04	0.08	0.05	0.10	0.10	0.07	0.11	0.09
	Standing Motors Dead—																		
37	Surface	0.07	0.06	0.10	0.11	0.10	0.12	0.11	0.10	0.12	0.05	0.05	0.16	0.15	0.20	0.17	0.17	0.13	0.12
38	Base course	....[c]	0.04	0.08	0.07	0.06	0.06	0.05	0.04	0.06	0.00	0.00	0.08	0.07	0.10	0.07	0.07	0.01	0.01
39	Subgrade	....	0.02	0.02	0.04	0.04	0.06	0.06	0.06	0.06	0.05	0.05	0.08	0.08	0.10	0.10	0.10	0.12	0.11
	(*e*) SAND-CLAY ITEM																		
	Taxiing—																		
40	Surface	....	....	....	....	....	....	0.40	0.50	....	....	....	....	....	....	....	....	....	....
41	Base course	....	....	....	....	....	....	0.20	0.17	....	....	....	....	....	....	....	....	....	....
42	Subgrade	....	....	....	....	....	....	0.20	0.33	....	....	....	....	....	....	....	....	....	....
	Standing Motors Running[b]—																		
43	Surface	....	....	....	....	....	....	....	....	....	....	....	....	....	0.87	....	....	....	....
44	Base course	....	....	....	....	....	....	....	....	....	....	....	....	....	....[c]	....	....	....	....
45	Subgrade	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....
	Towed Motors Dead—																		
46	Surface	0.30[d]	0.33	....	....	0.62[d]	0.64	0.39	0.39	....	0.33	0.33	....	....	0.85	0.87	....	0.85	0.80
47	Base course	0.06[d]	0.11	....	....	0.17[d]	0.23	0.18	0.21	....	0.07	0.06	....	....	0.37	0.38	....	0.23	0.28
48	Subgrade	0.24[d]	0.22	....	....	0.45[d]	0.41	0.21	0.18	....	0.26	0.27	....	....	0.48	0.49	....	0.62	0.52
	Standing Motors Dead—																		
49	Surface	0.40[d]	0.50	....	....	0.72[d]	0.78	0.57	0.37	....	0.42	0.39	....	....	1.11	0.94	....	0.79	0.80
50	Base course	0.08[d]	0.19	....	....	0.21[d]	0.33	0.32	0.22	....	0.23	0.13	....	....	0.52	0.40	....	0.30	0.32
51	Subgrade	0.32[d]	0.31	....	....	0.51[d]	0.45	0.25	0.15	....	0.19	0.26	....	....	0.59	0.54	....	0.49	0.48

[a] Line Nos. refer to Fig. 22. [b] Approximately 1,000 rpm. [c] No reading; gage inoperable. [d] Gages in line 2, Table 6(*e*), were destroyed by previous tests with the B-29 plane. These were replaced as additional gages in line 3.

TABLE 7.—Summary of Pressure Cell Data in the Flexible

(Average Recorded Pressures,

No.	Test conditions of plane	Sand-Asphalt Item[a]											
		W-1			W-2			W-3			W-4		
		30 in. N	Over	30 in. S	30 in. N	Over	30 in. S	30 in. N	Over	30 in. S	30 in. N	Over	30 in. S
	Depth of Pressure Cell Below Surface—												
1	Before testing	10.3 in.			24.5 in.			37.9 in.			51.2 in.		
2	After testing	10.7 in.			23.8 in.			37.7 in.			51.0 in.		
	(a) Single Wheel												
	Plane Empty; 20-Kip Load on Each Single Wheel:												
3	Towed, motors dead	...	42.4	...	...	18.4	...	...	8.6	...	...	5.6	...
4	Taxiing	...	30.4	...	...	14.5	...	...	6.2	...	...	4.4	...
	Standing—												
5	Motors running 1,000 rpm	...	36.9	...	...	15.8	...	...	6.8	...	...	4.5	...
6	Motors dead	1.0	42.7	0.9	3.2	20.7	3.0	2.8	9.1	2.1	2.6	5.7	2.3
	Plane Partly Loaded; 25-Kip Load on Each Single Wheel:												
7	Towed, motors dead	...	48.5	...	...	23.5	...	...	10.3	...	...	5.6	...
	Standing—												
8	Motors running 1,000 rpm	...	44.2	...	...	22.2	...	...	9.3	...	...	...	...
9	Motors dead	...	47.9	2.4	2.4	23.3	2.0	2.4	10.8	2.6	2.8	6.7	3.0
	Plane Loaded; 30-Kip Load on Each Single Wheel:												
10	Towed, motors dead	...	52.3	...	...	26.9	...	...	12.0	...	...	7.9	...
11	Taxiing	...	49.2	...	...	25.7	...	...	10.8	...	...	6.9	...
	Standing—												
12	Motors running 1,000 rpm	...	51.5	...	...	28.3	...	...	11.5	...	...	7.9	...
13	Motors running 2,500 rpm	...	...	...	...	22.0	...	...	9.6	...	...	4.8	...
14	Motors dead	1.6	52.7	2.4	4.1	26.2	4.2	3.1	12.9	2.8	4.2	8.2	2.4
	(b) Dual Wheel												
	Plane Empty; 40-Kip Load per Dual Wheel:												
	Towed, with Motors Dead—												
15	One tire over cell	...	27.4	...	...	16.4	...	...	6.7	...	...	5.7	...
16	Center of dual wheel over cell	...	8.8	...	...	8.5	...	...	7.4	...	...	6.5	...
17	Taxiing, with one tire over cell	...	29.3	...	...	17.6	...	...	7.5	...	...	5.4	...
	Standing, Motors Dead—												
18	One tire over cell	1.2	42.4	1.0	1.4	18.1	1.8	2.7	9.5	2.5	3.7	6.5	3.0
19	Center of dual wheel over cell	0.0	3.6	0.2	1.1	9.8	2.0	2.9	8.6	3.2	3.7	7.4	3.2
	Plane Loaded; 60-Kip Load per Dual Wheel:												
	Towed, with Motors Dead—												
20	Tire over cell	...	55.9	...	...	30.3	...	...	13.6	...	...	10.0	...
21	Center of dual wheel over cell	...	16.0	...	...	15.5	...	...	10.4	...	...	9.3	...
22	Taxiing, one tire over cell	...	...	...	...	...	...	...	...	...	...	...	...
	Standing, One Tire Over Cell—												
23	Motors running 1,000 rpm	...	56.8	...	...	26.8	...	...	14.6	...	...	10.2	...
	Standing; Motors Dead—												
24	One tire over cell	1.5	58.2	1.8	2.4	29.9	4.4	3.9	14.3	4.6	5.2	10.5	3.9
25	Center of dual wheel over cell	0.6	8.0	1.1	2.6	15.8	2.6	4.0	11.6	4.1	5.5	10.7	4.7

[a] Pressure cells are identified by letter prefixes W and V (see Fig. 22); the letters N and S refer to the distance actually 1 ft north of each V-cell. This did not affect moving tests as the maximum pressure shown on the oscillo- V-cell and consequently did not show maximum pressure. [b] The numerals in parentheses under "Medium by N. M. Newmark, M. ASCE, for the wheel load directly over each cell and 1 ft north of each cell and by V-cells need no correction because these distances were corrected to show the true relationship to the cells.

PAVEMENT BEHAVIOR TESTS, MARIETTA, GA.

in Pounds per Square Inch)

MEDIUM MACADAM ITEM[a]												
V-1			V-2			V-3			V-4			No.
42 in. N	Over[b]	18 in. S	42 in. N	Over[b]	18 in. S	42 in. N	Over[b]	18 in. S	42 in. N	Over[b]	18 in. S	
	34.4 in.			47.6 in.			61.2 in.			74.2 in.		1
	33.0 in.			46.3 in.			59.1 in.			72.0 in.		2
LOADS (B-24 PLANE)												
...	8.7	...	...	5.7	...	...	2.8	...	...	1.8	...	3
...	8.5	...	...	6.0	...	...	2.7	...	...	1.8	...	4
...	6.2(8.1)	...	...	4.0(4.7)	...	...	2.2(2.5)	...	...	1.6(1.7)	...	5
1.6	7.5(9.4)	4.9	0.9	4.3(5.0)	3.1	0.6	2.2(2.5)	1.6	1.3	2.1(2.2)	1.7	6
...	11.2	...	...	6.8	...	...	3.7	...	...	2.7	...	7
...	7.8(10.2)	...	...	5.2(6.0)	...	...	2.4(2.8)	...	...	2.4(2.5)	...	8
1.6	8.1(10.5)	6.5	1.2	5.2(6.0)	5.0	0.9	2.7(3.1)	2.3	1.2	2.2(2.3)	1.8	9
...	12.3	...	...	8.2	...	...	4.5	...	...	3.2	...	10
...	11.4	...	...	7.6	...	...	4.2	...	...	2.8	...	11
...	10.4(13.2)	...	...	6.2(7.2)	...	...	3.9(4.3)	...	...	2.8(2.8)	...	12
...	10.8(13.6)	...	...	6.4(7.4)	...	...	3.7(4.1)	...	...	2.4(2.5)	...	13
1.8	10.7(13.5)	6.9	2.2	7.0(8.0)	5.8	1.4	4.0(4.4)	3.3	1.6	3.1(3.2)	2.5	14
LOADS (B-29 PLANE)												
...	10.3	...	...	6.9	...	...	3.6	...	...	3.0	...	15
...	10.6	...	...	7.8	...	...	4.1	...	...	3.6	...	16
...	9.7	...	...	6.3	...	...	3.2	...	...	2.2	...	17
2.4	8.5(10.6)	7.7	1.9	5.5(6.3)	4.3	1.9	3.8(4.1)	3.6	2.3	3.5(3.6)	3.3	18
2.7	8.5(10.1)	6.9	2.3	5.5(6.4)	4.7	1.9	3.4(3.9)	3.4	1.7	3.3(3.5)	3.5	19
...	43.0	...	...	9.0	...	...	4.9	...	...	5.0	...	20
...	15.4	...	...	11.3	...	...	6.8	...	...	6.3	...	21
...	14.1	...	...	10.2	...	...	6.7	...	...	5.8	...	22
2.0	11.4(14.5)	10.9	2.9	9.4(10.6)	6.3	3.4	6.4(7.0)	5.3	2.7	5.2(5.4)	4.2	23
2.3	11.7(14.8)	10.0	3.1	8.8(10.0)	6.9	2.9	4.9(5.5)	5.0	2.2	4.9(5.1)	4.8	24
2.9	10.2(12.6)	10.0	3.7	9.6(10.9)	5.5	4.5	6.0(6.7)	5.8	3.4	6.1(6.5)	4.5	25

north or south of the cell indicated. Surface points supposedly directly above V-cells were, by erroneous location, grams was recorded; but on standing tests the plane was stopped over the surface points 1 ft away from each Macadam Item" are the corrected values obtained by computing pressures by the graphical solution introduced adding the difference to the observed readings. Readings shown for distances 42 in. north and 18 in. south of

General features of the pressure cells used in these tests are shown in Fig. 26. The cell contains a machined steel diaphragm to which is attached a resistance type strain gage. Pressure is transmitted from the face plate to the diaphragm by an oil-filled chamber. Pressure applied to the diaphragm causes a strain resulting in a resistance change in the strain gage. The effect of variation in strain-gage temperature is compensated for by use of a dummy strain gage which is the same as that mounted on the diaphragm. The dummy gage is mounted in the chamber and is not affected by diaphragm strains but only by temperature changes. It is placed in the electrical circuit in such a manner as to nullify the effects of temperature changes in the active strain gage. Pressure cells are calibrated in the laboratory prior to field use by being placed in a calibrating device in which known increments of pressure may be applied uniformly over the face of the cell through an air-filled rubber diaphragm. In this manner, changes in applied pressure may be related to changes in resistance of the strain gage.

Field measurements of deflections and pressures were made by recording oscillographs and associated electrical circuits which furnished permanent records of the resistance changes with application of loads. A typical oscillograph record (oscillogram) is shown in Fig. 27 for four of the pressure cells; deflection-gage records are similar. Known resistance changes indicated by decade resistors in the circuit furnish a method of converting scaled changes in the position of the trace on the oscillogram to changes in deflection or pressure. A 60-cycle timing wave is recorded on the bottom of the record, and changes in the amplitude of this wave indicated the passage of the wheel load over a timing box on the test section, the location of which was known. Thus, the speed of the moving wheel load could be determined.

Discussion of Tests and Test Results

Since accelerated traffic testing was not conducted, the pavement items were subjected to relatively light traffic when compared to that imposed on other test sections where accelerated traffic was applied. Test traffic was applied in an expedient manner for obtaining the desired deflection and pressure data, and the traffic pattern was not uniform as in accelerated traffic tests. The amount of applied traffic (in addition to the consolidation traffic discussed in a preceding paragraph) therefore can only be estimated, and is approximately as follows: (*a*) Line 1 received 50 coverages with the dual wheel over a traffic lane 6 ft in width, this being the width required to meet the test conditions for one wheel over the gage and for the center of dual wheels over the gage; (*b*) line 2 received 150 coverages, about 40% of which was first applied by the dual wheel over a lane 6 ft in width and the remainder was concentrated in a single track directly over the gage line for the single (B-24) wheel load traffic; and (*c*) line 3 received 120 coverages, about 70% of which was first applied in a single track by the single wheel load traffic, and the remainder was applied over about a 6-ft lane with the dual wheel traffic as previously described.

The three macadam items all successfully withstood the limited amount of traffic imposed during these tests. By reference to Table 6, it may be noted that, although there are variations in deflections as measured on the three gage lines of a given item, there are apparent overall-trends of logical deflection variation with load and with base thickness. It may also be noted that in general the condition of standing with motors dead is about as severe as any condition encountered. Deflections in the thin macadam item are considered to have been sufficiently high that the item probably would not have stood an indefinite number of traffic coverages. Pressure data shown in Table 7 and obtained in the subgrade under the medium macadam base compare favorably with check computations made in accordance with the theory of elasticity, and these data are believed to be valid. The CBR of the surface of the clay subgrade beneath the macadam items averaged about 4% as constructed, and at the completion of all testing the CBR of this clay subgrade still averaged about 4% at the surface. The moisture content of the surface of the clay subgrade averaged about 30% before and after plane testing. The dry density of this subgrade increased from an average of 89 lb per cu ft before plane testing to 92 lb per cu ft after plane testing. The average density of the macadam bases was about 140 lb per cu ft after all testing, and the CBR of these bases averaged about 80% throughout the test period. It may be noted that the tentative CBR design curves, presented elsewhere in this Symposium, require a total thickness of base and pavement of about 29 in. for a single wheel load of 30,000 lb. It is believed probable, therefore, that the thin macadam item would not have been satisfactory in an accelerated traffic test. Under the limited amount of traffic applied in these tests, however, only minor rutting occurred in this item and the ruts were limited primarily to the junction of the thin macadam item with the telford item.

The telford item was unsatisfactory for wheel loads of the magnitude of those used in these tests, and it was previously noted that visibly large deflections occurred under only a 10,000-lb wheel load. B-29 tests were first applied on line 1 of this item, and surface deflections of about 1 in. were measured on the first few trials after which the deflections increased and exceeded the 1-in. measuring capacity of the deflection gage. Deterioration of the telford item was rapid, and deep ruts and apparent lack of stability caused early abandonment of the scheduled testing. After a few test runs, oak planking was laid over the wheel lanes to provide access to the other items. The telford pavement reacted similarly under the single wheel loads of the B-24. One feature that should be noted is that free water became entrapped in the base prior to laying the wearing course, some of which remained in the sand cushion course throughout testing. Prior to plane testing, tests on the clay subgrade of this item indicated an average CBR of 5%, a dry density of 92 lb per cu ft, and a moisture content of 30%. On completion of plane testing, the CBR of the subgrade had decreased from 2% to 3%, and the dry density had increased to 95 lb per cu ft, whereas the moisture content remained about the same. The early failure of this item was probably due (*a*) to the absence of load-spreading characteristics of this base which was composed of a single layer of stone as previously described, and (*b*) to an inadequate base and pavement thickness for a subgrade having such a low bearing value. In general, it is believed that this type of base allows load transfer in a columnar manner directly to the subgrade with little chance for load distribution. The presence of free water in the sand cushion may have had some influence.

The sand-clay item also did not satisfactorily withstand the traffic imposed by these tests. Ruts began developing in the traffic lanes during the early stages of testing and became progressively worse but at a slower rate than in the telford item. Ruts developed which allowed the ponding of water and subsequent infiltration into the base around the deflection gage sleeves and through cracks in the 1-in. asphaltic pavement. In general, rutting in this item amounted to from about 2 in. to 3 in. in all traffic lanes on completion of all plane testing. Slight humps appeared generally along the edges of the traffic lanes. By reference to Table 6, it may be noted that surface deflections obtained in this item were greater than those obtained in the thin macadam item, but subgrade deflections were less than those found under the thin macadam base. Observations made in trenches excavated after traffic was completed indicated that progressive shear deformation of the base occurred during traffic. The sand-clay base had a CBR of 35%, a dry density of 122 lb per cu ft, and a moisture content of 10% prior to plane testing. After plane testing, this base had a CBR of 12%, a dry density of 127 lb per cu ft, and a moisture content of 11%. During the testing period the moisture content increased only slightly, the density increased appreciably, and the CBR dropped considerably. The CBR curves developed by the laboratory for this material show that optimum moisture decreases with increase of compactive effort. The final moisture content of 11% was well on the wet side of the optimum moisture for the increased compactive effort. Tests show that for such a condition a lowering of the CBR can be expected. This condition is probably associated with pore pressure development. Thus, even though the moisture content was optimum for the constructed density, the remolding action of this material under traffic with increasing density placed the material above optimum moisture for the increased compactive effort with a resultant loss in CBR value. The clay subgrade in this item had a CBR of 5%, a dry density of 92 lb per cu ft, and a moisture content of 31% prior to plane testing. After plane testing, the subgrade CBR was 3%; the dry density, 94 lb per cu ft; and the moisture content, 30%.

The sand-asphalt item satisfactorily withstood all imposed traffic. Deflection data were not obtained for this item, but pressure data were obtained in the sand base. These pressure data agree favorably with computed pressures and appear to be valid and useful. No rutting or any other detrimental factors, such as cracking or signs of pavement movement, occurred in any part of this item. The sand subgrade appeared to be adequate for the applied wheel loads although the CBR was lower than that required by the CBR design for a 6-in. thickness. Since a very limited number of coverages was applied at a time when the pavement was cold and therefore very strong, no conclusion is believed warranted as to the adequacy of this pavement. Prior to and after plane testing, the sand subgrade had a CBR of 24%, a moisture content of 10%, and a dry density of 110 lb per cu ft. Consolidation of the sand subgrade was noted only to a very minor extent at the surface.

It is believed that the deflection and the presssure data presented in Tables 6 and 7 furnish an adequate basis for the fulfilment of the basic purpose of these tests which was to

> * * * furnish data whereby design curves, similar to the California method tentative design curves in Chapter XX [later Chapter 2, Part XII] of the *Engineering Manual [for War Department Construction*[1]], could be developed for a dual wheel load of 60,000 lbs."

Thus, it was not considered necessary to conduct accelerated traffic tests on this test section.

CONCLUSIONS

The following conclusions are considered applicable to the data discussed in this paper:

a. The deflection and the pressure data obtained from the three macadam base items and the pressure data obtained from the sand-asphalt item are believed to be reliable and are considered valid for use in detailed analyses.

b. An 18-in. depth of telford base and pavement as constructed for these tests, and having a subgrade with a CBR of 5% or less, is unsatisfactory for even a few operations of B-24 or B-29 planes.

c. The sand-clay base as constructed for these tests is unsatisfactory for B-24 or B-29 plane operations. The base was compacted to optimum conditions practically obtainable by ordinary construction methods. The plane traffic caused additional compaction, and the moisture content of the material increased during plane testing. Laboratory tests show that under these conditions the bearing value of this material decreases.

OTHER ACCELERATED TRAFFIC TESTS

BY J. F. REDUS, JR.,[31] JUN. ASCE

In addition to the three major test sections described in previous papers of this Symposium, several accelerated traffic tests were conducted on existing flexible airfield pavements or on specially constructed test sections to furnish data related to the CBR method of design. The tests were conducted at the following locations:

- Corpus Christi Municipal Airport, Corpus Christi, Tex.
- Dothan Municipal Airport, Dothan, Ala.
- Fargo Municipal Airport, Fargo, N. Dak.
- Grenier Field, Manchester, N. H.
- Natchitoches Municipal Airport, Natchitoches, La.
- Eglin Field, Valparaiso, Fla.
- Langley Field, Virginia
- Waterways Experiment Station, Vicksburg, Miss.

DESCRIPTION OF TEST SECTIONS AND TEST PROCEDURES

Table 8 shows the design of the various sections, the traffic imposed, the behavior, and the test data before and after traffic for the various sections. Four of the test sections (Corpus Christi, Dothan, Natchitoches, and Waterways Experiment Station) had regular cross sections with uniform thicknesses, as shown in Table 8. The remaining four test sections were of special design with nonuniform thicknesses. To clarify the design of these sections, the data in Table 8 are supplemented by the longitudinal profiles shown in Fig. 28. A brief description of the four sections with nonuniform thicknesses is given in the following paragraphs. A description of the Waterways Experiment Station test section is also given, since it involved special design considerations.

Fargo.—The tests at Fargo Municipal Airport were conducted on a runway before the asphalt wearing course had been placed. The surfacing at the time of the tests consisted of 6 in. of soil cement and was underlain by an imported sandy soil which had a thickness at the edge of 12 in. and a thickness at the center line of 24 in. The subgrade was a highly plastic clay.

Grenier.—The test at Grenier Field was conducted on a specially constructed test section 150 ft long and 75 ft wide which was divided longitudinally into five areas having different thicknesses of base course. The wearing surface consisted of a treatment of tar prime, an application of sand, an application of tar, and a cover aggregate of either "pea gravel" or sand.

Eglin.—The tests at Eglin Field were conducted on a specially constructed test section with three traffic lanes as follows: Lane 1, 37,000-lb wheel load; lane 2, 50,000-lb wheel load; and lane 3, 15,000-lb wheel load. The thicknesses of wearing course and base course varied as shown in Fig. 28.

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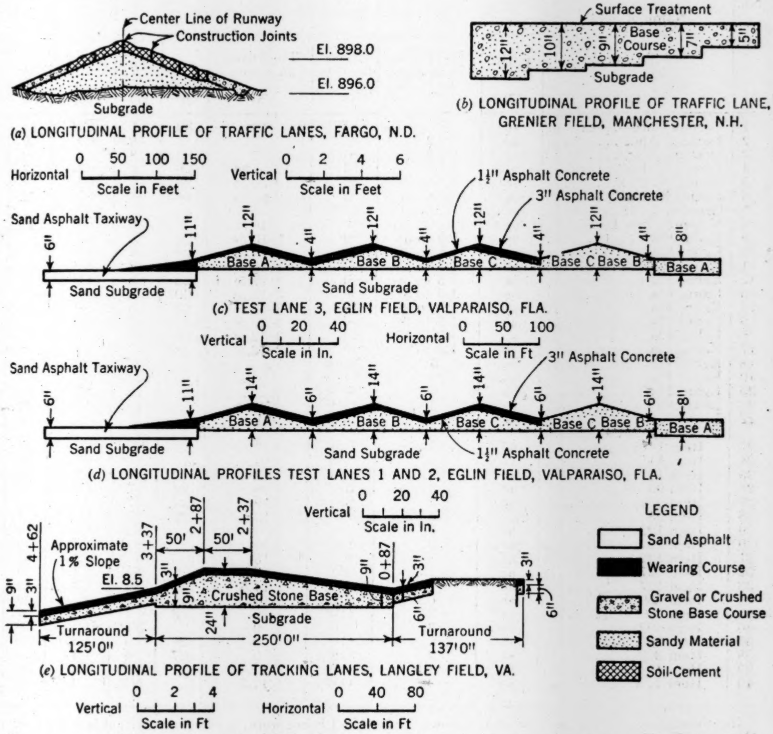
Langley.—The tests at Langley Field were conducted on a specially constructed test section. The first test was performed with a 20,000-lb wheel load, and when this did not produce the desired subgrade shear failures a second test was made using a 50,000-lb wheel load. The thickness of base course varied as shown in Fig. 28.


Center Line of Runway
Construction Joints
El. 898.0
El. 896.0
Subgrade
Surface Treatment
12"
10"
9"
Base Course
7"
5"
Subgrade
Horizontal 0 50 100 150 Scale in Feet
Vertical 0 2 4 6 Scale in Feet
1½" Asphalt Concrete
3" Asphalt Concrete
Sand Asphalt Taxiway
6"
11"
12"
4"
12"
4"
12"
4"
12"
4"
8"
Base A
Base B
Base C
Base C
Base B
Base A
Sand Subgrade
Sand Subgrade
Vertical 0 20 40 Scale in In.
Horizontal 0 50 100 Scale in Ft
3" Asphalt Concrete
Sand Asphalt Taxiway
6"
11"
14"
6"
14"
6"
14"
6"
14"
6"
8"
Base A
Base B
Base C
Base C
Base B
Base A
Sand Subgrade
Sand Subgrade
1½" Asphalt Concrete
Vertical 0 20 40 Scale in In.
LEGEND
Sand Asphalt
Wearing Course
Gravel or Crushed Stone Base Course
Sandy Material
Soil-Cement
4+62
3+37
2+87
2+37
50'
50'
Approximate 1% Slope
El. 8.5
9"
3"
3"
9"
Crushed Stone Base
Subgrade
24"
9"
0+87
3"
6"
3"
6"
Turnaround 125'0"
250'0"
Turnaround 137'0"
Vertical 0 2 4 Scale in Ft
Horizontal 0 40 80 Scale in Ft

(a) LONGITUDINAL PROFILE OF TRAFFIC LANES, FARGO, N.D.

(b) LONGITUDINAL PROFILE OF TRAFFIC LANE, GRENIER FIELD, MANCHESTER, N.H.

(c) TEST LANE 3, EGLIN FIELD, VALPARAISO, FLA.

(d) LONGITUDINAL PROFILES TEST LANES 1 AND 2, EGLIN FIELD, VALPARAISO, FLA.

(e) LONGITUDINAL PROFILE OF TRACKING LANES, LANGLEY FIELD, VA.

FIG. 28.—LONGITUDINAL PROFILES; FARGO, GRENIER, EGLIN, AND LANGLEY FIELDS

Waterways Experiment Station.—The tests at the Waterways Experiment Station were conducted on a specially constructed test section with wheel loads of 15,000 lb and 37,000 lb on single tires and of 60,000 lb on a set of dual tires. This test section was primarily constructed for a study of asphalt wearing course mixes, but several types of base course material were used in order to cover a range of types. The thicknesses of base and wearing course were designed by the CBR method; however 20% additional thickness of base was provided in the main test tracks (items 1–162) to insure adequate subgrade protection since this test section was built primarily to study asphaltic wearing courses.

The accelerated traffic tests were conducted using loaded earth-moving equipment traveling along designated lanes on the pavements. The tests were

TABLE 8.—SUMMARY OF

(a) CLASSIFICATION AND THICKNESS DATA

No.	Name of field	Pavement		Base[b]				Subgrade[b]		
		Type[a]	Thickness, in inches	Classification[c]	Thickness, in inches	w_L	I_P	Classification[c]	w_L	I_P
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
1	Corpus Christi, Tex.	ac	5	SF	10	30	11	CH	64	38
2	Dothan, Ala.	ac	2	SF	7	35	18	CL	29	12
3	Dothan, Ala.	ac	2	SF	7	35	18	CL	29	12
4	Fargo, N. Dak.	sc	6	SF	12 to 24[g]	28	13	CH	77	52
5	Fargo, N. Dak.	sc	6	SF	12 to 24	28	13	CH	77	52
6	Fargo, N. Dak.	sc	6	SF	12 to 24	28	13	CH	77	52
7	Grenier, N. H.	st	....	GW	5	....	NP	SW	....	NP
8	Natchitoches, La.	ac	2	GF	10	22	6	CH	46	28
9	Eglin, Fla.	ac	1.5 to 3[i]	SF	1–10.5[g]	22	3	SP	....	NP
10	Eglin, Fla.	ac	1.5 to 3[i]	SF	1–10.5[i]	20	3	SP	....	NP
11	Eglin, Fla.	ac	1.5 to 3[i]	SF	1–10.5[i]	22	3	SP	....	NP
12	Eglin, Fla.	ac	1.5 to 3[i]	SF	3–12.5[i]	22	3	SP	....	NP
13	Eglin, Fla.	ac	1.5 to 3[i]	SF	3–12.5[i]	20	3	SP	....	NP
14	Eglin, Fla.	ac	1.5 to 3[i]	SF	3–12.5[i]	22	3	SP	....	NP
15	Eglin, Fla.	ac	1.5 to 3[i]	SF	3–12.5[i]	22	3	SP	....	NP
16	Eglin, Fla.	ac	1.5 to 3[i]	SF	3–12.5[i]	20	3	SP	....	NP
17	Eglin, Fla.	ac	1.5 to 3[i]	SF	3–12.5[i]	22	3	SP	....	NP
18	Eglin, Fla.	sa	5	None	....	...	....	SP	....	NP
19	Eglin, Fla.	sa	6	None	....	....	....	SP	....	NP
20	Eglin, Fla.	sa	6	None	....	....	....	SP	....	NP
21	Langley, Va.	ac	3	GW	12	....	....	....	22	4
22	Langley, Va.	ac	3	GW	21	....	....	....	22	4
	Vicksburg, Miss.									
23	Item 14	ac	3	GW	6	....	NP	CL	39	14
24	Item 14	ac	3	GW	10	....	NP	CL	39	14
25	Item 14	ac	3	GW	13	....	NP	CL	39	14
26	Item 95	ac	3	SF	6	22	7	CL	39	14
27	Item 95	ac	3	SF	10	22	7	CL	39	14
28	Item 95	ac	3	SF	13	22	7	CL	39	14
29	Item 122	ac	3	SP	6	....	NP	CL	39	14
30	Item 122	ac	3	SP	10	....	NP	CL	39	14
31	Item 122	ac	3	SP	13	....	NP	CL	39	14
32	Item 127	ac	1.5	SP	7.5	....	NP	CL	39	14
33	Item 127	ac	1.5	SP	14.5	....	NP	CL	39	14
34	Item 160	ac	6	SP	10	....	NP	CL	39	14
35	Item 302	sa	2	GW	9	28	4	CL	39	14
36	Item 302	sa	2	GW	9	....	NP	CL	39	14
37	Item 300	sa	2	GF	9	29	9	CL	39	14
38	Item 300	sa	2	GF	9	27	10	CL	39	14
39	Item 1	ac	1.5	GW	11.5	....	NP	CL	39	14
40	Item 1	ac	1.5	GW	11.5	....	NP	CL	39	14

[a] Abbreviations in Col. 2 are defined as follows: ac is asphaltic concrete; sc is soil cement; st is surface treatment;
NP means "nonplastic." [c] According to Arthur Casagrande, M. ASCE, *Transactions*, ASCE, Vol. 113, 1948, p. 911.
expressed as percentages of the modified AASHO (American Association of State Highway Officials) density.
subgrade consolidation. [g] A double wedge-shaped base course; see Fig. 28. [h] To a depth of 24 in. through
[j] Also subgrade shear.

continued until failure occurred or a specific number of coverages was completed. Visual observations were made of pavement behavior during the tests and measurements of deformation of the pavement surface both during and after the traffic tests were taken. Field and laboratory tests were performed on the materials in the base and the subgrade after traffic had been stopped. In most cases these tests were made both outside and inside the traffic lanes to represent conditions before and after traffic. Soil classifications are based on the system devised by Professor Casagrande[7] and density relations are compared to the AASHO classification[25] as modified by the Corps of Engineers.

DATA FROM TEST SECTIONS

(b) TRAFFIC DATA

Performance	Wheel load, in pounds	Coverages	Base[d]				Subgrade				Adequate protection[e]	No.
			Before Traffic		After Traffic		Before Traffic		After Traffic			
			Density	CBR	Density	CBR	Density	CBR	Density	CBR		
(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)	(20)	(21)	(22)	
Base shear	15,000	285	99	23	....	....	96	13	....	....	Yes	1
Base shear and consolidation[f]	20,000	487	89	10	....	....	90	25	....	....	Yes	2
	50,000	1,166	89	10	....	....	90	25	....	....	Yes	3
Base and subgrade shear	30,000	570	89	9	....	....	83	3	....	....	No	4
	20,000	530	89	9	....	....	83	3	....	....	No	5
Subgrade shear[h]	12,500	1,470	89	9	....	....	83	3	....	....	No	6
Base consolidation[f]	15,000	3,710	104	....	109	....	95	35	100	....	Yes	7
Subgrade shear[f]	15,000	582	86	35	86	30	87	5	93	5	No	8
Base and subgrade consolidation	15,000	3,500	95	57	94	53	100	34	100	35	Yes	9
	15,000	3,500	91	58	93	95	100	....	101	....	Yes	10
	15,000	3,500	94	45	93	61	99	....	99	....	Yes	11
	37,000	3,500	96	48	97	71	98	22	98	45	Yes	12
	37,000	3,500	94	45	97	89	94	34	95	42	Yes	13
	37,000	3,500	96	28	98	61	99	23	96	31	Yes	14
	50,000	3,500	....	....	94	77	....	....	97	45	Yes	15
	50,000	3,500	96	38	96	76	94	43	96	41	Yes	16
	50,000	3,500	96	38	100	55	98	32	99	26	Yes	17
Subgrade consolidation	15,000	3,500	....	....	....	....	94	29	95	40	Yes	18
	37,000	3,500	....	....	....	....	95	25	100	14	Yes	19
	50,000	3,500	....	....	....	....	....	....	100	25	Yes	20
	20,000	6,667	....	....	....	....	88	15	94	30	Yes	21
	50,000	2,733	....	....	....	....	88	15	94	30	Yes	22
Satisfactory	15,000	3,500	....	81	91	78	....	....	84	28	Yes	23
	37,000	1,500	....	81	103	86	....	....	93	31	Yes	24
	30,000	1,500	....	81	....	87	....	....	98	35	Yes	25
Base consolidation	15,000	3,500	....	28	99	82	....	....	97	40	Yes	26
	37,000	1,500	....	28	94	49	....	....	97	34	Yes	27
	30,000	1,500	....	28	85	40	....	....	88	43	Yes	28
	15,000	3,500	....	33	102	51	....	24	95	33	Yes	29
	37,000	1,500	....	33	100	81	....	24	97	37	Yes	30
	30,000	1,500	....	33	99	89	....	24	95	31	Yes	31
Base and subgrade consolidation	15,000	3,500	....	35	102	44	....	21	94	29	Yes	32
	30,000	666	....	35	102	57	....	21	97	41	Yes	33
	30,000	666	....	34	99	94	...	20	92	30	Yes	34
Satisfactory	37,000	100	....	54	92	50	....	38	91	33	Yes	35
Subgrade shear	37,000	100	....	54	85	32	....	38	92	14	No	36
Satisfactory	30,000	100	....	54	95	44	....	38	97	27	Yes	37
Base shear[i]	30,000	100	....	54	92	13	....	38	93	10	No	38
Satisfactory	37,000	400	....	....	....	72	....	18	91	18	Yes	39
Subgrade shear	37,000	400	....	....	....	56	....	18	90	10	No	40

and sa is sand asphalt. [b] In Cols. 6 and 9, w_L is the liquid limit; and, in Cols. 7 and 10, I_P is the plasticity index.
[d] All density and CBR values are the results of "in-place" tests, and are percentages. The density values are
[e] Adequate protection afforded to the subgrade by the thickness of the base and the wearing course. [f] Also
pavement and base. [i] Two different thicknesses of ac on a double wedge-shaped base course, see Fig. 28.

TEST RESULTS AND DISCUSSION

In the tests at Corpus Christi, deformation of the pavement surface of about 1 in. in the traffic lane and a bulge of about 0.5 in. on either side indicated that the base course underwent a small amount of shear deformation and possibly a little consolidation. There was considerable shear deformation or consolidation of the base at only one localized area in which the deformations were about 4 in., and this was attributed to excessive moisture in the base and probably insufficient compaction.

In the test at Dothan, large deformations occurred in the 20,000-lb lane and considerably larger deformations in the 50,000-lb lane. Relatively small upheavals at the sides of the traffic lanes indicated distress which later observations showed to be the result of some shear deformation in the base. The large deformations in the traffic lanes were found to be the results of consolidation in the base and the subgrade.

At Fargo, considerable deformations in the traffic lanes and upheavals at the sides of the lanes occurred throughout the 30,000-lb and the 20,000-lb lanes and to a point in the 12,500-lb lane where total thickness of the soil cement and the underlying sandy soil was 24 in. From observations, it was concluded that shear deformations had taken place in both the sandy base material and the subgrade throughout the 20,000-lb and the 30,000-lb lanes. No shear deformation occurred in either the sandy base or the subgrade in the 12,500-lb lane where the total thickness above the subgrade was more than 24 in.

No in-place CBR values were determined on the base material at Grenier Field either before or after traffic, but the average laboratory remolded and soaked value for this material was 75%. Traffic produced no signs of shear deformation in either the base or the subgrade, but there was evidence of consolidation in the base. The surface treatment was inadequate to withstand the abrasive action of traffic.

In the test at Natchitoches, large deformations occurred in the tracking lanes with adjacent upheavals. The upheavals indicated that subgrade and possibly base shear deformation occurred. However, the upheavals were not large enough to compensate for the large deformations inside the tracking lanes, and it was concluded that subgrade consolidation also occurred.

No shear deformation occurred in the base or subgrade in any traffic lane of the Eglin Field test section. A thickness of base and pavement of 4 in. which was the minimum in the test section was adequate to prevent shear deformation in the subgrade under the 15,000-lb wheel load. A thickness of base and pavement of 6 in. was sufficient to prevent shear deformation in the subgrade for the 37,000-lb and the 60,000-lb wheel loads. The 6-in. thickness was the minimum in these lanes. Some consolidation occurred in all base and subgrade materials under each type of wearing course in each traffic lane. The maximum deformations were 5.3 in., 5.5 in., and 2.2 in. for lanes 1, 2, and 3, respectively. All wearing courses were adequate in all traffic lanes.

In the tests at Langley Field, considerable settlement had occurred in the 20,000-lb lane at the end of the test, which had a minimum total thickness of base and pavement of 12 in. This settlement was all due to consolidation, except in the turnarounds where subgrade shear occurred. The subgrade shear occurred under the 20,000-lb wheel load where the total thickness was only 9 in. Very large deformations had occurred in the 50,000-lb lane at the end of the test. These were partly caused by consolidation, but subgrade shear deformation also occurred where the total thickness of base and pavement was 22 in. or less. Excessive springing occurred in the tests, which indicates that excess hydrostatic pressure had developed in the pore water. The subgrade CBR at the start of the test was 15% and was 30% after the tests were completed. Analysis is difficult, since the CBR at the time the springing occurred is not known. It is considered that thicknesses of 12 in. and 24 in. are considerably more than adequate to prevent shear deformation in a subgrade with a CBR of 30 under 20,000-lb and 50,000-lb wheel loads, respectively, provided no hydrostatic excess pressure occurs in the pore water.

In the tests at the Waterways Experiment Station, the performance of pavements in items 14, 95, and 122 is shown as adequate for subgrade protection in Table 8, although some consolidation took place in both base and subgrade as evidenced by a slight deformation of the surface and increases in in-place CBR values. Rather large deformations occurred in items 127 and 160. On the basis of observations, it was concluded that they were caused by base and subgrade consolidation, since the deformations were not accompanied by upheavals at the sides of the tracking lanes, and since in-place CBR values of both base and subgrade increased. In items 1 and 302, subgrade shear occurred, which was evidenced by upheavals. In item 300, subgrade and base shear occurred, which was also evidenced by upheavals outside the trafficked area.

CONCLUSIONS

Since the data presented in this paper are to be used with other data in a summary analysis of the CBR method of design in a later paper, a few general but no specific conclusions will be made here. The following conclusions are based on the data shown in Table 8:

a. Shear failure did not occur in nonplastic to slightly plastic base or subgrade material;

b. Shear failures in either base or subgrade developed at a rapid rate once the condition for failure existed;

c. In general, nonplastic base courses were consolidated to more than 100% modified AASHO density by traffic, and bases having some plasticity were consolidated to more than 95% of modified AASHO density; and

d. All nonplastic subgrades were consolidated to more than 95% of modified AASHO density by traffic and, in general, plastic subgrades were consolidated to more than 90%.

DESIGN CURVES FOR SINGLE WHEEL LOADS

By C. R. Foster,[32] Assoc. M. ASCE

As explained in a previous paper, the CBR design curves for heavy wheel loads were extrapolated from highway data in the initial stages of the airfield construction program. Subsequently, the curves were correlated with the service behavior of pavements under both accelerated traffic and actual airplane traffic. The purpose of this paper is to present the data that have been accumulated on this subject and to show the correlations that have been made. A comparison between the data and the present CBR thickness design curves is presented.

The general method of correlating the curves with service behavior has been to select satisfactory pavements and pavements that failed and to compare the CBR of the subgrade, or layer being studied, and the total thickness of overlying base and pavement with the CBR thickness design curves. All comparisons are based on "in-place" CBR values which are considered to represent the condition of the soil at the time traffic is applied.

Pavements included in the "failed" category are limited to those in which shear deformation occurred in a subgrade or in a lower base course overlain by a high-quality base in which no shear deformation occurred. In no instance is information used where there was evidence of shear deformation in the base. For instance, data from items 1 and 2 in the Barksdale test (see Table 4(*c*)) are not used because shear deformation occurred in the base as well as in the underlying subgrade. The data from item 5 at Barksdale, where shear deformation occurred in the subgrade but not in the base, are used (see Table 9(*a*), line 49, and Table 9(*b*), line 25).

Certain criteria have been used in selecting pavements from the standpoint of intensity of traffic. A study of the available records where failure occurred showed that where the failure is caused by shear deformation it usually occurs after a relatively few repetitions. In approximately fifty cases for which reliable data are available, 96% of the failures occurred before 2,000 coverages had been applied. This amount of traffic is relatively small if the total that can occur at an airfield is considered. Traffic counts have shown that approximately 75% of the traffic on a 150-ft runway occurs in the central 50 ft. Assuming traffic in the central 50 ft to be uniformly distributed, a 30,000-lb wheel load with two tire prints 1.33 ft wide will produce a coverage in approximately nine landings and take-offs in the central 50 ft which would require 12 cycles of operation on the field. One landing and one take-off constitute a cycle. A reasonable value for the maximum number of cycles of operation per day for planes of this weight during the pilot training period was 270, which would produce slightly more than 20 coverages per day in the central 50 ft of the runway. On this basis, 2,000 coverages represent about one hundred days of operation.

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The failures under consideration are those resulting from shear deformation caused by overstress. Failures due to consolidation and to surface deterioration occur over a much longer period. There are also cases of failure from saturation of the subgrade or base that occur after several years of operation. However, when conditions become critical so that the subgrade is overstressed, failure occurs after a relatively few additional repetitions.

On the basis of the foregoing considerations, airfield pavements that have been subjected to more than 30,000 cycles of operation (approximately 2,000 coverages) without showing distress are considered satisfactory. Accelerated traffic test pavements that have successfully withstood 2,000 coverages are also considered adequate. No minimum traffic criteria have been applied to pavements that failed.

Table 9 shows the pertinent data that have been accumulated from the various studies. All pavements for which reliable traffic and CBR data are available are shown. Table 9 includes the data obtained from the accelerated traffic tests described in previous papers of this Symposium together with the information available from actual airfields. It gives the name of the field and facility; the identification of test area; the wheel load; the amount of traffic at the time of test; the type and thickness of wearing course, base, and subgrade; and the CBR of the various layers. The data are presented in two groups to facilitate plotting. One group gives data for wheel loads less than 30,000 lb; the other, for wheel loads of 30,000 lb and greater. Each line in the two groups is given a number for reference on the plots.

Fig. 29(*a*) shows CBR design curves for wheel loads of 15,000 lb, 20,000 lb, 25,000 lb, 30,000 lb, 40,000 lb, 50,000 lb, and 60,000 lb. The 15,000-lb, 40,000-lb, and 60,000-lb curves are the same as the runway curves presented in the *Engineering Manual for War Department Construction*[33] prepared by the Office of the Chief of Engineers. The others are interpolations from curves given in the manual. The interpolated curves are presented to facilitate comparison with the field data. These design curves represent a slight modification from the original extrapolations as presented in Symposium paper No. 3 in that the thickness requirements have been reduced slightly for the higher CBR values in the range of cohesionless materials. This modification was made on the basis of the results of accelerated traffic tests. Also, minimum thicknesses have been established for the various wheel loads. The minimum thicknesses are specified to insure satisfactory performance from settlement, weather, and other features, as well as to prevent shear in the subgrade. For comparison, the California design curves representing light and heavy highway experience used in the extrapolations are shown.

Fig. 29(*a*) is a plot of data for wheel loads less than 30,000 lb; Fig. 29(*b*) is a plot of data for wheel loads of 30,000 lb and more. The data from the accelerated traffic tests and the airfield pavements are identified by appropriate symbols. Variable thicknesses were usually tested in the accelerated traffic tests; therefore, three symbols were used for the data. Where it was possible to determine a thickness of base and pavement adequate to prevent shear deform-

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[33] "Airfield Pavement Design: Flexible Pavements," *Engineering Manual for War Department Construction*, Office of Chf. of Engrs., U. S. Army, Washington, D. C., July, 1946, Pt. XII, p. 11, Fig. 2.

TABLE 9.—CBR DATA FOR THE DEVELOPMENT OF DESIGN CURVES FOR SINGLE WHEEL LOADS

Line	Army airfield	Facility[a]	Pit No.	SUBGRADE OR SUBBASE[b]				OVERLYING BASE[b]					SURFACE		Total thickness, in inches	TRAFFIC		Performance satisfactory[d]
				Type[c]	w_L (%)	I_P (%)	CBR (%)	Type[c]	w_L (%)	I_P (%)	CBR (%)	Thickness, in inches	Type[e]	Thickness, in inches		Wheel load, in kips	Coverages	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)
								(a) WHEEL LOADS LESS THAN 30,000 LB										
1	Fargo, N. Dak.	Test section	....	CH	79	55	3	SF	28	13	9	18.0–24.0	sc	6.0	12.0–30.0	12.5	1,470	Yes[e]
2	Florence, S. C.	NE-SW	....	SP	...	...	13	la[f]	....	....	....	11.5	ac	2.5	14.0	13.0	3,134	Yes
3	Pan-American[g]	E-W	....	SP	...	...	38	lr[c]	....	....	....	6.0	st	0.5	6.5	13.0	4,051	Yes
4	Bergstrom Field, Austin, Tex.	NW-SE1	1	SC	...	NP	27	GW	17	1	108	10.0	ac	2.0	12.0	15.0	11,258	Yes
5		NW-SE1	1	CH	70	44	10	la	NP-17	NP-1	27–108	15.0	ac	2.0	17.0	15.0	11,258	Yes
6		NW-SE1	3	CH	70	44	6	la	NP-17	NP-1	20–76	15.0	ac	2.0	17.0	15.0	358	No[h]
7		NW-SE1	4	SC	...	NP	37	GW	17	1	150	10.0	ac	2.0	12.0	15.0	11,258	Yes
8		NW-SE1	4	CH	70	44	11	la	NP-17	NP-1	37–150	15.0	ac	2.0	17.0	15.0	11,258	Yes
9		NW-SE1	6	SC	...	NP	30	GW	17	1	135	10.0	ac	2.0	12.0	15.0	11,258	Yes
10		NW-SE1	6	CH	70	44	12	la	NP-17	NP-1	30–135	15.0	ac	2.0	17.0	15.0	11,258	Yes
11		T-1	7	ML	22	7	19	GW	17	3	150+	10.0	ac	2.0	12.0	15.0	23,700	Yes
12		T-1	7	CH	58	39	10	la	17–22	3–7	19–150+	15.0	ac	2.0	17.0	15.0	23,700	Yes
13	Berry Field, Nashville, Tenn.	West N-S	1	CH	71	37	23	GP	16	2	150+	9.0	ac	3.0	12.0	15.0	1,741	Yes
14		West N-S	2	ML	37	11	43	GP	16	2	124	9.0	ac	3.0	12.0	15.0	1,741	Yes
15		West N-S	3	CL	41	20	27	GP	16	2	139	9.0	ac	3.0	12.0	15.0	1,741	Yes
16		West N-S	4	CH	51	29	16	GP	16	2	93	9.0	ac	3.0	12.0	15.0	1,741	Yes
17	Dodge City (Kans.) airfield	T-4A	2	CL	28	13	37	GF	25	9	88	8.5	ac	3.5	12.0	15.0	8,888	Yes
18		T-4A	2	CL	47	32	14	la	25–28	9–13	37–88	15.5	ac	3.5	19.0	15.0	8,888	Yes
19	Eglin, Fla.	Test section	....	SP	NP	NP	35	SF	21	3	60	1.0–10.5	ac	1.5 and 3.0	4.0–12.0	15.0	3,500	Yes[i]
20		Test section	....	SP	NP	NP	35	none	....	....	....	....	sa	6.0	6.0	15.0	3,500	Yes
21	Grenier, New Hampshire	Test section	....	SW	NP	NP	30	GW	NP	NP	....	5.0–12.0	st	....	5.0–12.0	15.0	3,710	Yes[j]
22	Lawson, Fort Benning, Ga.	T-4	3	SC	19	1	22	SF	19	3	45	7.0	ac	3.0	10.0	15.0	3,475	Yes
23		T-4	4	SC	19	1	25	SF	19	3	32	8.0	ac	2.0	10.0	15.0	3,475	Yes
24	Waterways Experiment Station, Vicksburg, Miss.	Test section	14[k]	CL	39	14	28	GW	....	NP	78	6.0	ac	3.0	9.0	15.0	3,500	Yes
25		Test section	95[k]	CL	39	14	40	ML	22	7	82	6.0	ac	3.0	9.0	15.0	3,500	Yes
26		Test section	122[k]	CL	39	14	33	SP	....	NP	51	6.0	ac	3.0	9.0	15.0	3,500	Yes
27	Rocky Ford (Colo.) Field; Auxiliary to the La Junta (Colo.) Field	E-W	1	CL	28	17	18	GW	....	NP	60	6.0	ac	2.5	8.5	16.0	2,902	Yes
28		E-W	1	CL	30	20	6	la	NP-28	NP-17	18–60	14.0	ac	2.5	16.5	16.0	2,902	Yes
29		E-W	4	CL	24	16	42	GW	....	NP	103+	6.0	ac	2.5	8.5	16.0	2,902	Yes
30		E-W	4	CL	30	18	11	la	NP-24	NP-16	42–103	14.0	ac	2.5	16.5	16.0	2,902	Yes

31	Douglas, Ariz.	N-S	4	ML	24	3	62	GW	....	NP	116	6.5	ac	2.5	9.0	17.5	3,134	Yes
32		N-S	4	CL	43	21	24	la	NP-24	NP-3	62-116	16.5	ac	2.5	19.0	17.5	3,134	Yes
33		N-S	5	ML	24	3	46	GW	....	NP	122	9.5	ac	2.5	12.0	17.5	3,134	Yes
34		N-S	5	CL	43	21	30	la	NP-24	NP-3	46-122	16.5	ac	2.5	19.0	17.5	3,134	Yes
35		N-S	8	GP	30	11	77	GW	27	9	69	7.25	ac	1.75	9.0	17.5	3,134	Yes
36		N-S	8	CH	50	24	24	la	27-30	9-11	69-77	13.25	ac	1.75	15.0	17.5	3,134	Yes
37		N-S	9	GP	30	11	83	GW	27	9	60	7.25	ac	1.75	9.0	17.5	3,134	Yes
38		N-S	9	CH	50	24	24	la	27-30	9-11	60-83	12.25	ac	1.75	14.0	17.5	3,134	Yes
39		N-S	11	GP	30	11	77	GW	27	9	86	7.0	ac	2.0	9.0	17.5	3,134	Yes
40		N-S	11	CH	50	24	38	la	27-30	9-11	77-86	12.0	ac	2.0	14.0	17.5	3,134	Yes
41	La Junta, Colo.	E-W	1	CL	35	20	29	GF	24	9	136	6.0	ac	7.0	13.0	17.5	4,642	Yes
42		E-W	1	CL	35	17	9	la	24-35	9-20	29-136	15.0	ac	7.0	22.0	17.5	4,642	Yes
43		E-W	4	CL	27	11	22	GF	28	14	64	6.0	ac	4.5	10.5	17.5	4,642	Yes
44		E-W	4	CL	24	10	15	la	27-28	11-14	22-64	15.0	ac	4.5	19.5	17.5	4,642	Yes
45		T-5	6	CL	35	16	17	GC	20	4	70	6.0	ac	5.0	11.0	17.5	10,310	Yes
46		T-5	6	CL	32	15	7	la	20-35	4-16	17-70	17.0	ac	5.0	22.0	17.5	10,310	Yes
47	Columbia, S. C.[l]	NE-SW	....	CL	33	15	29	SW	29	9	50	8.0	sa	5.0	13.0	18.0	3,528	Yes
48	Kirtland[m]	N-S	....	SF	20	3	15	GF	24	8	26	9.0	ac	2.0	11.0	18.0	2,824	Yes
49	Barksdale[n]	Test section	5[o]	CL-CH	45	28	5	GW	NP	NP	65	8.0-26.0	ac	3.0	11.0-29.0	20.0	5,000	Yes[p]
50	Fargo, N. Dak.	Test section	....	SF	28	13	9	sc	....	....	....	6.0	none	....	6.0	20.0	530	No[q]
51	Langley Field, Va.	Test section	....	SF	22	4	30	GW	NP	NP	80	9.0-21.0	ac	3.0	12.0-24.0	20.0	6,670	Yes[r]
52	Birmingham, Ala.	NE-SW	....	CL	42	24	4	cs	....	....	....	4.0	ac	3.0	7.0	23.0	194	No[h]
53	Campbell, Camp Campbell, Ky.	N-S	1	CL	37	20	11	GW	NP	NP	56	9.0	ac	3.0	12.0	25.0	2,330	Yes[e]
54		N-S	2	CL	37	20	8	GW	NP	NP	75	9.0	ac	3.0	12.0	25.0	2,330	Yes[i]
55		N-S	3	CL	37	20	17	GW	NP	NP	110	9.0	ac	3.0	12.0	25.0	2,330	Yes[i]
56		N-S	4	CL	37	20	7	GW	NP	NP	58	9.0	ac	3.0	12.0	25.0	2,330	Yes[j]
57		N-S	5	CL	37	20	15	GW	NP	NP	48	9.0	ac	3.0	12.0	25.0	2,330	Yes[e]
58	Minden, Nev.	N-S[t]	....	CH	58	31	11	la	15-29	NP-10	....	25.0	ac	3.0	28.0	25.0	2,554	Yes
59		NE-SW	....	CL	34	17	5	la	18-26	NP-13	....	16.0	ac	2.0	18.0	25.0	385	No[h]
60	Smyrna, Tenn.	N-S	....	CH	62	36	11	GF	18	4	80+	11.0	ac	5.0	16.0	25.0	21,359	Yes
61	Stockton, Calif.	Test section	....	CH	57	34	5	GW	NP	NP	80+	6.0-42.0	ac	3.0	6.0-48.0	25.0	1,924	Yes[u]
62	Woodward, Okla.	T-3	1	SF	NP	NP	10	GW	NP	NP	28	9.0	ac	3.0	12.0	25.0	4,344	Yes
63		T-3	1	CL	23	9	11	la	NP	NP	10-28	14.0	ac	3.0	17.0	25.0	4,344	Yes
64	Gowen, Boise, Idaho	NW-SE	....	CL	46	24	14	GW	21	NP	80+	15.5	ac	3.0	18.5	26.0	5,219	Yes
65		T-"G"	....	CL	43	22	19	GP	....	NP	80+	12.5	ac	2.0	14.5	26.0	18,615	Yes

[a] Direction of runway, as "NW-SE1" is the northwest-southeast runway No. 1 and "West NS" is the west north-south runway; or "T-1" is taxiway No. 1 and "T-'G'" is taxiway "G"; etc. [b] NP denotes "nonplastic"; w_L, liquid limit; and I_P, plasticity index. [c] In Cols. 4 and 8 the capital letters refer to the Casagrande (Arthur Casagrande, M. ASCE) classification (*Transactions*, ASCE, Vol. 113, 1948, p. 901). The small-letter abbreviations in Cols. 8 and 13 are defined as follows: ac = asphaltic concrete; cs = choked slag; la = layered base; lr = lime rock; sa = sand asphalt; sc = soil cement; and st = surface treatment. [d] Performance with respect to shear deformation. [e] A thickness of 24 in. was adequate to prevent subgrade shear. [f] Layered base is the term applied to the base where there is an upper high-quality base and one or more subbases of lower quality. [g] 36th Street, Miami, Fla. [h] Subgrade shear. [i] A minimum thickness of 4 in. was adequate to prevent subgrade shear. [j] A minimum thickness of 5 in. was adequate to prevent subgrade shear. [k] These are item Nos. in Table 8. [l] Columbia Army Air Base. [m] At Albuquerque, N. Mex. [n] At Shreveport, La. [o] These are item Nos. in Table 4. [p] A thickness of from 18 in. to 21 in. was adequate to prevent subgrade shear. [q] Shear in sandy base material. [r] A minimum thickness of 12 in. was considerably more than adequate to prevent subgrade shear. [s] Data not considered reliable since the CBR tests were made after a 7-month period of practically no traffic. [t] Taxiway. [u] A thickness of from 23 in. to 28 in. was required to prevent subgrade shear. [v] A minimum thickness of 6 in. was adequate to prevent subgrade shear. [w] A thickness of from 32 in. to 36 in. was required to prevent subgrade shear. [x] A thickness of from 30 in. to 36 in. was required to prevent subgrade shear. [y] A thickness of 24 in. was considerably more than adequate to prevent subgrade shear.

(Table continued on next page)

TABLE 9.—(*Continued*)

Line	Army airfield	Facility[a]	Pit No.	SUBGRADE OR SUBBASE[b]				OVERLYING BASE[b]					SURFACE		Total thickness, in inches	TRAFFIC		Performance satisfactory[d]
				Type[c]	w_L (%)	I_P (%)	CBR (%)	Type[c]	w_L (%)	I_P (%)	CBR (%)	Thickness, in inches	Type[c]	Thickness, in inches		Wheel load, in kips	Coverages	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)
	(*b*) WHEEL LOADS OF 30,000 LB AND MORE (*for Footnotes, See Table 9(a)*)																	
1	Clovis, N. Mex.	N-S"B"	4	CL	23	9	28	GF	24	7	52	12.0	ac	1.5	13.5	30.0	1,709	Yes
2	Fargo, N. Dak.	Test section	....	SF	28	13	9	sc	....	....	....	6.0	none	....	6.0	30.0	570	No[g]
3	Kirtland[m]	T-2	1	SP	NP	NP	59	SF	22	3	175+	8.5	ac	2.5	11.0	30.0	1,824	Yes
4		N-S	3	ML	26	7	64	SF	24	6	79	8.0	ac	2.0	10.0	30.0	7,612	Yes
5		N-S	4	ML	25	4	47	SF	24	6	92	8.5	ac	2.5	11.0	30.0	7,612	Yes
6		N-S	6	SP	NP	NP	36	SF	24	5	62	9.0	ac	2.5	11.5	30.0	7,612	Yes
7	Pueblo, Colo.	E-W	3	CL	44	24	17	GF	21	3	100+	9.0	ac	2.0	11.0	30.0	2,951	Yes
8		E-W	4	CL	40	20	8	GF	21	3	27	9.0	ac	2.0	11.0	30.0	Edge pit	No[h]
9		E-W	5	CL	39	20	17	GF	21	4	85+	9.0	ac	2.0	11.0	30.0	2,951	Yes
10		E-W	7	CL	39	20	16	GF	21	4	100+	9.0	ac	2.0	11.0	30.0	2,951	Yes
11		T-6	9	CL	35	18	25	GF	18	1	91+	7.0	ac	5.0	12.0	30.0	4,716	Yes
12	Waterways Experiment Station, Vicksburg, Miss.	Test section	14[k]	CL	39	14	35	GW	NP	NP	87	13.0	ac	3.0	16.0	30.0	1,500	Yes
13		Test section	95[k]	CL	39	14	43	ML	22	7	38	13.0	ac	3.0	16.0	30.0	1,500	Yes
14	Yuma, Ariz.	N-S	3	SP	NP	NP	44	SW	NP	NP	46	9.0	ac	4.0	13.0	30.0	48,683	Yes
15		N-S	4	SP	NP	NP	50	SW	NP	NP	43	8.0	ac	4.0	12.0	30.0	48,683	Yes
16		N-S	7	SP	NP	NP	38	GP	NP	NP	62	10.0	ac	4.0	14.0	30.0	48,683	Yes
17	Eglin, Fla.	Test section	....	SP	NP	NP	35	SF	21	3	60	3.0–12.5	ac	1.5 and 3.0	6.0–14.0	37.0	3,500	Yes[v]
18		Test section	....	SP	NP	NP	35	none	....	....	....	....	sa	6.0	6.0	37.0	3,500	Yes
19	Waterways Experiment Station, Vicksburg, Miss.	Test section	14[k]	CL	39	14	31	GW	NP	NP	86	10.0	ac	3.0	13.0	37.0	1,500	Yes
20		Test section	95[k]	CL	39	14	34	ML	22	7	49	10.0	ac	3.0	13.0	37.0	1,500	Yes
21		Test section	122[k]	CL	39	14	37	SP	NP	NP	81	10.0	ac	3.0	13.0	37.0	1,500	Yes
22		Test section	1[k]	CL	39	14	10	GW	NP	NP	56	11.5	ac	1.5	13.0	37.0	400	No[h]
23		Test section	302[k]	CL	39	14	14	GW	NP	NP	32	9.0	sa	2.0	11.0	37.0	100	No[h]
24	Stockton, Calif.	Test section	....	CH	57	34	5	GW	NP	NP	80	6.0–42.0	ac	3.0	6.0–48.0	40.0	3,730	No[w]
25	Barksdale[n]	Test section	5[o]	CL-CH	45	28	5–6	GW	NP	NP	80	23.0–27.0	ac	3.0	24.0–43.0	50.0	5,000	No[x]
26	Eglin, Fla.	Test section	....	SP	NP	NP	35	SF	21	3	60	3.0–12.5	ac	1.5 and 3.0	6.0–14.0	50.0	3,500	Yes[v]
27		Test section	....	SP	NP	NP	35	none	....	....	....	....	sa	6.0	6.0	50.0	3,500	Yes
28	Langley Field, Va.	Test section	....	SF	22	4	30	GW	NP	NP	80	9.0–21.0	ac	3.0	12.0–24.0	50.0	2,733	Yes[y]

ation in the subgrade, a half black rectangle has been used. Inasmuch as the half black rectangles represent the thickness selected as necessary to prevent shear deformation, the design curves should pass through them. A solid black rectangle indicates that shear deformation occurred under the maximum thickness tested; hence, the symbol indicates that a greater thickness is required.

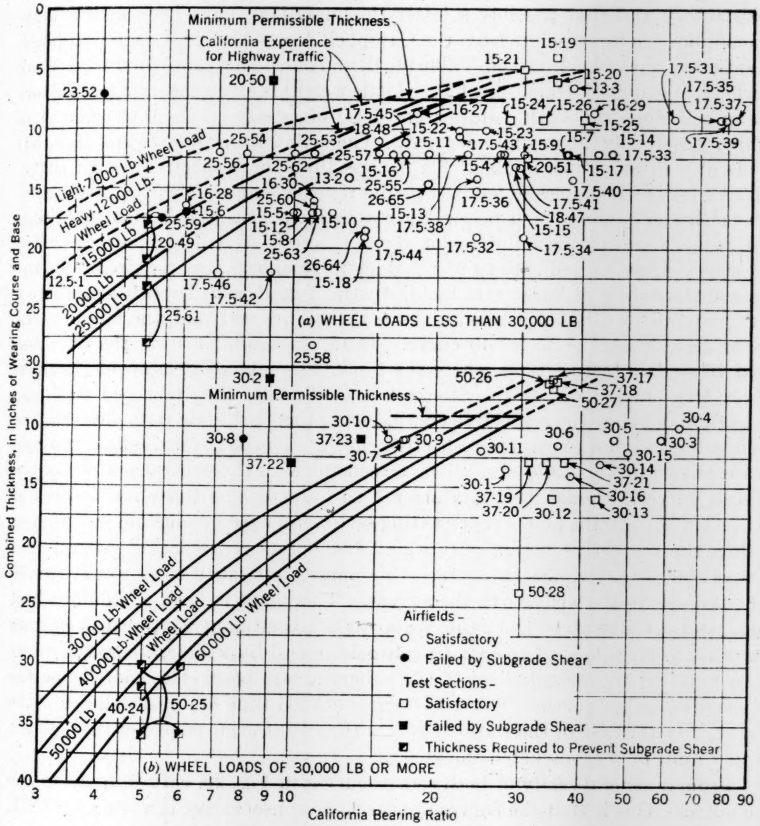


Fig. 29.—Correlation of CBR Design Curves with Field Data

An open rectangle is used where no shear deformation occurred in the subgrade at the minimum thickness of base and pavement tested; hence, a lesser thickness may be adequate. Variable thicknesses were usually not found at the airfields and only two symbols were used for the airfield data. A solid black circle denotes a pavement that failed and an open circle denotes a satisfactory pavement. The first number beside each point is the wheel load, in kips or thousands of pounds, and the second number refers to the appropriate line in Table 9.

A study of Fig. 29(*a*) reveals that the data for wheel loads of less than 30,000 lb agree with the general pattern of the design curves. In general, the points representing adequate pavements plot below the respective design curves, which is the proper relationship since the indications are that the thickness was satisfactory and that possibly a lesser thickness may have been satisfactory. Exceptions are three points from the Campbell Army Airfield, Camp Campbell, Kentucky (lines 53, 54, and 56, Table 9(*a*)), and one point from Woodward Army Airfield, Woodward, Okla. (line 62). Little weight is given the data from Camp Campbell since the CBR values were determined after a 7-month period in which practically no traffic occurred. The point from Woodward (line 62) plots at a depth which is about 3.5 in. less than the respective design curve. Except for the instances noted, the points representing satisfactory pavements plot either very close to, or below, the respective curves. It is also noted that the points representing failures plot above the respective design curves, which is proper since the indications are that more thickness was required. In the case of the failure at Bergstrom Field, Austin, Tex. (line 6, Table 9(*a*)), where a subgrade with a CBR of 6 failed under a 15,000-lb wheel load, the failure point plots directly on the 15,000-lb curve. Data are available from the Stockton and Barksdale test sections where the thickness required to prevent subgrade shear was bracketed. In the Stockton data (line 61) the bracketed thickness tends to fall below the respective curve, and in the Barksdale data (line 49) the points tend to plot above the respective curve. The data for the heaver wheel loads shown in Fig. 29(*b*) also are in agreement with the general pattern of the design curves although the data are not so numerous as those for the lighter wheel loads. All the points representing satisfactory pavements fall either on or below the respective design curves and the points representing failures plot above the respective design curve. One point plots fairly close to the curves but the others are well above the curves. This point is for a 37,000-lb wheel load and a CBR of 14 from the Waterways Experiment Station test section (line 23, Table 9(*b*)). The data from Stockton and Barksdale, where the thickness required to prevent shear in the subgrade was bracketed, plot below the respective design curve. The data from Stockton (line 61) plot well below the curve. In the case of the Barksdale data (line 49) the curve passes through the upper limit of the bracketed thickness.

From a general study of both sets of curves in Fig. 29, certain features can be noted. One is that the curves appear to be conservative in the higher CBR values. Except for the point for the 37,000-lb wheel load from the Waterways Experiment Station test section (line 23, Table 9(*b*)), there are no failure points which approach the design curves for the higher CBR values. Reductions in the thickness requirements have been made in this range, however; and, as shown in Fig. 29(*a*), the requirements are less than the original California highway experience. Also, any possible thickness reductions in this range would be small since minimum thicknesses have been established for the various wheel loads for other reasons than the prevention of subgrade shear.

Another feature that is noted is that the curves and data for the lower CBR values are in reasonably good agreement for wheel loads less than 30,000 lb; however, the agreement is not so good for the heavier wheel loads. There is evidence that the thickness requirements should probably be increased about 4 in. or 5 in. for the 50,000-lb and the 60,000-lb wheel loads.

In summary, the data that have been obtained from the accelerated traffic tests follow the general pattern of the design curves. There are indications that the design curves are slightly conservative for the higher CBR values. Any reduction in thickness requirements, however, would be slight, and its application would be limited as minimum thickness requirements prevail for CBR values above the range of from 20 to 30. For the lower CBR values, the design curves appear to be approximately correct for the lighter wheel loads, but there are indications that thickness requirements for the heavier wheel loads may have to be increased 4 in. or 5 in. No change in the present design curves is contemplated, however, until additional data are received.

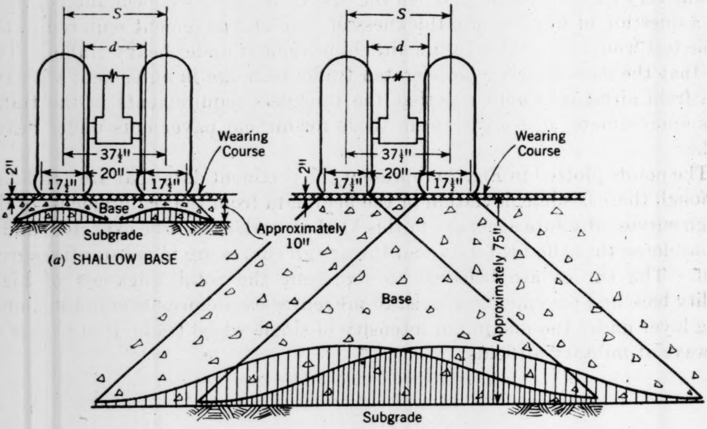
In the analysis, considerable weight was given to the data from the accelerated traffic tests. From the standpoint of test results, these data are probably the most reliable since the amount of traffic was carefully recorded and all testing was very carefully done. When the first traffic tests were conducted, there was a question of whether the thickness of base and pavement required in the traffic test would be correct for an airfield pavement under heavy traffic. The fact that the data from the accelerated traffic tests are in agreement with the data from airfields is evidence that the thickness requirements in the traffic tests approximate, reasonably well, those for airfield pavements under heavy traffic.

The points plotted in Fig. 29 represent the pertinent data that are available. Although there is some deviation of the field data from the requirements of the design curves, absolute accuracy in this kind of work cannot be expected, and it is considered that the field data and the design curves are in substantial agreement. The curves are believed to represent the total thickness of high-quality base and pavement required to prevent shear deformation in the underlying layer under the maximum intensity of single wheel traffic that occurs on runways at military airfields.

DESIGN CURVES FOR VERY HEAVY MULTIPLE WHEEL ASSEMBLIES

By W. K. Boyd,[34] M. ASCE, and C. R. Foster,[32] Assoc. M. ASCE

In the preceding Symposium papers, data have been presented leading to the development of design criteria for single wheel loads. When the B-29 plane was introduced with dual wheel assemblies, it was necessary to evaluate the effect of the dual wheel in comparison with the single wheel if maximum use of the plane was to be obtained. At the request of the U. S. Army Air Forces, the Corps of Engineers initiated a study to develop design curves for flexible pavement thickness and for other requirements of the B-29 plane.[35] These curves show difinitely that a considerable saving in base thickness can be effected by the use of dual wheels. Further study of the problem has evolved procedures whereby the method used in developing the B-29 curves can be extended and applied to any wheel load assembly. Curves for a given assembly can be developed readily which will permit the rapid evaluation of the efficiency of variations in the spacing of the wheels. The purpose of this paper is to present the method by which the B-29 curves were developed and to show the extension that may be applied to any given assembly.



(a) SHALLOW BASE

(b) DEEP BASE

Fig. 30.—Schematic Diagram of B-29 Dual Wheel Assembly

Although the method described herein was developed primarily for airplanes, it is also applicable in the determination of the effect of multiple truck axles on highway pavements. The method can be used to find axle spacings and loads that will produce no greater detrimental effect than is produced by a given load on a single axle.

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[34] Senior Engr., Chf., Flexible Pavement Branch, Soils Div., Waterways Experiment Station, Corps of Engrs., Vicksburg, Miss.

[35] "Certain Requirements for Flexible Pavement Design for B-29 Planes," U. S. Waterways Experiment Station, Vicksburg, Miss., 1 August 1945.

Fig. 30 is a schematic diagram of the B-29 dual wheel assembly on a thin flexible pavement and on a thick flexible pavement. The design load on the dual assembly of the B-29 plane is 60,000 lb or 30,000 lb on each tire. Fig. 30(*a*) refers to the case of a shallow base. In this case the bearing value of the subgrade must necessarily be high in order that the thin base may be adequate to prevent failure in the subgrade. Fig. 30(*b*) refers to the case of a thick base. The subgrade in this case would be poor, requiring a substantial thickness of base to prevent failure. On both diagrams, shading is shown under each tire which is intended to suggest the distribution of critical stresses induced in the subgrade. The distributions shown represent no particular case and are used only for the purpose of this discussion. Regardless of the accuracy of the distribution as drawn, it must be conceded that at some shallow base thickness the two wheels of the dual assembly will stress the subgrade as practically independent 30,000-lb units, with little or no overlapping of stresses. Under a very deep base (Fig. 30(*b*)), the stresses from the two wheels overlap considerably. Again, it must be recognized that there is a depth at which the overlapping of stresses would be so great that the stress induced in the subgrade from the dual wheel assembly, for all practical purposes, would be the same as that induced by a single 60,000-lb wheel load. Thus it was reasoned that the thickness design curves for the B-29 plane must range between the curves for the 30,000-lb and the 60,000-lb single wheels. As previously stated, thickness design curves were available for single wheels. The problem of determining design curves for the B-29 dual wheel was narrowed to finding:

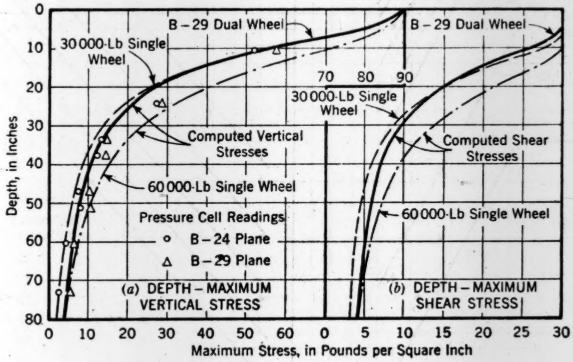

FIG. 31.—VERTICAL STRESSES AND SHEAR STRESSES

a. The thickness at which each tire stresses the subgrade as an independent unit.

b. The thickness at which the two tires stress the subgrade as one single unit.

It was further reasoned that, if those two thicknesses could be determined, thickness requirements between the two values would vary in an orderly manner.

The thickness at which each tire of the B-29 dual assembly acts as an independent unit, and the thickness at which the two tires act as a single unit, were determined by comparisons of vertical stresses, shearing stresses, and deflections. These comparisons are illustrated by Figs. 31 to 35. Details of the manner in which these data were obtained are not essential for present purposes; however, these details appear elsewhere.[35]

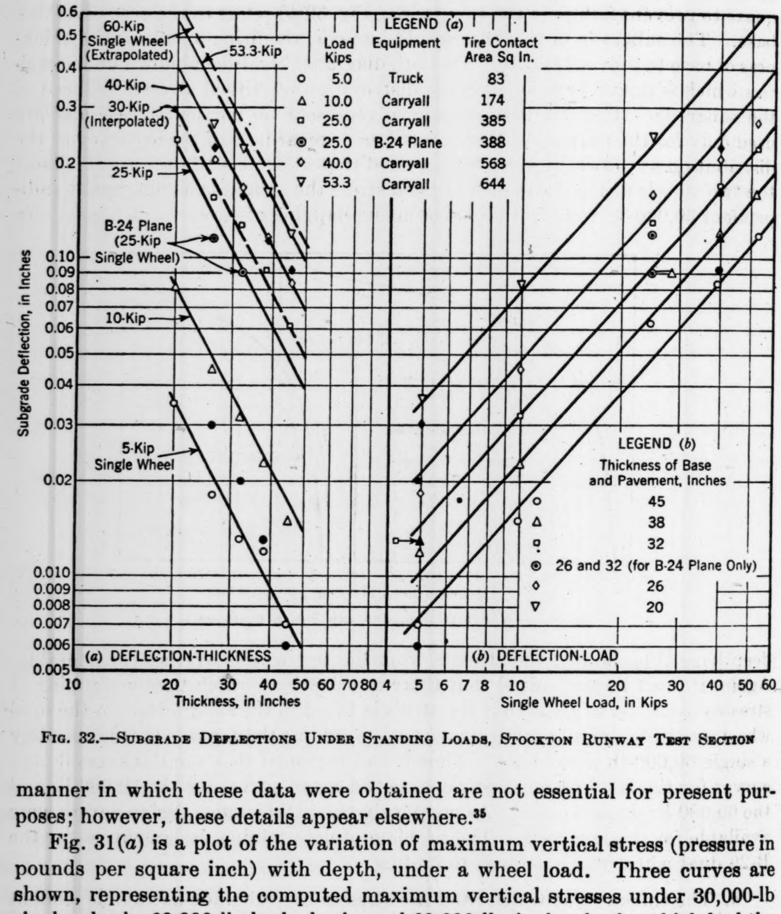

FIG. 32.—SUBGRADE DEFLECTIONS UNDER STANDING LOADS, STOCKTON RUNWAY TEST SECTION

Fig. 31(*a*) is a plot of the variation of maximum vertical stress (pressure in pounds per square inch) with depth, under a wheel load. Three curves are shown, representing the computed maximum vertical stresses under 30,000-lb single wheels, 60,000-lb dual wheels, and 60,000-lb single wheels, which had the physical characteristics listed in Table 10. Computations were made using Boussinesq's formulas assuming homogeneous material. Stress charts devised

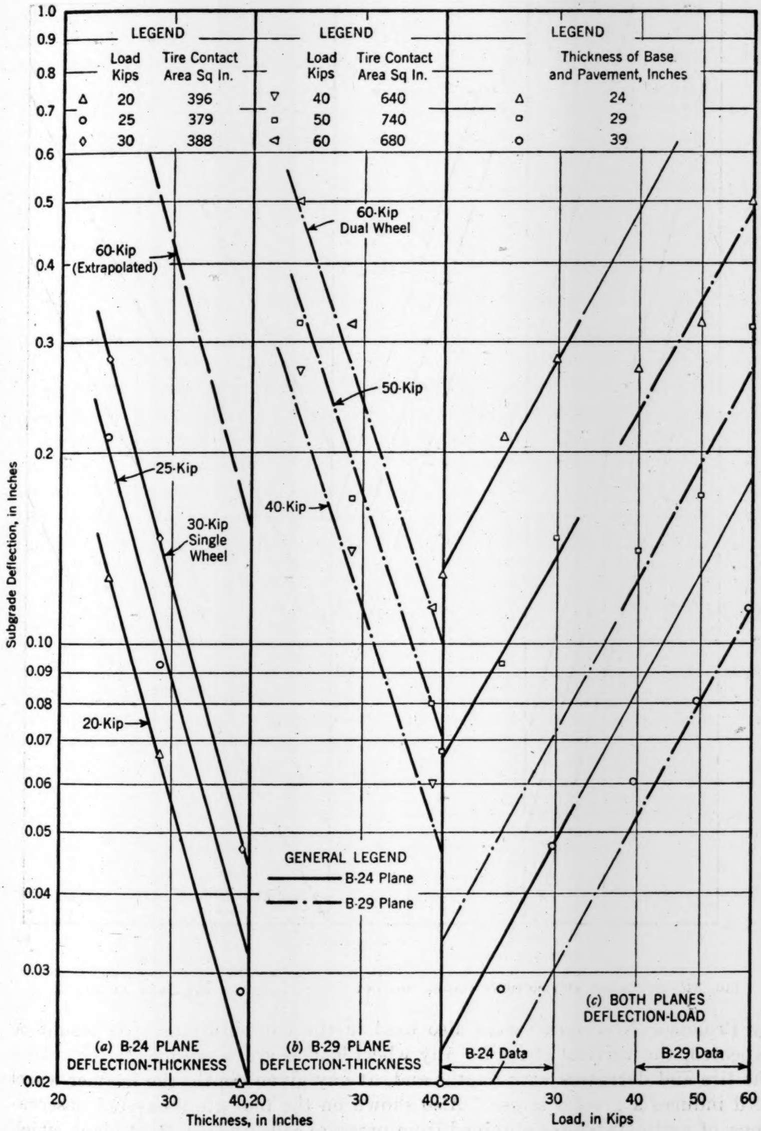

FIG. 33.—SUBGRADE DEFLECTIONS UNDER STANDING LOADS, MARIETTA PAVEMENT BEHAVIOR TESTS

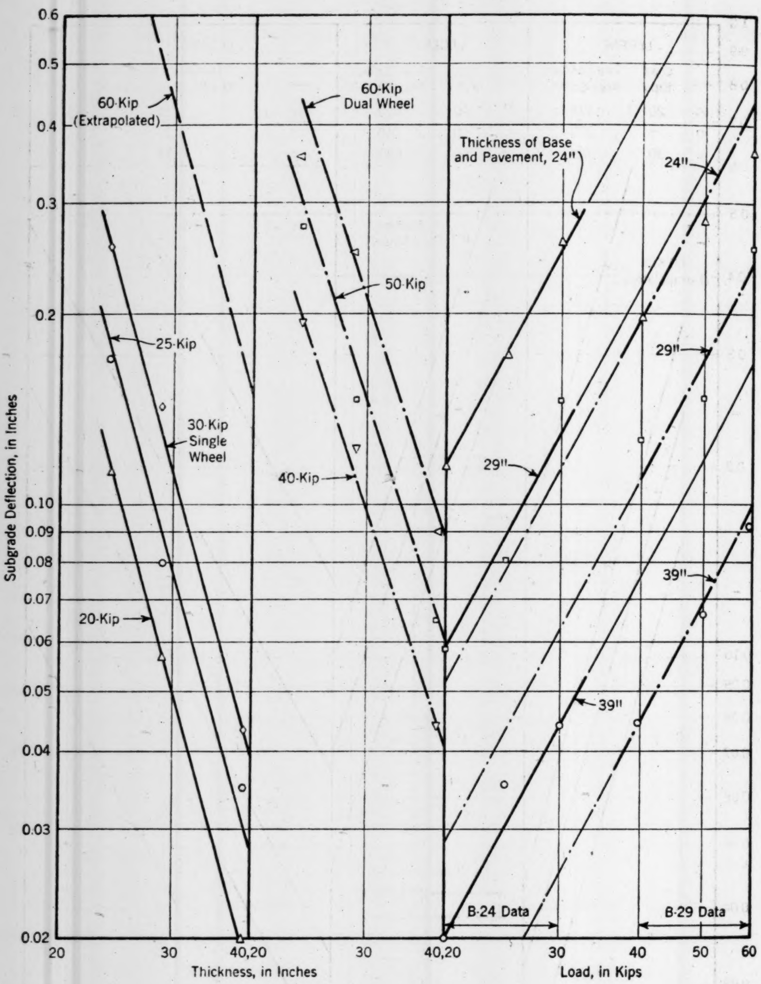

FIG. 34.—SUBGRADE DEFLECTIONS UNDER MOVING LOADS (LEGENDS THE SAME AS FIG. 33)

by Professor Newmark[36] were also used in the computations. As would be expected, the vertical stress for any wheel load is greatest immediately under the tire and decreases with depth; and, at any given depth, the heavier wheel load induces a greater stress. Also shown on the plot are measured observations of vertical stresses obtained from pressure cells under a B-24 plane which had a 30,000-lb load on a single wheel, and under a B-29 plane which had a 60,000-lb load on dual wheels. The circles and the light curve on the left, as well as the triangles and the heavy curve, represent comparable loading. The agreement between actual observation and theory in this case is considered reasonably close. Attention is directed to the fact that the theoretical 30,000-lb single wheel curve and the B-29 dual wheel curves are identical to a depth of about 17 in. Also, for all practical purposes, the B-29 dual load and the 60,000-lb single load have become identical at a depth of about 80 in. Thus, the study of maximum vertical stresses establishes the two thicknesses at about 17 in. and 80 in. At depths of 17 in. and less, the tires of the 60,000-lb dual assembly induce maximum vertical stresses as independent 30,000-lb units. At depths of about 80 in. and greater, they induce maximum vertical stresses as a single 60,000-lb wheel load.

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[36] "Influence Charts for Computation of Stresses in Elastic Foundations," by Nathan M. Newmark, *University of Illinois Bulletin*, Vol. 40, No. 12, Urbana, Ill., November 10, 1942.

TABLE 10.—Physical Characteristics of Wheel Loads

Wheel				Elliptical Contact Surface			
Type	Distance center to center (in.)	Plane	Load (lb)	Major axis (in.)	Minor axis (in.)	Area (sq in.)	Pressure (lb per sq in.)
Single........	....	B-24	30,000	27.12	15.73	335	89.55
Dual.........	37.0	B-29	30.000	27.12	15.73	670	89.55
Single........	....	....	60,000	38.35	22.24	670	89.55

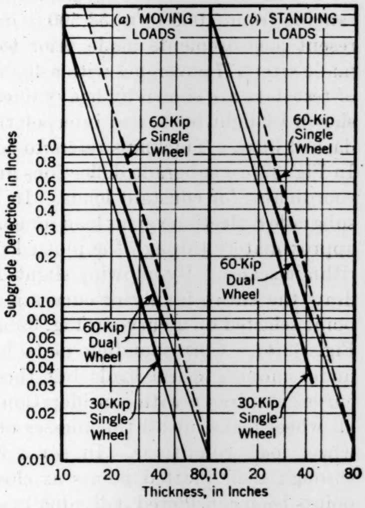

Fig. 35. — Subgrade Deflection-Thickness Curves; Single Wheel Versus Dual Wheels

Fig. 31(*b*) shows the variation of maximum shearing stress with depth under a wheel load. For a given wheel load, the stress is greatest at the surface and decreases with depth. Also, at any given depth the greater wheel load induces the greater stress. Curves are shown for the maximum shearing stresses for 30,000-lb single wheels, 60,000-lb dual wheels, and 60,000-lb single wheels. It is noted that the curves for the B-29 dual wheel load and the 30,000-lb single wheel load coincide at about 20 in. It is believed that the two curves should be identical for depths less than 20 in., and the fact that the curves cross with the B-29 dual wheel showing less stress than the 30,000-lb single wheel in the range below 20 in. is attributed to an erroneous assumption of a weightless soil mass that was made in computing the stresses. It is also noted that at 70 in. the stress is approximately the same for the B-29 dual wheels and the 60,000-lb single wheels. Thus, an analysis of maximum theoretical shearing stresses sets the limiting thicknesses at about 20 in. and 70 in. For depths of about 20 in. and less, the two tires on the B-29 dual wheel induce stresses as independent 30,000-lb wheel loads. For depths of about 70 in. and greater, the maximum shearing stresses induced by the B-29 dual wheel assembly are the same as those induced by a 60000-lb, single wheel load.

A comparison can also be made between single and dual wheels on the basis of subgrade deflections. In this paper the comparisons are made using a graphical method which is a special development. For this reason, it will be presented in more detail than were the comparisons which were based on vertical pressures and maximum shear stresses. Actual deflection data for a dual wheel load are available only from the Marietta test section. Although the Marietta data included comparisons between single wheels and dual wheels, none were obtained for a single wheel weighing 60,000 lb, equal to the combined maximum load on the B-29 dual wheel. Therefore, it was necessary to extrapolate the deflection data available to 60,000 lb for a single wheel to make the analysis. To justify such an extrapolation, deflection data will first be presented from the Stockton test section[37] in which data are available for single wheel loads between 5,000 and 53,500 lb.

Fig 32 is a plot of the subgrade deflections produced by standing wheel loads ranging from 5,000 lb to 53,500 lb in the Stockton tests.[38] (Open symbols represent measurements made prior to traffic and solid symbols represent tests made after 477 coverages with a 40,000-lb wheel load. Curves within the range of test data are shown by heavy lines, curves beyond the range of test data are shown by light lines; and interpolations and extrapolations are represented by dashed lines. Thickness refers to combined thickness of base and pavement.) In Fig. 32(*a*), subgrade deflections and pavement thicknesses are plotted as the coordinates for constant loads. In Fig. 32(*b*), the same data are plotted with subgrade deflection versus load for constant pavement thickness. Curves drawn approximately through the plotted points appear as straight lines on the logarithmic scales. By allowing slight deviations from the points, it is possible to draw the entire family of curves mutually parallel to each other so that any point selected on a curve in Fig. 32(*a*) will also lie on its corresponding curve in Fig. 32(*b*). Corresponding plots have been made for deflections measured under moving wheel loads but these are not presented in this paper. The curves are drawn with consideration of the entire group of points secured from all wheel loads, at all thicknesses of base, and for both moving and standing wheel load conditions. In some instances, individual curves do not pass through their plotted points as closely as might have been possible had the points been considered individually and not in connection with the entire group of deflection data.

The Stockton report demonstrates that the tire contact areas varied considerably for the different wheel loads; however, this variation apparently did not affect the deflections. In all cases, the depth of base was more than the width of the tire contact area and it is believed that under these conditions the size of the contact area has relatively little influence on the subgrade deflection. The same condition is noted in the Marietta deflection data.

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[37] "Report on Stockton Runway Test Section," U. S. Engr. Office, Sacramento, Calif., September, 1942.

[38] *Ibid.*, Figs. 9, 12, 16, 20, and 23.

By extending the curves for the pavement thickness shown in Fig. 32(*b*) from 53,500 lb to 60,000 lb as indicated, the theoretical deflections can be determined for each pavement thickness for the 60,000-lb wheel load. Replotting these points in Fig. 32(*a*), an extrapolated 60,000-lb wheel load can be drawn. The curve for the 30,000-lb load in Fig. 32(*a*) may be drawn by interpolating the deflections for a 30,000-lb load from the curves in Fig. 32(*b*).

The deflection data for moving wheel loads obtained at Stockton produce similar curves when plotted in this manner. The slopes of the curves for the moving wheel load deflections are the same as those for the standing wheel load deflections; however, the magnitudes of the deflections are less for the moving than for the standing wheel load.

The Stockton data show that the variation of subgrade deflection with thickness of base and pavement and the variation of subgrade deflection with wheel load can be expressed as straight lines on logarithmic plots for the range of wheel loads and thicknesses included in the study. With these straight-line curves interpolations are possible which are considered rational. Also, extrapolations are possible which are considered reasonable as long as they do not extend too far beyond the range of measured data.

Deflection measurements for single wheel and dual wheel loads from the Marietta data are shown in Figs. 33 and 34. (The plotted points are average values of all the standing wheel load tests—that is, with the loaded tire over the center of the gage—given in the report on the Marietta tests.[39] Fig. 33 shows deflections for a standing load and Fig. 34 those for a moving load. A 60,000-lb single wheel load curve obtained by extrapolation as discussed in preceding paragraphs is shown in Figs. 33 and 34. In the case of the B-29 dual wheel, the plots represent deflections secured with one tire, rather than with the center of a dual wheel, over a gage. For very thin pavements and base courses, this position of the tire over the gage would produce the greater subgrade deflection. At a greater thickness of pavement and base, the deflection under the center of the dual wheel is shown to be slightly greater than that directly under a tire. However, the difference is so small that it is within the range of the scattering of individual test data. Because the inclusion of deflection data for the case in which the center of the dual wheel is over the gage further complicates the analysis without appreciably changing the results, it has been omitted.

In Figs. 33(*a*) and 34(*a*), which are plots for single wheel loads, the curves drawn through the plotted points are mutually parallel. The slopes of the curves in Fig. 33(*a*) are the same as those in Fig. 34(*a*). They form parallel lines when plotted as indicated in Figs. 33(*c*) and 34(*c*). The plots for dual wheel loads, Fig. 33(*b*)and 34(*b*), are also all mutually parallel, but they are not parallel with the plots for single wheel loads, Figs. 33(*a*) and 34(*a*). Therefore, it is indicated, that, as the thickness of the base is increased, the deflections in the subgrade will decrease in a different ratio for single wheels than for dual wheels.

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[39] "Certain Requirements for Flexible Pavement Design for B-29 Planes," U. S. Waterways Experiment Station, Vicksburg, Miss., Appendix on "Flexible Pavement Tests, Marietta, Ga.," 1 August 1945, Tables 3 and 10.

In Figs. 33(*c*) and 34(*c*), which are deflection-versus-load plots for constant base and pavement thicknesses, the plotted curves for both single wheels and dual wheels are all mutually parallel. As the load on either a single or dual wheel is increased, the deflections produced by either will increase in the same ratio. The pavement, base, and subgrade therefore appear to be acting as an elastic medium in a comparable manner under either wheel.

Curves of deflection versus thickness for 30,000-lb and 60,000-lb single wheel loads and for 60,000-lb dual wheel loads are shown together in Fig. 35 for comparison. The curves for both the moving loads and the standing loads have been extended beyond the available data points to include thicknesses of base between 10 in. and about 80 in.

It is recognized that the extrapolation of the data to base thicknesses of 10 in. extends the curves to points where unreasonable deflections are indicated, because shear deformation would certainly have occurred if the subgrade had deflected the amounts shown by the extrapolated data. However, a basis of this extrapolation may be found in the Stockton test data, where deflections produced by wheel loads from 5,000 lb to 53,500 lb plot essentially as straight lines. If the test had been conducted on a subgrade with a sufficiently high CBR, it can be assumed that the deflections at a 10-in. depth of base would not have overstressed the subgrade. In this case the slope and the spacing of the curves would probably change, but it is believed that the change would be in the same proportion for the dual wheels as for the single wheels, with the result that the deflection-versus-thickness curves would still approach each other at about the same thicknesses as indicated by the available data. From Fig. 35, it can be noted that the curve for the B-29 dual wheel approaches the curve for a 30,000-lb single wheel at a thickness of base of about 75 in. Thus, for base courses 10 in. and less in thickness, a 60,000-lb, B-29, dual wheel tire produces subgrade deflections as an independent 30,000-lb single wheel; and, for bases thicker than 75 in., the 60,000-lb, B-29, dual wheel produces subgrade deflections as though it were a 60,000-lb single wheel.

On the basis of the deflection measurements and the theoretical analysis of stresses, the two tires of the B-29 dual wheel would act as a point load at a depth of about 75 in. Therefore, the assumption that, for all practical purposes, the 60,000-lb dual and single wheels produce stresses and strains in the subgrade at thicknesses of base and pavement greater than 75 in. is considered to be reasonably correct. In consequence, this assumption is used in the remainder of the analysis. The upper limit—the point where the effect of the B-29 dual wheel load (60,000 lb) is the same as that of a single 30,000-lb wheel load—is more difficult to determine. The deflection measurements indicate about 10 in.; the vertical pressures, about 17 in.; and the maximum shears, about 20 in. Since the 10-in. depth is slightly more conservative, it has been selected.

From these findings, it is concluded that the B-29 design curve should coincide with the 30,000-lb single wheel load curve at a depth of 10 in. and with the 60,000-lb single wheel load curve at a depth of 75 in. The thickness requirements between these two limits should vary in an orderly manner.

The next step is to develop the design curve from these limits. The single wheel load design curves presented in Symposium paper No. 9 are used for this development, rearranged with wheel load plotted against thickness to logarithmic scales as shown in Fig. 36. CBR values for single wheel loads are shown as a nest of curves.[34] To develop the B-29 curve, the point at a thickness of 10 in. with a wheel load of 30,000 lb is connected with the point at a 75-in. thickness with a 60,000-lb wheel load. A straight line is used to join these two points. Since the deflection data plot as straight lines on logarithmic scales, it was considered that a straight line between these points represented an orderly variation. The intersections of this straight line and the CBR curves determine the thickness requirements for the B-29 curve. The thickness required for a CBR of 3 is 41 in.; for a CBR of 4, 34 in.; etc. These thicknesses can be plotted to a semilogarithmic scale similar to the single wheel design curves presented in paper No. 9.

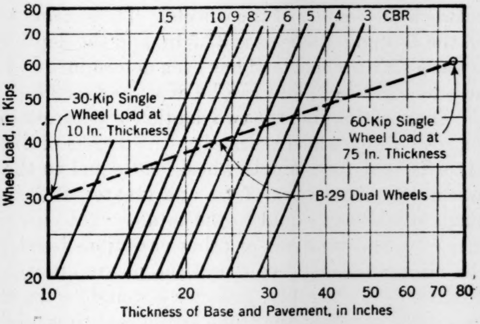

FIG. 36.—DEVELOPMENT OF DESIGN CURVES FOR A 60-KIP LOAD ON B-29 DUAL WHEELS

In translating thickness requirements from single to dual wheels in this manner, it is assumed that a dual wheel which produces the same maximum stresses and strains in the subgrade as a given single wheel will require the same thickness of base and pavement under actual traffic as does the single wheel. Theoretical considerations indicate that the requirements for the dual wheels will probably be slightly less than those for the single wheels.[35]

This method was extended to cover any multiple wheel assembly simply by resolving the two thicknesses—the shallow depth at which the dual wheels act as independent units and the deep depth at which they act as a single wheel—into ratios of appropriate dimensions of the assembly. In this manner the two depths can be determined for any assembly, and curves can be drawn.

From Fig. 30, it is readily apparent that the dimension of the assembly that governs the depth at which the dual wheel loads act as independent units is the spacing between the contact areas of the tires, which has been designated as *d*. If the two tires are moved closer together, the stress patterns would overlap at a shallower depth. If they are moved farther apart, the depth would be greater before they overlap. Therefore, the depth at which the tires act as independent units has been resolved into a ratio of the clear distance between the contact areas of the two tires. This distance on the B-29 plane is approximately 20 in. and the thickness at which the tires act as independent units has been determined as 10 in. The ratio of this thickness in terms of *d* is 1 to 2 and can be expressed as $d/2$.

By inspection it can also be noted that the depth at which the dual wheels act as a single wheel load is governed by the distance between the centers of the wheels, which has been designated *s*. The tire contact width would also affect this depth, as wider tires would produce greater overlapping of stresses. However, at the depth under consideration, the percentage increase in overlapping of the stresses caused by wider tires would be small. Consequently, the depth at which the dual wheels act as a single unit is resolved into a ratio of the distance between the centers of the wheels. The spacing *s* between the centers of wheels on the B-29 dual wheels is 37.5 in. and the depth at which the dual wheel loads act as a single wheel load has been determined as 75 in. The ratio of this thickness to *s* is 2 to 1, and can be expressed as 2 *s*.

These ratios give the two depths for determining the end points of the design curves for any wheel assemply. At depths of $d/2$ or less, the curve will be the same as that for a single wheel load equal to the individual wheel load in the assembly. At depths of 2 *s* and greater, the curve is the same as that for a single wheel load equal to the total weight on the assembly. In applying the ratios to dual tandems or other variations involving more than one dimension, the minumum distance *d* and the maximum distance *s* must be used. For a dual tandem assembly, distance *d* would be the clear spacing between the contact areas of either the front or the rear tires, and distance *s* would be the diagonal center-to-center spacing between a front tire and the opposite rear tire.

Extension of the existing single wheel design curves should be limited to multiple wheel assemblies which use tires with pressures comparable to those used in developing the single wheel curves. The existing single wheel curves are based on observations of the behavior of pavements subjected to traffic of tires inflated to pressures ranging from about 50 lb per sq in. to 110 lb per sq in. It is considered that the existing single wheel curves are applicable for tire pressures in the range of about 100 lb per sq in.

For a numerical illustration, consider the problem of a 300,000-lb (gross weight) plane. For all practical purposes, it is assumed that the total load will be carried on two wheel assemblies which are loaded to 150,000 lb each. Three different types of wheel assemblies are considered: Case I, a single tire; case II, dual wheels spaced 56 in. from the center to center; and case III, dual tandems with the wheels spaced 40 in. from center to center and the axles 61 in. from center to center. For each case it is assumed that the tires will have circular contact areas rather than the usual elliptically shaped areas. Studies have shown that this can be done without causing appreciable errors. A contact pressure of 100 lb per sq in. is used. With these data, values for $d/2$ and 2 *s* can be computed readily. For case I no values are obtained, since this is the single wheel. Values of $d/2$ are 12.6 in. and 9.1 in. for cases II and III, respectively. Values of 2 *s* are 112 in. and 146 in. for cases II and III, respectively.

In Fig. 37, the single wheel design curves are shown with wheel load plotted against thickness of base and pavement to logarithmic scales. The curves have been extended to a 150,000-lb wheel load, using the tentative 150,000-lb curve introduced in the *Engineering Manual for War Department Construction* in July, 1946.[33] For case II the point at 75,000 lb and 12.6 in. ($d/2$) was joined with the point at a 150,000-lb wheel load and 112 in. (2 *s*). The intersections of this line and the CBR curves determine points on the design curve for this assembly. For a CBR of 3, 68 in. would be required; for a CBR of 4, 55 in.; etc. The design curve for case III was determined in a similar manner by joining the point at a 37,500-lb wheel load and 9.1 in. with the point at 150,000-lb wheel load and 146 in. The intersections of this line and the CBR curves determine the thickness requirements for this assembly. For a CBR of 3, 56 in. is required; for a CBR of 4, 44 in.; etc. An interesting feature is that small variations can be made in the values of either *d*/2 or 2 *s* without appreciably changing the thickness values for the design curves. Variations as great as 10% in either *d*/2 or 2 *s* will not vary the thickness requirements more than about 1 in.

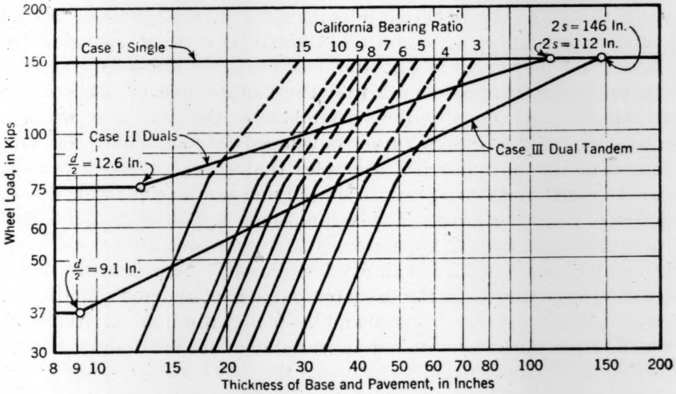

FIG. 37.—TENTATIVE METHOD OF COMPARING THICKNESS REQUIREMENTS FOR VARIOUS WHEEL LOAD ASSEMBLIES

Comparisons of the effectiveness of the various assemblies can be made directly from this plot; however, it is difficult to obtain the actual extent of savings in thickness reduction unless they are converted to percentages. Table 11 shows values illustrating the comparative effectiveness for two subgrades, one with a CBR of 4 and one with a CBR of 10. It shows the thickness of base and pavement required for each case and the savings as compared to the thickness required for case I, the single wheel. For a CBR of 4, a thickness of 63 in.

TABLE 11.—COMPARISON OF THE EFFECTS OF ASSEMBLIES

Case	Wheel assembly	CBR = 4		CBR = 10	
		Thickness[a] (in.)	Decrease[b] (%)	Thickness[a] (in.)	Decrease[b] (%)
I	Single	63	..	38	..
II	Dual	55	13	28	26
III	Dual tandem	44	30	21	45

[a] Required thickness of base and pavement. [b] Percentage decrease from case I.

is required for the single wheel, whereas the dual wheels would require 55 in. and the dual tandem wheels only 44 in. These values represent decreases of 13% and 30% from the 63 in. required for the single wheel assembly. For a CBR of 10 the savings are even greater. The thickness required for the single wheel is 38 in., whereas 28 in. is required for the dual wheels and 21 in. for the dual tandem wheels—decreases of 26% and 45% respectively. For either CBR value, the dual tandem assemblies show considerably more percentage decrease than do the dual wheels (30% as compared to 13%, and 45% as compared to 26%). From the standpoint of the airfield pavement design, the dual tandem assemblies are more effective for this plane than are the dual wheels, because they result in a greater decrease in thickness requirements.

The preceding example shows that the method can be used to evaluate, readily, the effectiveness of various assemblies or the spacing of wheels in any given assembly. The design curves developed for the three cases should be considered merely as numerical examples that may not apply to any given plane. As noted previously, certain simplifying assumptions were necessary to develop the various curves. The assumptions are discussed in detail elsewhere.[35] The assumptions appear reasonable; however, each person using the method should evaluate their validity for himself. Although this method has these and other limitations, the basic reasoning is considered sound. The ease with which the method can be used, and the fact that minor variations in setting the two limiting thicknesses do not appreciably affect the design curves, justify its use for determining flexible pavement thickness requirements for multiple wheel assemblies until a more rational method is developed.

APPRAISAL OF THE CBR METHOD

BY W. J. TURNBULL,[40] M. ASCE

The preceding papers have presented a review of the historical development of the CBR test and its application to flexible pavement design. They also contain individual summaries of the principal traffic test sections that were constructed to check the validity of the design curves. Finally, papers are included which verify the design curves for single wheel loads and which develop principles from which design curves for any desired multiple wheel load can be prepared. It is the purpose of this concluding paper to evaluate the several component parts of the design method and to present both its virtues and its shortcomings to the end that the maximum intelligent use may be made of the method in the laboratory, for design, and on construction. It is proposed to review the method as developed, thus pointing the way to its intelligent use.

The CBR test used in this method consists of forcing a 3-sq-in. circular piston into the soil and measuring the resistance to penetration. The resistance is converted to a bearing ratio by comparing it to the resistance obtained in penetrating a high-bearing material adopted as a standard. The penetration test is essentially a simple shear test, and the CBR is an index of the shearing strength of the soil. The CBR is used to determine, by design curves, the total thickness of base and pavement required to prevent shear deformation in the given subgrade soil for a specified wheel load.

After the initial trial of the CBR method by the Corps of Engineers, it became apparent that conditions and soil types, varying considerably from those encountered in California, made it necessary to initiate a program of laboratory research on the test used in the method. This project, explained in paper No. 4, was assigned to the Soils Division of the Waterways Experiment Station. After about 3 years of study, the test was developed to the state in which it is now (1948) used by the Corps of Engineers. Also, as presented in several previous papers, a comprehensive program of field testing was conducted to correlate the design curves with traffic requirements. Research and field correlation studies have been carried to the point where it is believed that the CBR method can be used for the practical and economical design of military airports.

The studies of the California method of design have resulted in considering the method to have three separate parts:

1. The design curves,
2. The penetration test, and
3. The preparation of laboratory samples.

THE DESIGN CURVES

The original design curves for airplane loadings were extrapolations of the basic highway loading curves developed by the California State Highway

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Department. These curves have been correlated with field behavior by measuring the CBR, in place, of test sections and existing pavements and by observing the effect of traffic. The curves have been adjusted to the point where it is considered that, if in-place CBR values for a specified base and subgrade are available, the curves will reflect their behavior under traffic. In other words, when the field in-place test indicates underdesign, overdesign, or correct design, by the present curves, the actual results of pavement behavior under traffic have generally substantiated the curves. This is particularly true for fine-grained cohesive soils. Some scattering of data has been experienced for cohesionless materials, especially for clean sands, and the design curves are generally considered conservative for these materials. As shown in paper No. 9, the curves, as drawn, conform closely to the available data and produce an orderly group of curves. Extreme accuracy should not be read into the curves; generally, they could be shifted an amount equivalent to 1 in. or 2 in. of pavement thickness and would still agree reasonably well with the data. In their present state, the curves are subject to minor modification from time to time as additional data become available. The final report on Stockton test section No. 2[41] in which loads up to 200,000 lb were tested (traffic tests completed in 1947) has not as yet (1948) been published, however the data indicate that curves extrapolated from the lower wheel loads (up to 70,000 lb) for the high loads used in the Stockton tests, were not conservative.

One criticism of the California method is that the base thicknesses obtained by this method are too great. As previously stated, the thicknesses indicated by the design curves are not excessively conservative. Also, it is noted that no safety factor of additional thickness has been included in the design curves. The criticism is partly the result of a lack in understanding the basic design premise with respect to permanent construction as used by the Corps of Engineers. Permanent construction may be defined as resulting in stable pavements for continuous use by the designated wheel load for a long period with only minor maintenance. Permanent construction is obtained by using a conservative design which (*a*) assumes maximum field moisture in the base and the subgrade, and (*b*) requires high-quality upper base and wearing courses. In general, permanent construction as defined is considerably more conservative than either highway or airport design in the years preceding World War II. This is not a criticism of flexible pavements designed by agencies other than the Corps of Engineers but does indicate a probable difference in the definition or interpretation of anticipated life. It is realized that many arguments can be advanced against a pavement design meeting the requirements for permanent construction as defined by the Corps of Engineers; however, it has been, and still is, considered feasible to design permanent military airports in the continental limits of the United States by this criterion. By clarifying the basic premise of design, it is hoped that engineers who have believed the CBR method to result in unduly costly and conservative pavements may reconsider the test in a different light.

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[41] "Flexible Pavement Test Section for 300,000-lb. Airplanes, Stockton, California," by Ralph A. Freeman and O. J. Porter, *Proceedings*, Highway Research Board, National Research Council, Vol. 25, 1925, p. 23.

Another feature of the CBR method of design which is sometimes criticized is that a base composed, throughout, of a high-quality material is not considered to have any advantage over the same thickness of layered base with inferior materials in the lower part. The present studies have progressed far enough to give good indications that base materials do not all have the same stress-distributing qualities and that low-quality bases are probably poorer stress-distributing media than are high-quality bases. It is entirely possible that the thickness indicated by the design curves should be increased if a low-quality base is used in the lower part of the pavement sections.

THE PENETRATION TEST

Part 2 of the method, as previously mentioned, deals with the penetration test. This test is relatively simple with very little chance for the personal element to affect the results. Experimentation on the mechanics and the procedures in making the penetration test, both in the laboratory and in the field, has indicated that it is reliable and that results can be reproduced reasonably well in the hands of ordinarily good technicians. Special details on technique are summarized in paper No. 4.[4] In testing clean sands, it is necessary to surcharge the surrounding area during the test to obtain CBR values that are indicative of the behavior of the sands. Difficulties are experienced in reproducing results on materials such as clay-gravels which contain large-size particles—because large particles directly under the piston affect the results. A large number of tests on these materials and an average of the results will assist materially in obtaining good test values. Soil is always a variable material and few uniform deposits exist. In using the CBR test, it is necessary, as with any other test, to conduct a sufficient number of tests to obtain good average results.

Admittedly, the penetration test has the weaknesses of all empirical-type tests. However, it has the following advantages:

a. It is supported by a background of practical experience;
b. It is applicable to a wide range of materials;
c. The equipment used in the test is light and mobile; and
d. The test can be used in the laboratory and field for correlation, design, evaluation, and construction control purposes.

Although they are not a part of the penetration test as such, mold effects have a direct bearing on the CBR values obtained. Mold effects have been harassing and are continuing to be studied. These effects are more noticeable in silts than in clays. It is necessary to exercise good judgment in weighing possible mold effects on CBR values for use in design.

PREPARATION OF LABORATORY SAMPLES

Part 3 of the method—that is, the preparation of the samples for the penetration test—is the most troublesome in the use of the CBR method as a design tool. The problem consists in preparing the sample so that it truly represents the condition of (1) consolidation, (2) moisture content, and (3) the structure that will be obtained in the prototype. If a true representation is obtained of these three conditions, then it has been proved that the CBR design curves can be used with assurance. It is admitted that an entirely satisfactory laboratory method of preparing specimens to duplicate the conditions in the prototype has not been developed. When tempered with judgment, however, the method can be used to give samples that will show CBR values which approximate reasonably well the CBR that can be expected in the prototype. As with the penetration test, greater trouble is experienced with sands and granular materials than with fine-grained cohesive soils.

The method of preparing specimens to the condition or consolidation expected in the prototype has consisted of compacting the specimens with a standardized compactive effort. It was desired to set the compactive effort high to produce specimens with a high degree of consolidation (which is beneficial if obtained in the prototype because the compaction increases the CBR of the material), thus permitting thinner base courses. This high compaction also reduces the possible consolidation hazards under the heavier wheel loads. Therefore, the Corps of Engineers increased the standard AASHO compactive effort to such a degree that the resultant maximum density was believed to be sufficiently high to meet the requirements necessary for satisfactory field behavior. The revised test is commonly called the modified AASHO compaction test. If the design CBR is based on a highly compacted sample and this compaction is not obtained in the prototype, the actual CBR obtained in construction will be less than the design value and the result will be a pavement section that is underdesigned. It must be admitted that the compactive effort used in the original CBR test as developed by the California Division of Highways and the compactive effort adopted by the Corps of Engineers produce densities in laboratory specimens that are very difficult to obtain with the usual construction equipment and methods. This fact is the source of some criticism.

In this connection it should be recognized that in the California method the emphasis has been placed on design thickness. This is not intentional, but results from the fact that a special test and a set of design curves are used for determining the required thickness. However, it is just as important that the subgrade and the base be compacted adequately to prevent detrimental consolidation. The CBR test does not provide information with respect to the amount of settlement to be expected in the pavement surface as a result of consolidation in the base course and in the subgrade. It is quite possible to design and to construct a pavement section containing a material compacted to a degree which may be satisfactory from the standpoint of shear deformation but which may allow consolidation to take place to the point that detrimental settlement will be produced at the surface. For this reason, it is necessary to supplement the CBR test with definite compaction requirements. The CBR method as used by the Corps of Engineers requires compaction to definite percentages of the density obtained from laboratory or field compaction tests. The percentages are varied in accordance with the wheel load and the depth below the surface to the given soil. It is recognized that for some soils the specified percentages cannot be obtained practicably. In these cases it is necessary to adjust the design to obtain a balance between the thickness requirements and the compaction requirements. Method 2, for preparing and testing CBR specimens, as described in paper No. 4, will furnish information that will permit ready adjustments of these requirements.

One of the favorable features of the CBR test is that it can be conducted on a laboratory specimen in which the moisture content can be adjusted to a given design value—condition (2) in the problem of preparing samples to meet prototype conditions. This phase is simplified, in that, for design the condition of maximum field moisture has been assumed to exist. Maximum field moisture is the maximum moisture content that a given soil can reach under conditions of heavy rainfall, high water table, and poor drainage. In general, under the condition of maximum field moisture, the voids in the soil, except for sands, are only from about 75% to 95% (by volume) filled with water. Consequently, the design condition is not as conservative as if it were based on a completely saturated condition where all the voids were filled with water. As information on moisture conditions under pavements was very conflicting, the Corps of Engineers had no other course than to assume the maximum condition to insure conservative design.

To simulate the maximum field moisture, specimens are soaked in the laboratory in water for 4 days prior to the penetration test. In general, 4 days of soaking approaches the maximum field moisture, especially near the surface where the penetration tests are made. In the prototype the condition of maximum field moisture may occur: (*a*) Immediately; (*b*) over a period of years; or (*c*) never, in some instances where ground water, drainage conditions, and soil types are particularly favorable. In conditions (*b*) and (*c*), the CBR method tends to be criticized because its use results in pavements which are too conservatively designed. The basis for this criticism in the case of condition (*b*) is not believed to be valid in that surplus strength is present in the pavement only until the design moisture condition occurs. However, in the case of condition (*c*), the criticism is believed to be valid. Long-range studies are being conducted to determine the moisture content that occurs under pavements in arid, semiarid, and wet regions.

It is true that, in general, the condition of maximum field moisture represents the lowest strength of a soil; however, when compacted to the degree called for by the design, the resulting strength may usually be relied on to fulfil the design requirements. Only in the cases of highly expansive soils, frost effects, or soils compacted to a much lower degree of density than called for by the design, are wholly inadequate strengths obtained.

At this point it is believed desirable to demonstrate the possible effects of consolidation on the CBR of a given subgrade soil or a base layer when a high percentage of the voids is filled with water. When consolidation occurs under this condition, hydrostatic excess pressures can be developed which may very materially reduce the strength of the soil. This reduction of strength can occur quite rapidly. The possibility of developing hydrostatic excess pressures is greatest for silts and fine silty sands, which are relatively impermeable but which, as they possess very little cohesion, have a tendency to consolidate readily under load. High water table, capillarity, pumping, vapor condensation, and poor drainage are critical factors in the development of hydrostatic excess pressures.

Under condition (3) in preparing samples to meet prototype conditions, the problem is to produce the same structure in laboratory samples as that produced by field compaction. As in most research and test programs, the research on the CBR test has demonstrated the need for additional lines of study—one of the most important being the possible variation in structure that a soil may attain as the result of variation in degree and type of compactive effort and of variation in compactive moisture. Variation in structure may indicate a considerable change in strength. Change in strength due to change in initial molding moisture has been demonstrated by the laboratory study on the CBR test, details of which may be obtained from a paper entitled, "Stress-Strain Characteristics of Compacted Soil Systems," by Joseph B. Eustis, Assoc. M. ASCE, and John L. McRae, presented at the ASCE Annual Meeting in New York, N. Y. on January 19, 1945. The Waterways Experiment Station engaged in a series of field compaction studies in an attempt to determine, among other things, any possible variation in structure produced by various types of field compaction equipment in comparison with that produced by various types of laboratory compaction equipment and methods. Soils studied were sand, sand-clay, and clay-silt. Structure effects were evident in the field studies on the two finer-grained materials. On one of them, reasonable correlation was obtained with the dynamic laboratory test, and on the other it was not. One very noticeable effect of the field "density-versus-moisture" curves as compared to those of the laboratory is that the former were transposed to the right, closer to the zero air-voids curve. Until this condition is corrected, complete correlation between field and laboratory compacted structures cannot exist.

Discussion

In applying the CBR method of design, it is necessay to make a thorough study of the existing subgrade soils and of the possible materials available for base course construction. The materials to be used must first be selected and then the proper CBR value determined for each. It is obvious that the selected CBR value must be based on the density and moisture conditions to be expected in the prototype if a satisfactory pavement is to be obtained. A high degree of field compaction is desirable, in that subsequent consolidation due to plane traffic is held to a minimum and higher CBR values are obtained.

In the selection of the proper CBR value for design, the Corps of Engineers resorts to one or the other of two test procedures. The first might be called the minimum procedure and is generally used, particularly in emergency construction, where time for design is limited. The second or longer procedure is a more complete development of design CBR values and is desirable if the time element is not too critical, and particularly where it is advantageous to have a complete picture of the variation of the CBR with both density and water content. Both of these methods are demonstrated in paper No. 4.

In the minimum (method 1) CBR test procedure, the compaction curve is fully developed for the modified effort and densities are obtained for two lesser compactive efforts at modified optimum moisture. CBR values on three soaked specimens molded at optimum water content for the three compactive efforts are obtained. Thus, at optimum moisture the range in CBR values with density is obtained. The variation of the CBR with moisture is not obtained; consequently, unless very close control of the moisture in field construction is maintained, considerable variation in strength would result.

The longer CBR test procedure (method 2) consists in developing compaction curves for the modified compactive effort, and for two lesser efforts, and determining the CBR of each compacted specimen. This procedure is demonstrated by Fig. 11. The data plotted in Fig. 11 will furnish design engineers and field engineers with a visual picture of the possible spread of CBR test values with both the water content and the density of the soil. The design engineer thus has a better method for choosing the design CBR value, and the field engineer has a better tool for the control of construction. By building up a background of experience and familiarity with the soils in a given area, the design engineer can evaluate the results of the CBR tests and can select a value that he can expect to obtain in the prototype—one which he can use with confidence. One important feature that should be emphasized is that tests on all soils except those for clean, granular materials tend to show a rapid decrease in soaked or unsoaked CBR values with increasing molding moisture above optimum. One must be impressed by the implication that field compaction and moisture control can be very critical in determining the final strength of the subgrade and the base course.

Briefly, it may be stated that, as a result of the laboratory research program, field traffic tests, and prototype experience behavior, the Corps of Engineers has found the CBR test to have broad utility, in that it can be used successfully as a laboratory design test, a field construction control test, and a pavement evaluation test.

Summary.—The penetration test and the design curves have been developed to the point where they are considered very reliable. It must be admitted that an entirely satisfactory method has not been developed for preparing laboratory specimens that will duplicate prototype conditions for all soils at all times, under all conditions, and for all areas. In particular the Corps is making a serious effort to determine in what general areas and specific localities the conservatism of design as represented by the 4-day soaking period need not be followed. However, with due consideration of the limitations just mentioned, considerable progress has been made in the preparation of specimens, and the test procedures and methods of data analysis now proposed give the design engineer very good information on the CBR that he can expect to obtain in the prototype. It is not considered amiss to point out that the difficulties encountered in preparing specimens for the CBR test would also be encountered in any other type of test.

In conclusion, it is desired to state that the Corps of Engineers believes that the empirical California method, as presently developed and when properly applied, is reasonable and satisfactory for the design of flexible military airport pavements. This statement does not imply that a more rational method of design employing actual strength characteristics of the soil is not desirable. It is a matter of record that the Corps has several long-range research and fact-finding studies under way from which it is hoped that a completely rational design procedure may be developed.

DISCUSSION

Miles S. Kersten,[42] **Assoc. M. ASCE.**—This Symposium has presented in a clear manner explanations of many of the features of the California bearing ratio (CBR) method of design of flexible pavements. The following discussion concerns just two points.

Extensive field testing and observations indicate that there is good agreement, for the most part, between the thickness requirements as given by the design curves and the pavement thicknesses giving satisfactory service. It appears that this agreement has been obtained by using the field, or in-place CBR test, results. The question arises as to how good an agreement would be obtained if laboratory CBR test values were used. Another way of posing this question would be to ask how much agreement is there between CBR values obtained on soaked laboratory samples and those on subgrades on which pavements have been in place for a long period of time.

Correlation between the laboratory and field tests is dependent on both moisture content and other factors such as soil structure and methods of compaction. It is the opinion of some engineers that laboratory bearing tests on soaked specimens always give smaller CBR values than tests on the prototype. This thought is probably based on their concept of the soaking procedure and the resultant high moisture content. However, it may be that such factors as inadequate field compaction, required speed in field construction, and differences in soil structure may quite often tend to give lower field CBR values for a subgrade than for the soaked laboratory specimen. Paper No. 4 by Messrs. Jervis and Eustis indicates that considerable work has been done to develop the laboratory procedure so that the results are applicable to field conditions. Additional data or comment as to how successful this has been would be enlightening.

Messrs. McFadden and Pringle, in paper No. 1, and Mr. Turnbull, in paper No. 11, mentioned studies in which data on subgrade moisture conditions are being collected. Some comment on the findings made thus far in such studies would be welcomed. There have been numerous general comments elsewhere on extremely wet subgrades being encountered in arid regions. However, very little actual data of such a nature are to be found. The writer was connected with some studies of subgrade moisture conditions made by the Highway Research Board of the National Research Council.[43,44] These studies indicated that for subgrades in arid or semiarid regions the moisture contents were markedly less than those in humid regions. For example, the data for forty-three airfields in western Texas, New Mexico, Arizona, and in parts of California with less than 11-in. annual precipitation show that less than 10% of the subgrade soil samples had moisture contents as great as their plastic limit. This was true even for clay soils which normally tend to exist at high moisture contents.

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[43] "Survey of Subgrade Moisture Conditions," *Proceedings*, Highway Research Board, National Research Council, Vol. 24, 1944.

[44] "Subgrade Moisture Conditions Beneath Airport Pavements," *ibid.*, Vol. 25, 1945.

This study was for airfields which, in many cases, had been paved for only a year or two. A test program sponsored by the Los Angeles (Calif.) District, and conducted with the cooperation of the Arizona and Nevada highway departments, consisted of taking samples of the subgrade at from three to six locations on the highways adjacent to, or as close as possible to, airfields. The pavements studied had been in place from 2 years to 30 years; the average age was about 12 years. The soils encountered were for the most part quite sandy. Of the ninety-three samples reported with plastic limits, only two had moisture contents in excess of this constant. Even the heavier textured soils (of which there were only about a dozen) did not tend to be wetter than their plastic limit. Studies in more humid sections of the United States indicated a much greater proportion of tests attaining high moisture contents.

If the test results of the current study of the Corps of Engineers, Department of the Army, indicate that moisture contents do tend to remain low beneath pavements in arid or semiarid regions it may be possible to alter the CBR procedure for soils from such areas. Some comment by the Symposium authors on the trend of their results would be of value.

F. C. TYRRELL,[45] Assoc. M. ASCE, AND L. A. PALMER.[46] —The eleven papers of the Symposium present, in concise form, the results of several years of research along a broad front in the comparatively new field of airport pavement design. Each paper is a definite contribution in itself and the entire group of papers has served very materially to advance the profession. Many readers will recognize the fact, however, that the information contained in this Symposium is actually but a small part of the total which the authors have made available and which deals with both classes of pavements, flexible and concrete.

For an organization that has several well equipped laboratories, stationed in different sections of the United States, there is a considerable advantage in using a type of test such as the North Dakota cone and the California bearing ratio (CBR) test. Because of their simplicity and the speed at which such tests can be made, it is possible to obtain, expeditiously, by their use, a thorough survey of soils and available aggregates in any locality. In the final analysis, however, a pavement and the supporting earth under it must be considered as a structure. What goes into the structure is important but the procedure whereby a material becomes an integral part of the structure—that is, the workmanship or construction—is a factor of equal importance. Any test that is a test of the finished product, the structure itself, reflects the combined influence of all the factors, including both materials and construction. Paving is a case of mass production and in all cases of mass production the final criterion is a test of the finished product. This may be in the form of actual traffic tests on pavement strips built especially for testing purposes or it may involve static load tests on several trial pavement sections constructed within the area of a proposed runway, parking area, etc., that is to be paved. The construction of trial pavement sections has proved to be in itself (and apart from load tests made on such sections) a most valuable source of practical information. By this method it has been found, in Navy experience at least, that certain difficulties encountered in the field compaction of certain subbase and base course materials could not have been predictable from the laboratory compaction tests.

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It is no doubt possible to establish an over-all statistical relationship between flexible pavement thickness and a laboratory test, such as the CBR test. Such statistical realtionships are exemplified by Fig. 7 in the paper by Mr. Middlebrooks. Such statistical curves may be improved and refined almost endlessly by obtaining more and more correlative data with respect to actual service performance. However, in any graphical representation of this kind the actual scattering of the points representing individual sections of pavements, runways, taxiways, or even entire airfields that do not receive special study cannot be shown. The exceptions that would fall on the unsafe side of any such curvilinear relationship, plotted from a great collection of statistical data, could be very costly. The general tendency, therefore, in all such statistical procedures, is to preclude exceptions that could be on the unsafe side by overconservatism in establishing the thickness design curves. This fact should be appreciated and thoroughly borne in mind by all who design flexible type pavements. The fact that such statistical relationships have considerable utility should also be appreciated since they tend generally to establish the upper limits in the design of pavement thickness.

For any specific case one cannot estimate the actual factor of safety by using the thickness versus the CBR values. A substantial factor of safety is required in a runway where sudden pavement failure may be hazardous to costly planes and human life. On taxiways and parking areas pavement failure usually involves nothing more than local patching of a paved area. Nevertheless, by reason of dynamic loading, taxiways, warm-up aprons, and parking areas are subjected to extra heavy loading. A factor of safety of 1.75 against overstressing of concrete pavement seems to be accepted generally as ample for unlimited operations. It is reasonable to suppose that the factor of safety against overloading a flexible pavement need not be more than this amount. The real job of engineering is to meet both requirements—safety and economy—in airport paving.

The Bureau of Yards and Docks, in the United States Department of the Navy, has considered that these requirements have demanded a realistic approach to the problem of flexible pavement design, that generalities should be avoided to a considerable extent, and that it is not economical to rely too much on statistical data and procedures in designing a flexible type of pavement for an individual airfield. It has concluded further, and after extensive field studies, that the assumption that the subgrade under the pavement, in time, will become as soft from the infiltration of water as the compacted laboratory samples subjected to 4 days of soaking, is absolutely untenable. The bureau has become increasingly aware also of a considerable degree of variation in the load-distributing properties of different types of subbase and base course materials and aggregates, a consideration that hitherto has been ruled out in using the CBR design yardstick. The ruling out of this factor was admittedly an error according to the observation recorded in the paper by Mr. Turnbull (under the heading, "The Design Curves"). This observation is a definite advance in the improvement and the use of the CBR design procedure. The use of design curves obtained by plotting concrete pavement thickness versus either the subgrade reaction modulus, or even the subgrade CBR, might seem to be warranted since concrete is a relatively standard material; but the use of such curves for any and all kinds of flexible pavements consisting of diverse types of subbases, base courses, and surfacings, irrespective of the extent to which each of the several layered components spreads the load, seems not to be warranted. Modification of the CBR thickness design curves in a manner such that they reflect the diverse load-spreading characteristics of different materials and of materials compacted differently is one of the next needed improvements in the procedure.

Another needed modification of the CBR design yardstick, according to the experience of the Bureau of Yards and Docks, is considerably less conservatism in estimating or anticipating the degree of softening or of loss of bearing value of the subgrade as caused by infiltration of surface water. The bureau has reported[47,48] the subgrade data obtained at thirty-two Naval and Marine Corps air stations, in various sections of the United States. The in-place moisture content and densities of the subgrades under the pavements were obtained in each case by Navy field personnel assigned to this task. The pavements had been subjected to traffic for variable periods of from 2 years to 6 years. The average in-place moisture content of the top 9 in. of subgrade did not exceed the average laboratory-determined optimum at seventeen airfields out of a total of twenty-six airfields for which laboratory optimum moisture data were available. At only one field was the average subgrade moisture above the average plastic limit of the soil and at this field the surfacing was in a very bad condition at the time of testing. At five out of twenty-seven fields having flexible type pavements the average in-place subgrade dry density exceeded the maximum laboratory-compacted dry density. This observation is in good agreement with the conclusions in paper No. 8 by Mr. Redus. A study of the tabulated data reported by the Bureau of Yards and Docks discloses very few individual instances where the maximum moisture content of the subgrade was above the plastic limit. The general trend was clear. There seemed generally to be a progressive diminution in moisture and a corresponding increase in subgrade density as traffic was continued and the pavements were kept in a reasonably good state of repair. Exceptions to this trend included pavements where the maximum ground-water level was less than 2 ft below the pavement. Similar subgrade and moisture data have been reported for Canadian airfields by Norman W. McLeod,[49,50] who further presents a direct

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[47] "Pavement Evaluation by Loading Tests at Naval and Marine Corps Air Stations," by L. A. Palmer and James B. Thompson, *Proceedings*, 2d International Conference on Soil Mechanics and Foundation Eng., Rotterdam, June, 1948, Vol. II, pp. 222 to 236.

[48] "Field Loading Tests for the Evaluation of the Wheel Load Capacities of Airport Pavements," by L. A. Palmer, in "Symposium on Load Tests of Bearing Capacity of Soils," *Special Technical Publication No. 79*, A.S.T.M., 1947.

[49] "Airport Runway Evaluation in Canada," by Norman W. McLeod, *Research Report No. 4B*, Research Correlation Service, Highway Research Board, National Research Council, 1947.

[50] "A Canadian Investigation of Load Testing Applied to Pavement Design," by Norman W. McLeod, in "Symposium on Load Tests of Bearing Capacity of Soils," *Special Technical Publication No. 79*, A.S.T.M., 1947.

comparison between the CBR wheel load rating and the actual traffic sustained by four Canadian airports. As an example of this comparison, Dorval Field at Montreal, Que., Canada, without excessive maintenance cost, sustained 23,000 operations of planes weighing more than 64,000 lb between January 1, 1942, and January 31, 1947, with a CBR load rating of 5,000 lb for a single tire. Quoting from Mr. McLeod's closure,[49]

> * * * "Canadian experience indicates that an airplane wheel load of any given magnitude can be carried by a pavement and base course which have only a fraction of the thickness specified by U.S.E.D. [United States Engineer Department] design."

The experience of the U. S. Navy is in 100% agreement with this statement. At the present time (1949) the pavements at sixty of its air stations have been studied and evaluated. These sixty air stations, with few exceptions, have flexible type pavements of from 6 in. to 12 in. in total thickness and, at comparatively low maintenance cost generally, they have sustained the traffic of planes having single wheel loads of from 5,000 lb to 60,000 lb. For the subgrades at these same fields and on the basis of the CBR design curves, the pavement thicknesses should have been from 12 in. to about 45 in. in thickness to have sustained this same traffic.

Fortunately, for the most part, overdesign by the CBR and other design yardsticks has not been too extravagant. It is understood that by taking the CBR equipment to the field in use and obtaining the in-place CBR values of the subgrades and base courses the wheel load rating of the pavement may be increased over its earlier design bearing value. This is a more realistic approach and in effect is a correction of the initial design load estimate. However, since a field CBR test is essentially a miniature loading test, the question arises as to why it would not be preferable to make actual loading tests on areas comparable to those of the tire prints? According to James B. Newman, Jr.,[51] M. ASCE, plate bearing tests have a definite place in the determination of strengths of completed pavements. The Bureau of Yards and Docks has found plate loading tests to be expedient and reliable both in the evaluation of flexible pavements after construction and in the design of new pavements. On the average, the loading test data are in good agreement with theory, but because of the exceptions no theory is relied on in bureau procedure. The wheel load rating of the runway, taxiway, etc., is made contingent on the data obtained from the weakest spots in the paved area, with the stipulation that such weak spots be truly representative of the entire paved section.

Traffic tests, either on actual pavements or on specially constructed test strips, should simulate actual and not exaggerated traffic conditions. In the Stockton Field tests in California, described in Mr. Porter's paper, 1,600 repetitions of a moving 40,000-lb wheel load "in one rut" would seem to give a quite distorted picture of the effect of normally distributed traffic. Even on the narrowest of taxiways there never is "tracking" to this extent. One can imagine the extreme punishment taken by a flexible pavement when 3,730 vehicles, each producing 40,000 lb of pressure on single tires, line up, one behind the other with all the tire loads traversing paths that are only three times the width of a tire. Field studies by the Bureau of Yards and Docks show definitely that traffic compacts the base and subgrade generally and under the entire taxiway and runway. This is compaction rather than shear and the data reported by Mr. Redus seem to be in good agreement with the Navy field studies. Mr. Redus reports progressive consolidation of both plastic and nonplastic base courses by traffic. Progressive base and subgrade compaction, all in one rut or narrow path, will inevitably cause failure of the bituminous surfacing which cannot act as a beam.

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[51] Discussion by James B. Newman, Jr., of "Military Airfields: A Symposium," *Transactions*, ASCE, Vol. 110, 1945, pp. 734-737.

Finally, a comment on the CBR test itself is required. The standard loads of 1,000 lb per sq in. and 1,500 lb per sq in. for 0.1-in. and 0.2-in. penetration, respectively, have reference to a "standard crushed stone," as described in Mr. Porter's paper. Aside from the fact mentioned by Mr. Turnbull that "large particles directly under the piston affect the results," it is obvious that the type of deformation that characterizes the standard crushed stone is vastly different from that which is characteristic of soaked, plastic clays, and silts. There can be no "modulus of shearing resistance" that is commonly characteristic of both of these extremely different types of materials, because the top inch of soaked clay is practically at its liquid limit and the penetration is due to displacement without either elastic compression or appreciable consolidation. On the other hand, the standard crushed stone is a brittle material that possibly fractures to some extent and surely must undergo elastic compression under the load of the piston. Again, variations of as much as 1,000% in "corrected" CBR values have been reported for the same soil at the same moisture content by the Vicksburg (Miss.) laboratory,[52] the variations being attributed primarily to different compaction procedures. A carefully controlled standard compaction procedure will eliminate all such extreme variations in measured CBR values, but is such a laboratory standard truly indicative of the actual field compaction? The summary of Mr. Turnbull's paper seems to express a reasonable doubt as to this possibility under diverse field conditions.

In conclusion, it is believed that although great progress has been made in using the empirical CBR design method, pending still further needed modifications of this method, more realistic procedures (such as direct loading tests or traffic tests that simulate actual traffic conditions) are required for the sake of economy in pavement design.

JACOB FELD,[53] M. ASCE.—Within about one hundred pages, the contributors have recorded 20 years of experience in developing, testing, and applying a method for designing soil layers to support moving loads. Each author treats his phase of the subject in detail, making available the story of the empirical deduction from field observations, followed by the confirmation from controlled tests for highway traffic, the extension of method to the design of airport pavements and, the program of field testing to confirm and modify the method so applied to airport pavement design. In all this detailed record, however, there is no clear definition of the California bearing ratio (CBR), or of the soil characteristic which it is supposed to represent.

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[52] "The California Bearing Ratio Test As Applied to the Design of Flexible Pavements for Airports," *Technical Memorandum No. 213-1*, U. S. Waterways Experiment Station, Vicksburg, Miss., 1 July 1945, plate 109.

[53] Cons. Engr., New York, N. Y.

This discussion is limited to the question of correlation of CBR value with soil identification and classification. In the "Foreword" the CBR of the soil is stated to be an index of strength obtained by a simple shear test. In paper No. 1, Messrs. McFadden and Pringle infer that a shear test is proper in the design of flexible pavements, although they also state that the CBR test may not supply the ultimate answer.

The basis of the design method is actually the following:

(1) There is a unique experimental procedure to be performed according to fixed rules—namely, a soil sample in a rigid mold, 6 in. in diameter by 5 in. high, is compressed by a static load of 2,000 lb per sq in. (144 tons per sq ft); then it is saturated while loaded with weights equal to the pavement and other static loads to be imposed above the soil layer under consideration; followed, finally, by a measurement of the load required to force a flat surfaced piston (area 3 sq in.) into the soil a distance of 0.1 in. The ratio of this load to the load required for the same penetration of the piston into crushed rock road base material is the "CBR" of the soil. For field tests, the static compression load is replaced by standard impact compaction methods.

(2) The result of this procedure is the major criterion of soil action under traffic loaded flexible pavements.

(3) Identity of test results represents identity of soil action, so that, in effect, for this design problem the CBR test is a complete method of soil identification and classification.

The primary purpose of engineering design is to avoid failures (sufficiency), and the requirement of economy (necessity) is a close second. Mr. Porter's original conclusions resulted from the observation of pavement failures (paper No. 2), which are classified as

(*a*) Lateral displacement of the soil as a result of softening by water absorption (viscous flow);
(*b*) Differential settlement (failure of the solid state); and
(*c*) Excessive deflections resulting from repetitions of load (failure of a plastic material).

The first two types of failure are assumed to be caused by insufficient compaction of the soil during construction, and the third is an indication of insufficient shear strength because of either the nature of the soil or poor compaction. Therefore, the CBR test seems to be limited to the strength determination and classification of such soil as will exhibit plastic characteristics (in the compacted state) under repetitions of load. The design method then becomes the choice of the necessary thickness of soil layers of proper CBR value to prevent the plastic deformation of the soils below such layers.

In a proper design by the CBR method, as is stated by Messrs. Middlebrooks and Bertram in paper No. 3, there is no shear deformation, but there are still the settlement of the subgrade, the compaction of the base and surface, and the elastic deformation. These three factors are not direct functions of the

CBR value. In addition, it might be argued that, even if the shearing stress does not exceed the shearing strength, there may also be a measurable strain. In any case, it must be made clear that the CBR method disregards the possibility of soil strains accumulating to failure from other causes, even if the necessary shear strength is provided.

A study of the test data reports brings up some questions along this same line of thought. In the Stockton Field tests in California, Mr. Porter reports pavement settlements of from ½ in. to ¾ in. during the traffic tests without causing pavement failure. These settlements are credited to the heavy wheel loads as added compaction over that obtained by the rollers during construction (paper No. 5 under the heading, "Description of Project"). However, the data in Table 1 indicate that subgrade deflections at the bottom of the crushed gravel base were substantial, and at least 70% of the measured values at the top of the pavement. Correspondingly, in the Barksdale Field test at Shreveport, La., Mr. Hansen reports (in Table 5, under the heading, "Summary of Test Results: Compaction Requirements to Prevent Excessive Consolidation," in paper No. 6) that the subgrade was compacted permanently by traffic, to depths of 6 in. for the 20,000-lb track and 12 in. for the 50,000-lb track, with no detrimental shear deformation.

In the Marietta (Ga.) test, Mr. Griffith (in paper No. 7, under the heading, "Discussion of Tests and Test Results") reports on two unsatisfactory pavement assemblies, in both of which the clay subgrade and the sand-clay base were consolidated to higher dry densities. In Table 7 there is an interesting phenomenon reported, for which Mr. Griffith may have an explanation. Although traffic caused additional consolidation of the subgrade, the pressure cells rose as much as 2 in. during the test. Such displacement of the recording cells must be accompanied by a local soil distortion, and readings of questionable accuracy. It seems pertinent to inquire why there is no dependence of soil changes and shear strength and, further, whether a decrease in CBR value denotes that shear failure, and only shear failure, is responsible.

The greater advance made in soil technology by the CBR method, often overlooked, is the premise that the soils as found are to be used, modified if necessary by physical manipulation, and covered with stronger layers for proper distribution of loading. Every soil can be used, sometimes at great expense, but no soil has zero value.

The weakness of the CBR method lies in the premise that only plastic failure must be guarded against and that other strains in the soil body will not cause failure of the pavement, either directly or by affecting the plastic range of action. It would seem more logical to set up a criterion that the soil characteristics (density, moisture content, and the CBR, or some other measure of shear resistance) are not altered permanently by traffic loading, for adequacy of design.

The respective authors should be complimented for making available this complete record of data and the correlation with the CBR design procedure, with special commendation for the work of Mr. Porter, so apparently visible in each phase of the Symposium report.

S. M. **FERGUS.**[54]—Messrs. Middlebrooks and Bertram in their paper state that one of the methods used in extrapolating the California bearing ratio (CBR) design curves to heavier wheel loads was the elastic theory. In their extrapolation it was the intent that sufficient thickness of base and wearing course be provided to prevent subgrade shear. The thicknesses obtained in this manner, however, were empirically decreased for the higher CBR values on the basis of the experience of the consultants. Mr. Foster presents data relative to the behavior of several pavements under traffic tests and compares the results by appropriate wheel load groups. It can be shown that the CBR design curves currently appearing in the *Engineering Manual for War Department Construction*[33] follow the elastic theory very closely and that, therefore, they may be expressed as a single curve which is applicable to any wheel load. Also, the traffic test data presented in Mr. Foster's paper may be plotted to the same scale and compared with this single design curve.

Fig. 38 represents any two independent uniform circular loads placed on the surface of a semi-infinite, elastic body. Point P′ corresponds to point P just as a point on a photographic enlargement corresponds to the same point on the original; that is, the depths and offset distances are the same in terms of the respective radii. Assuming that both loads have the same surface contact pressure, it is well known from elastic theory that the value of any particular stress (vertical, horizontal, principal, or shear) at point P′ is equal to the value of the same stress at point P. If a comparison is made of a series of such uniform circular loads which have the same contact pressure but are of increasing total magnitude, it can be noted that the value of any stress can be maintained constant at corresponding pionts such as P and P′ by maintaining the ratio of the depth to the radius of the loaded area at a constant value. Moreover, since P and P′ are any corresponding points, they can as well be taken at the location of the most critical stress at the depth under consideration.

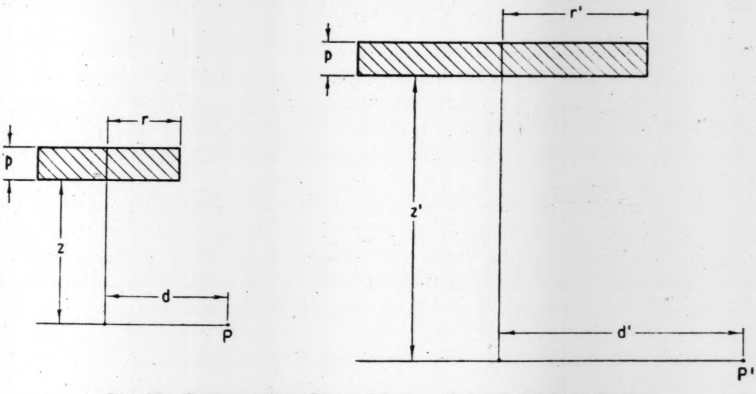

FIG. 38.—CORRESPONDING POINTS BENEATH UNIFORM CIRCULAR LOADS

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[54] Engr., Flexible Pavement Branch, Soils Div., Waterways Experiment Station, Corps of Engrs., Vicksburg, Miss.

The original CBR curves were extrapolated by the consultants in different ways. Messrs. Middlebrooks and Bertram made their extrapolations on the basis of the elastic theory, increasing the thicknesses for the different wheel loads sufficiently that the maximum shear stress for a given CBR value was held at a constant value. This result can be accomplished readily, by the photographic enlargement concept described in the preceding paragraph, by maintaining the ratio of depth to radius of loaded area constant for any given CBR value. As described in the latter part of the paper by Messrs. Middlebrooks and Bertram Mr. Porter's curves were based on an allowable deformation for the various loads, whereas those of A. Casagrande, M. ASCE, were based on the relative sizes of the loaded areas. However, since the curves obtained by the three different methods were in substantial agreement, it appears that the same relation between the thicknesses and the dimensions of the loaded areas is inherent in each and, therefore, a constant ration of depth to radius of loaded area can be assumed for any given CBR. This relation is expressed by the equation:

$$z = a\,r = a\sqrt{\frac{P}{\pi\,p}} = \frac{a}{\sqrt{\pi\,p}}\sqrt{P} = K\sqrt{P}\;\ldots\ldots\ldots\ldots(1)$$

in which z is the combined thickness of wearing course and base, in inches; r is the radius, in inches, of the loaded area treated as a circle; a is the ratio of the depth z to the radius r, a constant which depends on the CBR of the subgrade; p is the surface contact pressure, in pounds per square inch; P is the total wheel load, in pounds (equal to $\pi\,r^2p$); and K is a constant (equal to $a/\sqrt{\pi}\;p$) if p and CBR are constant.

The thicknesses for wheel loads of from 5,000 lb to 150,000 lb required by the CBR design curves published in the *Engineering Manual for War Department Construction*[33] are given in Table 12. The values shown for the 200,000-lb wheel load are those derived in 1948 from Stockton, Calif., test section No. 2. In each case the value for K has been computed from the relation:

$$K = \frac{z}{\sqrt{P}}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Reading from left to right in Table 12, it can be noted that, for any particular CBR, the values for K tend to be nearly equal and that there is no particular trend with the load, the values for the smaller loads being in some cases greater than those for the heavier loads. However, those for the 40,000-lb wheel load and the 70,000-lb wheel load are somewhat less than the others. To determine how much change in the design thicknesses would be involved by assuming that K is constant for a particular CBR, thicknesses for each of the wheel loads were computed from the average values for K shown in the right column of Table 12. These thicknesses are plotted and compared with the original curves in Fig. 39. A comparison of the two sets of curves shows that there is good agreement for the 5,000-lb curve and the 15,000-lb curve and for that part of the 150,000-lb curve with a CBR less than about 15. The agreement for the 40,000-lb curve and the 70,000-lb curve is not as good, the present design curves showing somewhat less thickness than the computed values. As stated by Mr. Foster, however, the behavior data indicate that the thicknesses for these two wheel loads may have to be increased. The computed thicknesses for the 150,000-lb curve are less than those required by the present design curve above a CBR of about 15. The thicknesses developed at Stockton for the 200,000-lb curve for these CBR values indicate that the lesser thicknesses for the 150,000-lb load may be more reasonable. In view of the generally good agreement between the two sets of curves, the assumption that *K* can be taken as a constant for a particular CBR seems reasonable.

TABLE 12.—PAVEMENT THICKNESS REQUIRED FOR GIVEN WHEEL LOADS (1 KIP = 1 "KILO"-POUND = 1,000 LB)

California bearing ratio	5 Kips		15 Kips		40 Kips		70 Kips		150 Kips		200 Kips		Average
	z	*K*	*z*	*K*	*z*	*K*	*z*	*K*	*z*	*K*	*z*	*K*	*K*
3	14.0	0.198	25.0	0.204	37.5	0.188	49.6	0.187	75.0	0.194	89.5	0.200	0.195
4	12.2	0.173	21.4	0.175	31.7	0.159	40.7	0.154	64.0	0.165	75.5	0.169	0.166
5	11.0	0.156	18.8	0.154	27.7	0.139	36.0	0.136	56.5	0.146	66.5	0.149	0.147
6	9.9	0.140	16.9	0.138	24.9	0.125	32.2	0.122	50.7	0.131	60.0	0.134	0.132
7	9.2	0.130	15.4	0.126	22.6	0.113	29.3	0.111	46.0	0.119	54.5	0.122	0.120
8	8.5	0.120	14.1	0.115	20.7	0.104	26.9	0.102	42.5	0.110	50.5	0.113	0.111
9	8.0	0.113	13.0	1.016	19.2	0.096	25.0	0.094	39.8	0.103	47.0	0.105	0.103
10	7.5	0.106	12.1	0.099	17.9	0.090	23.3	0.088	37.3	0.096	44.4	0.099	0.096
15	5.4	0.076	8.9	0.073	13.4	0.067	17.9	0.068	29.0	0.075	35.0	0.078	0.073
20	....	....	6.7	0.055	10.6	0.053	14.7	0.056	25.0	0.065	28.8	0.064	0.059
30	....	....	....	....	6.9	0.035	10.5	0.040	19.5	0.050	20.9	0.047	0.043
40	....	....	....	....	4.4	0.022	7.6	0.029	16.5	0.043	16.3	0.036	0.033
50	....	....	....	....	....	....	5.6	0.021	14.0	0.036	13.3	0.030	0.029
60	....	....	....	....	....	....	....	....	12.5	0.032	11.2	0.025	0.029
70	....	....	....	....	....	....	....	....	....	....	9.5	0.021	0.021
80	....	....	....	....	....	....	....	....	....	....	8.3	0.019	0.019
90	....	....	....	....	....	....	....	....	....	....	7.4	0.017	0.017
100	....	....	....	....	....	....	....	....	....	....	6.6	0.015	0.015

The use of the single factor *K* for the expression of the thicknesses for any wheel load makes it possible to plot all the available traffic test data to the same scale. Using values for *z* and *P* as shown in Table 9, the thicknesses at which the several pavements failed or were satisfactory were converted to *K*-values using Eq. 2. These points are shown in Fig. 40 and are compared with a curve plotted from the average values for *K* taken from Table 12. The curve divides the failed from the satisfactory fields in a reasonable manner. The fact that in one case a point indicating a failure falls within the satisfactory zone indicates that the curve is not unduly conservative. It also is clear that the shape of the curve for CBR values greater than 15 has not been fully substantiated.

Throughout this analysis no value has been assigned to the surface contact pressure; it has been designated simply as a constant. Messrs. Middlebrooks and Bertram mention a pressure of 60 lb per sq in. for their extrapolations and it is known that contact pressures at the Stockton test section No. 2 exceeded 100 lb per sq in. The contact pressures reported for the airfields and the test sections listed in Table 9 varied between approximately 50 lb per sq in. and 100 lb per sq in. In view of the fact that the curve presented in Fig. 40 is applicable to the loading conditions represented by the plotted data, it is considered that this curve is valid for contact pressures up to and including at least 100 lb per sq in.

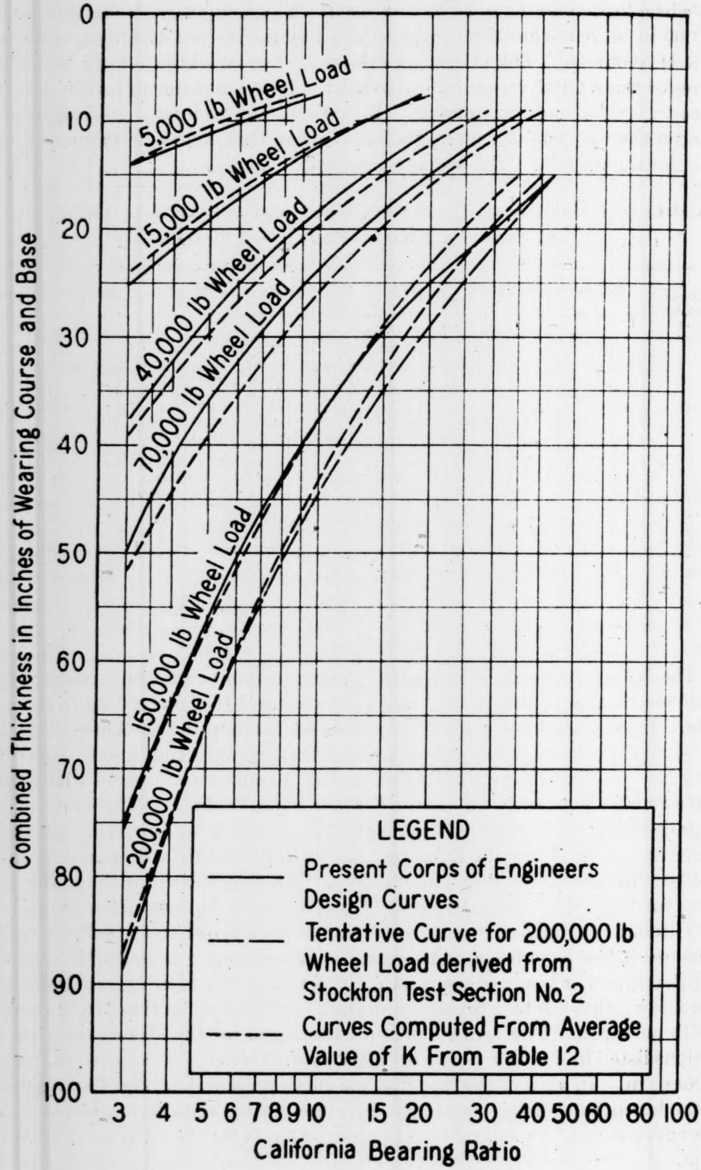

Fig. 39.—Comparison of Present Design Curves with Curves Computed From Average *K*-Value

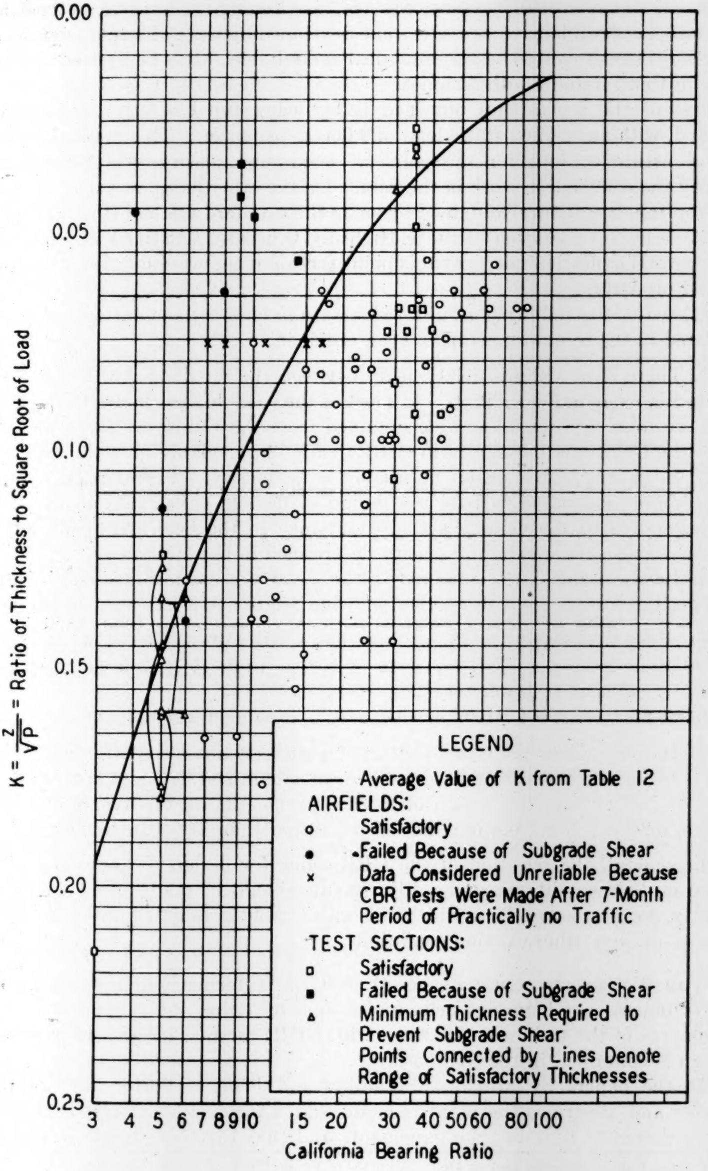

FIG. 40.—CORRELATION OF AVERAGE DESIGN CURVE WITH AIRFIELD EVALUATION DATA

Raul Valle-Rodas,[55] Assoc. M. ASCE.—Among the various methods for the design of flexible pavements, the one developed by the California Division of Highways has proved to be very practical and reliable. It is being used successfully in Central and South America.

One of the advantages obtained is by using the California compaction method, with an applied static load of 2,000 lb per sq in. This method gives a low moisture content for the point of maximum density, which facilitates greatly the compaction work in the field. On the other hand, it was found that the "optimum moisture" obtained by using the standard method (the method of the American Association of State Highway Officials (AASHO)) is sometimes at or near the plastic limit. At this point the soil is too wet and cannot be compacted properly.

Likewise, the following results (which are in close agreement with the ones reported in the Symposium) have been obtained:

1. There is no definite relationship between the modified AASHO and the California compaction method. In most of the cases higher values (up to 6%), for maximum dry densities, were obtained from the California static method. Also, the California bearing ratio (CBR) varied appreciably in both cases.
2. In general, CBR varies noticeably with density. Therefore, it appears to be more reasonable to base the design of flexible pavements on the CBR corresponding to the density actually obtained in the field after compaction, rather than on the maximum density determined in the laboratory.
3. Between the CBR values at 0.1 in., and the expansion obtained after soaking the sample for 4 days, the following relationship has been established:
   - (*a*) Soils that expanded 3% or more had CBR values of less than 9%.
   - (*b*) Soils having a CBR equal to or higher than 15% showed expansions equal to or lower than 2%.
   - (*c*) Soils having a CBR higher than 30% expanded less than 1%.
4. In many cases the stress-penetration curve is not a smooth one. If the test is rerun and the same nonsmooth curve is obtained again, it appears to be more logical to base the design of flexible pavements on the average CBR at 0.1 in., 0.2 in., 0.3 in., 0.4 in., and 0.5 in., rather than at 0.1 in., or 0.2 in.

In general, the selection of the CBR value for design purposes should be based on field conditions, and readjustments should be made according to the density, moisture, depth at which the water table is found, rainfall, drainage conditions, and other existing conditions.

Tom W. Stallworth,[56] Assoc. M. ASCE.—Information clarifying a number of points that have not been understood by either the proponents or the opponents of the California bearing ratio (CBR) method for airfield pavement design is contained in the Symposium.

In the papers by Messrs. McFadden and Pringle, Porter, and Middlebrooks and Bertram, the point is made that a form of shear test is required in the design of flexible type pavements and that the CBR test is a measure of shearing resistance of soils. From a knowledge of the test details it would seem that the resistance of the soil sample to penetration of the piston consists of two reaction components comparable to those which resist the penetration of a friction pile. These components are (1) the resistance to punching shear around the periphery of the piston and (2) the point resistance which is distributed over the area of the contact face of the piston. Since the modulus of shearing resistance is determined by the resistance at 0.1-in. penetration, it probably can be argued that the first of the components is responsible for a predominant part of the resistance to penetration for such a low value of deformation where the second component has not yet been mobilized to a significant degree. This explanation seems logical if the soil is a cohesive material; but the value of the modulus of shearing resistance obtained in this manner, for coarse-grained base materials that are of very low compressibility, would appear to be a measure of penetration resistance, not shearing resistance. The question arises relative to the limitation as to type of soil for which the test is valid. It is understood that the California Division of Highways does not use the test to determine a measure of the shearing resistance of all types of soils, including flexible base materials, and some discussion of this fact in the closure will be appreciated.

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[56] Asst. Prof., Soil Mechanics Section, Civ. Eng. Dept., Agri. & Mech. College of Texas, College Station, Tex.

In the description of the adaptation of the CBR method to the design of airfield pavements the extrapolation of the original curves has been made on the basis of the elastic theory. It is significant that shear stresses were used as a guide in making the extrapolation and, for the first time to the writer's knowledge, the pressure used in this process is stated.

Published data on airfield test projects indicate that tire inflation pressure and tire contact pressure are not exactly the same. Therefore, it would be more accurate to state that contact pressure of 60 lb per sq in. was used in the extrapolation because that was the approximate value of the tire inflation pressure.

Fig. 6 illustrates a fact which the authors recognize, but which has not received the attention it deserves in limiting the use of the design curves. This figure shows that the maximum shearing stress for the 70-kip wheel load is no greater than that produced by a 12-kip load. If the contact pressure remains constant the increased load is reflected in an increased bearing area which does not increase the magnitude of the maximum shearing stress but does increase the depth at which it occurs. The important part of this discussion is that, unless the qualifying contact pressure is given, the designer is not aware of the effect of contact pressure and considers gross wheel load alone as the significant factor in the design. The trend in inflation pressures for aircraft tires is upward to a marked degree and the maximum shearing stress of 38 lb per sq in. induced by a 12-kip load at a contact pressure of 120 lb per sq in. is double the maximum shear stress that develops when this same load is applied at a pressure of 60 lb per sq in. This increase in shear stress resulting from changes in aircraft landing gear design probably will require some modification of the empirical design curves now in use.

This deficiency of the CBR design curves has been noted independently by several writers, but it is believed that the implications of greatly increased contact pressures have not reached the engineers who are responsible for pavement design.

Robert Horonjeff,[57] Assoc. M. ASCE.—The eleventh Symposium paper by Mr. Turnbull is an excellent summary on the advantages and shortcomings of the California bearing ratio (CBR) method. The writer wishes to elaborate on some of the points mentioned by Mr. Turnbull. The comments are not to be considered as criticism of the CBR method but are merely presented to make the practicing engineer fully aware of certain phases of the method.

As indicated in the fourth Symposium paper, CBR values are extremely sensitive to the density of the soil and this fact is illustrated in Fig. 11(*c*). With a moisture content of 6% and a molded dry density of 120 lb, the CBR value is approximately 38. If the density is increased to 125 lb, the CBR of the same material is increased to 78, or more than double the original value. Extreme variations also occur in the highly plastic soils with CBR values of from 3 to 10. In one airport the writer found that the CBR value was 6 when the dry density was 106 lb per cu ft; however, when the density was increased to 115 lb, the CBR was increased to 20. Since there is no evidence that precise correlation exists between the compaction of a given soil by any laboratory procedure and the compaction of the same soil in a fill, the density of the fill could easily vary as much as 5 lb per cu ft from the laboratory compacted sample.

Referring to the design curves in Fig. 7, paper No. 3, it is readily seen that small variations in CBR values for subgrade soils with CBR values of 30 or greater will not result in appreciable differences in the total thickness of pavement and base required to support a specified wheel load. However, the same is not true for subgrade soils whose CBR values range from 3 to 10.

To illustrate this point further assume that a flexible pavement is to be designed for a wheel load of 70,000 lb. If the CBR of the subgrade is 38, the required thickness of pavement and base is 10; however, if the CBR is increased to 78, the thickness of pavement and base is reduced to 6 in. In this instance only a 4-in. variation in thickness results. Now, assume that the CBR of the subgrade soil is 3. The total thickness of pavement and base required for a 70,000-lb wheel load is 50 in. If the CBR value is increased to 5, the total thickness of pavement and base is reduced to 36 in. In this case the CBR changed by only 2 but the thickness varied approximately 14 in. It should be remembered that 2% is often not within the accuracy of the test.

From the foregoing discussion it can be readily seen that the CBR values for the weaker subgrade soils must be established with greater care and accuracy for design purposes than for soils with a CBR of, say, 25 or greater. The design CBR for airports is usually established from explorations and laboratory tests prior to initiation of construction. It is often very costly to revise the thickness of pavement and base after construction has begun, since it would normally delay the contractor until revised grades could be established. For this reason extreme care must be taken to insure that the density and moisture content of the laboratory sample conform as closely as possible to the density and moisture content of the fill in place. Most of the time this is difficult to achieve.

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Another point which the writer desires to elaborate on is the 4-day soaking period. It is gratifying to know from Mr. Turnbull's discussion (paper No. 11) that

> "* * * the Corps [of Engineers] is making a serious effort to determine in what general areas and specific localities the conservatism of design as represented by the 4-day soaking period need not be followed."

In several instances the writer found that the moisture content of the subgrade several years after completion was less than the moisture content of the corresponding laboratory soaked samples. It is recognized that in many parts of the United States this condition would not be true. Furthermore, the moisture content can vary considerably throughout the year depending on the season. However, in the interest of economy, consideration should be given to modifying the soaking period in localities where it can be definitely established that the moisture content will not vary appreciably from conditions at the time of construction—especially for the soils where a small variation in CBR results in large differences in total thickness of pavement and base. This is well illustrated by the data contained in Table V of the paper by Norman W. McLeod on load testing applied to pavement design.[50]

John J. O'Mara,[58] Assoc. M. ASCE.—In the last and summary paper of the Symposium Mr. Turnbull relates that considerable criticism has been made of the procedure for the preparation of the test specimens in that the 4-day soaking period is excessive and leads to pavement thicknesses which are too conservative. One aspect of the comparison of the degree of saturation of the laboratory sample to that of the prototype has not been discussed fully. The spring breakup period in many sections of the United States is the critical point of saturation for thousands of miles of airfield and highway pavements. During this period (which occurs in all regions of moderate to deep frost penetration) the ice lenses melt and leave the subgrade in a saturated or, perhaps, a "supersaturated" (excess hydrostatic pressure) condition. The reduction in shearing resistance due to such moisture conditions is much more severe than that created by the 4-day soaking period specified for the laboratory specimens because the latter rarely results in degrees of saturation approaching 100%.

In view of the conditions prevalent under pavements as frost leaves the ground, it would be well to continue the specified soaking of laboratory specimens of subgrades in frost regions until such time as experience and the long-range studies referred to by Mr. Turnbull develop qualifying procedures. In this connection it is suggested that caution be exerted in comparing pavement performance during a few springs with the required thickness of pavements. First, critical spring breakup periods occur about only once in every 5 years, and with no regularity. Second, pavement failure during the period is caused by a combination of the moisture weakened subgrade and the application of heavy loads on the pavement. On major highways the loads coincide with the subgrade condition; on airfields, repetitive wheel loads are distributed over a wide area, and operations are limited to a few repetitions per hour or may be suspended during the critical period. Consequently, an airfield pavement may not fail during the spring breakup for several years, and judgment should be withheld until sufficient time has elapsed to evaluate the pavements properly.

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The writer questions the unqualified rule of adjusting the stress-penetration curves to correct initially concave upward curves. It is true that it is practically impossible to produce a stress-deformation curve of any kind which begins at the origin and has the correct shape. For this reason most materials testing requires the application of a small load before extensometers are attached and set. In some cases this is not feasible and the stress-strain curve is adjusted to correct for uneven bearing and other faults at the beginning of the test. However, these adjustments amount to no more than one or two thousandths of an inch in ordinary materials testing. In a test conducted by the writer in May, 1949, the adjustment amounted to more than 0.1 in. and resulted in a CBR number of 50 compared to a CBR of 10 for the unadjusted curve. Some factor other than uneven seating of the piston must be involved in such an adjustment. This may be the actual shearing deformation prior to consolidation of the soil immediately below the piston. If so, it is of importance and should not be disregarded.

However, this deformation may be due to eccentric or other inconsequential movement occuring between the frame of the testing machine and the top of the mold. Some research should be devoted to arranging the strain gage to measure the deformation of the soil, not the movement between the frame and a point on the mold.

Until such a procedure is developed, better data could be secured by using two strain gages, in the same fashion as the present one, but arranged to record the movements of diametrically opposed points on the rim of the mold. The readings of the gages, then, should be averaged to secure the penetration. Still, caution should be exerted in any adjustment of the resulting stress-penetration curve, and a limit should be set for such adjustment—that is, a maximum offset from the original origin of this curve.

D. P. Krynine,[59] M. ASCE.—An elementary design rule requires that a structure must be safe and economical. In its turn, an elementary principle of engineering economics states that in comparing two different solutions of an engineering problem from the economical point of view, not only should the original cost of a structure be considered, but also the capitalized cost of its repairs and maintenance during its span of life.

Pavements are structures. That flexible pavements designed using the California bearing ratio (CBR) method are safe is beyond doubt. Accelerated traffic tests and heavy war service have proved it. Sometimes it is claimed, however, that flexible pavements designed using the CBR are "overdesigned," because of the excessive soaking of samples in the test. Possibly this is true in some instances; but a record of such instances has yet to be presented. The Corps of Engineers, United States Army, who introduced the CBR during the emergency period clearly understood it. In his paper Mr. Turnbull states,

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"Long-range studies are being conducted to determine the moisture content that occurs under pavements in arid, semiarid, and wet regions." This initiative is commendable as a step in the right direction.

In the technical press during 1948 and 1949 statements have appeared indicating that wheel loads of any given magnitude can be carried by a pavement and base course which have only a fraction of the thickness specified by the Army design.[49] Again in some cases this is true. The writer, however, does not discern any answer to two questions that make him rather uneasy: (*a*) How long will these pavements with a fraction of the thickness specified by the Army design remain adequate? and (*b*) How long will they stay in a really good condition under heavy traffic in the case of a sudden emergency? This recalls the necessity of working out the theory of the economic life span of a runway. For many years highway people have had an analogous theory for pavements. A pavement has only a predetermined life span, after which it is more economical to replace it with a new one, instead of continuing the repairs. Only when such a theory is elaborated and put to work is it possible to determine a correct comparison between two pavements on the basis of sound engineering economics, as stated at the beginning of this discussion.

Accumulation of Moisture Under Pavements.—Apparently few engineering questions are as controversial as the question concerning the accumulation of moisture under pavements. Some engineers claim that a 100% subgrade saturation combined with high moisture content is quite possible in all cases, whereas others state that there is a tendency for decrease in subgrade moisture under pavements with traffic and time. Both schools of thought base their opinions on field observations. Unfortunately for them, and for the profession, their observations are incomplete and reveal only a part of the true picture. At the annual meeting of the Highway Research Board, National Research Council, held in Washington, D. C., in December, 1948, L. D. Hicks, chief soil engineer of the North Carolina State Highway and Public Works Commission, read a paper entitled, "Observations of Moisture Contents and Densities of Soil Type Bases and Their Subgrades." Among other things, Mr. Hicks determined the moisture content of the subgrades of flexible pavements at twenty points (stations) of the State of North Carolina each month of the year 1947, using samples of soil spaced from 5 ft to 10 ft along the length of the highway, and taken at a distance of about 30 in. from the edges of the pavement. Presumably, his work may have only a relative value for runways that are considerably wider than highways; but his results are instructive and interesting for any pavement engineer. A few of Mr. Hicks' data follow:

Station 1.—The subgrade soil at this station was designated Wadesboro B and C (approximately A-7 in the Public Roads Administration (PRA) classification). The pavement was constructed in 1942. The observed moisture content of the subgrade was 33.0% in December, 32.2% in May, and only 21.8% in January and February. The degree of saturation in May was 96.1%.

Station 2.—This was a subgrade soil designated Colfax B (approximately A-7 in PRA classification). The pavement was constructed in 1938. In April the moisture content was 32.5% and the degree of saturation, 99.7%; and in September the moisture content was only 13.8%.

Station 3.—The subgrade soil at this point was designated Tirzah B and C (approximately A-7 in PRA classification). The moisture content in June was 26.6% with 100% saturation. The lowest moisture content value at this station was 22.7% in December.

Station 8.—The subgrade soil at this point was Lloyd B and C (approximately A-5 and A-7 in PRA classification). The pavement was constructed in 1939. The 100% saturation occurred in September (moisture content 25.9%). The lowest moisture content was in February (15.1%).

Station 13.—The subgrade soil was Wadesboro B and C (approximately A-5, A-6, and A-7 in PRA classification). The pavement was constructed in 1937. The 100% saturation occurred in January (moisture content 25.0%). The lowest moisture content was in April (19.1%).

Mr. Hicks' data show how careful one should be in interpreting field observations. Consider a hypothetical case of an investigator who in January of a certain year determined the moisture content at station 13, and came back, say, six years later at a wrong month, say, April. If seasonal fluctuations of the soil moisture are not considered, an erroneous conclusion may be drawn that the soil moisture has decreased by about 1% per yr "due to traffic and time."

It is obvious that 100% saturation under a runway pavement may be exceedingly dangerous when a heavy plane is landing, no matter what the value of the moisture content. In such a case the total load is taken up by the pore water with a possibility of a "quick" state of the subgrade.

Moisture Under Pavements Under Desert Conditions.—Besides interesting data on soil moisture under desert conditions, gathered by the California Division of Highways, there are observations and suggestions by long time residents of desert regions. Of course, these observations require criticism and thorough analysis; their value cannot be denied, however. In a letter to the writer J. M. Lackay, district engineer of the Asphalt Institute, Los Angeles, Calif., called the attention of the writer to a case when—

> "* * * several miles of pavement in the driest part of the desert about three miles west of Salton Sea (Imperial Valley, Calif.) failed so rapidly that it could not be repaired and had to be abandoned, although a test hole put down a few feet from the edge of the pavement, some 20 or 30 feet in depth, did not disclose as much as a trace of moisture in the soil while at the same time the moisture accumulated under the pavement reduced its supporting power to that of a thin liquid."

Mr. Lackay furthermore describes an opposite case of a paved street through Balboa Park at San Diego, Calif., where an asphalt pavement about 1.5 in. in thickness laid directly on a local yellowish adobe clay was in perfect condition at the age of 14 years, whereas adjoining concrete pavements on both ends had been pounded to pieces because of the accumulation of water under them.

Thermo-Osmosis.—Engineers using the new term, "thermo-osmosis," describe the phenomenon as the natural tendency of water to flow from a relatively warm to a relatively cold region within a saturated or partly saturated soil mass. According to them, black pavements absorb much heat in warm seasons, resulting in a downward movement of soil water. To clarify the situation, the writer wishes to state that what is termed by some engineers "thermo-osmosis" is not a new phenomenon but simply a particular case of a phenomenon controlled by the second law of thermodynamics established by Carnot more than a hundred years ago. That law[60] may be formulated thus: "No self-acting device unaided by an external agency can transfer heat from a body of lower temperature to one of higher temperature."

In applying this law, its limitations must be clearly understood; and, if colder water moves to warmer regions of a soil mass, as is sometimes the case in nature, an "external agency" should be sought. Thus, when colder water moves toward hot asphaltic pavements (as apparently occurs sometimes) an "external agency" should be operative. Without field and laboratory investigations along these lines the writer hesitates to define the physical nature of such an agency even approximately. However, an analogy is that of cold soil water moving to the top of a tree, sometimes to incredible heights. This movement is caused by the strong negative pressure, developed by the leaves of the tree, which permits the tree to maintain a diffusion pressure of water in the roots lower than in the surrounding soil.[61]

Conclusions.—(*a*) Designers and builders of runways should consider the economic span life of pavements in comparing two types of pavements or two different pavement thicknesses of the same type; and (*b*) a thorough study of the accumulation of moisture under pavements, including the factors that possibly influence such accumulation, is an engineering problem of major importance.

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[60] "Thermodynamics of Soil Moisture," by N. E. Edlefson and Alfred Anderson, *Hilgardia*, February, 1943, p. 60.

[61] "Fracture of Liquids: Nucleation Theory Applied to Bubble Formation," by J. C. Fisher, *Scientific Monthly*, June, 1949, p. 415.

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AND W. J. TURNBULL,[74] M. ASCE.—In view of the nature of the questions posed in the discussions, the writers have elected to present a single concluding discussion. The closure was prepared by the Flexible Pavement Laboratory at the Waterways Experiment Station in Vicksburg, Miss. It has been reviewed by and has the approval of each author.

As noted by Mr. Krynine, the technical press of 1948 and 1949 carried articles on Canadian experience[49] which indicate that pavements with only a "fraction" of the thickness indicated by the California bearing ratio (CBR) design method have given satisfactory performance. Commander Tyrrell and Mr. Palmer state that the experience of the Navy is in 100% agreement with the Canadian experience.

An attempt was made in the Symposium to show that the development of the CBR design procedure actually involved two problems. One was the correlation of wheel load with pavement thickness based on in-place CBR values; the other was the development of laboratory procedures for preparing samples to meet a future prototype condition. Each of these problems is discussed in the following paragraphs with respect to the relative conservatism of the CBR method of design.

As stated in paper No. 9, pavements which have withstood 2,000 coverages without shear deformation are considered adequate. Since this coverage could be experienced in 100 days, the amount of traffic was considered relatively small. It appears, however, that it is large in comparison with the traffic experienced in other regions. For the example quoted in paper No. 9, 2,000 coverages represent 27,000 cycles of operation of a 30,000-lb wheel load. The 23,000 operations of the 64,000-lb or heavier planes experienced at Dorval Field in Canada (see discussion by Commander Tyrrell and Mr. Palmer) are equivalent to 11,500 cycles of a 32,000-lb (or heavier) wheel load, since Mr. McLeod[50] states that each landing or take-off constitutes an operation. Even if all these operations had been on one runway (which is scarcely possible since there are three runways at Dorval), this traffic would have produced less than 1,000 coverages. When it is considered that the traffic was spread over a 5-year period with the subgrades frozen from 4 months to 5 months each year as compared to the 100-day freezing period for military fields in the United States, it is evident that there is considerable difference in the volume of traffic. The next most heavily used field, as reported by Mr. McLeod, is Toronto, Ont., Canada, with 3,400 operations (about 300 coverages) of planes weighing more than 50,000 lb (25,000-lb wheel load). Traffic of heavy planes at all other fields is reported as "several hundred."

The Corps of Engineers has prepared design curves for airfield pavements in theaters of operation where life expectancy is measured in terms of days and weeks rather than in terms of years. The time element in military construction in theaters of operation is of primary importance, and many elements considered essential in the permanent type of airfield pavements are eliminated to permit speedy construction. The resulting high maintenance can be accepted to achieve an earlier completion date. Because of the short life

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TABLE 13.—Coverages Versus Failure Study

Line (1)	Identification (2)	Wheel load (lb) (3)	Thickness (in.) (4)	Coverages to produce failure (5)	California bearing ratio (6)	Design thickness (in.) (7)	Percentage of design (8)
(a) Stockton (Calif.) Site No. 1							
1	....	25,000	12	200	5	23.5	51
2	....		14.5	300	5	23.5	62
3	....		18	500	5	23.5	77
4	....		22	1,000	5	23.5	94
5	....		24.5	2,000	5	23.5	104
6	....		25	3,000	5	23.5	106
7	....	40,000	20	200	5	28.5	70
8	....		26.5	500	5	28.5	93
9	....		31	1,000	5	28.5	109
10	....		36	2,000	5	28.5	125
11	....		38	3,000	5	28.5	133
(b) Stockton (Calif.) Site No. 2							
12	Item 1	200,000	39	150	6	60	65
13	Item 2*a*		44	1,700	9	48	92
14	Item 2*b*		46.5	2,000	10	45	103
15	Item 5*a*		18	10	14	37	49
16	Item 5*b*		20.5	60	16	34	60
17	Item 6		24.5	360	13	40	61
18	Item 7		30	1,500	13	40	75
19	Item 8		34	1,140	17	33	103
20	Item B		30	1,300	8	50	78
(c) Barksdale Field, La.							
21	Item 5	20,000	10.5	250	5	21.5	48
22			13	500	5	21.5	60
23			15.5	1,000	5	21.5	73
24			17.5	3,000	5	21.5	81
25			18	5,000	5	21.5	84
26		50,000	17.5	200	5.5	29	61
27			20.5	500	5.5	29	71
28			24	1,000	5.5	29	82
29			26	3,000	5.5	29	90
30			26.5	5,000	5.5	29	92
(d) Test Section; Waterways Experiment Station, Vicksburg, Miss.							
31	Item 1	37,000	13	400	10	17.5	74
32	Item 302		11	100	14	14	78
33	Item 4[a]		13	About 400	13	15	87
34	Item 4[b]		16	Prior to 400	5	27	59
35	Item 49[c]		16	About 350	4	31	52
36	Item 60[d]		16	Prior to 260	2	45	36
(e) Airfield at Minden, Nev.							
37	NE–SW	25,000	18	385	5	23.5	77
(f) Bergstrom Field, Austin, Tex.							
38	NW–SE[e]	15,000	17	358	6	17	100
(g) Airfield at Birmingham, Ala.							
39	NE–SW	23,000	7	194	4	26	27

[a] 1A-2-1, lane b. [b] 1A-2-1, lane c. [c] 2A-2-1, lane c. [d] 3A-2-3, lane c. [e] Pit 3.

expectancy at certain fields in the theaters of operation, the volume of traffic is small, and reductions in total thickness of base and wearing course are permitted. Table 13 presents available information on the coverages at which failures occurred at several test sections and airfields, where it is known definitely that failure was due to shear deformation in the subgrade. Table 13 shows the wheel load, the thickness of base and pavement above the subgrade, and the CBR value of the subgrade. The wheel load and the CBR have been used to determine a design thickness from the CBR design curves, and the actual thickness is given as a percentage of the design. Fig. 41 shows a plot of the percentage of design thickness versus coverages required to produce failure. The numbers beside each point refer to the respective lines in Table 13. Although there is considerable scatter of the data, there is a definite trend for failures to occur at a low number of coverages where the thickness is low as compared to the design, and at a higher number of coverages as the thickness approaches the design. Independent studies were made of these data by Arthur Casagrande, M. ASCE, P. C. Rutledge and T. A. Middlebrooks, Assoc. Members, ASCE, and engineers of the Flexible Pavement Laboratory and the Airfields Branch of the Office, Chief of Engineers. The results were in good agreement, and the curve shown in Fig. 41 represents a reasonable average of the five independent analyses. The curve was prepared for theater design and therefore is not conservative, because it is believed that the importance of

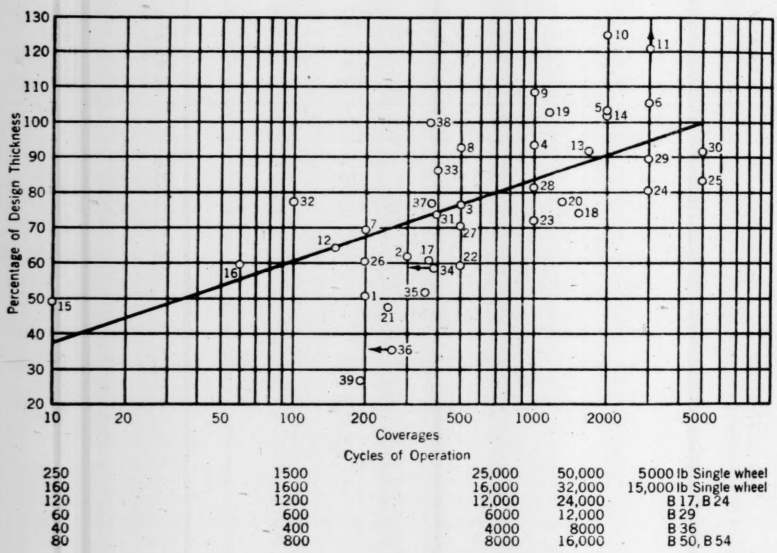

FIG. 41.—PERCENTAGE OF DESIGN THICKNESS VERSUS COVERAGES

the time element, and the fact that high maintenance can be accepted, warrant a reasonable element of nonconservatism.

Commander Tyrrell and Mr. Palmer state that the factor of safety cannot be estimated by the use of the CBR curves. It is true that no procedure has been developed for expressing the adequacy of the design in a numerical ratio with 1 as a base; however, Fig. 41 indicates the degree of adequacy for a given volume of traffic. Failures have occurred where the thickness was equal to the design value, which is evidence that, for unlimited traffic, the factor of safety is small, as no extra thickness has been added. Furthermore, only small reductions in thickness are justified until the volume of traffic is very small. Fig. 41 also indicates that a pavement constructed to even a small percentage of the design thickness would have a fairly high factor of safety for the case of "sudden pavement failure," mentioned by Commander Tyrrell and Mr. Palmer, since a slab with even one half of the design thickness would withstand about 35 coverages of a failing load. On the basis of the data previously presented and the data plotted in Fig. 41, it is concluded that the CBR design curves are not overconservative for unlimited operation.

The second problem in the development of the CBR design procedure is the development of laboratory procedures for preparing samples to meet a future prototype condition. Primarily, this requires the adjustment of the soil sample to a moisture content anticipated as maximum at some future time. It is recognized that other factors, such as thixotropic set, molding moisture content, method of compaction, and future density changes, also will influence the future strength of a soil. As Mr. Krynine states in his discussion, the accumulation of moisture under pavements is probably the most controversial subject in pavement design. Mr. Turnbull states in paper No. 11 that the use of the 4-day soaking period results in pavements which are conservatively designed until the design moisture content is reached. If it could be established that the moisture content at a given location always would be less than that obtained in the 4-day soaking procedure, a substantial saving could be made in the thickness. The accumulation of moisture under pavements, therefore, is not only a controversial point but also one of vital interest. Mr. Krynine's comments on this subject present a typical cross section of the available information and observations on the accumulation of moisture under subgrades.

The Corps of Engineers recognized this lack of positive information and in 1944 established a study of the accumulation of moisture under airfield pavements. Sites were selected in New Mexico, in Texas, and in the Mississippi Valley. These areas represent rainfall regions of less than 15 in., 15 to 35 in., and more than 35 in., respectively. Sites were selected in each region to give a range of subgrade materials. Moisture content readings were made at the center of the runway, between the edge and the center, at the edge, and out on the shoulder at each site. In addition, readings were made near an artificial crack cut in the pavement. "Bouyoucos" plaster-of-Paris moisture cells were installed in the sites in New Mexico, but did not prove satisfactory. All moisture values have been based on direct sampling. Initial observations were made at the fields in New Mexico in October, 1945. In July, 1946, these fields were sampled and initial measurements were made at the fields in Texas

TABLE 14.—Results of Field Moisture Content Survey Through 1948

Line	Average annual rainfall	Base Course ("NP" Denotes Nonplastic)				
		Type	Liquid limit	Plastic limit	Plasticity index	Water content[a]
(1)	(2)	(3)	(4)	(5)	(6)	(7)
	Less Than 15 In.—					
1	Clovis, N. Mex.	Caliche	24	17	7	2.5
2	Kirtland, N. Mex.	Sand caliche	17 to 20	14 to 15	3 to 5	1.5
3	Santa Fe, N. Mex.	Caliche gravel	26 to 29	15 to 17	11 to 12	3
	15 In. to 35 In.—					
4	Bergstrom, Tex.	Crushed stone	NP to 18	....[b]	NP to 1	2
5	Goodfellow, Tex.	Caliche	27 to 29	18	9 to 11	2
6	South Plains, Tex.	Caliche	30 to 38	28 to 31	2 to 7	1
	More Than 35 In.—					
7	Keesler, Miss.	Shell	NP	NP	NP	2
8	Memphis, Tenn.	Clay gravel (GC)[c]	22 to 31	11 to 15	9 to 16	3 to 5

TABLE 14.—(*Continued*)

Line	Thickness of base course (in.)	Subgrade ("NP" Denotes Nonplastic)				Range of Moisture Contents	
		Class[c]	Liquid limit	Plastic limit	Plasticity index	Base course	Surface of subgrade
(1)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
(<15 In.)—							
1	12	CL	22 to 28	14 to 15	8 to 13	8.5 to 12.2	9.0 to 12.7
2	9	SM to SC	18 to 21	14 to 16	4 to 7	2.5 to 6.7	6.8 to 11.0
3	8.5	CL	27 to 34	17 to 24	10 to 12	3.4 to 5.1	11.4 to 17.5
(15 In. +)—							
4	16	CL to CH	48 to 52	21 to 23	28 to 29	4.1 to 6.3	19.0 to 27.1
5	16	CH	55 to 60	23 to 28	32 to 34	9.6 to 13.9	20.4 to 23.6
6	10	SC	27 to 28	14 to 15	10 to 12	11.9 to 18.3	11.1 to 14.0
(>35 In.)—							
7	11	SP	NP	NP	NP	4.9 to 5.9	3.2 to 4.5
8	8 to 14	ML to CL	40 to 45	23 to 24	16 to 22	5.0 to 8.8	16.2 to 23.2

TABLE 14.—(*Continued*)

Line	Range of Moisture Contents (*Cont'd*)	Range of Percentage of Saturation		Trends in Moisture Content			
				Variation from Center to Edge		Variation with Time	
	At 12 in. to 16 in. in subgrade	Base	Subgrade	Base	Subgrade	Base	Subgrade
(1)	(15)	(16)	(17)	(18)	(19)	(20)	(21)
(<15 In.)—							
1	8.9 to 12.2	70 to 100	70 to 100	Edge[d]	Edge[d]	Increase[e]	Increase[e]
2	7.4 to 11.0	<50	<50	Center	Center	Decrease[f]	Decrease[f]
3	12.6 to 16.4	<50	<50	No trend	Edge	Increase[g]	No trend
(15 In. +)—							
4	19.0 to 28.3	<50	75 to 100	No trend	No trend	No trend	No trend
5	19.0 to 23.6	75 to 100	75 to 100	Edge	Edge	Decrease[g]	Decrease[g]
6	12.0 to 14.0	60 to 100	70 to 100	Edge	Edge	Decrease[f]	Decrease[f]
(>35 In.)—							
7	3.2 to 4.5	60 to 80	<25	No trend	No trend	No trend	No trend
8	19.2 to 25.9	70 to 100	70 to 100	Center	No trend	Increase[g]	No trend

[a] Moisture content expressed as a percentage of dry weight. [b] Approximately the same as the liquid limit. [c] *Transactions*, ASCE, Vol. 113, 1948, p. 912, Table 4. [d] Moisture is definitely higher at the edge in both the base and the subgrade. [e] About two percentage points. [f] Slight to moderate. [g] Slight.

and in the Mississippi Valley. Subsequent sampling was accomplished in June, 1947, and November, 1948. As stated by Mr. Krynine, careful interpretation is needed where the investigations do not cover the entire year; however, the sampling was not as frequent as desired because of lack of funds. It is anticipated that sampling will be more frequent in the future. These circumstances and the short period covered by the readings limit the information that can be obtained from the investigations at the present time. Table 14 includes information on the base course and subgrade characteristics at the several sites, together with the range of moisture contents experienced to date. There has been a fairly consistent tendency for the moisture content to be higher at the edges than in the center of the runways. The change with time has tended to be relatively minor except at Clovis, N. Mex., where the moisture content has increased about two percentage points. It should be noted that in the driest region two of the sites, Kirtland and Santa Fe, N. Mex., have shown very low moisture contents as compared to the plastic limit. The range of percentage saturation also has been low at these two sites. The moisture contents at Clovis have been fairly high even if the sampling was done in a relatively dry season. In the other two regions the moisture contents of the bases and subgrades have been relatively high except where the material is sandy.

As the result of a detailed study of the data collected to date from this project, together with an analysis of other available data, it appears that there may be a systematic variation in both moisture content and density under flexible pavements that may be significant, especially since it probably produces changes in the strength of the subgrade. The scope of the field moisture content investigation has been enlarged, therefore, to include measurements of density and CBR as well as moisture.

Although the study of moisture under pavements has not yielded definite information to date, other data can be studied to obtain an idea of the relative merits of the 4-day soaking procedure. Mr. Kersten asks specifically about the agreement between laboratory-soaked and in-place CBR values. This subject also is controversial, with data available to support both views. As a typical example, information from the asphalt-stability test section is presented. A complete description of the test section is given in a report published by the Waterways Experiment Station.[75] This example is used because more information is available on this particular subgrade than on any other. The subgrade is a lean, silty clay classified as ML to CL by the Corps of Engineers classification. Fig. 42 shows the results of the laboratory design CBR tests conducted in accordance with method 2 as specified in the *Engineering Manual*[17] (described in paper No. 4). Incidentally, the tests shown in Fig. 42 were conducted in 1944 during the design of the test section. Optimum moisture and design density (95% of modified AASHO (American Association of State Highway Officials) specifications) for this soil are 15% and 108 lb per cu ft, respectively. The soaked CBR for this case, as indicated by Fig. 42, would be about 18%. In actual construction, however, a range of values would occur.

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[75] "Investigation of the Design and Control of Asphalt Paving Mixtures," *Technical Memorandum No. 3-254*, Waterways Experiment Station, Vicksburg, Miss., May, 1948.

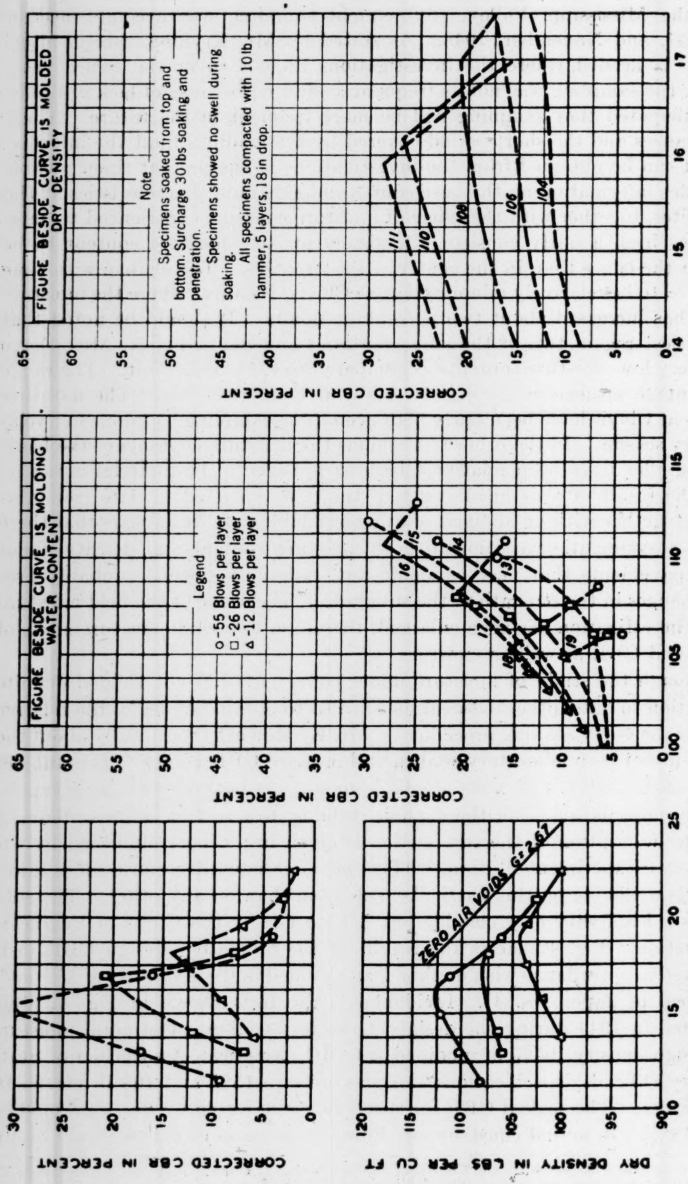

FIG. 42.—DESIGN CALIFORNIA BEARING RATIO TESTS; SUBGRADE

In the particular case of this subgrade, tests made in the fall of 1944 immediately after construction at fifteen locations (three or more tests being made at each location) showed an average moisture content of 16.1% and a density of 106 lb per cu ft. Ranges of about 3% in moisture content and of about 5 lb per cu ft in density occurred. The soaked CBR values for all the combinations of moisture and density conditions at the fifteen locations would range between 10% and 20%; and, as may be determined from Fig. 41, the soaked CBR value for the average location (16.1% moisture and 106 lb per cu ft density) would be 15%. The value to use in design would be a matter of judgment. If the designer were familiar with the soil and certain that construction would be adequate, the average probably would be the best design value. Under less favorable circumstances, the designer might lean toward the lower values. The actual design of the asphalt test section was based on a subgrade CBR of 20%, because it was considered that traffic testing would be completed before the subgrade accumulated any appreciable amount of moisture.

To continue with the history of the subgrade conditions: The in-place CBR of the subgrade at the fifteen locations tested in the fall of 1944 was 28%, which is well above the range for the soaked values. Tests made at fifty-three locations in the summers of 1945 and 1946, during and immediately after traffic testing, showed an average CBR of 32%. Moisture content averaged 16.6%, and the density showed some increase from the effect of traffic with values ranging up to 112 lb per cu ft. Between 1946 and 1949 the test section has been subjected to incidental traffic of cars, trucks, and rubber-tired construction equipment. Maintenance of drainage has been fair; but the surfacing over the earth island between the two tracks of the test section has deteriorated, and it is known that water has entered the test section at several locations. In the summer of 1949, in-place CBR tests were made at ten locations in preparation for additional traffic tests, with results shown in Table 15. Each CBR value is the average of three tests. It can be noted that in five cases the in-place CBR values are within the range predicted by the laboratory tests on samples subjected to the 4-day soaking test. In the other five cases, the in-place CBR is above the range predicted by the tests on soaked samples. The variation in moisture content generally is consistent with the range in CBR; and, where the moisture content has increased, the CBR has decreased. The percentage saturation in these cases also is high.

TABLE 15.—RESULTS OF TESTS MADE IN 1949

Section	In-place CBR[a]	Moisture	Density	Saturation (%)
1C-3-3	11	17.3	112.4	96
1A-1-2	15	18.2	....	....
1A-3-1	16	17.2	....	....
1C-1-1	20	17.2	111.5	93
2C-1-2	20	18.5	110.7	98
1B-2-1	23	16.8	....	....
4C-1-1	24	16.2	112.5	90
1A-3-3	25	16.5	....	....
1A-1-1	32	16.4	....	....
4A-1-3	32	16.0	107.4	77

[a] California bearing ratio.

Two features in connection with the subgrade in the asphalt stability test section appear worthy of note. The first is that the CBR value as constructed was well above the design based on the soaked test, and the CBR remained high for a period of about 2 years. Since that time, it appears that the moisture content has increased at some locations, and the CBR values now are approaching those based on the soaked CBR test. At other locations the moisture has not changed appreciably, and the CBR value still is high. Intermediate conditions also were found. The other feature that should be noted is that in all cases a range of values occurred and in no instance was it possible to state precisely that the CBR was a certain specific value. It should be recognized that the design procedure of compacting and soaking samples yields a CBR value that represents a considered estimate of the conditions that may occur rather than a precise value. It is true that design method 1 (as presented in paper No. 3) results in a single CBR value. Method 2 will result in curves similar to Fig. 42 from which CBR values can be determined from the range of estimated conditions.

Specific instances also can be cited to show where the in-place CBR value has fallen well below that predicted by the tests on soaked samples, but these cases have been associated generally with pavements that are considerably underdesigned and have been subjected to large deflections. In general, the over-all picture indicates that in-place values obtained to date have been slightly to moderately higher than those based on laboratory-soaked samples. The pattern of the points shown in Fig. 29 for airfield pavements illustrates this trend. All the pavements represented in Fig. 29 were designed for specific wheel loads and CBR values (based on soaked samples). If the in-place value had equaled the design value, each point would fall on its respective design curve. There is a definite tendency for the satisfactory points to plot to the right of their respective design curves, which indicates that the in-place CBR value was higher than the design value at the time of sampling. The time between construction and sampling generally was in the range of from 1 year to 5 years, and complete adjustment of moisture content may not have occurred; therefore, the weight that should be attached to an analysis of this type is a matter of judgment. The fact that in-place values may be higher than the laboratory-soaked values in certain instances has been recognized by the Corps of Engineers. Reductions in thickness are permitted in the drier regions where drainage conditions are good. Evaluations of existing airfields are based on in-place values where the pavements have been in place for fairly long periods, and there is adequate reason to believe that the in-place value is about as low as it ever will be.

Summarizing, it is desired to stress that any conservatism which exists in the CBR design procedure is not a result of the design curves that are based on in-place values, as the thicknesses indicated by the curves are not excessive for the volume of traffic anticipated at military airfields. For substantially smaller volumes of traffic, lesser thicknesses may be satisfactory. It is realized, as frankly admitted by Mr. Turnbull in paper No. 11, that the laboratory procedure of compacting samples and subjecting them to a 4-day soaking test may not duplicate the final prototype condition and may be conservative. As noted, this is a matter of concern and is being studied. Present knowledge, however, does not permit satisfactory predictions for cases where the in-place value will be higher than the soaked value except in the drier regions where drainage is good. These features are not peculiar to the CBR test. It is known definitely that the strength of subgrades changes after construction and any satisfactory design must be based on the weakest condition that will occur, even if the pavement may be overdesigned for a part of its life.

From the statements made in several of the discussions, there appears to be some difference of opinion as to just what the CBR test measures. Positive proof of the specific factors measured in the CBR test cannot be given; however, an indication can be obtained by inference. In paper No. 4 it was pointed out that the trends obtained with the CBR test also were found with the triaxial test if a low percentage of strain is used in the triaxial test. These results were obtained on both sands and clayey soils. By the very nature of the test, the CBR is a measure of the resistance of the penetration of the piston. In other words, deformation must occur to accommodate the piston. This deformation can be either consolidation or plastic movement. It has been found that, when consolidation occurs, an upward swing usually occurs in the load-deformation curve. In practice, this effect of the consolidation is removed by correcting the curve to a straight line. The test is not used by the Corps of Engineers to design against future settlement (consolidation), as this feature is provided for by specifying compaction during construction to a degree which will prevent future settlement. Therefore, it is considered that the resistance obtained in the CBR test is an index of strength due to resistance to shear deformation or plastic flow.

Commander Tyrrell and Mr. Palmer question the use of values of 1,000 lb per sq in. and 1,500 lb per sq in. for "standard crushed stone." The use of these values is merely a device to bring the readings into the range of from 0 to 100, 0 representing material with no strength and 100 representing a highly satisfactory material. The device is not new. The Public Roads Administration classification system of rating soils from A1 to A7 in general order of decreasing satisfactoriness is well known. A similar scheme has been used by F. N. Hveem, Assoc. M. ASCE, in connection with "stabilometer" readings. It should be noted, however, that the CBR procedure could be used without this device with the arbitary selection or either 0.1-in. penetration or 0.2-in. penetration. Such a change would produce very little difference in the procedures, as the CBR values at 0.1 in. and 0.2 in. are generally in good agreement. For this reason it is considered that the referencing of the CBR values to a "standard" has very little effect on the procedure.

In the studies of the CBR method conducted at the Flexible Pavement Laboratory, it has been noted in many instances that the plastic soils seem to be in a different category from the other soils. When the design curves are plotted as in Fig. 36, the curves are a family of parallel lines up to a CBR value of about 15. The curves for values higher than about 15 are not parallel lines, thus indicating a difference in characteristics. The important point is that, even if the nonplastic soils show different characteristics, the CBR procedure with the penetration test and the design curves automatically accepts and compensates for this difference. Recent experience with high-quality base courses subjected to traffic with high-pressure tires has strengthened the belief that the CBR can be correlated with experience even above the presently used range of CBR values. In summation on this point, it is considered that, even if plastic and nonplastic soils appear to be different, the index measured by the CBR test is the particular strength effective in resisting shear deformation under repetitive application of airplane wheel loads.

With respect to some of the specific questions, the following statements are presented.

Commander Tyrrell and Mr. Palmer criticize the CBR design procedure because variation in load-distributing characteristics of various materials has been ruled out. This decision was made as a matter of engineering judgment, because information was not available (and still is not available) to permit satisfactory assignments to different materials. It is not a fault of the CBR procedure, however, for as soon as reasonable variations can be proved and agreed on they can be included in the CBR method by assigning the thickness indicated by the design curves to a specific material and requiring adjustments upward or downward for materials of varying stress-distributing characteristics.

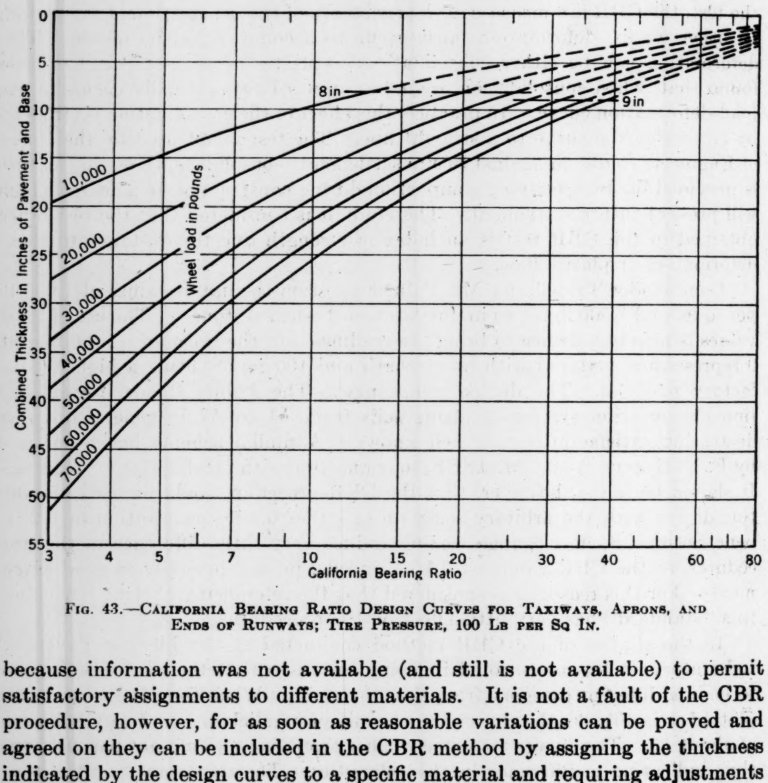

FIG. 43.—CALIFORNIA BEARING RATIO DESIGN CURVES FOR TAXIWAYS, APRONS, AND ENDS OF RUNWAYS; TIRE PRESSURE, 100 LB PER SQ IN.

Both Messrs. Feld and Valle-Rodas include statements inferring that the CBR test is conducted on statically compacted samples. The original California procedure was based on static compaction, but it is desired to emphasize that the procedures recommended by the Corps of Engineers are based on samples compacted dynamically.

Mr. Feld questions the "rise" of the pressure cells at Marietta, Ga. Apparently he has misinterpreted these data in that the values shown are not elevations, but, as noted in Col. 2, Table 7, the values are for "Depth of Pressure Cell Below Surface" before and after testing. In all cases except one the depth of cell was less after testing than before testing, which simply indicates consolidation of the material above the cell, possibly coupled with slight shear deformations causing small lateral movements of soil from beneath the traffic lanes to adjacent areas. This occurrence appears to be logical and would have been much more difficult to explain had the depth been greater after testing than before testing, as happened to a slight degree in the one exception noted. In the one case where cell depth was greater after testing than before testing, it is not known whether there was an error in measurement of the elevations or whether shearing displacements at the shallow depth of this cell caused this phenomenon; it is believed that the latter were the probable cause of this nonconformity.

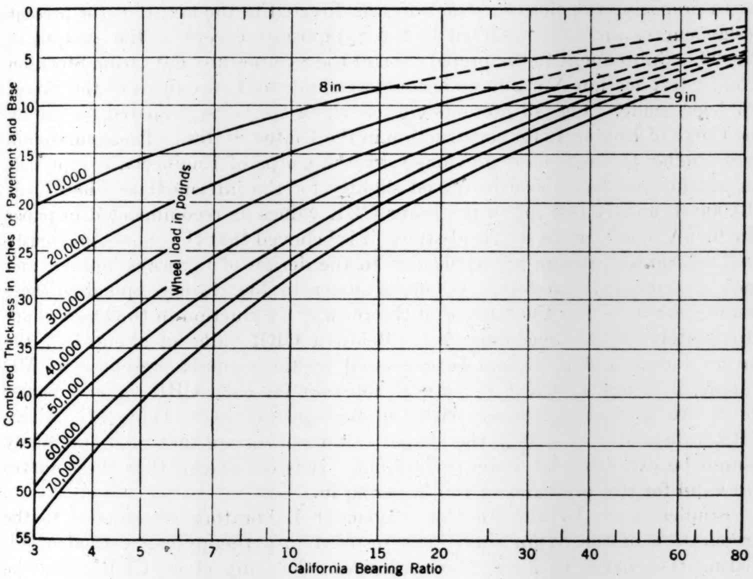

FIG. 44.—CALIFORNIA BEARING RATIO DESIGN CURVES FOR TAXIWAYS, APRONS, AND ENDS OF RUNWAYS; TIRE PRESSURE, 200 LB PER SQ IN.

In regard to Mr. Feld's question as to whether a decrease in CBR denotes shear failure, in analyzing the results shear deformation was determined by a study of the cracking of the pavement, the rate of permanent deformation, the variation of deflection with coverages, and the relative position of the surface of the subgrade before and after traffic. Upheaval of the subgrade outside the traffic lane is a definite indication of shear deformation, and this feature was present in most instances where the pavements were classed as "failed" at Marietta. A reduction in CBR for a remolded material subjected to traffic immediately after construction, as in the case at Marietta, is evidence that the stresses induced are sufficient to produce compaction, coupled with a material sufficiently saturated so that hydrostatic excess pressure is produced in the pore water. Whether shear deformation would occur depends on the degree of overload, rate of drainage, and possibly other factors.

In preparing the closure, attention was directed to the fact that the present design curves are not presented in the Symposium except in the analysis in Fig. 29. Subsequent to the preparation of the Symposium, data from Stockton (Calif.) test section No. 2 have become available and reanalysis of the curves has been made. Fig. 43 presents curves which are being prepared for use by the Corps of Engineers for construction in the United States. These curves are very similar to those used previously by the Corps of Engineers, except that the thicknesses have been increased slightly for the intermediate wheel loads (40,000 lb and 70,000 lb) in the lower CBR values as recommended in paper No. 9. Also as a result of recent study, it is believed that the curves previously used for runway design are applicable to the design of taxiways, aprons, and ends of runways; therefore, the curves shown in Fig. 43 correspond to these runway curves. For the interior of the runway a reduction in thickness of approximately 10% is recommended. Below a CBR value of about 15% the curves shown in Fig. 43 can be expressed by the formula presented by Mr. Fergus, in which $z = K \sqrt{P}$, with K constant for each CBR value. Values for K are in close agreement with the averages shown in Table 12. Above CBR values of about 20%, the shapes of the curves are such that the family cannot be expressed by a simple formula. It is considered that these curves are valid for tire pressures of 100 lb per sq in.

Studies made by the Flexible Pavement Laboratory subsequent to the preparation of the Symposium have resulted in a theoretical method of adjusting the curves to higher tire pressures. For any given CBR value the thickness for a given load on high-pressure tires has been increased so that the maximum shearing stress induced in the subgrade directly under the center of the load would not be any greater than that for the corresponding load on a 100-lb-per-sq-in. tire. Fig. 44 shows the curves obtained by this adjustment for contact pressures of 200 lb per sq in. The method is considered reasonably satisfactory for the lower CBR values (below about 15%) where substantial thicknesses are involved because the stress patterns at the subgrade level for the two cases are sufficiently similar to permit the assumption that the behavior will be practically equal if the maximum shearing stresses induced in the subgrade are equal. At shallow depths, the stress patterns are dissimilar, and this method is not considered satisfactory. The relationships shown in Fig. 44 for CBR values above about 15% are based on judgment tempered with limited data from accelerated traffic tests made with high-pressure tires. The curves in Fig. 44 should be considered as tentative pending the results of tests currently in progress.

The writers wish to express their appreciation for the interest shown by the discussers. Those of the writers who still are connected with the engineering investigational program conducted by the Corps of Engineers especially wish to express their appreciation of the criticisms and suggestions, as these will be helpful in planning and conducting future investigations.

AMERICAN SOCIETY OF CIVIL ENGINEERS
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TRANSACTIONS

Paper No. 2407

LIGHT-WEIGHT PUMICE CONCRETE

BY A. E. NIEDERHOFF,[1] M. ASCE

WITH DISCUSSION BY MESSRS. MILTON A. KARP, G. B. DRUMMOND, J. P. HAWKE, AND A. E. NIEDERHOFF.

SYNOPSIS

Pumice has been used as an aggregate in concrete to give it light weight, thermal insulating value, and certain other characteristics not usually associated with hard rock concrete. The experience of the United States Navy at Inyokern, Calif., has confirmed the fact that pumice concrete can be mixed and placed like any other concrete except that presaturation of the pumice is required and care must be exercised not to overvibrate the mixture. The strength of pumice concrete is controlled by the volume of cement used per cubic yard. Generally, one additional sack of cement is required per cubic yard to obtain the same strength as hard rock concrete.

On the west coast where pumice deposits are prevalent, the cost of light-weight concrete is no more than the cost of hard rock concrete. Because of its light weight and insulating qualities it has been found feasible and economical to ship pumice as far as 300 miles for use in building construction.

Pumice concrete has a low modulus of elasticity, which influences the design of reinforced members. Considerable saving in materials results when correctly designed structures of pumice concrete are used in earthquake-proof structures. Field tests of buildings made of pumice concrete have verified laboratory findings.

This material weighs about one half as much as conventional concrete, can be designed for strengths up to 3,500 lb per sq in., has low heat conductivity, and is nailable in about the same degree as certain woods. It will not spall under influence of direct flames and has sound absorption values which can be controlled by the type of surface finish that is used. Caution is indicated when using this concrete in structures subject to abrasion or erosion.

INTRODUCTION

The use of pumice concrete for building construction in California and Mexico is not new; but, because of its recent acceptance and widespread use by the United States Army, the United States Navy, and private structural

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engineers on important multistory buildings, it is well to review the characteristics of this material. A rare opportunity is afforded by experience on the $6,500,000 Michelson Research Laboratory built for the U. S. Navy at Inyokern. During the construction of the laboratory[2] it was found that coarse aggregate of local pumice and fine aggregate of sand would produce workable concrete having the required strength of 2,800 lb per sq in. and would weigh only 110 lb per cu ft. The thermal qualities of this material as an insulator against desert heat were superior to those of hard rock concrete and there appeared to be a reduction in noise reverberations in rooms whose walls and ceilings were composed of pumice concrete because of the numerous, minute air holes in the surface.

Pumice is an igneous rock, white or gray in color, formed in prehistoric time by the extravasation of water vapor from molten lava. This highly vesicular glass, produced by the inclusion of air bubbles in the magma during consolidation, will float on fresh water. The specific gravity ranges from 1.12 to 1.60 but flotation occurs because the many channels are sealed, preventing the water from entering. From one half to two thirds of the volume of pumice rock is pore space. The hardness ranges from 4.0 to 5.0 on the Moh scale of hardness, making pumice about as hard as fluorspar of feldspar.

Anywhere in the vicinity of volcanoes one can expect to find pumice. Many western states have large deposits but in the midwest and east there are no volcanoes and there is no pumice. Pure, large deposits that are free from clays, volcanic ash, and dust, and have the strength, particle size, shape, and proper strength-weight ratio for concrete aggregates are relatively hard to find. Such deposits occur in the Coso Mountain Range about 30 miles north of Inyokern and about 50 miles from the borderline of Death Valley National Monument in California. These deposits are ideally situated so that daily truck shipments of hundreds of yards are made to Los Angeles, Calif., and its vicinity, each load identical with the preceding load, and each load meeting the standards set up by the purchaser.

The U. S. Navy has specified pumice aggregate for concrete poured in many of the buildings constructed at the Ordnance Testing Station at Inyokern. The requirements are that: Pumice must have a minimum hardness of 4.0 on the Moh scale; it must have a specific gravity of 1.17 or less; it must be free of clay or other deleterious matter; it must not disintegrate in water; and it must have the following gradation (percentage passing, by weight):

Sieve size	Fines	Coarse
1 in.	100	95–100
¾ in.	100	50–85
⅜ in.	95–100	20–50
No. 4	90–95	5–10
No. 8	70–80	0–7
No. 16	50–60	....
No. 28	35–45	....
No. 50	25–35	....
No. 100	15–25	....
No. 200	5–15	....

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[2] "Naval Testing Lab in Desert," by A. E. Niederhoff, *Western Construction News*, September, 1946, p. 90.

Materials for concrete houses have been taken from pits at Bishop, Calif. Tests on combined coarse and fine aggregate showed a specific gravity of 1.62 and a weight, as it came from the mine, of 53.2 lb per cu ft. The dry, loose weight per cubic foot was 8 lb less. Chemical analysis showed the following constituents (in percentages):

Silica (SiO_2)	70.22
Aluminum oxide (Al_2O_3)	22.56
Iron oxide (Fe_2O_3)	0.48
Titanium oxide (TiO_2)	trace
Calcium oxide (CaO)	0.54
Magnesium oxide (MgO)	0.24
Sodium oxide (Na_2O)	2.70
Potassium oxide (K_2O)	0.11
Loss on ignition	3.10
Total	99.95

Sieve analysis on this mine run pumice showed a large percentage of fines:

Sieve size	Percentage passing
¾ in.	100
⅜ in.	88.8
No. 4	67.1
No. 8	53.3
No. 14	46.4
No. 28	40.5
No. 50	32.7
No. 100	19.0
No. 200	11.3

PROPORTIONING PUMICE CONCRETE

Using pumice aggregates or pumice coarse aggregate and clean sand, it is possible to make light-weight structural concrete having 28-day crushing strengths ranging from 1,000 lb per sq in. to 3,500 lb per sq in. by varying the proportion of cement added to the mix. As an illustration, the mixtures shown in Table 1 were used for three characteristic strengths of concrete. The weights are based on saturated and surface dry materials. Pumice coarse aggregate is usually sprayed with water for 48 hours prior to mixing or until it has absorbed 33% of its dry weight of water.

Fig. 1 shows the age-strength relationship of several pumice concrete mixes including the three shown in Table 1. Tests were made at the U. S. Naval Testing Laboratory in Inyokern. In common with hard rock concrete the ultimate strength of pumice concrete increases with age. Test results of 1-year-old cylinders are not available; but, judging from the difficulty in driving nails in old pumice concrete walls as compared with the ease with which this same operation can be performed at the age of 28 days, it is assumed that the gain in strength is equal to that of conventional concrete.

Table 2 gives recommended mixes for pumice concrete based on desired strength. It is assumed that the pumice conforms generally to the standards adopted at the U. S. Naval Ordnance Test Station at Inyokern.

Experience with "Pozzolith" as an admixture has shown that, although it increases the workability of a harsh mix, it does not have much effect on the ultimate strength. If the Pozzolith is added in a dry form, it is difficult to control, and it has been known to retard the setting of concrete to such an extent that formwork could not safely be removed even 7 days after pouring. If the Pozzolith is added to a solution of water, a much greater dispersion is obtained throughout the mix; and, if the addition of the Pozzolith is accompanied by a reduction in the total water, thereby decreasing the water-cement ratio, the strength of 28-day-old cylinders will be increased about 20%. However, the same workability and increased strength can be obtained for the same cost by increasing the cement content and presaturating the coarse aggregate.

TABLE 1.—TYPICAL MIXES OF PUMICE CONCRETE FOR CHARACTERISTIC STRENGTHS

(Quantities Are in Pounds per Cubic Yard of Concrete)

Material	Specific gravity	STRENGTHS (POUNDS PER SQUARE INCH)		
		1,700	2,800	3,200
Pumice fines	1.57	868	....	532
Sand	2.60	....	1,471	482
Pumice coarse	1.12	670	682	690
Portland cement	3.1	564	658	658
Admixture (Pozzolith)	....	....	4.2	4.2
Water (in gallons)	....	(42)	(36)	(44)

TABLE 2.—SPECIFICATION FOR PUMICE CONCRETE CONSTRUCTION

Weight (pounds per cubic foot) (1)	Estimated 28-day compressive strength (pounds per square inch) (2)	Cement factor (sacks per cubic yard[a]) (3)	Maximum water (gallons per sack of cement[b]) (4)	APPROXIMATE WEIGHTS OF SATURATED SURFACE DRY AGGREGATES (POUNDS PER CUBIC YARD)		
				Sand (5)	Pumice fine aggregate (6)	Pumice coarse aggregate[c] (7)
75	1,500	5.5	6.0	....	898	697
83	2,000	6.5	6.0	....	875	650
92	2,500	7.0	7.0	463	512	663
110	3,000	7.2	5½	1,471	....	682
100	3,500	8.0	6½	482	532	690

[a] Number of 94-lb sacks of cement per cubic yard of concrete freshly mixed. [b] Maximum gallons of water per 94-lb sack of cement. [c] Saturated to 30% moisture content.

A commercial laboratory (Smith-Emery Company) made a series of tests on pumice concrete with and without an admixture. Trial mixes were first studied to decide the proportion of water required to produce a 4½-in. slump. The same total volume of mixing water was then used for samples with and without admixtures, thus keeping the yield and the cement factor the same for each design. In the field, the same mix design is used in many comparative compression tests, without the use of an admixture. An admixture that reduces the total water to produce the same slump reduces the yield per sack of cement or increases the cement content per cubic yard, thereby showing a higher percentage increase in compressive strength with the use of admixture. The laboratory control of the design avoided this error.

In making the test specimens the natural sand and pumice aggregate were saturated fully and correction was made for free water. Thus, all mix designs are based on saturated surface dry material, which experience has shown to be the only proper procedure for using light-weight aggregate without trouble during placing. All materials were weighed and all batches were one-sack batches, mixed in a power mixer for 4 min.

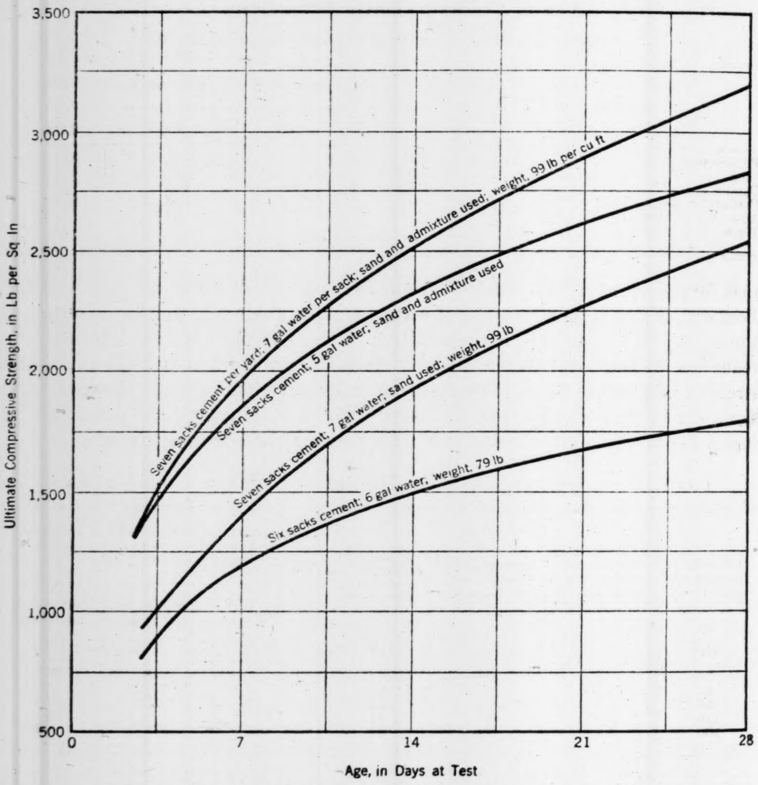

FIG. 1.—AGE-STRENGTH RELATIONSHIP FOR SOME PUMICE CONCRETE MIXES

To accelerate the initial set of pumice concrete or to prevent freezing before concrete has cured sufficiently, some contractors have successfully used a small percentage of calcium chloride in the mix. In a mix for a 28-day specified strength of 1,500 lb per sq in., contractors have used 9½ lb of calcium chloride per yard of concrete. This concrete was poured in the wintertime when desert temperatures go below freezing nearly every night. The addition of calcium chloride had no appreciable effect on the final strength or on the durability of the concrete.

Mixing and Placing

There are some peculiarities about mixing and placing pumice concrete which were learned through experience in placing thousands of cubic yards of various consistencies and strengths of the material. Best results are obtained when the pumice is soaked with water for 48 hours before use—or until it absorbs water up to 30% of its dry weight. It was also found advantageous to increase the mixing time to 4 min, particularly if the coarse aggregate had not been presaturated adequately. Mixing in a conventional, revolving drum-type mixer is not as satisfactory as in a rotating paddle-type mixer. The latter machine produces a homogeneous concrete in less time and causes less wear and breakage of the coarse aggregate than does the conventional mixer.

External vibrating of pumice concrete is necessary to force the material to flow under openings in wall forms. On flat slabs, particularly in pre-cast work, external vibration is desirable to produce consolidation, but care must be exercised not to overvibrate. In pouring concrete houses on the same project[3] both internal vibration and external vibration were required. More internal vibrators were needed for wall concrete placed below the window sills, but above these openings external vibration (and very little of it) was used exclusively. Concrete was placed and vibrated with discretion, and in a certain sequence, painfully learned through experience, to prevent aggregate from floating to the top and to eliminate honeycomb and voids in the pour.

Curing

Pumice concrete should be cured in the same manner as hard rock concrete. The coal-tar pitch cutback sealing process for curing has been widely used on the desert where water is scarce[4] or where surrounding work would be damaged by ponding or spraying. On some pre-cast work[2] initial curing was accomplished in the forms by covering with wet burlap bags. After 12 hours the bags were removed from the forms and the concrete was sprayed with water. In precasting slabs for concrete houses, a contractor in California cured the slabs in a steam tunnel for from 6 hours to 8 hours and obtained an equivalent of the usual 7-day strength of the concrete.[5] A combination of a high temperature of 170° F and a saturated atmosphere accelerated the hardening process. Even after removal from the steam tunnels the slabs continued to gain strength beyond that of unsteamed concrete.

Cost of Pumice Concrete

The cost of pumice concrete is dependent on the distance between the deposits and the point of ultimate use of the aggregate. For the favorable situation that exists at the Naval Ordnance Test Station at Inyokern, the comparison between hard rock concrete and pumice concrete of equal strength shows a small saving in cost of materials in favor of the light-weight aggregate.

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[3] "Machine-Built Desert Houses," by A. E. Niederhoff, *Western Construction News*, September, 1947, p. 89.

[4] "The Metropolitan Water District of Southern California, History and First Annual Report," Los Angeles, Calif., 1939, p. 180.

[5] "Prefabricated Pumice Concrete Houses," by H. L. Mathews, *Journal*, A.C.I., May, 1948, p. 797.

Based on 1946 job prices of aggregate delivered to one contractor's mixer at Inyokern, for light-weight and conventional concrete having 28-day strengths of 2,800 lb per sq in., costs were as shown in Table 3(*a*). Because of the extra care in presaturating pumice, the extra mixing time, and the pains with which the batch must be vibrated, the unit cost, in the forms, probably amounts to 10% more than that of hard rock concrete of the same strength. On a prefabricated concrete house job made with pumice aggregate having a specified crushing strength of 1,500 lb per sq in., the cost in dollars per cubic yard of dry materials at the batcher, in 1947, is given in Table 3(*b*).

TABLE 3.—COMPARISON OF COSTS, PUMICE CONCRETE AND HARD ROCK CONCRETE

Material	PUMICE CONCRETE			HARD ROCK CONCRETE		
	Quantity	Unit price	Sum	Quantity	Unit price	Sum
(*a*) LIGHT-WEIGHT AND CONVENTIONAL CONCRETE; INYOKERN, CALIF.; 28-DAY STRENGTH, 2,800 POUNDS PER SQUARE INCH; 1946 JOB PRICES						
Sand, in cubic feet	9.039	$0.104[a]	$0.94	8.7	$0.104[a]	$0.905
Rock, in tons	....	....	....	0.965	3.31[b]	3.195
Portland cement, in sacks	7	0.80	5.60	5.2	0.80	4.16
Pumice, in cubic yards	0.352	3.59[c]	1.30	....	....	....
Admixture, in pounds	4.2	0.081[d]	0.34	....	....	....
Total, per cubic yard	....	....	$8.18[e]	....	....	$8.26[e]
(*b*) PREFABRICATED CONCRETE HOUSE; SPECIFIED CRUSHING STRENGTH; 1,500 POUNDS PER SQUARE INCH; 1947 PRICES						
Pumice fines, in cubic yards	0.87	$3.00	$2.61	....	....	....
Pumice coarse, in cubic yards	0.574	3.00	1.72	....	....	....
Portland cement, in sacks	6	0.80	4.80	4.2	$0.80	$3.36
Sand, in cubic feet	....	....	....	14.5	0.12	1.73
Rock, in tons	....	....	....	0.97	3.75	3.64
Total, per cubic yard	....	....	$9.13[e]	....	....	$8.73[e]

[a] $\frac{\$2.75}{\text{yd}^3} + 2.5\%$ California state sales tax $= \frac{\$2.81}{27\ \text{ft}^3} = \$0.104$ per cu ft. [b] Coarse hard aggregate, classified as ¾-in. rock, cost $0.81 per ton f.o.b. (free on board) Los Angeles, Calif., plus $1.94 transportation plus $0.06 federal tax on rail hauls, plus $0.50 per ton for stock piling at the batcher. [c] Pumice aggregate cost $3.50 per cu yd, plus 2.5% sales tax delivered at Inyokern, or $3.59 per cu yd. [d] Pozzolith admixture, bought in quantities, cost $0.081 per lb. [e] Total cost of dry ingredients at the mixer.

Merely comparing unit costs for conventional and for pumice concrete does not give a true picture of the relative economy of the two materials when made into structural slabs, beams, and girders. Comparisons have been made on the sizes of slabs and beams to support a given live load. Because of the smaller dead weight of pumice concrete, and the lower modulus of elasticity, it was cheaper to build the ten-story building in Los Angeles for the Prudential Insurance Company of this material. Foundation conditions and cost dictated the use of pumice concrete on the new telephone building also in Los Angeles. The General Petroleum Building, a ten-story structure in Los Angeles, showed appreciable savings in reinforcing and structural steel used in and under pumice concrete floors. The sizes of the foundations were reduced because the dead weight of pumice concrete is only two thirds that of conventional concrete.

It has been shown elsewhere[6] that a pumice concrete beam can carry a greater live load than can a hard rock concrete beam designed for the same bending moment because of the smaller dead load. Additional advantages of the pumice concrete beam were: Increased headroom because the beam was shallower, and a saving of 21% in concrete.

In the design of pumice concrete slabs there is also a possibility of saving materials.[7] For the same bending moment and the same allowable unit stresses the pumice concrete slab requires an effective depth to steel only 80% of that which was necessary for hard rock concrete. Slightly more steel was necessary for the shallower slab, but the live-load carrying capacity was greater because the dead weight of pumice concrete was only 75% of the dead weight of conventional concrete. Considering the prevailing market price of materials in Los Angeles, multistory buildings with relatively poor foundation material can be built of pumice concrete for less than can buildings of conventional concrete.

THERMAL QUALITIES

The thermal qualities of pumice concrete have been adequately proved on the Mojave Desert in California. On this subject Capt. H. L. Mathews has written:[8]

> "Because of extremes in desert temperatures and the necessity for air cooling in all buildings during 8 months of the year, the thermal characteristics of the roof and walls of the houses are of utmost importance. Operational expense in keeping a poorly insulated building cool in summer and warm in winter could be prohibitive and would continue unabated during the life of the building. Accordingly, it was specified that the conductivity or *k* factor for concrete should not be greater than 1.95 BTU [British thermal units] per sq ft per in. thickness per hour per degree. This is one-sixth that of hard rock concrete, four-tenths the conductivity of cinder concrete, and half that of Haydite concrete. Reference is made to Fig. 2[9] which gives the insulating values of various building materials. Similarly the thermal transmission coefficient or *U* factor for the 5-in. thick pumice concrete wall was specified as not greater than 0.30. * * *"

The Pittsburgh (Pa.) Testing Laboratory made a thermal conductivity test to determine the *k*-value for a sample of 2,500-lb concrete, with the following results:

Thickness tested, average in inches	2.049
Density as tested, in pounds per cubic foot	93
Hot plate temperature, in degrees Fahrenheit	71.3
Cold plate temperature, in degrees Fahrenheit	29.5
Mean temperature, in degrees Fahrenheit	50.4
k-value	2.69

The thermal conductivity *k* is expressed in British thermal units per hour, per square foot, per 1-in. thickness, per 1° F temperature.

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[6] "Chart for Pumice-Concrete Beams," by A. E. Niederhoff, *Engineering News-Record*, September 18, 1947, p. 89.

[7] "Nomograph Chart Aids Pumice-Concrete Slab Design," by A. E. Niederhoff, *Civil Engineering*, October, 1947, p. 47.

[8] "Prefabricated Pumice Concrete Houses," by H. L. Mathews, *Journal*, A.C.I., May, 1948, p. 800.

[9] *Ibid.*, p. 801.

Light-weight concrete having much less strength, and made of one part of cement to eight parts of pumice, had a k-value of 1.61. Apparently the more pumice used and the less sand and cement, the better the k-value. A k-value of 1.92 has frequently been specified in connection with 2,000-lb concrete.

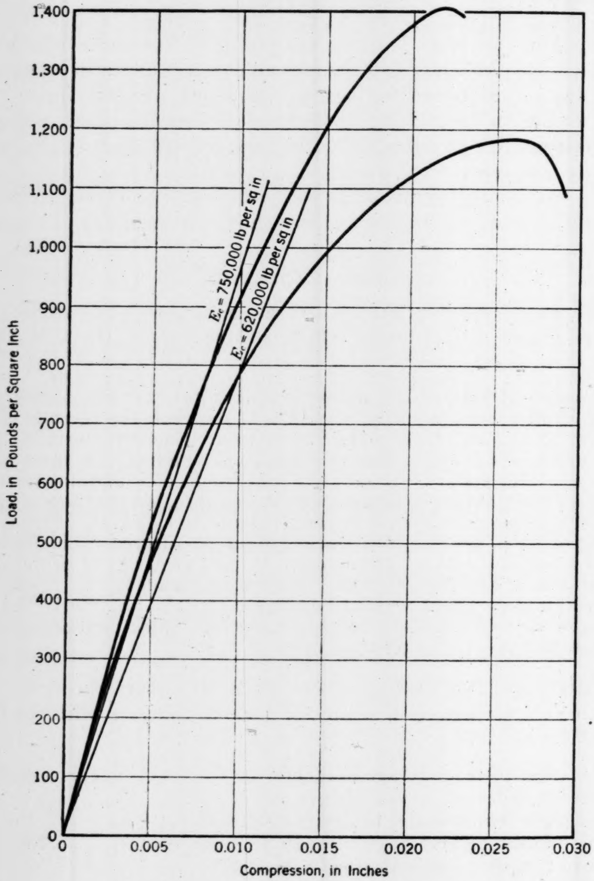

FIG. 2.—STRESS-STRAIN CURVES FOR PUMICE CONCRETE CYLINDERS, 6 IN. BY 12 IN., 28 DAYS OLD

MODULUS OF ELASTICITY

The stress-strain diagrams obtained from measurements taken on thirty-two cylinders, 6 in. by 12 in., of pumice concrete were curves throughout their length (see Fig. 2), indicating greater increase in rate of strain for increased stress. Tests were made at the U. S. Naval Laboratory at Inyokern during the first quarter of 1948. Loadings were applied relatively fast to avoid plastic flow and to eliminate permanent set; but, since pumice concrete is no more elastic than conventional concrete, there was considerable variation in the observed modulus.

TABLE 4.—MODULUS OF ELASTICITY (SECANT), IN POUNDS PER SQUARE INCH, TO BE USED IN DESIGN

Type of concrete	RANGE OF ULTIMATE COMPRESSIVE STRENGTHS (POUNDS PER SQUARE INCH)				
	1,000 to 1,499	1,500 to 1,999	2,000 to 2,499	2,500 to 2,999	3,000 to 3,499
Conventional	....	....	2,000,000	2,500,000	3,000,000
Pumice	750,000	830,000	1,000,000	1,250,000	1,500,000

The secant modulus of elasticity was used and the slope of a straight line was obtained drawn from the origin through a point on the curve at a compressive stress of 800 lb per sq in. It was generally observed that the stronger the pumice concrete the higher would be the modulus. Also, it was found that the modulus of elasticity increases with age. For purposes of comparison, the recommendations of the 1940 Joint Committee on Reinforced Concrete Design[10] are included together with the findings of the Smith-Emery and U. S. Navy laboratories on the moduli of elasticity of conventional and pumice concrete. The values in Table 4 are based on a pumice aggregate having a hardness of from 4.0 to 5.0 on the Moh scale and a specific gravity ranging from 1.12 to 1.57. For harder or more dense pumice aggregate used in concrete it is expected that the modulus of elasticity would increase.

Ordinary tables and design charts for the design of reinforced concrete are not usable for design in pumice concrete. If the modulus of elasticity of reinforcing steel is 30,000,000 lb per sq in., then the ratio of $\frac{E_s}{E_c}$ (called n in reinforced concrete design parlance) ranges from 40 for 1,000-lb pumice concrete to 20 for 3,000-lb pumice concrete. Table 4 indicates n-values twice those of conventional concrete of equivalent ultimate strength.

OTHER TEST RESULTS

The Smith-Emery Company made many tests on pumice concrete, some of which are of interest to the builder and others which are primarily of interest to pumice concrete designers. The test summary given in Table 5 is based on several specimens made with sand, pumice fines, and pumice coarse aggregate. Test 2 had concrete of the same mixture and consistency as test 1 except that Pozzolith was added and other ingredients were varied to obtain a 4.25-in. slump.

Enlightening field tests on pumice concrete structures have been made from time to time. It is well known that a fire in a conventional concrete building will cause aggregate to pop, and then the concrete will spall and crack. Extensive spalling may expose reinforcing steel, which, when damaged by fire, makes the entire structure unfit for occupancy. Pumice concrete, on the other hand, will not spall. It can be heated to fusion and will not crack or explode. Early in 1940 a pumice concrete building was soaked with gasoline and waste, ignited, and allowed to burn. When the fire had burned out, the building was inspected and the only evidence of the fire was smoke stain on the walls and ceiling. The experiment was repeated except that after the fire was well started the second time it was put out by water from fire hoses. Inspection of the building again showed no structural damage.

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[10] "Recommended Practice and Standard Specifications for Concrete and Reinforced Concrete," *Proceedings*, ASCE, June, 1940, Pt. 2.

TABLE 5.—SUMMARY OF DATA

Test No.	Sacks of cement per cubic yard (1)	Water-cement ratio; $\frac{w}{c}$ (gallons per sack) (2)	Admixture (3)	Slump (inches) (4)	Workability (5)	Segregation (6)	WEIGHT (POUNDS PER CUBIC FOOT)		Modulus of elasticity (pounds per square inch) (9)
							As tested (7)	Dry (8)	
1	7.0	7.0	None	4.25	Good	None	99.5	91.75	1,296,440
2	7.0	6.25	Pozzolith	4.25	Excellent	None	99.0	99.4	1,529,799

In rainy climates where buildings have pumice concrete exterior walls and roofs, they can be waterproofed by ordinary methods with any one of a dozen or more commercial products on the market. Smooth exterior walls have been pleasingly waterproofed by three coats of colored, unthinned paint consisting of vinyl resins, pigments, and solvents. Pumice concrete roof decks coated with fiberized asphalt emulsions are inexpensive and effectively waterproofed.

The capability of pumice concrete members to accept internal strain without structural damage is demonstrated every time carpenters drive nails into pumice concrete. It is common practice to scab 2-in. by 4-in. pieces of lumber onto pumice concrete walls by nailing with 16-penny nails. Many concrete buildings have the baseboard and moldings nailed on with finishing nails. At least one motel, recently constructed with pre-cast pumice concrete roof slabs 3 in. thick, had the roof nailed through to the supporting rafters below. By actual test with pumice concrete slabs 1.5 in. thick it was found that 16-penny nails could be driven through them into wood supports below without cracking the unreinforced concrete. Nails had to be kept at least 6 in. from the edge of the thin slab.

The holding power of nails in pumice concrete is almost as good as that of nails driven in wood. In fastening pipes or air ducts by hangers to a ceiling or wall there is a large saving in time by merely nailing the hangers in place.

CONCLUSIONS

Laboratory and field tests and job experience involving thousands of cubic yards of reinforced pumice concrete warrant the following conclusions with respect to this material:

1. Pumice concrete varies in weight from 75 lb to 110 lb per cu ft, dependent, to some extent, on the specified strength. It is about one half to two thirds the weight of sand and gravel concrete. The light weight reduces the dead weight of structures and there is a marked saving in reinforcing steel Extra stories can be added to existing buildings where footings are designed for specified dead loads. During construction, the formwork for light-weight concrete is not required to be as strong as that used for conventional concrete, resulting in a lower initial cost for the building. Where earthquake considerations are vital, the low dead weight of the structure and, consequently, the reduced horizontal forces to be taken care of in design, are exceedingly advantageous. In placing pre-cast units by hand, the light-weight pumice concrete saves time, energy, equipment, and money.

on Tests of Pumice Concrete

Pounds per Square Inch						Shrinkage (total inches per inch)	Absorption (percentages)	Bleeding (pounds of water per cubic yard of concrete)	Test No.
Modulus of rupture	Ultimate Compressive Strength (Average, Three Cylinders)		Bond Strengths (Pull Out)						
			Vertical	Horizontal					
	7 days	28 days		Bottom	Top				
(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	
418.9	1,345	2,565	403.9	418.7	297.0	0.00061	11.1	2.87	1
425.2	1,953	3,278	461.6	537.8	350.9	0.00060	7.10	1.21	2

2. Strength of pumice concrete can be varied to suit design conditions within practical limits of from 1,000 lb per sq in. to 3,500 lb per sq in. Many inexpensive mixes have been designed to specified strengths and many data are available to aid the engineer in a predetermination of pumice concrete mix and cost.

3. Pumice concrete has a low heat conductivity that puts it in the same category with some of the best insulating materials. This property is inherent and, when combined with light weight and good structural strength, makes wall construction with pumice concrete less expensive than that with other materials that must have applied insulation. Because of the poor heat conductivity of pumice concrete there is little condensation, allowing large savings in building construction because furred walls are not necessary. Air conditioning of pumice concrete buildings is facilitated as proved in houses constructed on the Mojave Desert. For walk-in refrigerators, pumice concrete floors, walls, and roofs have been specified without further applied insulation, thus lessening the initial cost of construction

4. Nails can be driven into pumice concrete so that they have a holding power almost equal to that of wood. This fact combined with experience in sawing thin slabs of lean pumice concrete presents definite advantages over ordinary concrete construction. The low modulus of elasticity allows this internal strain without catastrophic structural damage. In concrete masonry and pre-cast units the low modulus of elasticity decreases breakage during production, transportation, stock piling, and final placement. Expansion and

FIG. 3.—IN A FIRE TEST, THE TRANSMISSION OF HEAT WAS RETARDED: NO SPALLING OF PUMICE CONCRETE

FIG. 4.—THERE WAS NO DAMAGE IN THIS PUMICE PREFABRICATED HOUSE, AFTER FIRE WAS QUENCHED BY WATER

contraction, vibration, concussion, and earthquake strains are ameliorated by this relatively inelastic material.

5. Tests have shown that pumice concrete can be heated to fusion (about 2,400° F) without spalling or cracking (see Fig. 3). Sudden quenching of hot concrete in cold water produces no cracks and the structural integrity of pumice concrete buildings remains practically unchanged even after a 4-hour fire (see Fig. 4). Welding steel reinforcement together right on top of a pumice concrete slab has not spalled the concrete. In fact, pumice concrete has been used successfully as a backup for boiler and firebox installations.

6. The quality of the mix and surface texture of pumice concrete determines the absorption of sound waves. In this respect it differs from conventional concrete building construction where other methods are necessary to reduce noise reverberations. A lean pumice concrete with a "popcorn" finish will absorb up to 50% of sound. This advantage is useful not only in small homes, but in auditoriums, theaters, and factories, where sound is controlled only by expensive processes for conventional concrete structures. Values of sound transmission are decreased from eight decibels to twelve decibels.

7. All the equipment used for mixing, hauling, and placing light-weight pumice concrete is less heavily loaded than when the same operations are performed with sand and gravel concrete. Monetary savings are immediately reflected in more efficient labor, less time involved in doing a job, and less wear and tear.

8. Pumice concrete offers a wide selection of textures or finishes ranging from a rough "popcorn" to a smooth "sack" finish. The labor involved in obtaining any particular finish costs no more than does that required for conventional concrete finishing.

9. Except for presaturation of coarse aggregate, longer mixing time, and discretion in the use of a vibrator when placing, pumice concrete is handled the same as conventional concrete.

10. Wearing surfaces should not be made of pumice concrete. Light-weight concrete floor slabs in multistory office buildings are usually covered with asphalt tile or other wearing surfaces. In this case full advantage can be taken of the light-weight material.

ACKNOWLEDGMENTS

Grateful acknowledgment is given to the Bureau of Yards and Docks, United States Navy, and in particular to Captain Mathews, officer in charge of construction at the U. S. Naval Ordnance Test Station at Inyokern, for the privilege of collecting data and preparing this paper. C. S. McKenzie, design superintendent at Inyokern and the writer's immediate superior, lent encouragement and assistance on many of the phases of design and construction.

Paul R. Splane contributed photographs, test data, and many job records available to him as a former pumice producer in Los Angeles.

Many others have assisted indirectly or directly and to these unnamed persons the writer offers grateful thanks.

DISCUSSION

Milton A. Karp,[11] **Jun. ASCE.**—Tests of pumice concrete by the Bureau of Reclamation,[12] United States Department of the Interior, published in April, 1949, indicate that drying shrinkage is of such magnitude as to warrant careful consideration. High shrinkage values are reported, in one case almost four times the amount cited by Mr. Niederhoff. This sharp disagreement in test results is perhaps explained by dissimilar curing conditions, but more probably it derives from a difference in the age of the specimens when the shrinkage reading was obtained.

This points out a common failing of test reports in that a listing of the conditions under which the results were obtained is often missing from the report itself. In this instance, the curing conditions and the age of the specimen, both of which have important bearing on the shrinkage, will "remain in the dark" until Mr. Niederhoff (it is hoped) casts light in that direction in his closing discussion.

Of more importance than "tying down" a shrinkage coefficient is the actual experience with drying shrinkage in pumice concrete on the completed job. There has been considerable discussion, pro and con, concerning the shrinkage behavior of pumice concrete in actual structures. Therefore, it would be extremely beneficial if Mr. Niederhoff, who no doubt has had invaluable opportunities for first-hand observation, were to devote part of his closing discussion to this aspect.

Engineers who propose to use pumice concrete in future buildings should be made aware of the limitations, if of any consequence, of this material and the special precautions required in its use.

G. B. Drummond,[13] **Assoc. M. ASCE.**—It is fortunate that research and experience as described by Mr. Niederhoff have focused attention on pumice as a light-weight building aggregate. The days are passing when the small operator with a broken-down truck could haul a load of what he called pumice, untested and ungraded, and "palm it off" as a satisfactory material. In the present stage, however—the serious investigation of the properties of this material—there is the possibility that research is conducted to prove some preconceived ideas rather than to determine, disinterestedly, the basic data required for an honest design.

In New Mexico, where it is estimated that two fifths of the nation's source of pumice is found, there is pumice and also there is "pumice." Generally, pumice is found in extinct volcanic areas under an overburden of as much as 150 ft; but within these areas are encountered deposits of various chemical analysis. For example, the compositions in Table 6 come from three deposits in the same vicinity. A physical characteristic of pumice is that it requires special handling when being mixed. Its cellular structure, which causes it to float in fresh water (although its specific gravity is greater than unity), produces a tendency for the aggregate to separate from the mix, unless the water content of the aggregate prior to designing the mix has been carefully determined. Presaturation will fill the pores of the pumice aggregate sufficiently to cause all particles to approach the same specific gravity. Also, better curing of the concrete will result when a portion of the water of hydration comes from within. Contractors in Albuquerque, N. Mex., have also learned that more uniform mixes have been obtained when the water and the cement are introduced into the mixer first, and the aggregate is added later. Some difficulty has been encountered from those experienced only in working conventional concrete, because of the tendency of pumice concrete to "bleed." Troweling generally should be delayed until a slight set has been achieved.

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[11] Chf. Engr., Ellis Wing Taylor Archts. and Engrs., Los Angeles, Calif.

[12] "Tests of Lightweight-Aggregate Concrete Designed for Monolithic Construction," by Walter H. Price and William A. Cordon, *Journal*, A.C.I., April, 1949, pp. 581–600.

[13] Civ. Engr., Albuquerque, N. Mex.

TABLE 6.—COMPARISON OF PUMICE CHEMICAL CHARACTERISTICS

Element	Valencia	Cochiti	Jemez
Silica	77.90%	70.28%	71.67%
Alumina	11.28	13.54	12.64
Ferric oxide	0.86	2.06	2.30
Lime	0.80	1.96	0.60
Magnesium	0.36	1.18	0.70
Soda	3.64	4.14	8.00
Potash	4.38	3.33	Trace
Titanium	0.06	0.26	Trace
Loss on ignition	5.20	3.73	4.46
	104.28	100.48	100.37

TABLE 7.—DESIGN DATA FOR TENTATIVE MIXES OF PUMICE CONCRETE

Description	COMPRESSION STRENGTH AT 28 DAYS, IN POUNDS PER SQUARE INCH					
	3,000	2,500	2,000	1,500	1,000	500
Cement factor, in sacks per cubic yard	11.9	9.0	7.5	6.3	5.0	3.7
Ratio, w/c, for a 4-in. slump, in gallons per sack	4.6	5.9	7.5	9.5	11.4	17.0

Experience has shown that an all-pumice mix is limited to about 3,200 lb per sq in. because of the structural limitations of the aggregate. Consequently, a design basis of 2,000 lb per sq in. or 2,500 lb per sq in. is indicated. This may be obtained with approximately seven sacks of cement, with due regard to the water-cement ratio. Table 7 has been utilized in Albuquerque for the design of tentative mixes.

As noted by Mr. Niederhoff, ordinary tables for the design of reinforced concrete are not usable for the design of pumice concrete because the values of n are nearly twice those for ordinary aggregates. Table 8 for pumice concrete is based on investigations involving aggregates from the vicinity of Cochiti, N. Mex., but the data are applicable for other pumice materials of similar chemical content and proper gradation.

Acknowledgment.—In Table 7 the data were supplied by the Albuquerque Gravel Products Company and in Table 8, by the Pumice Aggregate Sales Corporation, both of Albuquerque.

TABLE 8.—Design Data for Pumice Concrete

Col. 1. f = specified compressive stress, in pounds per square inch;

Col. 2. γ_w = unit weight of wet pumice concrete mix, in pounds per cubic foot;

Col. 3. γ_d = unit weight of dry pumice concrete mix, in pounds per cubic foot;

Col. 4. f_c = working stress in compression, in pounds per square inch;

Col. 5. $\begin{cases} n = \text{modular ratio} = E_s/E_c; \\ E = \text{modulus of elasticity, in pounds per square inch;} \end{cases}$

Cols. 6. $K = \frac{f_c}{2} k\, j;$

Cols. 7. $k = \frac{1}{1 + f_s/n\, f_c};$

Cols. 8. $j = 1 - \frac{k}{3};$

Cols. 9. $p = \frac{f_c}{2\, f_s} k;$

Cols. 6 to 9. $\begin{cases} f_s = \text{tensile unit stress in longitudinal reinforcement, in pounds per square inch; and} \\ a = \frac{f_s}{12{,}000} \times (\text{average value of } j) \text{ for use in } A_s = \frac{M}{a\, d}. \end{cases}$

f	γ_w	γ_d	f_c	n	(a) f_s = 16,000; a = 1.333 j				(b) f_s = 18,000; a = 1.500 j			
					K	k	j	p	K	k	j	p
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(6)	(7)	(8)	(9)
250	73.0	53.0	113	81.1	18	0.364	0.879	0.0013	....	....	....	....
500	74.5	54.5	225	54.1	42	0.432	0.856	0.0030	....	....	....	....
1,000	77.5	57.5	450	38.7	97	0.521	0.826	0.0073	93	0.492	0.836	0.0062
1,500	80.5	60.5	675	30.9	155	0.566	0.811	0.0119	149	0.537	0.821	0.0101
2,000	83.6	63.6	900	25.8	214	0.592	0.803	0.0167	206	0.563	0.812	0.0141
2,500	87.0	67.0	1,125	22.1	273	0.608	0.797	0.0214	263	0.580	0.807	0.0181
3,000	92.0	72.0	1,350	19.4	332	0.621	0.793	0.0262	321	0.593	0.802	0.0222

TABLE 8.—(*Continued*)

f	(c) f_s = 20,000; a = 1.667 j				(d) f_s = 22,000; a = 1.833 j				(e) f_s = 24,000; a = 2.000 j			
	K	k	j	p	K	k	j	p	K	k	j	p
(1)	(6)	(7)	(8)	(9)	(6)	(7)	(8)	(9)	(6)	(7)	(8)	(9)
1,000	88	0.465	0.845	0.0052	85	0.442	0.853	0.0045	81	0.421	0.860	0.0039
1,500	143	0.511	0.830	0.0086	138	0.487	0.838	0.0075	133	0.465	0.845	0.0065
2,000	198	0.537	0.821	0.0121	192	0.514	0.829	0.0105	185	0.492	0.836	0.0092
2,500	254	0.554	0.815	0.0156	246	0.531	0.823	0.0136	238	0.509	0.830	0.0119
3,000	310	0.567	0.811	0.0191	300	0.543	0.819	0.167	291	0.522	0.826	0.0147

J. P. **Hawke**,[14] Assoc. M. ASCE.—As with dense rock concrete, the best advantages of pumice concrete become available only with proper laboratory control of concrete production and placement. The laboratory control of pumice concrete has been materially aided by the formation of the Pumice Producers Association which was established to stabilize the quality and uniformity of the pumice aggregates produced by its members. In making preliminary laboratory design mixes particular consideration should be given to the practical factors of linear shrinkage and to the cement content required for proper workability.

Difficulties in placement due to the harshness of the mix when low cement contents are used have been cited by the author. The very successful use of air entrainment in improving the workability of lean and intermediate mixes of light-weight burned shale or expanded slag concrete has been reported.[15]

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[14] Berkeley, Calif.

[15] "Properties of Some Lightweight-Aggregate Concrete With and Without Air-Entraining Admixture," *Building Materials and Structures Report BMS 112*, National Bureau of Standards, U. S. Dept of Commerce, Washington, D. C.

This report would indicate that the workability of pumice concrete may be improved by air entrainment so that rich mixes are not required for maximum workability. The leaner mixes of pumice concrete would improve the economics of its use and also tend to reduce the linear shrinkage to be provided for in design.

The author has referred to several large buildings in the Los Angeles (Calif.) area in which light-weight pumice concrete was used. Light-weight concrete was also used in San Francisco, Calif., on the addition to the Matson Building, the Pacific Gas and Electric Company Annex Building, and the Standard Oil Company Annex Building. The use of light-weight concrete in these buildings has made possible very significant reductions in the structural dead loads. Since the lateral force requirements for earthquake resistance, in the local building code, are based on a stated fraction of the total column loads for each story, this reduction in dead load reduced the lateral forces to be provided for in the design. The reduced dead load was also important in connection with the foundation design. The engineers in charge of the foregoing projects selected a commercial type of expanded shale rather than pumice aggregates. This selection was made after extensive tests and studies and may have been influenced, to some extent, by the greater local experience in the use of haydite aggregate and by the readily available local source of supply.

The 1949 adjustments in the freight rate on pumice aggregates to San Francisco, together with improved technology in the use of pumice aggregates, may be expected to result in an increased interest in pumice concrete by engineers in this area.

A. E. NIEDERHOFF,[16] M. ASCE.—The profession shows a surprising lack of interest in light-weight pumice concrete. Although it was realized that the use of pumice is confined to the western states where deposits occur, it was thought that engineers in Los Angeles, Calif., would contribute more fully to the discussion. In Los Angeles and vicinity this material has probably been used more often than elsewhere, and the personnel concerned with design and construction have experienced the difficulties of shrinkage. The problem of shrinkage can be solved, however, as evidenced by the expanding market for pumice and other volcanic rocks for concrete aggregates. A 1950 news item[17] calls attention to the exploitation of 11,000,000 tons of volcanic ash deposited in the desert between Mojave and Lone Pine, Calif. The volcanic ash, like pumice, will be used for concrete aggregates and for manufacturing building blocks.

The account of the laboratory tests that Mr. Karp cites[12] contains the following statement:

> "It has already been mentioned that the aggregates used in these tests were batched in a room-dry state. This dry batching is contrary to the procedure of saturating the aggregate before batching, as recommended by some light-weight aggregate associations."

---

[16] Senior Engr., International Eng. Co., San Francisco, Calif.

[17] "Coming! The Ash Age," *Engineering News-Record*, February 16, 1950, p. 15.

It was noticed by the writer, when he was employed by the United States Navy at Inyokern, Calif., that without presaturating the pumice to 35% of its dry weight, unpredictable and lower compressive strengths would result. Good evidence supports the belief that presaturation of the pumice also reduces the amount of shrinkage of the concrete, obviates placing difficulties, and enhances the future permanence of the finished concrete.

The results of the aforementioned laboratory tests and the conclusions drawn therefrom closely check the writer's field experience where materials and conditions were comparable. The account states that the flexural strength of light-weight concrete is higher than that of regular concrete of the same compressive strength. Also, it was stated that none of the concretes made with light-weight aggregates showed distress in the test specimens resulting from drying shrinkage. The wide difference in the amount of shrinkage recorded in the tests conducted at two different laboratories may be due to dissimilar curing, to tests made at different ages of the specimens, or, more likely, to the effect of presaturating the aggregates in the one case and mixing them in "room-dry" condition in the other case.

The following data on shrinkage are quoted from a report of a commercial laboratory (Smith-Emery Company) for the enlightenment of interested engineers.

> "Molds: The steel gang molds which we used for the shrinkage specimens provided for a 2x2-inch test specimen with a 10-inch plus or minus 0.1 inch effective gage length between the innermost points of the $\frac{1}{4}$-inch reference pins. These reference pins were set so that their principal axis coincided with the principal axis of the test specimen, and extended into the specimen $\frac{5}{8}$ inch.
>
> "Length Comparator: The changes in the length of the test specimens were measured by a length comparator, as called for in the 'Method of Test for Autoclave Expansion of Portland Cement' (ASTM C 151–43). This instrument has been checked by the Bureau of Standards' Cement Reference Laboratory. In taking our readings, the instrument was checked frequently with the reference bar.
>
> "Three specimens were made (1 gang mold) from the same batch of which the 6x12-inch compression cylinders were made. The original length readings of the specimens were taken twenty-four hours after molding. The specimens were air dried at a temperature of 70 degrees F. plus or minus two degrees. Readings were taken every twenty-four hours until the specimens were fourteen days old, after which the specimens were oven dried to constant weight, air cooled to 70 degrees F. and the final reading taken [Table 9].
>
> "The pumice lightweight aggregates should be saturated before mixing because of the control and other detrimental effects produced by the high absorption of lightweight aggregates.
>
> "Mixing time in general is longer than with natural aggregates, and should be determined by the equipment to be used on the job."

It is heartily agreed that the important element when dealing with light-weight concrete is not the determination of "coefficients" from laboratory samples, but the field behavior of the material in structures. Here the writer is confronted with two sets of seemingly opposed facts. On monolithic construction of pumice concrete houses built in the desert, numerous cracks appeared. The walls and roofs of these houses were poured as a unit, without joints. In contrast, where pumice concrete houses were built of pre-cast slabs, steam cured at a central plant, and hauled to the site for erection, practically no cracks appeared. In the first case, no provision was made for expansion and contraction in the box-like residence. Curing consisted of superficial painting of the surfaces with a coal-tar pitch cutback sealing material. The slab houses had expansion joints and the slabs were inspected for cracks at the central plant before shipment to the site. Curing was done in a steam chamber for 6 hours. Thus, the opposed facts of cracking versus noncracking of pumice concrete can be rationalized if all conditions are known and properly evaluated. Pumice concrete, more than hard-rock concrete, requires expansion joints and adequate curing if cracks are to be prevented. The important point is that they can be prevented.

TABLE 9.—CHANGES OF LENGTH OF TEST SPECIMENS OF PUMICE CONCRETE

Age, in days	INCHES PER INCH		Age, in days	INCHES PER INCH		Age, in days	INCHES PER INCH	
	Test No. 1	Test No. 2[a]		Test No. 1	Test No. 2[a]		Test No. 1	Test No. 2[a]
(1)	(2)	(3)	(1)	(2)	(3)	(1)	(2)	(3)
1	0.00000	0.00000	6	0.00018	0.00013	11	0.00037	0.00033
2	0.00003	0.00001	7	0.00023	0.00017	12	0.00039	0.00036
3	0.00006	0.00003	8	0.00027	0.00022	13	0.00041	0.00039
4	0.00010	0.00006	9	0.00031	0.00026	14	0.00043	0.00041
5	0.00014	0.00009	10	0.00034	0.00030	Total[b]	0.00061	0.00060

[a] Average of three specimens. [b] Total, or dry.

The writer attempted to apprize engineers of the limitations of light-weight pumice concrete when he stated (under the heading, "Conclusions") that it could not stand abrasion, that it required presaturation of the aggregate, that it needed longer mixing in a paddle-type mixer, and that it demanded the skill of experienced men on the vibrators when placing. The one characteristic not emphasized by the writer has been pointed out by Mr. Karp. Pumice concrete, like some other light-weight concretes, shrinks.

Mr. Hawke mentions that the harshness of a pumice concrete mix can be ameliorated by the use of an air-entraining agent. Several federal agencies have incorporated requirements for air entrainment in their specifications for concrete. The beneficial results have received almost universal acclaim. In laboratory tests of light-weight aggregate concrete, reported in 1949[12] concrete mixes containing an air-entraining agent were made and compared with similar mixes without such an agent. It was found that the unit weight of concrete was not increased by air entrainment and that there was no reduction in strength. Workability was increased.

Mr. Hawke also points out that in the San Francisco area the local preference in light-weight aggregate is for a commercial type of expanded shale. It is the writer's firm belief that the widest publicity should be given to all research in light-weight aggregates, so that the engineer may pick the most suitable and economical material for the job at hand.

Mr. Drummond's contribution of constants for use when designing reinforced pumice concrete is a significant addition to the discussion. To realize the full benefits of pumice concrete, it is good common sense to recognize the implications of the low modulus of elasticity. Naturally, if the engineer uses these constants, he fully expects the same standards of quality of the pumice on which these data were based. Several organizations have been formed to establish these qualities physically and chemically. These standards will eliminate many small prospectors who were not too careful about the claims they made for the material they had for sale. As Mr. Drummond points out, disinterested and honest research by competent investigators will clarify some of the confusion which now exists about pumice light-weight concrete.

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CONTINUOUS ARCHES AND BENTS ANALYZED BY COLUMN ANALOGY

By T. M. Yu[1]

With Discussions by Messrs. Jack R. Benjamin, Leslie J. Meszaros, A. A. Eremin, J. J. Polivka, Powell P. W. Tsu, and T. M. Yu.

Synopsis

Basically, the method of column analogy, as introduced[2] by Hardy Cross, Hon. M. ASCE, in 1930, is applicable to beams, arches, and bents of not more than three redundant elements. This paper offers an extension of the method to continuous beams, arches, and bents, provided that there are no closed panels.

The procedure is built around four broad steps:

(*a*) Select one span (any span) at a time;
(*b*) Consider all parts adjacent to that span as its elastic foundations;
(*c*) Replace the elastic foundations (item (*b*)) by substitute members with characteristics such that the transformed span is equivalent to a span on elastic foundations; and
(*d*) Determine the elastic properties of substitute members and proceed with the application of the fundamental column analogy method.

The ideas presented, clarified by numerical examples, demonstrate that column analogy is a convenient tool for sketching influence lines. The writer concedes in advance that there may be better methods of structural analysis for specific cases but he offers the hope that, by excluding extensive discussion of such alternate methods, it may be possible to "sharpen" the column analogy "tool" for use in its particular fields of application.

Introduction

In its original form,[2,3] the method of column analogy introduced by Professor Cross is applicable only to single-span arches and bents with not more

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Note.—Published in February, 1949, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Prof. of Civ. Eng., Chiao-Tung Univ., Shanghai, China.

[2] "The Column Analogy," by Hardy Cross, *Bulletin No. 215*, Univ. of Illinois Eng. Experiment Station, Urbana, Ill., 1930.

[3] "Continuous Frames of Reinforced Concrete," by Hardy Cross and N. D. Morgan, John Wiley & Sons, Inc., New York, N. Y., 1932.

than three redundant elements; and the analogy lies in a "* * * mathematical identity between the moments produced by continuity in a beam, bent, or arch and the fiber stresses in a short column eccentrically loaded."[4] In this paper the writer explores the possibility of extending the column analogy to the analysis of continuous arches or bents. It is requested that discussion be confined to an objective appraisal of this effort.

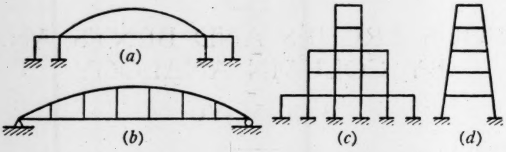


FIG. 1

The extension is not applicable to frames with closed panels (Fig. 1), and is therefore not intended to replace the moment distribution method. It is expected, however, to compare favorably with other methods in the determination of influence lines for continuous arches or bents.

Notation.—The letter symbols adopted for use in this paper are defined where necessary, in the text or by illustration, and are assembled alphabetically for convenience of reference in the Appendix.

Structural Formulas and Expressions.—The following expressions are used without further definition in this paper:

$$\text{Flexural rigidity} = E\,I$$

First moments of an elastic area are

$$A\,\bar{x} = \int \frac{x\,ds}{E\,I} \qquad (1a)$$

and

$$A\,\bar{y} = \int \frac{y\,ds}{E\,I} \qquad (1b)$$

Second moments of elastic area are

$$I_x = \int \frac{y^2\,ds}{E\,I} \qquad (2a)$$

$$I_y = \int \frac{x^2\,ds}{E\,I} \qquad (2b)$$

and

$$I_{xy} = \int \frac{x\,y\,ds}{E\,I} \qquad (2c)$$

---

[4] "Continuous Frames of Reinforced Concrete," by Hardy Cross and N. D. Morgan, John Wiley & Sons, Inc., New York, N. Y., 1932, p. 46.

Expressions for horizontal and vertical reactions, and for moments, due to "continuity," respectively, are: For symmetrical structures—

$$h_i = \frac{M_x}{I_x} \qquad (3a)$$

$$v_i = \frac{M_y}{I_y} \qquad (3b)$$

and

$$m_i = \frac{P}{A} + \frac{M_y}{I_y}x + \frac{M_x}{I_x}y \qquad (3c)$$

and, for asymmetrical structures—

$$h_i = \frac{M'_x}{I'_x} \qquad (4a)$$

$$v_i = \frac{M'_y}{I'_y} \qquad (4b)$$

and

$$m_i = \frac{P}{A} + \frac{M'_y}{I'_y}x + \frac{M'_x}{I'_x}y \qquad (4c)$$

in which

$$M'_y = M_y - \frac{I_{xy}}{I_x}M_x \qquad (5a)$$

$$M'_x = M_x - \frac{I_{xy}}{I_y}M_y \qquad (5b)$$

$$I'_y = I_y - \frac{I^2_{xy}}{I_x} \qquad (5c)$$

and

$$I'_x = I_x - \frac{I^2_{xy}}{I_y} \qquad (5d)$$

The final values of total horizontal and vertical reactions, and moments are, respectively,

$$H = h_s - h_i \qquad (6a)$$

$$V = v_s - v_i \qquad (6b)$$

and

$$M = m_s - m_i \qquad (6c)$$

Sign Convention.—According to Mr. Cross,[5]

> "* * * the bending moments in the column analogy are taken as positive when they produce tension on the bottom of a beam or on the inner side of an arch or bent. * * * Positive rotations are such as would be produced by positive bending moments. * * *
>
> "For horizontal forces, pull at the abutments (tension in girder of a bent) is positive.
>
> "For vertical forces, use the customary convention for shear in girders. (Positive shear is up on the left, down on the right.)

---

[5] "Continuous Frames of Reinforced Concrete," by Hardy Cross and N. D. Morgan, John Wiley & Sons, Inc., New York, N. Y., 1932, pp. 55–56.

"The correct signs for changes in reaction are then automatically determined by the column analogy, and the correct values for the reactions follow algebraically from $V = v_s - v_i$; $H = h_s - h_i$.

"Positive displacements correspond to positive forces. Positive bending moment is positive force times positive distance (distance measured up, or to the right, from the force to the section)."

The Substitute Member

The beam, arch, or bent AB in Fig. 2(*a*) is shown in Fig. 2(*b*) supported at the right end by the substitute member BB′. The characteristics of this method are such that the two structures shown in Fig. 2 are equivalent as far as redundant elements (and therefore deflections) in span AB are concerned, provided that external loads act on span AB only. (In Fig. 2(*a*) if column BC is to be analyzed, instead of span AB, it is assumed to be supported at point B by the "elastic foundation" ABDE. Similarly, the beam-and-column span BDE is assumed to be supported at point B by the "elastic foundation" ABC.)

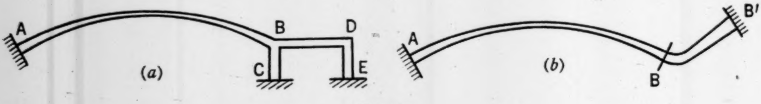


Fig. 2

Elastic Properties of the Substitute Member

In order that the substitute member BB′, Fig. 2(*b*), shall be equivalent to the elastic foundation BCDE (Fig. 2(*a*)), it is necessary (and sufficient) that the original members and the substitute member give the same deflections and rotations at point B when equal loads are applied at point B. Member AB is temporarily assumed to be cut at point B, making the elastic foundation BCDE (or the substitute member BB′) an independent member. This principle is illustrated in Fig. 3.

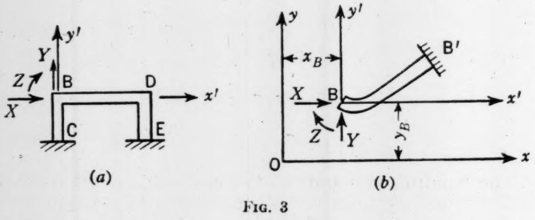


Fig. 3

The deflections of the elastic foundation BCDE (Fig. 3(*a*)) at joint B caused by unit influence loads at that point can be readily computed, making use of the column analogy or other methods. Similarly, it is possible to write the corresponding deflections for the substitute member BB′. These deflections are given in the most general form, in Table 1. Equating the corresponding deflections, lines 9 and 8, Table 1, yield, respectively,

TABLE 1.—FORMULAS FOR A SUBSTITUTE MEMBER AT POINT B, FIG. 3

Load at B, Fig. 3	Line	Deflection at B	Elastic foundation (BCDE)	Substitute member (BB′)
$X = 1$ $Y = Z = 0$	1	Horizontal	δ_{xx}	$\int \frac{(y')^2\,ds}{E\,I}$
	2	Vertical	δ_{xy}	$-\int \frac{x'\,y'\,ds}{E\,I}$
	3	Angular	δ_{xz}	$-\int \frac{y'\,ds}{E\,I}$
$Y = 1$ $X = Z = 0$	4	Horizontal	$\delta_{yx} = \delta_{xy}$	$-\int \frac{x'\,y'\,ds}{E\,I}$
	5	Vertical	δ_{yy}	$\int \frac{(x')^2\,ds}{E\,I}$
	6	Angular	δ_{yz}	$\int \frac{x'\,ds}{E\,I}$
$Z = 1$ $X = Y = 0$	7	Horizontal	$\delta_{zx} = \delta_{xz}$	$-\int \frac{y'\,ds}{E\,I}$
	8	Vertical	$\delta_{zy} = \delta_{yz}$	$\int \frac{x'\,ds}{E\,I}$
	9	Angular	δ_{zz}	$\int \frac{ds}{E\,I}$

$$\int \frac{ds}{E\,I} = \delta_{zz} \qquad \text{(7a)}$$

and

$$\int \frac{x'\,ds}{E\,I} = \delta_{yz} \qquad \text{(7b)}$$

in which *ds* is an elementary segment of the substitute member, BB′, whose coordinates are x', y', referred to joint B as the origin, and whose flexural rigidity is $E\,I$ (E is the modulus of elasticity; and I is the moment of inertia of the cross section). Dividing Eq. 7*b* by Eq. 7*a*,

$$\bar{x}' = \frac{\delta_{yz}}{\delta_{zz}} \qquad \text{(8a)}$$

By a similar logic, dividing line 7, Table 1, by line 9,

$$\bar{y}' = -\frac{\delta_{xz}}{\delta_{zz}} \qquad \text{(8b)}$$

Eqs. 8*a* and 8*b* give the coordinates of the centroid of the elastic area of the substitute member.

The second moments referred to the elastic center of BB′ $\left(\text{or the centroid of } \int \frac{ds}{E\,I}\right)$ are

$$I_{gy} = \int \frac{(x')^2\,ds}{E\,I} - (\bar{x}')^2 \int \frac{ds}{E\,I} = \delta_{yy} - \frac{\delta^2_{yz}}{\delta_{zz}} \qquad \text{(9a)}$$

$$I_{gx} = \int \frac{(y')^2\,ds}{E\,I} - (\bar{y}')^2 \int \frac{ds}{E\,I} = \delta_{xx} - \frac{\delta^2_{xz}}{\delta_{zz}} \qquad \text{(9b)}$$

and

$$I_{gxy} = \int \frac{x'\, y'\, ds}{E\, I} - \bar{x}'\, \bar{y}' \int \frac{ds}{E\, I} = -\, \delta_{xy} + \frac{\delta_{xz}\, \delta_{yz}}{\delta_{zz}} \quad \text{(9c)}$$

Translating the axes from point B to the new origin O, Fig. 3(*b*),

$$\bar{x} = x_B + \bar{x}' \quad \text{(10a)}$$

and

$$\bar{y} = y_B + \bar{y}' \quad \text{(10b)}$$

from which

$$I_y = \delta_{yy} - \frac{\delta^2{}_{yz}}{\delta_{zz}} + (x_B + \bar{x}')^2\, \delta_{zz} \quad \text{(11a)}$$

$$I_x = \delta_{xx} - \frac{\delta^2{}_{xz}}{\delta_{zz}} + (y_B + \bar{y}')^2\, \delta_{zz} \quad \text{(11b)}$$

and

$$I_{xy} = -\, \delta_{xy} + \frac{\delta_{xz}\, \delta_{yz}}{\delta_{zz}} + (x_B + \bar{x}')\, (y_B + \bar{y}')\, \delta_{zz} \quad \text{(11c)}$$

In using Eqs. 9 to 11, positive unit loads or moments at point B act in the directions or sense shown in Fig. 3. The same directions or sense is assumed positive for deflections or rotation, respectively. Then the final results will automatically give the correct algebraic signs, regardless of the side of the beam, arch, or bent on which the elastic foundation lies.

It is not necessary (nor is it possible) to know the flexural rigidity $E\,I$ at any particular section of the substitute member; it is not necessary (nor is it possible) to know even the location of its axis. With the properties determined by the formulas in Table 1, and by Eqs. 9 to 11, the designer can proceed with the application of the column analogy.

EXAMPLE 1

The frame shown in Fig. 4 is solved in Tables 2 and 3, which are self-explanatory to anyone who has a proper understanding of the column analogy method. The tabulation and the notations follow those of Mr. Cross[2,3] except that I_x, I_y, M_x, and M_y represent, respectively, I_y, I_x, M_y, and M_x in the Cross notation. These changes are made to be consistent with those commonly used in textbooks on engineering mechanics.

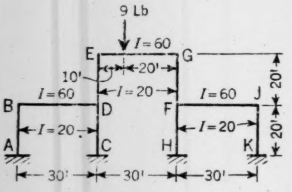


FIG. 4

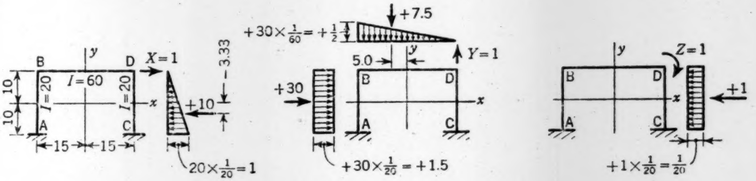
Tables 2 constitute the complete analysis of moments, by column analogy, in the elastic foundation ABDC (which is the same as HFJK) when unit influence loads are applied at joint D, Fig. 4. It can be noted by inspection of Table 2*B* that the analysis of moments due to the unit influence load $Y = 1$ is unnecessary; but this has been included to make the example reasonably complete. The deflections at joint D, namely, δ_{xx}, δ_{xy}, δ_{xz}, δ_{yy}, and

δ_{zz} are then computed (Table 2*C*(*b*)). It is convenient to note that $\delta_{xz} = \delta_{zx} = +0.38$ gives a check to the correctness of the analysis. Finally, the properties of the substitute member DD′ are computed by formulas in Table 1 and

TABLE 2*A*.—PROPERTIES OF BENT ABDC, EXAMPLE 1

No.	Bar (see Fig. 4)	GIVEN DATA				COMPUTED PROPERTIES					
		L	I	x	y	Elastic Area	First Moments		Second Moments		
						a	$a\,x$	$a\,y$	$a\,x^2 + i_y$	$a\,y^2 + i_x$	$a\,x\,y + i_{xy}$
1	AB	20	20	−15	0	1.0	−15	0	225	33.33	0
2	BD	30	60	0	+10	0.5	0	+5	37.5	50.00	0
3	DC	20	20	+15	0	1.0	+15	0	225	33.33	0
4	Correction to elastic centroid					....	....	....	0	−10.0	0
5	Σ	....	....	$\frac{0}{2.5} = 0 = \bar{x}$	$\frac{+5}{2.5} = +2 = \bar{y}$	2.5	0	+5	$487.5 = I_x$	$106.7 = I_y$	$0 = I_{xy}$

TABLE 2*B*.—ELASTIC LOADS AND FORMULAS FOR m_i, BENT ABDC, EXAMPLE 1


B
D
$X=1$
y
x
$I=60$
$I=20$
$I=20$
10
10
A
C
15
15
$+10$
-3.33
$20\times\frac{1}{20}=1$
$+30\times\frac{1}{60}=+\frac{1}{2}$
$+7.5$
$+30$
5.0
y
$Y=1$
B
D
A
C
x
$+30\times\frac{1}{20}=+1.5$
y
$Z=1$
B
D
A
C
x
$+1$
$+1\times\frac{1}{20}=\frac{1}{20}$

No.	Bar (see Fig. 4)	HORIZONTAL FORCE $X = 1$					VERTICAL FORCE $Y = 1$					MOMENT $Z = 1$				
		Given			Computed		Given			Computed		Given			Computed	
		P	x	y	M_y	M_x	P	x	y	M_y	M_x	P	x	y	M_y	M_x
1	AB	....	....	....	....	....	+30.0	−15	0	−450	0	..	....	..	....	....
2	BD	....	....	....	....	....	+ 7.5	− 5	10	− 37.5	+75	..	....	..	....	....
3	DC	+10	+15	−3.33	+150	−33.3	....	....	....	....	....	1	+15	0	+15	0
4	Corrected[a]		....	....	0	−20.0	....	....	....	0	−75	..	....	..	0	−2
5	Σ	+10	....	....	+150	−53.3	+37.5	....	....	−487.5	0	1	....	..	+15	−2
6	m_i	$= \frac{10}{2.5} + \frac{150}{487.5}x - \frac{53.3}{106.7}y$					$= + \frac{37.5}{2.5} - \frac{487.5}{487.5}x + 0$					$= \frac{1}{2.5} + \frac{15}{487.5}x - \frac{2}{106.7}y$				

[a] Correction to elastic centroid.

TABLE 2C.—COMPUTATION OF MOMENTS AND DEFLECTIONS, BENT ABDC, EXAMPLE 1

No.	Point (see Fig. 4)	x	y	HORIZONTAL FORCE $X = 1$			VERTICAL FORCE $Y = 1$			MOMENT $Z = 1$		
				m_s	m_i	M	m_s	m_i	M	m_s	m_i	M
(a) MOMENTS, $M = m_s - m_i$												
1	A	−15	−12	0	+ 5.38	−5.38	+30	+30	0	0	+0.163	−0.163
2	B	−15	+ 8	0	− 4.62	+4.62	+30	+30	0	0	−0.212	+0.212
3	D_{DB}	+15	+ 8	0	+ 4.62	−4.62	0	0	0	0	+0.712	−0.712
4	D_{DC}	+15	+ 8	0	+ 4.62	−4.62	0	0	0	+1	+0.712	+0.288
5	C	+15	−12	+20	+14.62	+5.38	0	0	0	+1	+1.087	−0.087
(b) DEFLECTIONS AT JOINT D, FIG. 4												
6	Horizontal			$\delta_{xx} = +20.5$			$\delta_{yx} = 0$			$\delta_{zx} = +0.38$		
7	Vertical			$\delta_{xy} = 0$			$\delta_{yy} = 0$			$\delta_{zy} = 0$		
8	Rotation			$\delta_{xz} = +0.38$			$\delta_{yz} = 0$			$\delta_{zz} = +0.1005$		

TABLE 3A.—ELASTIC LOAD AND FORMULA FOR m_i, BENT DEGF, EXAMPLE 1 (FIGS. 5 AND 6)

GIVEN DATA					COMPUTED PROPERTIES						ELASTIC LOAD				
Bar	L	I	x	y	Area[a]	First Moments		Second Moments			P	x	y	$M_y = Px$	$M_x = Py$
					a	$a\,x$	$a\,y$	$a\,x^2 + i_y$	$a\,y^2 + i_x$	$a\,x\,y + i_{xy}$					
DE	20	20	−15	0	1.000	−15.00	0	225.0	33.33	0	....	....	....	....	....
EG	30	60	0	+10	0.500	0	+5.00	37.5	50.00	0	+15.0	−1.67	+10.0	−25.0	+150
GF	20	20	+15	0	1.000	+15.00	0	225.0	33.33	0	....	....	....	....	....
DD′[b]	....	....	−15	....	0.1005	− 1.51	−1.39	22.6 0	19.1 19.1	+9.4 0	....	....	....	....	....
FF′[b]	....	....	+15	....	0.1005	+ 1.51	−1.39	22.6 0	19.1 19.1	−9.4 0	....	....	....	....	....
....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....
Correction to centroidal axis					....	....	....	0	−1.8	0	....	....	....	0	−12.3
Σ	....	....	0[c]	0.822[c]	2.701	0	+2.22	532.7[d]	191.3[d]	0	+15.0	....	....	−25.0[e]	+137.7[e]

$$m_i = \frac{15}{2.701} - \frac{25}{532.7}x + \frac{137.7}{191.3}y$$

[a] Elastic area. [b] See text (Example 1). [c] $\bar{x}$ and $\bar{y}$, respectively. [d] I_y and I_x, respectively. [e] M_y and M_x, respectively.

TABLE 3B.—MOMENTS $M = m_s - m_i$, BENT DEGF, EXAMPLE 1 (FIGS. 5 AND 6)

No.	Point	x	y	m_s	m_i	M (see Fig. 5(b))
1	D	−15	−10.82	0	− 1.52	+ 1.52
2	E	−15	+ 9.18	0	+12.86	−12.86
3	G	+15	+ 9.18	0	+11.44	−11.44
4	F	+15	−10.82	0	− 2.94	+ 2.94

by Eqs. 7 to 9, as follows:

$$\int \frac{ds}{I} = \delta_{zz} = +\,0.1005$$

$$\bar{x} = \frac{\delta_{yz}}{\delta_{zz}} = 0$$

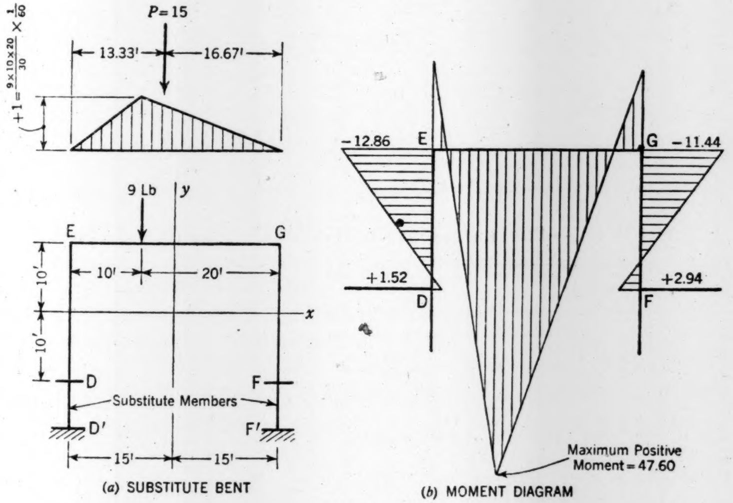
$$\bar{y} = -\,\frac{\delta_{xz}}{\delta_{zz}} = -\,3.78$$

$$I_{oy} = \delta_{yy} - \frac{\delta^2{}_{yz}}{\delta_{zz}} = 0$$

$$I_{ox} = \delta_{xx} - \frac{\delta^2{}_{xz}}{\delta_{zz}} = 19.1$$

$$I_{oxy} = -\,\delta_{xy} + \frac{\delta_{xz}\,\delta_{yz}}{\delta_{zz}} = 0$$

Similarly to Tables 2, Tables 3 contain the complete analysis, by column analogy, of the moments in bay DEGF, Fig. 4, considered to be supported by DD′ and FF′. The moments in bents ABDC and HFJK are not evaluated.


+1 = 9×10×20/30 × 1/60
P=15
13.33′
16.67′
9 Lb
y
E
G
10′
20′
10′
10′
x
D
F
Substitute Members
D′
F′
15′
15′
−12.86
E
G
−11.44
+1.52
D
+2.94
F
Maximum Positive Moment = 47.60

(a) SUBSTITUTE BENT (b) MOMENT DIAGRAM

FIG. 5

If desired, they can be readily computed by finding the moment and shear transmitted by column DE at point D (or by column FG at point F), Fig. 5, and making subsequent computations by superposition (using the results in Table 2C, with proper attention to signs).

SKETCHING INFLUENCE LINES; EXAMPLE 2

The determination of influence lines by the principle of virtual work has been treated elsewhere,[6] and the application of the column analogy to the same problem is also well known.[2,3] A simple example will be given to illustrate the usefulness of the method.

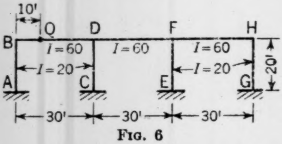


FIG. 6

A three-bay continuous bent is analyzed for bending moment, shear, and axial tension at point Q, Fig. 6. The analysis begins with the determination of the properties of the substitute member DD′ (see Tables 4) and the moments due to virtual displacements at point Q, Fig. 6 (see Tables 5).

In Table 4*C*(*b*), $\delta_{xz} = 0.317$ and $\delta_{zx} = 0.312$, indicating an error as the result of slide rule inaccuracies. In subsequent computations the designer would use the average of these two values. Finally, the properties of the substitute member DD′ are computed by formulas in Table 1 and by Eqs. 7 to 9, as follows:

$$\int \frac{ds}{E\,I} = \delta_{zz} = +\,0.097$$

$$\bar{x} = \frac{\delta_{yz}}{\delta_{zz}} = 0$$

$$\bar{y} = -\,\frac{\delta_{xz}}{\delta_{zz}} = -\,3.25$$

$$I_{gy} = 0$$

$$I_{gx} = \delta_{xx} - \frac{\delta^2_{xz}}{\delta_{zz}} = 12.58$$

$$I_{gxy} = 0$$

TABLE 4*A*.—PROPERTIES OF BENT CDFF′, EXAMPLE 2 (FIG. 6)

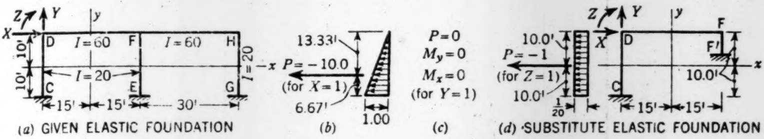
GIVEN DATA						COMPUTED PROPERTIES					
No.	Bar[a]	L	I	x	y	Elastic Area	First Moments		Second Moments		
						a	$a\,x$	$a\,y$	$a\,x^2 + i_y$	$a\,y^2 + i_x$	$a\,x\,y + i_{xy}$
1	CD	20	20	−15	0	1.00	−15	0	225.0	33.3	0
2	DF	30	60	0	+10	0.50	0	+5.0	37.5	50.0	0
3	FF′	....	....	+15	+ 6.22	0.1005	+ 1.51	+0.63	22.6	$\begin{Bmatrix} 3.9 \\ 19.1 \end{Bmatrix}$	+ 9.4
4	Second moments corrected to elastic centroid							....	−113.7	−19.8	+47.5
5	Σ	....	....	−8.43[b]	+3.52[b]	1.60 = A	−13.49	+5.63	171.4 = I_y	86.5 = I_x	+56.9 = I_{xy}
6	Corrected for dissymmetry					....	....	....	134.0 = I'_y	67.6 = I'_x	....

[a] See Fig. 6 and Table 4*B*. For the properties of the substitute member FF′, see lines 4 and 5, Table 3*A*, and the text under Example 1. [b] Values of $\bar{x}$ and $\bar{y}$, respectively.

---

[6] "Structural Theory," by Hale Sutherland and H. L. Bowman, John Wiley & Sons, Inc., New York, N. Y., 3d Ed., 1942.

TABLE 4*B*.—ELASTIC LOADS AND FORMULAS FOR m_i, BENT CDFF′, EXAMPLE 2


Z
Y
y
X
D
I = 60
F
I = 60
H
10'
10'
I = 20
C
E
G
I = 20
15'
15'
30'
−x
13.33'
P = −10.0
(for X = 1)
6.67'
1.00
P = 0
M_y = 0
M_x = 0
(for Y = 1)
10.0'
P = −1
(for Z = 1)
10.0'
1/20
Z
Y
y
X
D
F
F'
x
10.0'
C
15'
15'

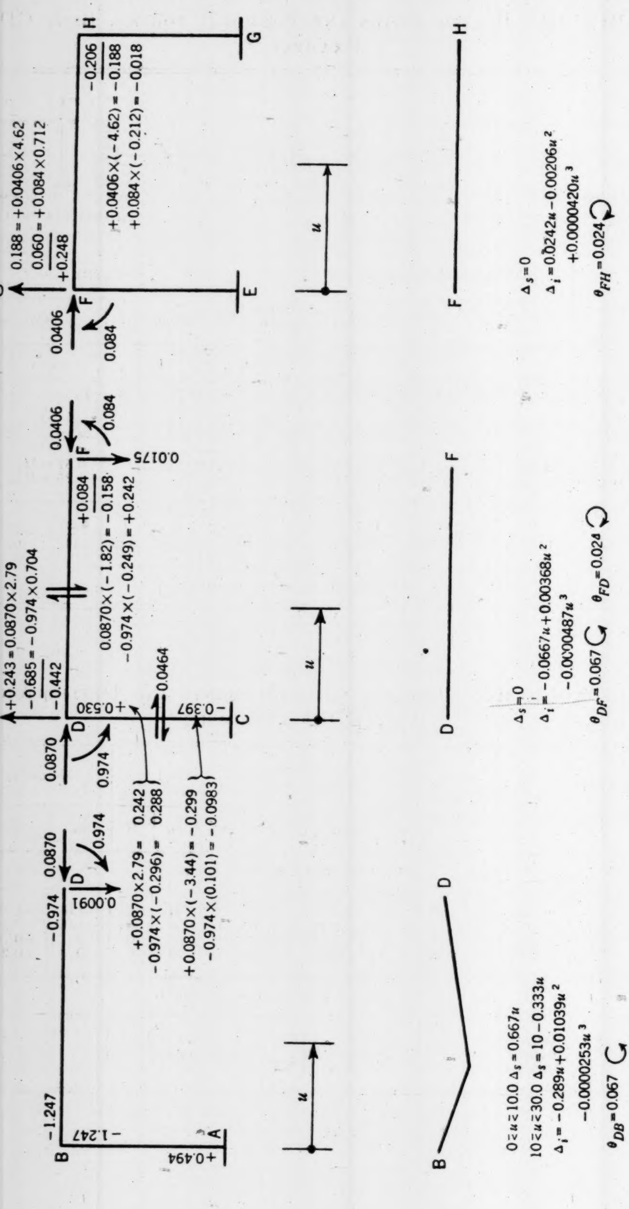
(*a*) GIVEN ELASTIC FOUNDATION (*b*) (*c*) (*d*) SUBSTITUTE ELASTIC FOUNDATION

No.	Bar (see Fig. 6)	Horizontal Force $X = 1$					Vertical force $Y = 1$	Moment $Z = 1$				
		Given			Computed			Given			Computed	
		P	x	y	M_y	M_x		P	x	y	M_y	M_x
1	CD	−10.0	−15	−3.33	+150	+33.3		−1.00	−15	0	15	0
2	DF	....	....	....	....	....	$P = 0$ $M_y = 0$ $M_x = 0$	....	....	..	....	....
3	FF′	....	....	....	....	....		....	....	..	....	....
4	Correction[a]	....	....	....	−84.3	+35.2		....	....	..	−8.43	+3.52
5	Σ	−10.0	....	....	65.7	+68.5		−1.00	....	..	$+6.57 = M_y$	$+3.52 = M_x$
6	Corrected for dissymmetry				+20.6	+46.7		....	....	..	$+4.25 = M'_y$	$1.34 = M'_x$
7	m_i	$= -\frac{10}{1.6} + \frac{20.6}{134}x + \frac{46.7}{67.6}y$					$= 0$	$= -\frac{1}{1.6} + \frac{4.25}{134}x + \frac{1.34}{67.6}y$				

[a] Correction to elastic centroid.

TABLE 4*C*.—COMPUTATION OF MOMENTS AND DEFLECTIONS, BENT CDFF′, EXAMPLE 2

No.	Point (see Fig. 6)	x	y	Horizontal Force $X = 1$			$Y = 1$			Moment $Z = 1$		
				m_s	m_i	M	m_s	m_i	M	m_s	m_i	M
	(*a*) Moments, $M = m_s - m_i$											
1	C	− 6.57	−13.52	−20	−16.56	−3.44	0	0	0	−1.00	−1.101	+0.101
2	D_{CD}	− 6.57	+ 6.48	0	− 2.79	+2.79	0	0	0	−1.00	−0.704	−0.296
3	D_{DF}	− 6.57	+ 6.48	0	− 2.79	+2 79	0	0	0	0	−0.704	+0.704
4	F	+23.43	+ 6.48	0	+ 1.82	−1.82	0	0	0	0	+0.249	−0.249
	(*b*) Deflections at Joint D, Fig. 6											
7	Horizontal			$\delta_{xx} = +13.6$			$\delta_{yx} = 0$			$\delta_{zx} = +0.312$		
8	Vertical			$\delta_{xy} = 0$			$\delta_{yy} = 0$			$\delta_{zy} = 0$		
9	Angular			$\delta_{xz} = +0.317$			$\delta_{yz} = 0$			$\delta_{zz} = +0.097$		


A
B
+0.494
-1.247
-1.247
-0.974
D
0.0870
0.974
0.0091
+0.0870×2.79 = 0.242
-0.974×(-0.296) = 0.288
+0.0870×(-3.44) = -0.299
-0.974×(0.101) = -0.0983
u
0.0870
0.974
D
C
0.0091
+0.243 = 0.0870×2.79
-0.685 = -0.974×0.704
-0.442
+0.530
-0.397
0.0464
0.0870×(-1.82) = -0.158
-0.974×(-0.249) = +0.242
+0.084
0.0406
F
0.084
0.0175
u
0.0406
F
0.084
E
0.0175
0.188 = +0.0406×4.62
0.060 = +0.084×0.712
+0.248
+0.0406×(-4.62) = -0.188
+0.084×(-0.212) = -0.018
-0.206
H
G
u
B
D
0 ≤ u ≤ 10.0 Δs = 0.667u
10 ≤ u ≤ 30.0 Δs = 10 - 0.333u
Δi = -0.289u + 0.01039u²
-0.0000253u³
θDB = 0.067
D
F
Δs = 0
Δi = -0.0667u + 0.00368u²
-0.0000487u³
θDF = 0.067
θFD = 0.024
F
H
Δs = 0
Δi = 0.0242u - 0.00206u²
+0.0000420u³
θFH = 0.024

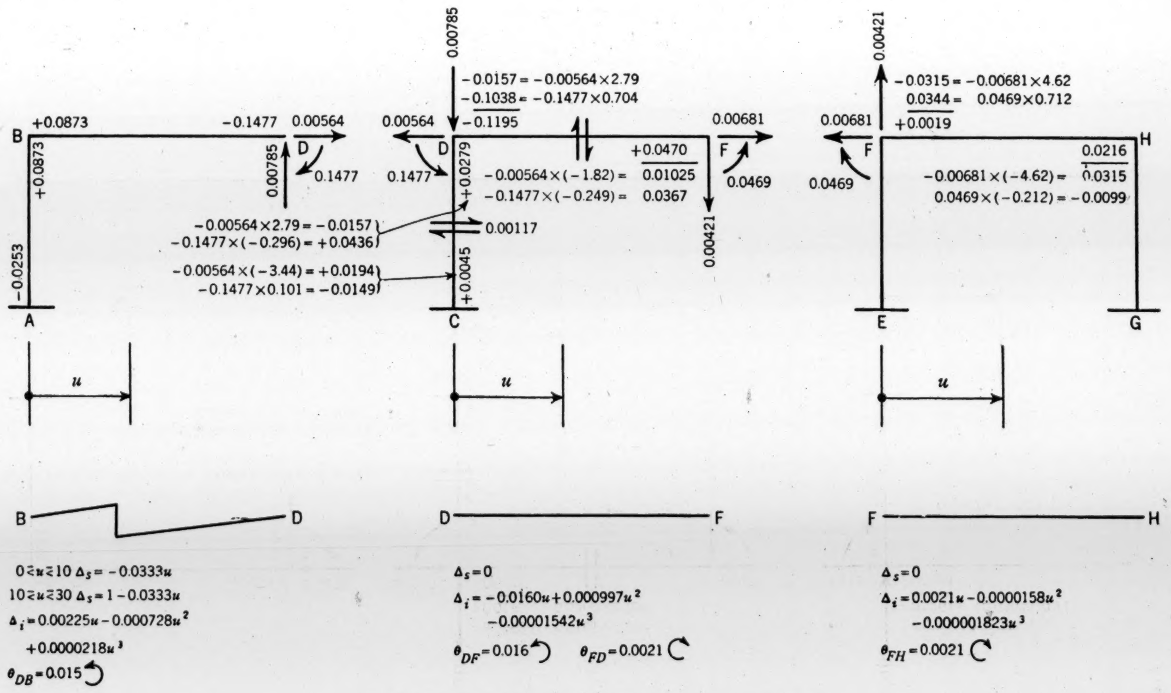

B
+0.0873
+0.0873
-0.0253
A
-0.1477
0.00564
D
0.00785
0.1477
-0.00564×2.79 = -0.0157
-0.1477×(-0.296) = +0.0436
-0.00564×(-3.44) = +0.0194
-0.1477×0.101 = -0.0149
0.00564
D
0.1477
0.00785
-0.0157 = -0.00564×2.79
-0.1038 = -0.1477×0.704
-0.1195
+0.0279
0.00117
+0.0045
C
-0.00564×(-1.82) = 0.01025
-0.1477×(-0.249) = 0.0367
+0.0470
0.00681
F
0.0469
0.00421
0.00681
F
0.0469
0.00421
-0.0315 = -0.00681×4.62
0.0344 = 0.0469×0.712
+0.0019
H
-0.00681×(-4.62) = 0.0315
0.0469×(-0.212) = -0.0099
0.0216
E
G
u
u
u
B
D
D
F
F
H
0 ≦ u ≦ 10 Δs = -0.0333u
10 ≦ u ≦ 30 Δs = 1 - 0.0333u
Δi = 0.00225u - 0.000728u²
+0.0000218u³
θDB = 0.015
Δs = 0
Δi = -0.0160u + 0.000997u²
-0.00001542u³
θDF = 0.016
θFD = 0.0021
Δs = 0
Δi = 0.0021u - 0.0000158u²
-0.000001823u³
θFH = 0.0021

Fig. 7.—Formulas for Virtual Displacements, Example 2

(Continued on next page)

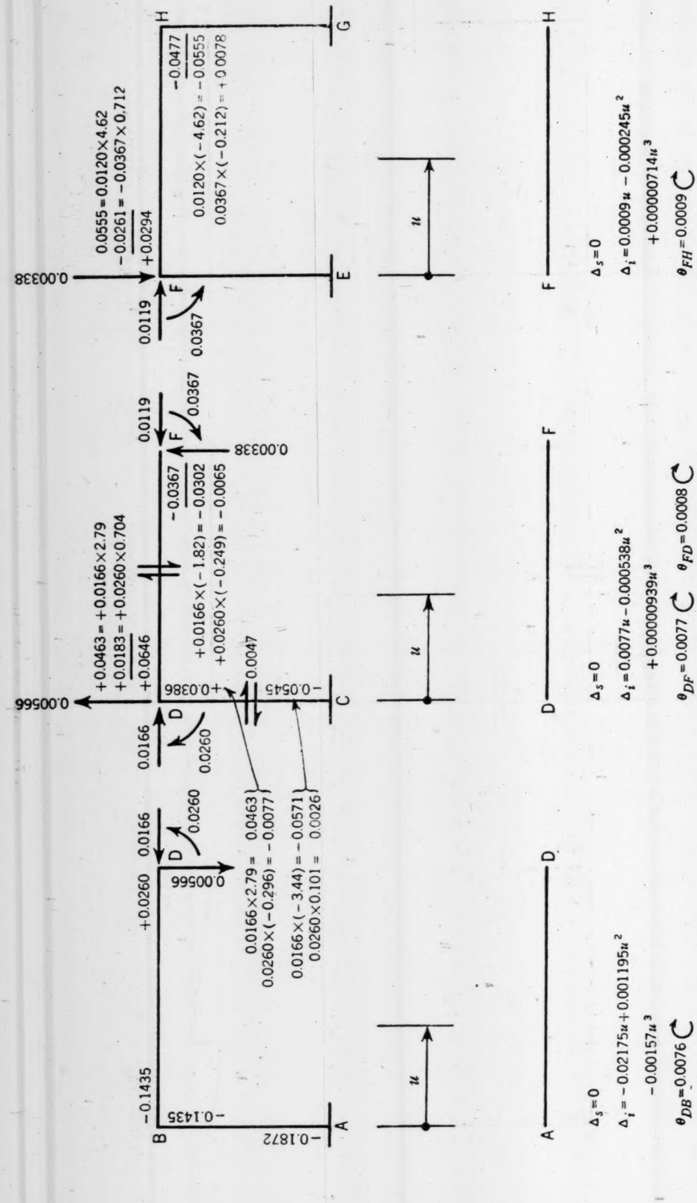
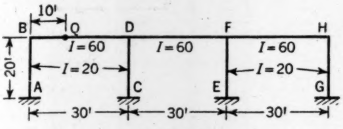
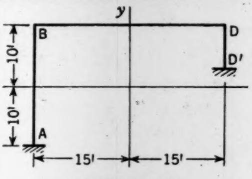

A
B
C
D
E
F
G
H
-0.1435
-0.1435
-0.1872
+0.0260
0.0166
0.0260
0.00566
0.0166 × 2.79 = 0.0463
0.0260 × (-0.296) = -0.0077
0.0166 × (-3.44) = -0.0571
0.0260 × 0.101 = 0.0026
0.0166
0.0260
0.00566
+0.0463 = +0.0166 × 2.79
+0.0183 = +0.0260 × 0.704
+0.0646
+0.0386
0.0047
-0.0545
+0.0166 × (-1.82) = -0.0302
+0.0260 × (-0.249) = -0.0065
-0.0367
0.0119
0.0367
0.00338
0.0119
0.0367
0.00338
0.0555 = 0.0120 × 4.62
-0.0261 = -0.0367 × 0.712
+0.0294
0.0120 × (-4.62) = -0.0555
0.0367 × (-0.212) = +0.0078
-0.0477
u
u
u
$\Delta_s = 0$
$\Delta_i = -0.02175u + 0.001195u^2 - 0.00157u^3$
$\theta_{DB} = 0.0076$
$\Delta_s = 0$
$\Delta_i = 0.0077u - 0.000538u^2 + 0.0000939u^3$
$\theta_{DF} = 0.0077$
$\theta_{FD} = 0.0008$
$\Delta_s = 0$
$\Delta_i = 0.0009u - 0.000245u^2 + 0.00000714u^3$
$\theta_{FH} = 0.0009$

FIG. 7.—(*Continued*)

TABLE 5*A*.—Properties of Bent ABDD′ (For Computation of Moments Due to Virtual Displacements), Example 2



GIVEN BENT

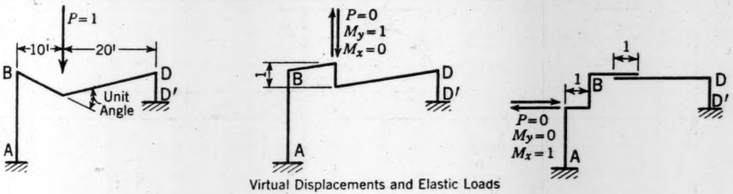


SUBSTITUTE BENT

No.	Given Data					Computed Properties					
	Bar[a] (see Fig. 6)	L	I	x	y	Elastic Area	First Moments		Second Moments		
						a	$a\,x$	$a\,y$	$a\,x^2 + i_y$	$a\,y^2 + i_x$	$a\,x\,y + i_{xy}$
1	AB	20	20	−15	0	1.000	−15.00	0	225.0	33.3	0
2	BD	30	60	0	+10	0.500	0	+5.000	37.5	50.0	0
3 4	DD′	..	..	+15	+ 6.75	0.097	+ 1.45	+0.655	21.8	$\begin{Bmatrix}4.4\\12.6\end{Bmatrix}$	+ 9.8
5	Second moment correction to centroid					....	....	....	−114.9	−20.1	+48.0
6	Σ	..	..	− 8.49[b]	+ 3.55[b]	1.597	−13.55	5.655	$169.4 = I_y$	$80.2 = I_x$	$+57.8 = I_{xy}$
7	Corrected for dissymmetry				....	....	....	....	$127.8 = I'_y$	$60.5 = I'_x$	....

[a] For properties of substitute member DD′, see text under Example 2. [b] $\bar{x}$ and $\bar{y}$, respectively.

TABLE 5*B*.—Elastic Loads on Bent ABDD′ and Moments Due to Virtual Displacements, Example 2



Virtual Displacements and Elastic Loads

No.	Bar[a] (see Fig. 6)	$P = 1$					$M_y = 1$					$M_x = 1$				
		Given			Computed		Given			Computed		Given			Computed	
		P	x	y	M_y	M_x	P	x	y	M_y	M_x	P	x	y	M_y	M_x
1	AB	....	....	....	....	....	..	..	..	....	....	..	..	..	....	+1.00
2	BD	+1.0	−5	+10	−5.0	+10.0	0	..	..	+1.00	0	..	..	..	....	....
3	DD′	....	....	....	....	....	..	..	..	....	....	..	..	..	....	....
4	Correction[b]	....	....	....	+8.49	−3.55	..	..	..	0	0	..	..	..	0	0
5	Σ	1.0	....	....	+3.49[c]	+ 6.45[c]	0	..	..	+1.00[c]	0[c]	0	..	..	0[c]	+1.00[c]
6	Corrected for dissymmetry				−1.16[d]	+ 5.26[d]	..	..	..	+1.00[d]	−0.341[d]	..	..	..	−0.721[d]	+1.00[d]
7	m_i	$= \frac{1}{1.597} - \frac{1.16}{127.8}x + \frac{5.26}{60.5}y$					$= \frac{1}{127.8}x - \frac{0.341}{60.5}y$					$= -\frac{0.721}{127.8}x + \frac{1}{60.5}y$				

[a] For properties of substitute member DD′, see text under Example 2. [b] Correction to elastic centroid. [c] These are values of ΣM_y and ΣM_x, respectively. [d] These are values of M'_y and M'_x, respectively.

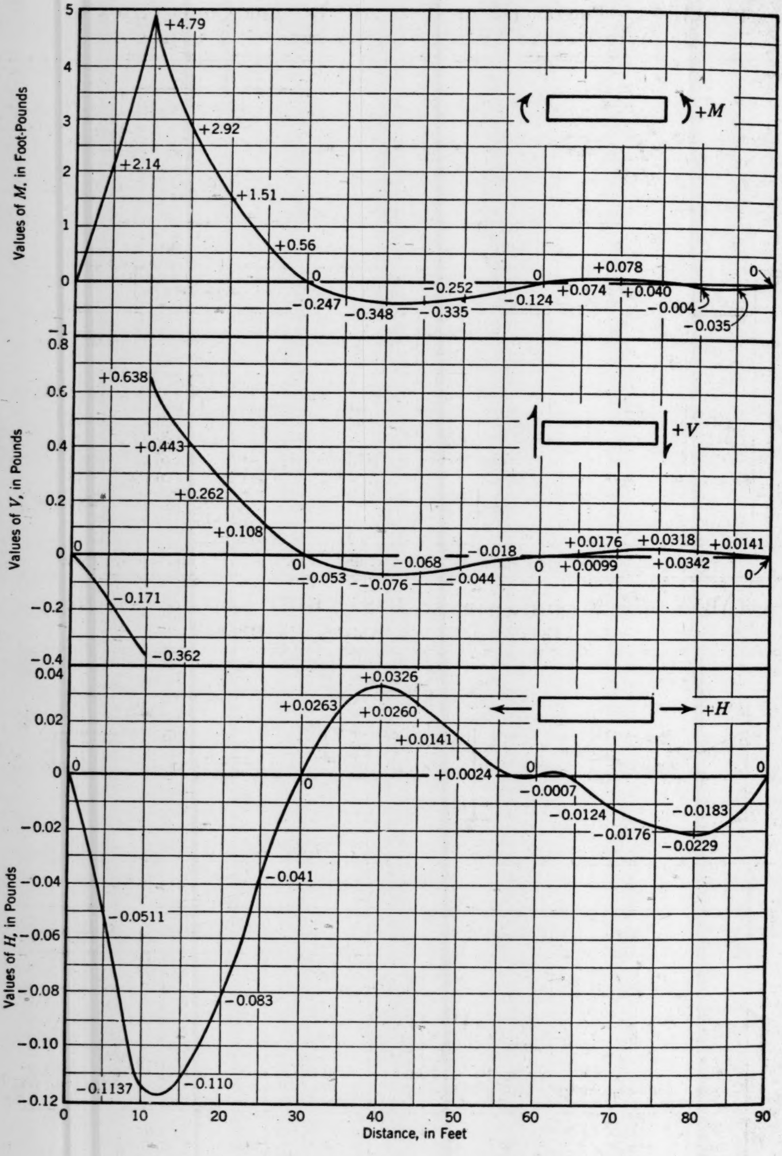


Fig. 8.—Influence Ordinates for Moment, Shear, and Axial Tension at Q, Example 2

TABLE 5C.—COMPUTATION OF REACTIONS AND MOMENTS DUE TO VIRTUAL DISPLACEMENTS, BENT ABDD′, EXAMPLE 2

No.	Point (see Fig. 6)	x	y	$P = 1$			$M_y = 1$			$M_z = 1$		
				m_s	m_i	M	m_s	m_i	M	m_s	m_i	M
				MOMENTS $M = m_s - m_i$								
1	A	− 6.51	−13.55	0	−0.494	+0.494	0	+0.0253	−0.0253	0	−0.1872	+0.1872
2	B	− 6.51	+ 6.45	0	+1.247	−1.247	0	−0.0873	+0.0873	0	+0.1435	−0.1435
3	D	23.49	+ 6.45	0	+0.974	−0.974	0	+0.1477	−0.1477	0	−0.0260	+0.0260

Fig. 7 gives the moments of the three spans. The moments in girders DF and FH, which belong to the elastic foundation, are found by superposition, making use of the results of Tables 4 and Tables 2. The ordinates of the influence lines, represented by

$$y = \Delta_s + \Delta_i \ldots\ldots\ldots\ldots\ldots\ldots (12)$$

are then computed and given in Fig. 8, in which Δ_s is the static deflection, and Δ_i is the deflection due to continuity.

CONCLUSIONS

In this paper the writer has attempted to do for the column analogy what T. Y. Lin,[7] Assoc. M. ASCE, J. A. Wise,[8] and others did for the method of moment distribution. Each of these efforts, including the present paper, can be properly criticized on the ground that apparent savings of time in computations are effected at the expense of preliminary work. Nevertheless, the ideas offered for discussion in this paper seem to possess the following merits: By avoiding simultaneous equations, computation work can be done more quickly and accurately; and, as a consequence, it is especially useful as a device for reviewing designed structures.

APPENDIX. NOTATION

The letter symbols used in this paper conform essentially with those adopted by Professor Cross in his exposition of the column analogy,[2,3] and also with tentative American Standard Letter Symbols for Structural Analysis (ASA—ZIOH—1947).

A = total elastic area = $\int \frac{ds}{E\,I}$;
a = elastic area of a member;
E = modulus of elasticity;

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[7] "A Direct Method of Moment Distribution," by T. Y. Lin, *Transactions*, ASCE, Vol. 102, 1937, p. 561.

[8] "A Precise Moment Distribution Method," by Joseph A. Wise, *Journal*, A.C.I., November, 1938, p. 93.

H = total horizontal reaction of the structure (Eq. 6*a*);

h = horizontal reaction of a structure, with subscripts s and i denoting, respectively, "statically determinate structure" and "due to continuity";

I = second moment of the area of the cross section of a member ($E\,I$ = flexural rigidity);

I_x, I_y, and I_{xy} = the second moments of the elastic area, about axes x, y, and xy, respectively; for asymmetric members only, I', with appropriate subscript, denotes the second moment corrected for dissymmetry;

i = second moment of elastic area for a single member only, with subscripts corresponding to I;

M = total bending moment at any point of a structure (Eq. 6*c*);

M_x and M_y = the total static moments of the elastic load about axes x and y, respectively; for asymmetric members only, M', with appropriate subscript, denotes the total static moments of the elastic load, corrected for dissymmetry;

m = bending moment at any point of a structure, with subscripts s and i, respectively, denoting "statically determinate structure" and "due to continuity";

P = elastic load = $\int \frac{m_s\, ds}{E\,I}$;

s = length; ds denotes any elementary segment of any member or substitute member, whose coordinates are x and y, referred to a given point as origin, and whose flexural rigidity is $E\,I$;

V = total vertical reaction of a structure (Eq. 6*b*);

v = vertical reaction of a structure, with subscripts s and i denoting, respectively, "statically indeterminate structure" and "due to continuity";

X = horizontal influence load;

x = distance parallel to the x-axis; x', or x with appropriate subscript, refers to some other origin; $\bar{x}$ denotes a centroidal value of x;

Y = vertical influence load;

y = distance parallel to the y-axis; y', or y with appropriate subscript, refers to some other point as origin; $\bar{y}$ denotes a centroidal value of y;

Z = influence torque;

Δ = static linear deflection, with subscripts s and i denoting "in statically indeterminate structure" and "due to continuity," respectively;

δ = deflection at the supporting point of the "elastic foundation" due to unit influence load or torque acting at the same point; subscripts indicate the directions of the influence load and of the deflections, thus δ_{xy} denotes the y-component of the linear deflection due to influence load X = unity; and

θ = static rotary deflection in the structure.

DISCUSSION

JACK R. BENJAMIN,[9] Assoc. M. ASCE.—The simplicity of the examples somewhat handicaps the paper. The two examples are easier to solve by other techniques; and, in order to show the method, obvious short cuts are not used. As an illustration, note by inspection in Example 1 that:

$$\delta_{yy} = 0 \qquad (13a)$$

$$\delta_{xy} = \delta_{yx} = 0 \qquad (13b)$$

and

$$\delta_{yz} = \delta_{zy} = 0 \qquad (13c)$$

However, to explain the method, simple problems are normally desirable, but not problems that can be solved more simply by other techniques.

Normally the column analogy considers only moment distortion. This is not a requirement. The δ-terms can be made to include direct stress and shear deformations. If the elastic properties are considered sufficiently defined by moment distortions, the load distortions can still be made to include direct stress or shear terms. The modification to consider direct stress can be applied very easily. In Example 1, the direct stress deformation in column CD could be significant. It can be considered as a shortening of CD, and the moments resulting from it can be assumed to be the same as if the support at C settled an equal amount.

The imposition of deformations rather than of forces to determine the rigidity of the substitute member will shorten the work in some cases and give additional information. In the application of relaxation procedures, forces resulting from unit displacements are desired. Structures such as those in Fig. 1 can be solved by a combination of the column analogy with relaxation, or by algebraic procedures in which not only forces but deflections will be simultaneously obtained.

The author is to be complimented in bringing the column analogy again to the attention of the profession. The author's procedure seems to offer the most competition to current techniques and to be at its best when applied to continuous arches on tall flexible piers and similar complicated problems.

LESLIE J. MESZAROS,[10] Assoc. M. ASCE.—Mr. Yu has developed a more direct and easily understood approach to a problem that has been solved in several ways. Among such treatments, for the record, a parallel development by Hardy Cross,[11,12] Hon. M. ASCE, the work of Mr. Cross with N. D. Morgan,[13] M. ASCE, and the work of Alexander Hrennikoff,[14] Assoc. M. ASCE, should be given.

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[9] Asst. Prof. of Civ. Eng., Stanford Univ., Stanford University, Calif.

[10] Civ. Engr., Piedmont, Calif.

[11] *Transactions*, ASCE, Vol. 88, 1925, pp. 1196–1206.

[12] *Ibid.*, Vol. 90, 1927, pp. 1137–1145.

[13] "Continuous Frames of Reinforced Concrete," by Hardy Cross and N. D. Morgan, John Wiley & Sons, Inc., New York, N. Y., 1932, p. 227 and p. 316 et seq., particularly p. 338.

[14] "Analysis of Multiple Arches," by Alexander Hrennikoff, *Transactions*, ASCE, Vol. 101, 1936, p. 388.

A. A. **Eremin**,[15] Assoc. M. ASCE.—In this paper the author has developed the column analogy method of stress analysis in continuous rigid frames and arches. The advantage of the method is that it eliminates a solution of simultaneous equations. It is regrettable that, in illustrating the method, the author has introduced a rigid frame with straight members which may be solved with less effort by various graphical or analytical methods. To illustrate this point, and also to check the method, the writer has solved the frame in Fig. 4 ("Example 1") by the graphical distribution of moments.

The distance a to the point of inflection (in Fig. 9) is determined by

$$a = s\,L \quad \text{(14)}$$

in which s is a constant that varies with the shape and elastic properties of members and can be computed analytically[16] or graphically.[17] The moments in the frame with the joints restrained from translation were constructed by cross lines in Fig. 9.

The basic moments in a member, produced by the transverse displacement of a joint, are

$$M = 6\,r\,N\,a \quad \text{(15)}$$

in which $N = \dfrac{1}{L\,(L - a_1 - a_2)}$; $r = \dfrac{1}{L}$; and $E = 1$. The moments produced by the horizontal displacement of joint D after distribution are shown in Fig. 10. The final moments are obtained by superposing the moments in Figs. 9 and 10, which are close to the moments computed by the author.

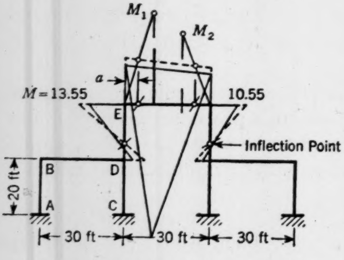

Fig. 9.—Diagram of Bending Moments (Kip-Feet)

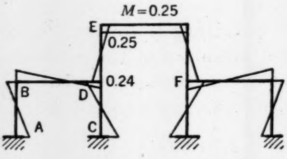

Fig. 10.—Effect of Sidesway at Points D and F

The advantage of the column analogy method applied to a single span is in the simplicity of the rule of signs. In Mr. Yu's method, applied to continuous frames, the rule of signs is quite involved and therefore his method is more complex than other methods in common use.

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[15] Associate Bridge Engr., California State Highways, Bridge Dept., Sacramento, Calif.

[16] *Transactions*, ASCE, Vol. 111, 1946, p. 887.

[17] "Simplified Graphical Distribution of Moments in Rigid Frames," by A. A. Eremin, Sacramento, Calif., 1948.

J. J. Polivka,[18] M. ASCE.—As an effort to simplify the analysis of structures that are statically indeterminate to a high degree, this paper is a valuable contribution. It constitutes an extension of the column analogy method to structures of more than three redundant elements by replacing all parts adjacent to a selected span (or to a structural part) with three or less redundant elements—substituting members of definite elastic properties. As a refinement of the column analogy method, this procedure is new; but the principles enumerated by the author in the "Synopsis" (steps (*a*) to (*d*)) have been available since 1920.[19] Furthermore, the column analogy method is basically an adaptation of the ellipse of elasticity method introduced and developed by C. Culmann[20] in 1875 and W. Ritter[21] in 1888. The author limits his application to a general solution of continuous arches and bents and the tracing of influence lines, whereas a more basic approach[19] will permit the solution of cases in which the effect of shear and direct stress on elastic deformations is to be appraised.

In the paragraph preceding Example 1, Mr. Yu states that: "* * * it is not necessary (nor is it possible) to know even the location of * * *" the axis of the substitute member. That it is possible to know the location of this axis is demonstrated by Fig. 11, which represents the central frame DEGF in Fig. 4. This free body rests on elastic foundations at points D and F, which have been replaced by the substitute members DD′ and FF′. The elastic area a_d—that is, the elastic weight of the substitute member DD′—is equal to 21/208 = 0.100961. (The author computed 0.1005.) It is concentrated at the centroid O_D, along the vertical axis of the supporting column CD

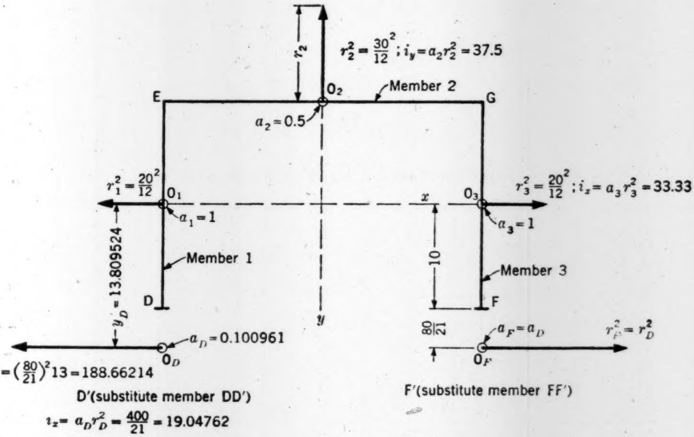
r_2
$r_2^2 = \frac{30^2}{12}; i_y = a_2 r_2^2 = 37.5$
E
O_2
Member 2
G
$a_2 = 0.5$
$r_1^2 = \frac{20^2}{12}$
O_1
$a_1 = 1$
x
O_3
$r_3^2 = \frac{20^2}{12}; i_x = a_3 r_3^2 = 33.33$
$a_3 = 1$
$y_D = 13.809524$
Member 1
10
Member 3
D
F
y
$a_D = 0.100961$
O_D
$\frac{80}{21}$
$a_F = a_D$
$r_F^2 = r_D^2$
O_F
$r_D^2 = (\frac{80}{21})^2 13 = 188.66214$
D′(substitute member DD′)
$i_x = a_D r_D^2 = \frac{400}{21} = 19.04762$
F′(substitute member FF′)

Fig. 11

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[18] Cons. Engr., Berkeley, Calif.

[19] "Continuous Arches on Elastic Supports (Elastic Foundations)," by J. J. Polivka, *Der Brueckenbau*, Heidelberg, 1920, Vols. 4, 5, and 6.

[20] "Graphic Statics," by C. Culmann, Zurich, Meyer & Zeller, 2d Ed., 1875.

[21] "Application of Graphic Statics," by W. Ritter, Meyer & Zeller, Zurich, 1888.

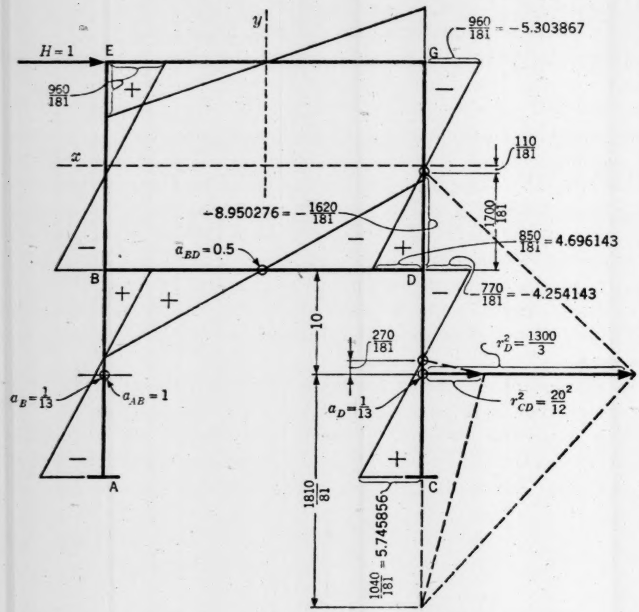
y
H = 1
E
G
$-\frac{960}{181} = -5.303867$
$\frac{960}{181}$
+
−
x
$\frac{110}{181}$
$\frac{1700}{181}$
$-8.950276 = -\frac{1620}{181}$
$a_{BD} = 0.5$
−
−
+
$\frac{850}{181} = 4.696143$
B
D
$-\frac{770}{181} = -4.254143$
+
+
−
10
$\frac{270}{181}$
$r_D^2 = \frac{1300}{3}$
$a_B = \frac{1}{13}$
$a_{AB} = 1$
$a_D = \frac{1}{13}$
$r_{CD}^2 = \frac{20^2}{12}$
−
+
A
C
$\frac{1810}{81}$
$\frac{1040}{181} = 5.745856$

FIG. 12

TABLE 6.—ELASTIC LOAD AND FORMULA FOR m_i, EXAMPLE 3

Bar	a	x	y	$a\,x$	$a\,y$	$a\,x^2 + i_y$	$a\,y^2 + i_x$	P	x	y	$P\,x$	$P\,y$
BE	1.00	−15	0	−15	0	225	$\frac{100}{3}$	−10	−15	$-\frac{10}{3}$	+150	$+\frac{100}{3}$
EG	0.50	0	+10	0	+5	37.5	50	...	...	...	...	...
GD	1.00	+15	0	+15	0	225	$\frac{100}{3}$	...	...	...	...	...
BB′	$\frac{1}{13}$	−15	−20	$-\frac{15}{13}$	$-\frac{20}{13}$	$\frac{225}{13}$	$+\frac{400}{13} + \frac{100}{3}$	$-\frac{30}{13}$	−15	$-\frac{310}{9}$	$+\frac{450}{13}$	$+\frac{3{,}100}{39}$
DD′	$\frac{1}{13}$	+15	−20	$+\frac{15}{13}$	$-\frac{20}{13}$	$\frac{225}{13}$	$+\frac{400}{13} + \frac{100}{3}$	...	...	...	...	...
Σ	$\frac{69}{26}$	0	$\frac{50}{69}$	0	$\frac{25}{13}$	$\frac{13{,}575}{26}$	$\frac{9{,}550}{39}$	$-\frac{160}{13}$	...	...	$+\frac{2{,}400}{13}$	$+\frac{4{,}400}{39}$
Correction to centroidal axis........						...	$-\frac{1{,}250}{897}$	...	...	$+\frac{8{,}000}{897}$	...	$\left\{ \begin{array}{l} +\frac{500}{69} \\ +\frac{1{,}500}{897} \end{array} \right.$
Corrected values to centroid.........						...	$+\frac{5{,}600}{23}$	...	...	...	...	$+\frac{2{,}800}{23}$

Moments of internal stresses (forces): $m_i = -\frac{320}{69} + \frac{64}{181}\,x + \frac{1}{2}\,y$

(Fig. 4), the ordinate y_D being equal to $-290/21 = -13.80952$. This value can be computed from Table 3*A*. For example, the second moment $a\,y^2$ of the substitute member DD′ is 19.1. Since $a_D\,y^2_D = 19.1$, $y_D = \sqrt{19.1/0.1005} = 13.785$, which checks the author's computation. Similarly, from Fig. 11, $r^2_D = (80/21)^2 \times 13 = 188.66214$, checking with the author's value of $19.1/0.1005 = 190.04975$.

TABLE 7.—MOMENTS $M = m_s - m_i$, EXAMPLE 3

Moment	x	y	m_s	m_i	$M = m_s - m_i$
M_{EB}................	-15	$+\frac{640}{69}$	0	$-\frac{960}{181}$	$+\frac{960}{181}$
M_{BE}................	-15	$-\frac{740}{69}$	-20	$-\frac{2{,}770}{181}$	$-\frac{850}{181}$

$$M_{BA} = \frac{60}{13} - \frac{850}{181 \times 13} = +\frac{770}{181};\; M_{AB} = -\frac{70}{13} - \frac{850}{181 \times 13} = -\frac{1{,}040}{181}$$

In introducing Fig. 1, the author states: "The extension [of the column analogy] is not applicable to frames with closed panels * * *." Fig. 12 demonstrates the analysis of a two-story frame with rectangular panels ABEGDC, one of which is closed. A horizontal force $H = 1$ acts at point E, as shown. The characteristics of the substitute members BB′ and DD′ can be computed from data in Table 2*C*. In this case also, a more accurate result is obtained by ratios. For example, referring to Fig. 10, $a_B = a_D = (17 - 9)/104 = (30 - 22)/104 = 1/13 = 0.0769231$. The author reports: $0.163 - 0.087 = 0.076$. Since $17 - 9 = 30 - 22$, the elastic centroids of the substitute members are at the midpoints of columns AB and CD.

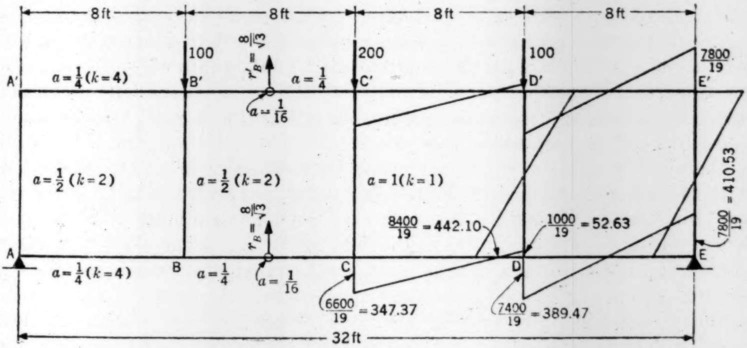
8 ft
8 ft
8 ft
8 ft
100
200
100
$\frac{7800}{19}$
$r'_B = \frac{8}{\sqrt{3}}$
A′
$a = \frac{1}{4}(k=4)$
B′
$a = \frac{1}{4}$
C′
D′
E′
$a = \frac{1}{16}$
$a = \frac{1}{2}\,(k=2)$
$a = \frac{1}{2}\,(k=2)$
$a = 1(k=1)$
$\frac{7800}{19} = 410.53$
$r_B = \frac{8}{\sqrt{3}}$
$\frac{8400}{19} = 442.10$
$\frac{1000}{19} = 52.63$
A
$a = \frac{1}{4}(k=4)$
B
$a = \frac{1}{4}$
$a = \frac{1}{16}$
C
D
E
$\frac{6600}{19} = 347.37$
$\frac{7400}{19} = 389.47$
32 ft

FIG. 13

The second moment, i_x, of the elastic area is equal to $a_B\,r^2_B$. The displacement δ_{xx} along BD is equal to $\frac{400}{3} - \frac{1{,}200}{13} = \frac{1{,}600}{39} = 41.025641$, as compared with $2 \times 20.5 = 41$ in Table 2*C*. With the knowledge that $\delta_{xx} = 1 \times 10 \times a_B \times d$, and $(d - 10)\,10 = r^2_D$, one can compute: $d = 160/3$;

$r^2_B = 1{,}300/3$; and $i_x = 100/3$. The complete analysis of the two-story frame is given in Tables 6 and 7. Table 6 is similar to Table 3*A*.

Another example of a continuous frame with closed panels is the Vierendeel open web truss (see Fig. 13). The characteristics of the substitute members are $r^2_B = 64/3$; i_x, the second moment, is 4/3; and, a, the elastic area concentrated at the midpoints of the chords, is 1/16. The analysis is similar to that in Fig. 12, and the results in Fig. 13 check the solution of this problem presented by A. Amirikian, M. ASCE, in 1942.[22]

Powell P. W. Tsu.[23]—The analysis of an indeterminate structure is essentially a study of the relation between the elastic deformations of each member comprising it, and the selection of sections that will maintain structural continuity. In the case of plane structures, these elastic deformations occur in response to moment, axial thrust, and shear. Except where the member is short and deep, the elastic deformations resulting from axial thrust and shear are generally small and negligible as compared to those produced by bending moment. Following this line of reasoning, the column analogy, as developed by Hardy Cross, Hon. M. ASCE, results in a simple formula for continuity moments, in which the deformations from axial thrust and shear are neglected. The method is useful in the solution of a three-redundant structure because it is susceptible to a systematic computation when arranged in tabular form.

The extension of the column analogy to the analysis of continuous structures, as presented by Mr. Yu, is a contribution to engineering literature despite the fact that its range of application is limited to certain cases, the same as many other methods and modifications in the field of structural analysis.

In its application to the analysis of arch spans (particularly when the arch is flat), the column analogy is good only for preliminary investigations because it neglects the effect of axial deformation, an omission that seems unwarranted. In other words, unless axial deformation can be taken into consideration, it is questionable if the author's conclusion (that his method is especially useful as a device for reviewing designed structures) is justified in the case of arch spans.

In Example 2, the author offers a continuous bent for analysis. The same problem was solved by the writer using the method of moment distribution. With the aid of an ordinary slide rule, it required only 8 hours to finish the entire computation, including the values of influence ordinates for moment, thrust, and shear, corresponding to the data in Fig. 8. Although a comparison with the author's working time is of no great consequence (the speed of computation depending to a large measure on the degree of acquaintance with each particular method), the relative accuracy of results is a proper criterion. Since presentation of other methods is beyond the scope of discussion, the writer's computations cannot be presented; but the results show, invariably, a smooth curve in influence ordinates for each stress function whereas the author's results show marked divergence in the remotest spans. The errors are most likely mistakes in computation because systematic errors in computation would produce influence ordinates that revealed a general disagreement, rather than the isolated ones actually produced. It is clear that the column analogy is limited as a basis for analysis because it offers no method of applying an independent check. Mistakes in computation will be carried along and thrown into the final result unless (as demonstrated by the author) they can be discovered by the equality of reciprocal deflections, which is a check not available until the computation is completed. For the case of a long series of continuous spans, the successive combination of spans necessary to arrive at a single span wherein the particular stress functions are to be studied would entail so many tabulations that a slight mistake in computation, if not rigidly checked at the end of each process of combination, may result in a final, cumulative, gross error.

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[22] "Analysis of Rigid Frames (An Application of Slope Deflection)," by A. Amirikian, U. S. Govt. Printing Office, Washington, D. C., 1942, p. 129.

[23] Design Engr., U. S. Naval Port Facilities, Shanghai, China.

Furthermore, in getting influence lines by using the Mueller-Breslau principle, the computation of deflection ordinates in remote spans, due to unit strains corresponding to the desired influence functions, permits a rather high degree of precision as shown by the significant data in Fig. 7. In the writer's solution of the same problem, a uniform degree of precision was controlled by using an ordinary slide rule. In tracing influence lines for stress functions at sections other than those illustrated by the author, it is necessary to find as many independent sets of indeterminate moments by column analogy (moments due to applied unit strains at a particular section) as there are numbers of sections intersected, and then to compute the deflections. By moment distribution, it is only a matter of simple arithmetic applied to the principle of statics, to compute influence ordinates at any number of sections, using only the influence lines for fixed-end moments and the joint moments distributed from a joint with an arbitrary unbalanced moment.

The column analogy computation gives the combined effects of sidesway and joint rotation but it affords no clue as to the proportions between them. Moment distribution yields that result directly and is much preferred.

Example 1 in the paper shows clearly the merit of Mr. Yu's method. The directness with which the substitute members are found simplifies the solution. The same problem, if solved by the method of moment distribution, would require two sidesway corrections and two simultaneous equations, making it perhaps less favorable than the author's method.

To summarize, the writer feels that, although the author's contribution to engineering literature is highly commendable, its field of application in the analysis of continuous structures is limited, being neither as rapid nor as accurate as many other methods.

T. M. Yu.[24]—The several discussers have added valuable material and constructive criticism to this paper.

Replying to Mr. Benjamin—an author is justified in demonstrating the application of a proposed method by simple examples, even if those examples, by themselves, can be solved more easily by other techniques. Implicit in the use of simple examples is the claim that, having mastered the application of a method by given examples, the analyst can himself apply the basic procedure to more complicated cases. The merits or demerits of a particular method were on trial in this paper, and an extensive description of other methods, necessarily, must be considered beyond the scope of discussion. Nevertheless, it would have given readers a basis for making their own comparisons if Mr. Benjamin had simply named the "other techniques" that he had in mind.

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[24] Prof. of Civ. Eng., Chiao-Tung Univ., Shanghai, China.

The presentation might have appeared in a better light had the writer elected to outline the short cuts inherent in his method; but to do so might also have been at the expense of a firmer basic understanding of all facets of a new subject. Short cuts come with practice, and in a competition with "other techniques," the method described in this paper will be found adaptable in this sense also.

Mr. Benjamin points out correctly that the deflection terms can be made to include direct stress and shear deformations. Messrs. Polivka and Tsu also mention these deformations. In fact, the writer makes no assumption as to the effect of these two distortions; if the effect of "arch shortening" is considered, it is believed that the computation will be similar to that in Table 5*B*, under $M_x = 1$.

Mr. Benjamin prefers the relaxation method[25] by the use of a "unit problem"—that is, he prefers to find the component forces resulting from the imposed unit distortions. It is correct that influence coefficients or block properties can often be computed easily by the column analogy method, which the writer is inclined to believe is merely a modification of the relaxation method. Since the labor of computation involved in the relaxation procedure depends almost entirely on the number of joints, a simpler solution is hardly to be expected.

Mr. Meszaros contributes four references of which the first two, by Hardy Cross, Hon. M. ASCE, are the discussions in connection with the "neutral point method"; the third, by Mr. Cross and N. D. Morgan, M. ASCE, is "a modification of the method of distributing fixed-end moments and thrust";[26] and the fourth, by Alexander Hrennikoff, Assoc. M. ASCE, is again, "based on the well-known principle of moment distribution." These references contain much valuable information, but they do not extend the column analogy method in the sense defined in the paper. In reviewing designs by different methods, it is not unusual to find points of similarity and correlation. For instance, the concept of the substitute member (which is basic to the theory of this paper) has been mentioned one way or another by the late C. B. McCullough, M. ASCE, and E. S. Thayer,[27] Fang-Yin Tsai, M. ASCE, and Mr. Polivka. Mr. Tsai published a paper on the subject (in Chinese) in about 1936. The particular development in the present paper was the independent work of the writer.

Mr. Eremin presents a graphical method for the solution of Example 1. To those who prefer graphical methods to analytical methods, this should be very valuable. He concludes his discussion with the statements: "The advantage of the column analogy method applied to a single span is in the simplicity of the rule of signs," but "* * * applied to continuous frames, the rule of signs is quite involved * * *." The writer disagrees with this view and would like to emphasize the fact that the sign convention in the paper is exactly according that used by Mr. Cross.[5] So far as the sign is concerned, there is no difference between the column analogy method for a single span, and its extension when applied to the continuous frame. In this connection, the writer believes that the sign convention will be no more involved in its application to the examples of the paper than when applied to a single span.

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[25] "Relaxation Methods in Engineering Science," by R. V. Southwell, Clarendon Press, Oxford, England, 1940.

[26] "Continuous Frames of Reinforced Concrete," by Hardy Cross and N. D. Morgan, John Wiley & Sons, Inc., New York, N. Y., 1932, p. 316.

[27] "Elastic Arch Bridges," by C. B. McCullough and E. S. Thayer, John Wiley & Sons, Inc., New York, N. Y., 1931.

Mr. Polivka is correct in stating that it is possible to locate the centroid of the elastic area of the substitute member. The writer's slide rule calculations appear in his demonstration (Example 1) of the formulas in Table 1 and in Eqs. 7 to 9; thus: $\bar{x} = \frac{\delta_{yz}}{\delta_{zz}} = 0$; $x_D = 0 - 15 = -15$; $x_F = 0 + 15 = +15$; $\bar{y} = -\frac{\delta_{xz}}{\delta_{zz}} = -3.78$; and $y_D = y_F = -3.78 - 10 = -13.78$.

However, it is impossible to know the location of the axis of the substitute member itself, since the hypothetical shape of the substitute member is immaterial. Thus, in Fig. 2(*b*), the substitute member BB′ can be of any shape (Fig. 14, for example).

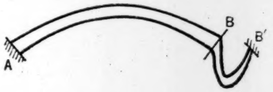

FIG. 14

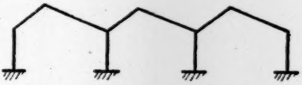

FIG. 15

Mr. Polivka's application of the method to frames with closed panels is a very welcome addition to this paper. He avoids simultaneous equations entirely, and his computation is straightforward and automatic.

In private correspondence with the writer, Mr. Polivka points out that all principles and rules of the ellipse of elasticity hold for the "extended" column analogy. The ellipse of elasticity is defined, in general, by two principal semiaxes. The short semiaxis of the ellipse for a substitute member is equal to zero as the effect of axial thrust is not considered in this paper.

Mr. Tsu, in demonstrating the effect of rib shortening, states that:

> "* * * (particularly when the arch is flat), the column analogy is good only for preliminary investigations because it neglects the effect of axial deformations * * *."

In this statement Mr. Tsu is correct; but the writer wishes to stress the fact that, as in the case of suspension bridges of small sag, flat arches (continuous or single span) should be analyzed by the "deflection theories" of E. Melan, H. Engesser, S. Kasarnowsky, B. Fritz,[28] A. Freudenthal,[29] M. ASCE, and others. In Mr. Tsu's example, neither the column analogy method nor the moment distribution method would be of much value.

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[28] "Theorie und Berechnung vollwandiger Bogentraeger bei Beruecksichtigung des Einflusses der Systemverformung," by B. Fritz, Springer, Berlin, 1934.

[29] "Methods of the Theory of Plasticity in the Investigation of Statically Indeterminate Supporting Structures of Reinforced Concrete," by Alfred Freudenthal, ***Memoires***, International Assn. for Bridge and Structural Eng., Zurich, Vol. 2, 1933–1934, pp. 180–192.

Mr. Tsu's remarks on the working speed, relative accuracy of results, and the danger of arithmetical errors, as stated in the fourth paragraph of his discussion, are probably correct. As the column analogy is a tool the writer believes it should be as systematic and mechanical as possible to increase the working speed and to diminish errors.

Mr. Tsu's criticism of the relative degrees of accuracy attained in the example given in Fig. 7 is rather beside the point. In the example, the writer endeavored to give a fairly accurate picture of the influence line, not only for the span BD but also for the adjacent span DF and the next span FH. To the practical engineer, probably only span BD (and possibly span DF as well) needs to be investigated for influence lines (see Fig. 6). The degree of accuracy warranted and, therefore, the amount of labor involved depend on whether an accurate picture of the influence line of the remotest span FH is desired, but not on whether "column analogy" or "moment distribution" is applied in computing it.

Mr. Tsu points out very clearly that the moment distribution method is rather involved in its application to Example 1. The writer is also inclined to believe that the extended column analogy explained in the paper may be advantageous in solving such continuous bents as shown in Fig. 15. A series of roof bents is a case in point; and the problem of continuous arches on slender piers is only a special case of such continuous bents.

Acknowledgment.—Under current difficulties of communication, the problem of making a direct response to discussers has been an extremely complex one. The writer acknowledges with thanks the willingness of S. S. Kuo of the Architect's Office, Ohio State University at Columbus, to act in his stead in the compilation of this reply from such letters as he has been enabled to transmit in the discussion period. The writer is also indebted to George E. Large, M. ASCE, and C. B. Smith, Assoc. M. ASCE, of Ohio State University, for valuable assistance in the preparation of this closure.

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DIFFUSION OF SUBMERGED JETS

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WITH DISCUSSION BY MESSRS. JAMES S. HOLDHUSEN, DUILIO CITRINI, STANLEY CORRSIN, W. DOUGLAS BAINES, ABRAHAM STREIFF, HAROLD R. HENRY, AND M. L. ALBERTSON, Y. B. DAI, R. A. JENSEN, AND HUNTER ROUSE.

SYNOPSIS

As the direct result of turbulence generated at the borders of a submerged jet, the fluid within the jet will undergo both lateral diffusion and deceleration, and at the same time fluid from the surrounding region will be brought into motion. The approximate characteristics of the corresponding mean flow pattern are derived analytically, with the exception of a single experimental constant, through assumptions that: (1) The pressure is hydrostatically distributed throughout the flow; (2) the diffusion process is dynamically similar under all conditions; and (3) the longitudinal component of velocity within the diffusion region varies according to the normal probability function at each cross section. Experimental data are presented which justify the analysis and provide the necessary coefficients for flow from both slots and orifices. All results are reduced to a form immediately useful for design purposes.

STATEMENT OF THE PROBLEM

As far as most engineers are concerned, high-velocity flow from a submerged outlet represents merely an irrecoverable loss of power, for a basic axiom of hydraulics states that the entire kinetic energy of such a jet will be dissipated through reaction with the surrounding fluid. In more explicit terms, the difference in velocity between a jet and the region into which it is discharged will give rise to a pronounced degree of instability, the kinetic energy of the oncoming flow steadily being converted into kinetic energy of turbulence, and the latter steadily decaying through viscous shear. However, such conversion is by no means restricted to the vicinity of the outlet, for any reduction in kinetic energy necessarily represents a decrease in the velocity of flow, and even the

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NOTE.—Published in December, 1948, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

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most elementary considerations of continuity indicate that the area of the flow section must become great as the velocity becomes small. In view of the Newtonian principle of action and reaction, moreover, it will be realized that deceleration of the fluid in the jet can occur only through simultaneous acceleration of the surrounding fluid, so that the total rate of flow past successive sections of the jet will actually increase with distance from the outlet.

The phenomenon of "exit loss" so simply treated in hydraulics evidently involves far more than a local energy conversion. Indeed, a detailed knowledge of the velocity distribution within such an expanding stream is essential to many problems of diffusion. Flow of a liquid suspension from a submerged pipe into a settling basin is a case in point, although a closely related phase of mixing is found in the efflux of heated or cooled air from a ventilation duct in the wall of a room; in either case the engineer is interested in the shape and the "carry" of the jet, and in the amount of entrainment by (and hence dilution of) the jet from section to section. Directly related, although more closely restricted by boundary (that is, free-surface) conditions, is the diffusion of plunging nappes from spillways or of submerged jets from sluice gates, because information as to the accompanying rate of deceleration is often necessary in the prevention of scour beyond the apron. Finally, since the slip stream of a propeller differs little from any other type of jet, the velocity distribution in the wake of aircraft and watercraft (a matter of vital importance, for example, in submarine detection) should be subject to the same method of analysis.

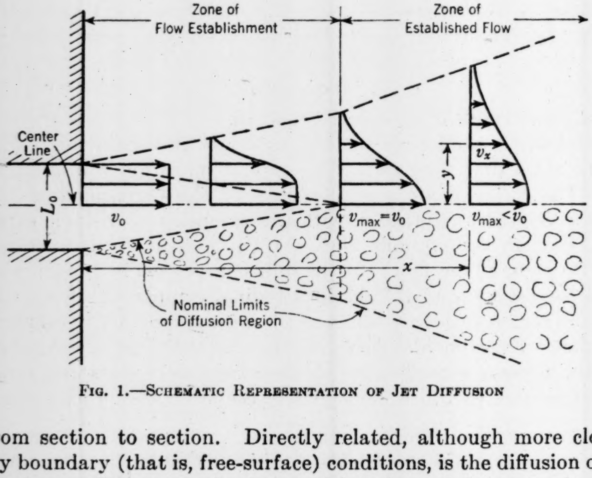

FIG. 1.—SCHEMATIC REPRESENTATION OF JET DIFFUSION

With reference to Fig. 1, it will be seen that an initial zone of flow establishment must exist beyond the efflux section of either a two-dimensional or a three-dimensional submerged jet. Since the fluid discharged from the boundary opening may be assumed to be of relatively constant velocity, at the efflux section there will necessarily be a pronounced velocity discontinuity between the jet and the surrounding fluid. The eddies generated in this region of high shear will immediately result in a lateral mixing process which progresses both inward and outward with distance from the efflux section. Such lateral mixing produces a necessarily balanced action and reaction: On the one hand, the fluid within the jet is gradually decelerated; on the other hand, fluid from the surrounding region is gradually accelerated or entrained. As a result, the constant-velocity core of the jet will steadily decrease in lateral extent, whereas both the rate of flow and the over-all breadth of the jet will steadily increase in magnitude, with distance from the efflux section. The limit of this initial zone of flow establishment is reached when the mixing region has penetrated to the center line of the jet.

Once the entire central part of the jet has become turbulent, the flow may be considered as fully established, for the diffusion process continues thereafter without essential change in character. Further entrainment of the surrounding fluid by the expanding eddy region is now balanced inertially by a continuous reduction in the velocity of the entire central region. Such variation approaches (but never actually attains) a limit as the center-line velocity becomes of negligible magnitude a very great distance from the original efflux section, the lateral extent of the eddy region—and hence the quantity of fluid within the jet—then being wholly out of proportion to conditions at the initial section.

Just as the theoretical limit of the zone of established flow is evidently nonexistent, the border section between this zone and the zone of establishment (indicated by the vertical broken line in Fig. 1) can be considered sharply defined only to the extent to which one may neglect the zone of transition which necessarily exists between two distinct flow regimes. There is, in fact, no precise point at which the eddies from opposite sides of the jet can be said to meet, for the statistical nature of the mixing process makes it impossible to place more than an arbitrary limit on its lateral range. The boundaries of the diffusion region, like the border between the two zones, must therefore be accepted merely as convenient nominal designations.

Conditions within the zone of flow establishment and the zone of established flow were first investigated theoretically by W. Tollmien in 1926,[5] on the assumptions that: (*a*) The sole effective force was the tangential shear expressed in terms of the lateral momentum transport of the mixing process; (*b*) the mixing length varied with the first power of the longitudinal distance from the efflux section; and (*c*) the velocity of the turbulence was proportional to the product of the mixing length and the mean velocity gradient. With the exception of a numerical coefficient which had to be evaluated experimentally, Mr. Tollmien was able to derive a stream function which indicated with good approximation the actual pattern of the mean flow in each region.[6] Later studies by other investigators[7,8,9] introduced either the concept of vorticity transport or the assumption of constant eddy viscosity across any section of the diffusion zone, but did not add materially to the agreement with experimental measurements. As a matter of fact, the sensitivity of the mean velocity-distribution function to the type of turbulence structure assumed is not great; the comparison between measurement and theory has, as a result, never been sufficient to provide a conclusive check on the accuracy of the assumed mixing characteristics. Indeed, not until detailed measurements of the turbulence within two-dimensional and three-dimensional jets were made by S. Corrsin[10] and by H. W. Liepmann and J. Laufer[11] was the considerable discrepancy between assumption and fact fully realized.

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[5] "Berechnung turbulenter Ausbreitungsvorgänge," by W. Tollmien, *Zeitschrift für angewandte Mathematik und Mechanik*, Vol. 6, 1926, p. 468.

[6] "Turbulente Ausbreitungsvorgänge im Freistrahl," by P. Ruden, *Die Naturwissenschaften*, Vol. 21, 1933, p. 375.

[7] "Investigations of the Turbulent Mixing Regions Formed by Jets," by A. M. Keuthe, *Journal of Applied Mechanics*, September, 1935, p. A-87.

[8] "Application of the Modified Vorticity Transport Theory to the Turbulent Spreading of a Jet of Air," by S. Tomotika, *Proceedings*, Royal Soc. of London, Vol. 165, Series A, March-April, 1938, p. 65.

[9] "Concerning the Velocity and Temperature Distribution in Plain and Axially Symmetrical Jets," by L. Howarth, *Proceedings*, Cambridge Philosophical Soc., Vol. 34, 1938, p. 185.

Since an approximate analysis of the mean velocity distribution within either zone actually requires no assumption as to the distribution of turbulence, and since the characteristics of jet diffusion have not yet been published in a form readily usable by engineers, this paper seeks to fulfil this twofold purpose. The dimensional aspects of the phenomenon are first discussed, followed by an elementary physical analysis of the mean flow pattern. Experimental data covering a considerable range of each independent variable for both two-dimensional and three-dimensional flow are then presented to justify the analysis and to provide the single numerical coefficient still required for each case. All results are finally summarized in a convenient dimensionless form.

ANALYSIS OF THE MEAN FLOW PATTERN

General Considerations.—If the Reynolds number for fluid efflux from a submerged boundary outlet is not too low, the mean velocity v at any point (see Fig. 1) should depend only on the coordinates x, y, and z, on the efflux velocity v_0, and on a linear dimension L_0 characterizing the particular outlet form. These variables may be grouped in the dimensionless relationship:

$$\frac{v}{v_0} = f_1\left(\frac{x}{L_0}, \frac{y}{x}, \frac{z}{x}\right) \qquad (1)$$

This relationship must be considered to involve the magnitude and the direction of the vector v, the components of which may further be related through the differential equation of continuity:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 \qquad (2)$$

The rate of flow or volume flux Q past successive normal sections may be written as the integral of the differential flux $v_x\,dA$ over any normal section. Since, because of entrainment, Q will vary with the longitudinal distance x from the efflux section, its ratio to the efflux rate Q_0 may be expected to have the functional form:

$$\frac{Q}{Q_0} = \frac{\int_0^\infty v_x\,dA}{v_0\,A_0} = f_2\left(\frac{x}{L_0}\right) \qquad (3)$$

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[10] "Investigation of Flow in an Axially Symmetrical Heated Jet of Air," by S. Corrsin, *Advance Confidential Report No. 3L23*, National Advisory Committee for Aeronautics, Washington, D. C., 1943 (*Wartime Report W-94*).

[11] "Investigations of Free Turbulent Mixing," by H. W. Liepmann and J. Laufer, *Technical Note No. 1257*, National Advisory Committee for Aeronautics, Washington, D. C., August, 1947.

in which A_0 is the cross-sectional area of the outlet. Similarly, since the momentum flux M may be written as the integral of the volume flux $v_x\,dA$ times the longitudinal component of momentum per unit volume $\rho\,v_x$, ρ being the fluid density, the ratio of M for any section to M_0 for the efflux section should be

$$\frac{M}{M_0} = \frac{\int_0^\infty (v_x)^2\,dA}{(v_0)^2\,A_0} = f_3\left(\frac{x}{L_0}\right) \qquad (4)$$

Finally, since the energy flux E is expressible as the integral of the volume flux $v_x\,dA$ times the kinetic energy per unit volume $\rho\,v^2/2$, the ratio of E for any section to E_0 for the boundary outlet should take the form:

$$\frac{E}{E_0} = \frac{\int_0^\infty v^2\,v_x\,dA}{(v_0)^3\,A_0} = f_4\left(\frac{x}{L_0}\right) \qquad (5)$$

Such general considerations will yield no further clue to the actual functional relationships without specific statements as to the dynamics of the flow. For example, on the basis of experimental evidence it may safely be presumed that the pressure distribution is essentially hydrostatic throughout the zone of motion—in other words, that the sole force producing the deceleration of the jet and the acceleration of the surrounding fluid is the tangential shear within the mixing region. Because this process is wholly internal, it follows at once that the momentum flux must be a constant for all normal sections of a given flow pattern:

$$\frac{M}{M_0} = \frac{\int_0^\infty (v_x)^2\,dA}{(v_0)^2\,A_0} = 1 \qquad (6)$$

If, moreover, viscous action is presumed to have no influence on the mixing process, the diffusion characteristics—and hence the characteristics of the mean flow—should be dynamically similar under all conditions. Thus, in effect, the same velocity function must characterize every section within the diffusion region. As a matter of fact, experimental data follow the general trend of the Gaussian normal probability function:

$$\frac{v_x}{v_{\max}} = \exp\left(-\frac{y^2}{2\,\sigma^2}\right) \qquad (7)$$

in which the right-hand member expresses the value e^n, the quantity in parentheses being the value of n. (A similar simplification applies subsequently in Eqs. 12, 19, 24, and 29.) The use of Eq. 7 in the analysis permits the characteristics of the entire flow pattern to be expressed in terms of the two parameters which define the proportions of the curve: the velocity $v_{\max}$ and the standard or root-mean-

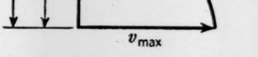

FIG. 2.—CHARACTERISTICS OF THE NORMAL-PROBABILITY CURVE

square deviation σ (see Fig. 2). Moreover, not only does Eq. 1 then reduce to

$$\frac{v_{\max}}{v_0} = f_5\left(\frac{x}{L_0}, \frac{\sigma}{x}\right) \qquad (8)$$

but the condition of dynamic similarity simultaneously requires that at all cross sections, regardless of the efflux velocity,

$$\frac{\sigma}{x} = C \qquad (9)$$

In other words, the angle of jet diffusion must be constant.[11a]

Through use of Eqs. 6, 7, and 9, all characteristics of the mean flow pattern for any specific boundary condition may now be determined analytically, with the exception of a single experimental coefficient. Such characteristics include the variation in the magnitude and direction of the velocity vector with coordinate space, and the variation of volume flux and energy flux with distance from the outlet section. In the following pages these characteristics are evaluated for the zone of flow establishment and for the zone of established flow under both two-dimensional and three-dimensional conditions.

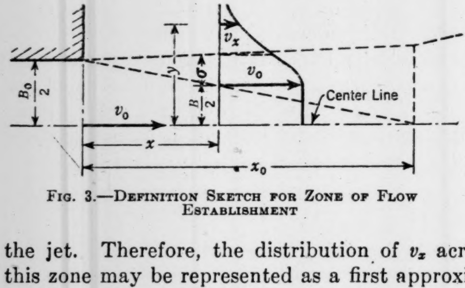

FIG. 3.—DEFINITION SKETCH FOR ZONE OF FLOW ESTABLISHMENT

Zone of Flow Establishment.—In the case of efflux from a long slot, it will be noted from Fig. 3 that $v_{\max}$ will correspond to v_0 as long as the eddies have not penetrated to the central plane of the jet. Therefore, the distribution of v_x across any normal section within this zone may be represented as a first approximation by the two symmetrical halves of the probability curve connected with a straight line through the constant-velocity core. Evaluation of the momentum-flux integral of Eq. 6, for the condition that $\sigma/x = C_1$, will be found to yield at once the specific relationship:

$$\frac{x_0}{B_0} = \frac{1}{\sqrt{\pi}\, C_1} \qquad (10)$$

in which x_0 represents the distance from the efflux section to the end of the zone of establishment. Evidently, the nominal inner boundary of the diffusion region is a plane satisfying the expression:

$$\frac{B}{B_0} = 1 - \frac{x}{x_0} \qquad (11)$$

The distribution function for v_x in the diffusion region then takes the explicit form:

$$\frac{v_x}{v_0} = \exp\left[-\frac{\left(y + \sqrt{\pi}\, C_1 \frac{x}{2} - \frac{B_0}{2}\right)^2}{2\,(C_1\, x)^2}\right] \qquad (12)$$

[11a] See revised statement in the closing discussion.

Similar evaluation of the volume-flux integral of Eq. 3, after introduction of v_x from Eq. 12, will result in the relationship:

$$\frac{Q}{Q_0} = 1 + \sqrt{\pi}\,(\sqrt{2} - 1)C_1 \frac{x}{B_0} \ldots\ldots\ldots\ldots\ldots \quad (13)$$

Inasmuch as the rate of change in volume flux with longitudinal distance dQ/dx must equal twice the velocity of entrainment (that is, the velocity v_y at a considerable distance y from the jet), it follows that

$$\operatorname*{Limit}_{y \to \infty} \frac{v_y}{v_0} = -\tfrac{1}{2}\sqrt{\pi}\,(\sqrt{2} - 1)C_1 \ldots\ldots\ldots\ldots\ldots \quad (14)$$

The same result would be obtained as a limit through solution of the differential equation of continuity for v_y as a function of x and y; however, since the latter function must be shown graphically rather than algebraically, it is not introduced at this point. (All detailed evaluations of this nature have been deposited with the Engineering Societies Library for reference.)

Although the energy-flux integral properly involves the magnitude of the velocity vector in the term for kinetic energy per unit volume, the vector magnitude may be expected to differ little from that of its longitudinal component except in the outer regions, and the contribution of these regions to the total flux should be relatively small. The energy-flux ratio may therefore be approximated in terms of v_x, as follows:

$$\frac{E}{E_0} = 1 + \sqrt{\pi}\,\left(\sqrt{\tfrac{2}{3}} - 1\right)C_1 \frac{x}{B_0} \ldots\ldots\ldots\ldots\ldots \quad (15)$$

Flow from a circular orifice should differ from the foregoing only in the substitution of the orifice diameter D_0 for the slot width B_0 and the substitution of the radial distance r for the lateral distance y in the basic equations, the integration being performed over the corresponding circular areas for the condition that $\sigma/x = C_2$. At the limit $x = x_0$, the momentum relationship at once results in

$$\frac{x_0}{D_0} = \frac{1}{2\,C_2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (16)$$

which is a counterpart of Eq. 10. However, the general solution of the momentum relationship yields an expression which—in contradistinction to Eq. 11 for the two-dimensional case—indicates that the diameter of the nominal inner border of the diffusion region for three-dimensional flow is not a linear function of x; that is,

$$\frac{D}{D_0} = \sqrt{1 + (\pi - 4)(C_2)^2 \frac{x^2}{(D_0)^2}} - \sqrt{\pi}\, C_2 \frac{x}{D_0} \ldots\ldots\ldots \quad (17)$$

Fortunately, the difference between this curvilinear function and the linear approximation

$$\frac{D}{D_0} = 1 - \frac{x}{x_0} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (18)$$

is probably no greater than the discrepancy between the actual velocity distribution and the assumed probability curve. For all practical purposes it is sufficient, therefore, to write the velocity distribution within the region of diffusion in the simplified form:

$$\frac{v_x}{v_0} = \exp\left[- \frac{(r + C_2 x - D_0/2)^2}{2\,(C_2 x)^2} \right] \dots\dots\dots\dots\dots (19)$$

In terms of this simplification, the equation for the variation in the volume-flux ratio with distance from the orifice then becomes

$$\frac{Q}{Q_0} = 1 + 2\,(\sqrt{2\,\pi} - 2)C_2 \frac{x}{D_0} + 4\,(3 - \sqrt{2\,\pi})(C_2)^2 \frac{x^2}{(D_0)^2} \dots\dots (20)$$

Although, as in the two-dimensional case, the rate of entrainment may now be expressed as the derivative of the volume flux with longitudinal distance, for reasons of continuity the radial velocity v_r beyond the diffusion region will vary inversely with radial distance from the center line; in other words, the product $r\,v_r$ rather than v_r itself approaches a finite limit in such three-dimensional flow:

$$\underset{r \to \infty}{\text{Limit}} \frac{v_r}{v_0} \frac{r}{D_0} = -\tfrac{1}{4}\,(\sqrt{2\,\pi} - 2)C_2 - (3 - \sqrt{2\,\pi})(C_2)^2 \frac{x}{D_0} \dots\dots (21)$$

The variation of v_r within the diffusion zone may be evaluated from the differential equation of continuity; since this function must again be shown graphically rather than algebraically, it is not presented herein. Finally, as in the case of flow from a slot, it may be assumed that the difference between the magnitude of the velocity vector and that of its longitudinal component will have little effect on the expression for the energy flux past successive sections beyond an orifice, which thus becomes

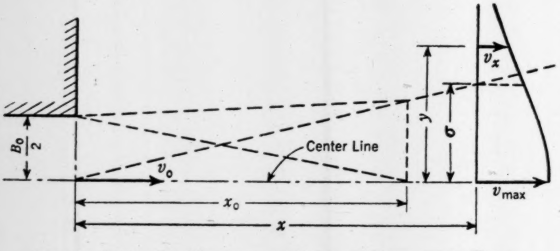

FIG. 4.—DEFINITION SKETCH FOR ZONE OF ESTABLISHED FLOW

$$\frac{E}{E_0} = 1 + 2\left(\sqrt{\frac{2\,\pi}{3}} - 2\right) C_2 \frac{x}{D_0} + 4\left(\frac{5}{3} - \sqrt{\frac{2\,\pi}{3}}\right)(C_2)^2 \frac{x^2}{(D_0)^2} \dots (22)$$

Zone of Established Flow.—As will be noted from Fig. 4, at all sections beyond the zone of flow establishment the probability curve indicating the velocity distribution will be continuous across the line of symmetry, and the ratio σ/x will therefore become directly indicative of the rate of outward spread of the eddy region. Assuming, for the two-dimensional case, that $\sigma/x = C_1$, evaluation of the momentum flux will lead at once to the expression:

$$\frac{v_{\max}}{v_0} = \sqrt{\frac{1}{\sqrt{\pi}\,C_1}\frac{B_0}{x}} \qquad (23)$$

in which the quantity $B_0/(\sqrt{\pi}\,C_1)$ represents the same parametric distance x_0 as that in Eq. 10. In other words, determination of the single constant C_1 from experimental measurement will suffice for both zones. The distribution of the longitudinal velocity component in the zone of established flow may then be written as

$$\frac{v_x}{v_0} = \sqrt{\frac{1}{\sqrt{\pi}\,C_1}\frac{B_0}{x}}\exp\left[-\frac{1}{2(C_1)^2}\frac{y^2}{x^2}\right] \qquad (24)$$

From the equation for volume flux it will be found, for the established zone, that

$$\frac{Q}{Q_0} = \sqrt{2\sqrt{\pi}\,C_1\frac{x}{B_0}} \qquad (25)$$

and differentiation with respect to x will yield the rate of entrainment and the corresponding lateral velocity at a considerable distance from the jet:

$$\operatorname*{Limit}_{y\to\infty}\frac{v_y}{v_0} = -\sqrt{\frac{\sqrt{\pi}\,C_1}{8}\frac{B_0}{x}} \qquad (26)$$

Eq. 26 will also be the limit of the general function for v_y obtained by graphical solution of the equation of continuity and reproduced at a later point. Evaluation of the energy flux in terms of $v_x \approx v$ finally leads to the approximate expression:

$$\frac{E}{E_0} = \sqrt{\frac{2}{3\sqrt{\pi}\,C_1}\frac{B_0}{x}} \qquad (27)$$

Once more replacing y and B_0 by r and D_0, and integrating over the corresponding areas, the equivalent expressions for the three-dimensional case may be evaluated for the condition that $\sigma/x = C_2$. The momentum equation at once reduces to

$$\frac{v_{\max}}{v_0} = \frac{1}{2\,C_2}\frac{D_0}{x} \qquad (28)$$

in which the quantity $D_0/(2\,C_2)$ corresponds to the same parametric distance x_0 as that in Eq. 16. Evidently, determination of the single experimental constant C_2 will suffice to define all characteristics of both zones, just as in the two-dimensional case. The remaining expressions then become

$$\frac{v_x}{v_0} = \frac{1}{2\,C_2}\frac{D_0}{x}\exp\left[-\frac{1}{2\,(C_2)^2}\frac{r^2}{x^2}\right] \qquad (29)$$

$$\frac{Q}{Q_0} = 4\,C_2\frac{x}{D_0} \qquad (30)$$

$$\operatorname*{Limit}_{r\to\infty}\frac{v_r}{v_0}\frac{r}{D_0} = -\frac{C_2}{2} \qquad (31)$$

and

$$\frac{E}{E_0} = \frac{1}{3\,C_2}\,\frac{D_0}{x} \quad\text{. . . . .}\quad (32)$$

Characteristics of the Mixing Process.—The foregoing analysis of the pattern of mean flow was made with complete disregard for the detailed characteristics of the diffusion pattern on which the phenomenon depends. If the assumed distribution of mean velocity and the resulting diffusion rate are in even approximate agreement with actual flow conditions, however, it should be possible to obtain at least a rough indication of the diffusion characteristics.

Since the pressure distribution was assumed to be hydrostatic, the only force to be considered in analyzing the interaction of the jet and the surrounding fluid must be the shear exerted along any longitudinal plane. The cumulative effect of such shear will necessarily disappear as the distance from the jet axis becomes great (that is, a zero net force corresponds to a zero longitudinal gradient of the total momentum flux), but within the zone of diffusion the intensity of shear τ may be expected to vary considerably. In fact, its local magnitude may be evaluated in terms of the change in momentum flux which occurs as fluid passes through an elementary part of the diffusion zone.

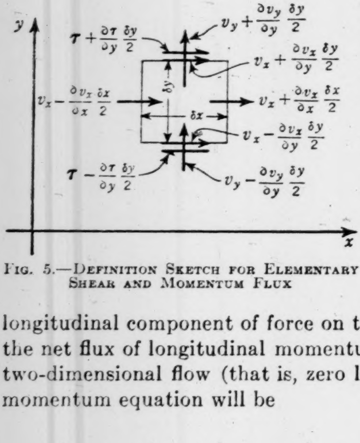
y
$\tau + \frac{\partial \tau}{\partial y}\frac{\delta y}{2}$
$v_y + \frac{\partial v_y}{\partial y}\frac{\delta y}{2}$
$v_x + \frac{\partial v_x}{\partial y}\frac{\delta y}{2}$
δy
$v_x - \frac{\partial v_x}{\partial x}\frac{\delta x}{2}$
$v_x + \frac{\partial v_x}{\partial x}\frac{\delta x}{2}$
δx
$v_x - \frac{\partial v_x}{\partial y}\frac{\delta y}{2}$
$\tau - \frac{\partial \tau}{\partial y}\frac{\delta y}{2}$
$v_y - \frac{\partial v_y}{\partial y}\frac{\delta y}{2}$
x

FIG. 5.—DEFINITION SKETCH FOR ELEMENTARY SHEAR AND MOMENTUM FLUX

With reference to Fig. 5, which represents the longitudinal cross section of a cubic element of space within the expanding jet, the flux of longitudinal momentum may be expressed as the product of the elementary volume flux at right angles to any face and of the corresponding longitudinal component of momentum per unit volume. The net longitudinal component of force on the faces of the element should then equal the net flux of longitudinal momentum through all faces of the element. For two-dimensional flow (that is, zero longitudinal shear on the $x-y$ faces), the momentum equation will be

$$\left(\tau+\frac{\partial \tau}{\partial y}\frac{\delta y}{2}\right)\delta x-\left(\tau-\frac{\partial \tau}{\partial y}\frac{\delta y}{2}\right)\delta x=\rho\left(v_x+\frac{\partial v_x}{\partial x}\frac{\delta x}{2}\right)^2\delta y-\rho\left(v_x-\frac{\partial v_x}{\partial x}\frac{\delta x}{2}\right)^2\delta y$$
$$+\rho\left(v_x+\frac{\partial v_x}{\partial y}\frac{\delta y}{2}\right)\left(v_y+\frac{\partial v_y}{\partial y}\frac{\delta y}{2}\right)\delta x-\rho\left(v_x-\frac{\partial v_x}{\partial y}\frac{\delta y}{2}\right)\left(v_y-\frac{\partial v_y}{\partial y}\frac{\delta y}{2}\right)\delta x \;. . .\; (33)$$

which reduces to the following statement of conditions at a point:

$$\frac{\partial \tau}{\partial y} = \rho\left(v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y}\right) \quad\text{. . . . .}\quad (34)$$

Evidently, the local intensity of shear will involve the integral of Eq. 34 with respect to y from the axis to the point in question:

$$\tau = \rho \int_0^y \left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} \right) dy \qquad (35)$$

Because of the difficulty of expressing v_y by other than graphical methods, evaluation of this relationship must also proceed graphically, the results being shown in plotted form subsequently. It is at once possible, on the other hand, to express certain characteristics of the turbulence in terms of this function.

From the Reynolds equations of acceleration, the intensity of longitudinal shear at any point in a turbulent fluid may be written as the sum of a mean viscous stress and an apparent turbulent stress:

$$\tau = \mu \frac{\partial v_x}{\partial y} - \rho \overline{v'_x v'_y} \qquad (36)$$

in which μ is the dynamic viscosity of the fluid. Under conditions of such pronounced mixing as exists within the eddy region of a spreading jet, it appears reasonable to presume that the term for mean viscous shear will be relatively unimportant. Indeed, this was prerequisite to the basic assumption of dynamic similarity. The necessarily finite magnitude of the mean product of the turbulent velocity components $\overline{v'_x v'_y}$, on the other hand, indicates that a finite degree of correlation must exist between the components v'_x and v'_y. The turbulence must be definitely nonisotropic, therefore, and in all probability the root-mean-square values of these components will differ from one another at every point. Because of the arbitrary nature of the assumed mean velocity distribution, it cannot be expected that the resulting mean flow pattern will yield any useful clue as to the relative magnitudes of these root-mean-square terms, and as a rough approximation it must necessarily be assumed that they are proportional. Further assuming that the Prandtl concept of proportionality between these components and the product of a so-called mixing length l and the mean velocity gradient is not greatly in error, it follows that, qualitatively,

$$\tau = \rho \sqrt{\overline{(v'_y)^2}}\, l \frac{\partial v_x}{\partial y} = \rho\, l^2 \left(\frac{\partial v_x}{\partial y} \right) \left| \frac{\partial v_x}{\partial y} \right| \qquad (37)$$

in which the product $\sqrt{\overline{(v'_y)^2}}\, l$ represents the diffusion coefficient ϵ (that is, the so-called kinematic eddy viscosity). Once Eq. 35 has been evaluated, it should be possible to express the approximate variation of l, $\sqrt{\overline{(v'_y)^2}}$, and ϵ as dimensionless functions of coordinate space, just as in the case of such other variables as v, Q, and E. The results are shown graphically in the following section.

EXPERIMENTAL RESULTS

Equipment and Test Procedure.—An experimental program resulting from a wartime fog-dispersal project of the Iowa Institute of Hydraulic Research (at the State University of Iowa, Iowa City, Iowa) involved the measurement of the mean velocity distribution in both two-dimensional and three-dimensional jets over a considerable range of each independent variable. Purely for convenience, all measurements were made in air, but there is no reason to presume that comparable results would not be obtained in any other fluid—whether liquid or gas—of moderately low viscosity. Essentially the same experimental equipment was used in each phase of the project, the flow being produced by a 2-hp centrifugal fan with suitable ducts and stilling devices. The two-dimensional slot (Fig. 6) was formed from two 6-ft strips of 1¼-in. by ⅜-in. cold-rolled steel, carefully rounded along the inner edges and accurately adjustable in spacing from $\frac{1}{32}$ in. to ¼ in. This slot was set flush with the surface of a 6-ft by 12-ft level floor, and was bounded at the ends by 6-ft by 12-ft vertical walls. (Because attempts to measure the velocity distribution in the zone of establishment for even the ¼-in. slot proved futile, a special 1-in. slot was made for this purpose.) The circular orifice (Fig. 7) was set into the top of a 4-ft by 4-ft table, and consisted of three interchangeable brass sections having outlet diameters of ¼ in., ½ in., and 1 in. with well-rounded transitions from the same 60° convergent approach section. In both cases, gage carriages

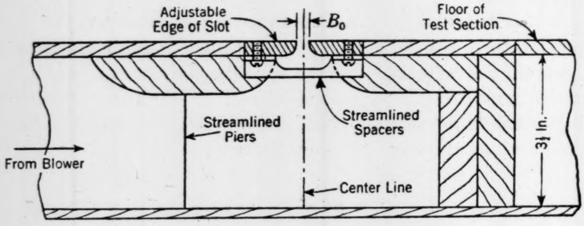
Adjustable Edge of Slot
B_0
Floor of Test Section
Streamlined Piers
Streamlined Spacers
From Blower
Center Line
3¼ In.

FIG. 6.—DETAILS OF SLOT AND APPROACH

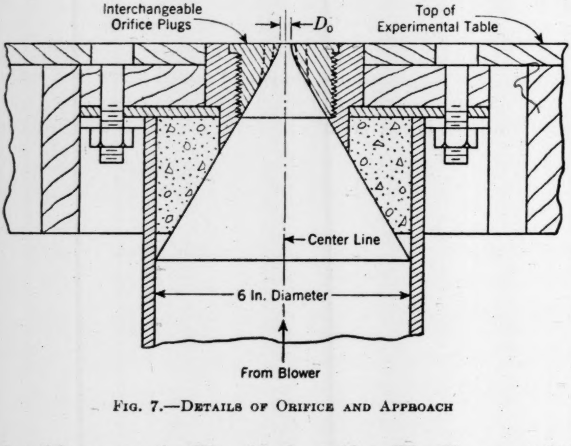
Interchangeable Orifice Plugs
D_0
Top of Experimental Table
Center Line
6 In. Diameter
From Blower

FIG. 7.—DETAILS OF ORIFICE AND APPROACH

permitted motion of measuring instruments in three coordinate directions, scales with verniers which read to 0.001 ft indicating the coordinate position.

Because of the extreme variation in velocity with location, it was necessary to utilize several different methods of velocity measurement. The efflux velocity of the jet and the velocity distribution in the outlet vicinity were determined with a No. 20 hypodermic needle used as a stagnation tube and connected directly to a differential water manometer read by point gage to 0.001 ft. In zones of moderate velocity the water manometer was replaced by a sensitive two-liquid gage, which was read by micrometer to 0.001 in. of alcohol. The lowest velocities were measured by a midget spoke-vane anemometer with jeweled bearings, which had been calibrated in an air tunnel. Except in the zones of flow establishment, the velocity instruments were oriented with respect to flow direction in accordance with prior indications of a directional vane consisting of a thin plate with a stagnation tube mounted on each side a short distance behind the leading edge. The vane was rotated about an axis through the leading edge until a zero differential reading was obtained on a manometer

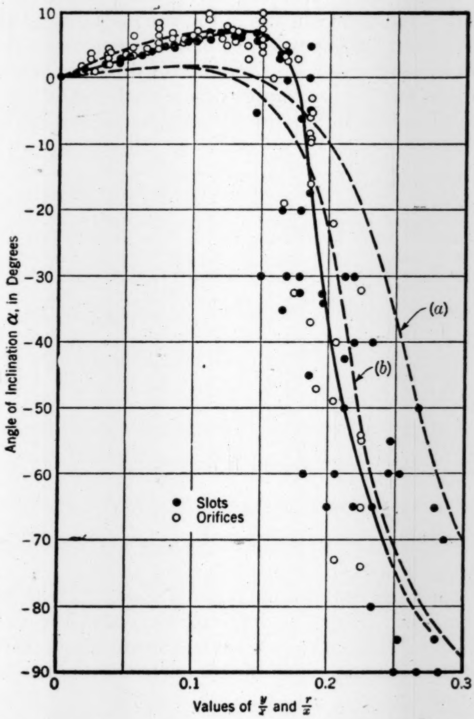

FIG. 8.—VARIATION IN DIRECTION OF FLOW WITHIN JETS FROM BOTH SLOTS AND ORIFICES

connected to the two stagnation tubes. In zones of very low velocity the vane was replaced by a fine jet of smoke for visual observation.

Since the distribution of velocity at any cross section was assumed to be governed solely by the efflux velocity, the outlet dimension, and the longitudinal distance to the section, these three independent variables were generally so controlled in the tests that the maximum practicable range of each was investigated in systematic steps as the other two were held constant. There were three exceptions to this general procedure: First, because the zone of established flow was of primary interest, only a few exploratory measurements of necessarily limited range were made in the zone of establishment. Second, as a result of the evident similarity of the distribution curves at all sections of the established flow, only a sufficient number of direction measurements (see Fig. 8) were made to establish the approximate variation of vector inclination for the two-dimensional and the three-dimensional cases beyond reasonable doubt, and this was then used to orient the velocity instruments for all subsequent measurements. Third, in order to determine the variation in v_{max} over the greatest possible range, independent center-line measurements were made far beyond the region in which the complete lateral distribution could accurately be determined.

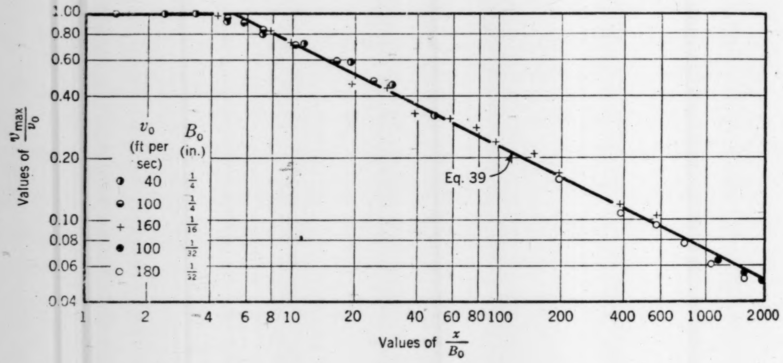

FIG. 9.—DISTRIBUTION OF CENTER-LINE VELOCITY FOR FLOW FROM SLOT

Results for the Two-Dimensional Jet.—Measurements of the center-line velocity from the efflux section over a longitudinal distance equal to several thousand times the slot width are shown as a dimensionless logarithmic plot in Fig. 9. Except for a limited zone of transition, the experimental points for all slot widths and efflux velocities follow closely either a horizontal line with the ordinate magnitude of unity for the zone of flow establishment, or a 1:2 sloping line for the zone of established flow. Since these two lines intersect at the point $x_0/B_0 = 5.2$, it follows at once that the constant C_1 for the equations of the zone of establishment of any two-dimensional jet will have the magnitude $C_1 = B_0/(\sqrt{\pi}\, x_0) = 0.109$.

Introduction of this value into Eq. 12 yields the specific relationship for the velocity distribution in the diffusion region of the two-dimensional zone of establishment

$$\log_{10} \frac{v_x}{v_0} = -18.4 \left(0.096 + \frac{y - B_0/2}{x} \right)^2 \ldots\ldots\ldots\ldots (38)$$

which is plotted in Fig. 10, using the lateral location of the slot boundary as the zero reference on the ordinate scale. Superposed on the analytical curve are the measured values obtained in the exploratory study of the velocity distribution in this zone. The agreement between analysis and experiment is quite satisfactory.

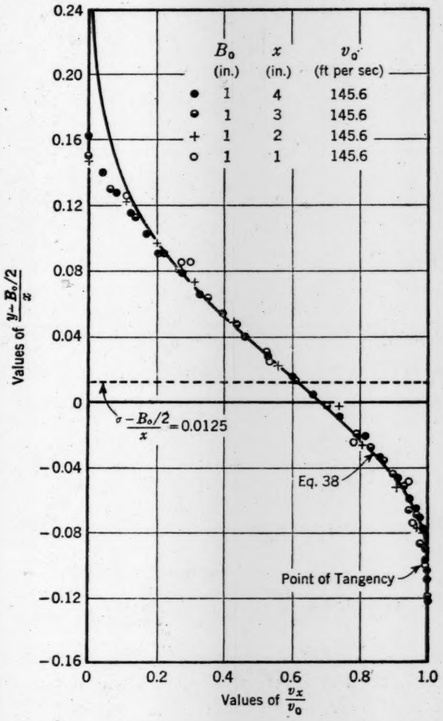

FIG. 10.—DISTRIBUTION OF LONGITUDINAL VELOCITY IN ZONE OF ESTABLISHMENT OF FLOW FROM SLOTS

For the zone of established flow, introduction of the same experimental coefficient $C_1 = 0.109$ into Eq. 23 leads to the specific expression for the sloping line of Fig. 9,

$$\frac{v_{\max}}{v_0} \sqrt{\frac{x}{B_0}} = 2.28 \ldots (39)$$

Eq. 39 represents the limiting form of Eq. 24 after introduction of the constant C_1:

$$\log_{10} \frac{v_x}{v_0} \sqrt{\frac{x}{B_0}} = 0.36 - 1.84 \frac{y^2}{x^2} \ldots (40)$$

Eq. 40 is plotted in Fig. 11, together with all measured values for the longitudinal velocity component. Despite appreciable experimental scatter, the analytical curve is in reasonable agreement with the measurements over the entire range of the independent variables. The graphical solution of the continuity relationship for v_y, in terms of the constant C_1, is shown in the same general manner in Fig. 12, together with experimental values obtained by multiplying the measured velocities by the sines of the corresponding angles of inclination of Fig. 8. Although the data follow the general trend of the theoretical curve, the values computed from the measurements are at least 100% greater than those indicated by the plotted function—the discrepancy also being apparent from comparison of the analytical curve (curve (*a*)) of Fig. 8 with the measured values. No explanation can be given for this disagreement; neither do the computations disclose any likelihood of mathematical error, nor have repeated measurements by different observers led to essentially different results. Fortunately, the angles involved were so small as to result in a negligible error in the longitudinal velocity. Because of the lack of dependable velocity measurements in the outer region, it cannot be stated whether or not the experimental data approach the velocity-of-entrainment limit given by the following specific form of Eq. 26:

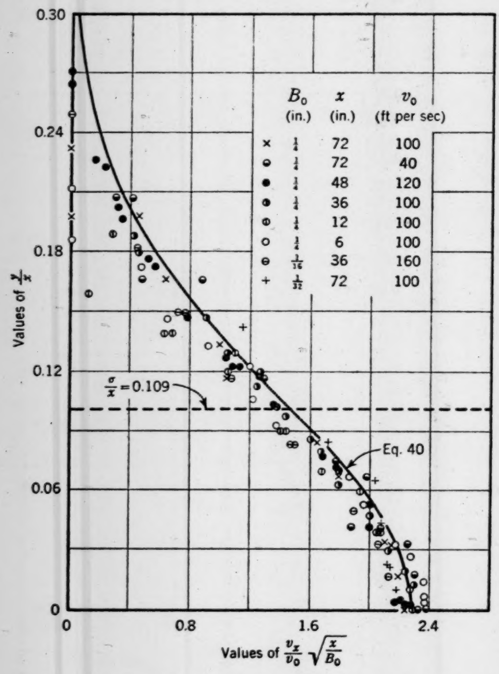

FIG. 11.—DISTRIBUTION OF LONGITUDINAL VELOCITY IN ZONE OF ESTABLISHED FLOW FROM SLOTS

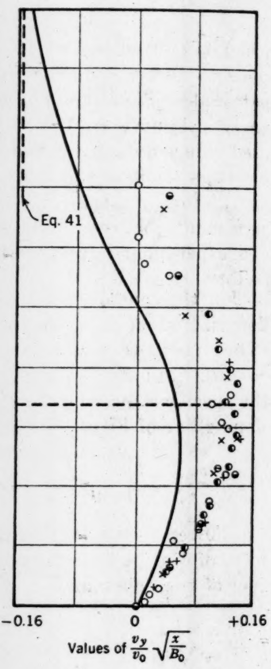

FIG. 12.—DISTRIBUTION OF LATERAL VELOCITY IN ZONE OF ESTABLISHED FLOW FROM SLOTS

$$\operatorname*{Limit}_{y\to\infty} \frac{v_y}{v_0}\sqrt{\frac{x}{B_0}} = -\,0.155 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (41)$$

Eqs. 13 and 15 for the volume-flux ratio and the energy-flux ratio in the zone of flow establishment become, in terms of the constant C_1,

$$\frac{Q}{Q_0} = 1 \mp 0.080\,\frac{x}{B_0} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (42)$$

and

$$\frac{E}{E_0} = 1 - 0.035\,\frac{x}{B_0} \quad \text{.....................(43)}$$

Likewise, Eqs. 25 and 27 for the zone of established flow will reduce to

$$\frac{Q}{Q_0} = 0.62\,\sqrt{\frac{x}{B_0}} \quad \text{.....................(44)}$$

and

$$\frac{E}{E_0} = 1.86\,\sqrt{\frac{B_0}{x}} \quad \text{.....................(45)}$$

Eqs. 42, 43, 44, and 45 (as well as the basic expression $M/M_0 = 1$) are plotted in Fig. 13, together with values obtained by integration of the measured velocity-distribution curves for typical runs with x as the independent variable.

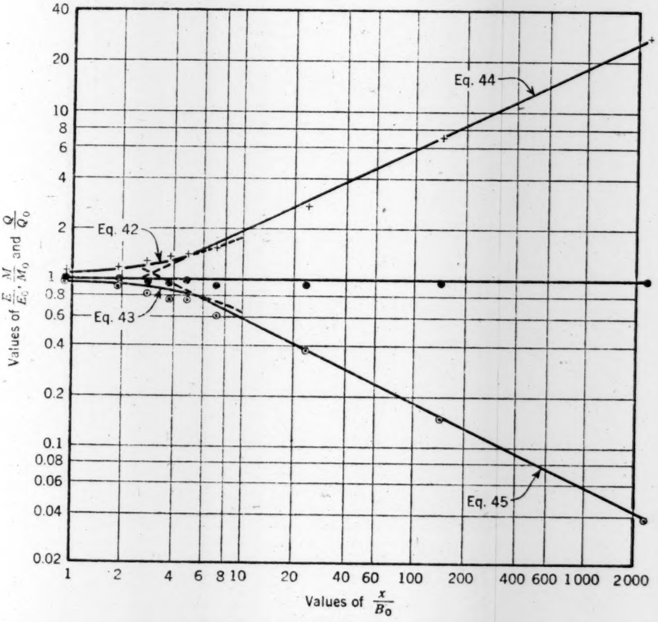

FIG. 13.—DISTRIBUTION OF VOLUME, MOMENTUM, AND ENERGY FLUX DOWNSTREAM FROM SLOT

Although no measurements of turbulence were made in this experimental program, it is nonetheless pertinent to note the results of the approximate analysis of the eddy pattern. Fig. 14 includes the simultaneous graphical solutions of Eqs. 35 and 37 as dimensionless curves for τ, $\sqrt{\overline{(v'_y)^2}}$, l, and ϵ at any section of the zone of established flow, the assumed curve for v_x being included for purposes of orientation. As will be noted therein, the zone of maximum intensity of shear, and hence maximum intensity of turbulence, lies somewhat inside the point of inflection or maximum slope of the velocity curve (that is, where $y = \sigma$). The scale of the eddies is a minimum in this general zone, whereas the mixing coefficient attains a maximum at the center line. The indications are, of course, merely qualitative, but nevertheless quite significant.

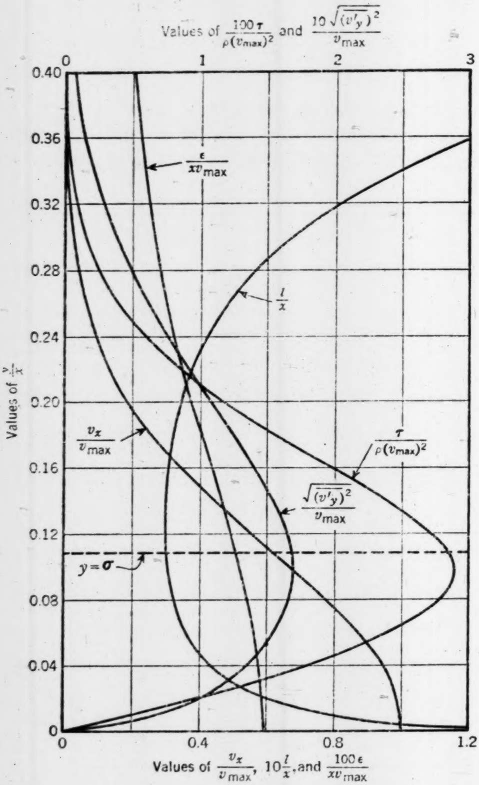

FIG. 14.—APPROXIMATE DISTRIBUTION OF TURBULENCE CHARACTERISTICS IN TWO-DIMENSIONAL ZONE OF ESTABLISHED FLOW

Results for the Three-Dimensional Jet.—As in the two-dimensional case just discussed, a logarithmic plot of all center-line velocity measurements (Fig. 15) yields a horizontal and a sloping line for the two zones of flow, that for the established zone now having a 1:1 slope. The intersection of these lines at the point $x_0/D_0 = 6.2$ at once yields $C_2 = D_0/(2\,x_0) = 0.081$. In terms of this constant, Eq. 19 for the distribution of the longitudinal velocity component in the zone of establishment assumes the specific form:

$$\log_{10} \frac{v_x}{v_0} = -33 \left(0.081 + \frac{r - D_0/2}{x} \right)^2 \ldots\ldots\ldots\ldots\ldots (46)$$

which is plotted, together with measured data, in Fig. 16. Unlike the case of the slot, the agreement between analysis and measurement is not wholly satisfactory. As shown by the broken line, considerable improvement would result through use of the smaller border value $x_0/D_0 = 4.5$ (that is, $C_2 = 0.111$).

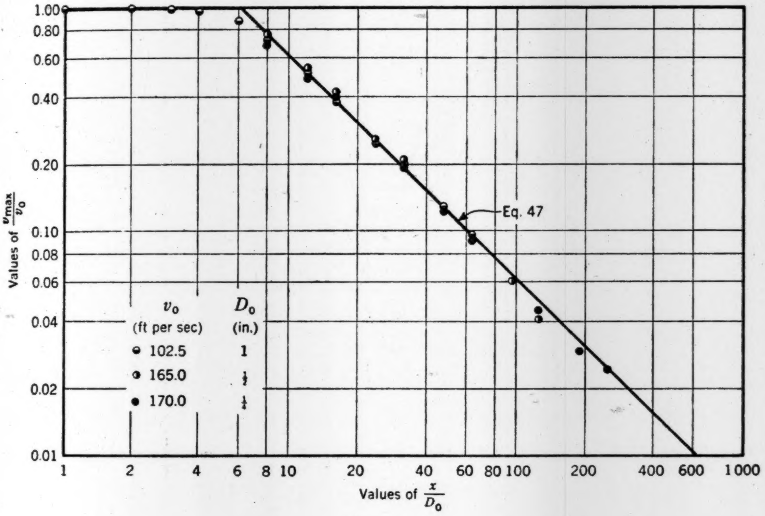

Fig. 15.—Distribution of Center-Line Velocity for Flow from Orifice

However, since close agreement in the limited zone of establishment is considered less significant than evaluation of both zones (as in the two-dimensional case) in terms of the same coefficient, the value $C_2 = 0.081$ is used throughout. Thus, Eq. 28 is seen to take the specific form:

$$\frac{v_{\max}}{v_0} \frac{x}{D_0} = 6.2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (47)$$

which corresponds to the sloping line of Fig. 15. This is simply the limiting form of the general relationship for v_x of Eq. 29, which accordingly becomes

$$\log_{10} \frac{v_x}{v_0} \frac{x}{D_0} = 0.79 - 33 \frac{r^2}{x^2} \ldots\ldots\ldots\ldots\ldots\ldots (48)$$

and is plotted in Fig. 17, together with measured data. Despite considerable experimental scatter due to the great range of the independent variables, the points will be seen to follow the probability curve with good approximation. Fig. 18 gives the graphical solution of the continuity equation for v_r, supplemented by experimental results obtained through multiplying the measured vector magnitudes by the sines of the corresponding angles of inclination. As in the two-dimensional case, the data follow the trend of the theoretical curve, but the values computed from the velocity measurements are at least twice as great as those of the plotted function (see also curve (*b*), Fig. 8). It is evidently impossible to discover whether or not they finally approach the hyperbolic curve corresponding to the following specific form of Eq. 31 for the radial velocity of entrainment:

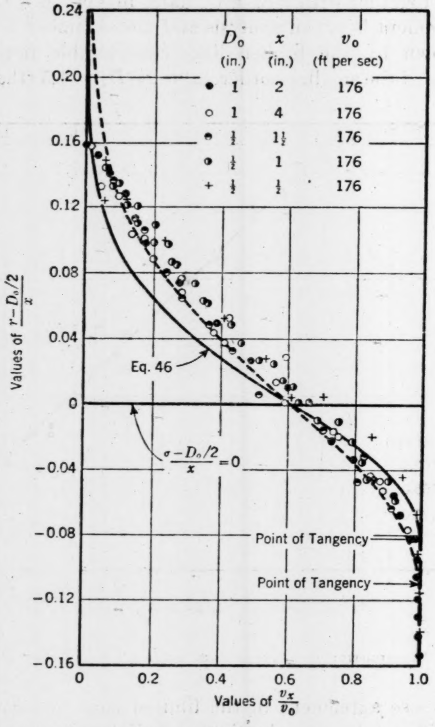

FIG. 16.—DISTRIBUTION OF LONGITUDINAL VELOCITY IN ZONE OF ESTABLISHMENT OF FLOW FROM ORIFICES

$$\operatorname*{Limit}_{r \to \infty} \frac{v_r}{v_0} \frac{x}{D_0} = -0.040 \frac{x}{r} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (49)$$

Introduction of the experimentally determined coefficients in Eqs. 20 and 22 for the volume-flux and energy-flux ratios in the zone of flow establishment gives

$$\frac{Q}{Q_0} = 1 + 0.083 \frac{x}{D_0} + 0.0128 \frac{x^2}{(D_0)^2} \quad \text{.............. (50)}$$

and

$$\frac{E}{E_0} = 1 - 0.090 \frac{x}{D_0} + 0.0058 \frac{x^2}{(D_0)^2} \quad \text{.............. (51)}$$

Similar evaluation of Eqs. 30 and 32 for the zone of established flow yields

$$\frac{Q}{Q_0} = 0.32 \frac{x}{D_0} \quad \text{........................ (52)}$$

and

$$\frac{E}{E_0} = 4.1 \frac{D_0}{x} \quad \text{........................ (53)}$$

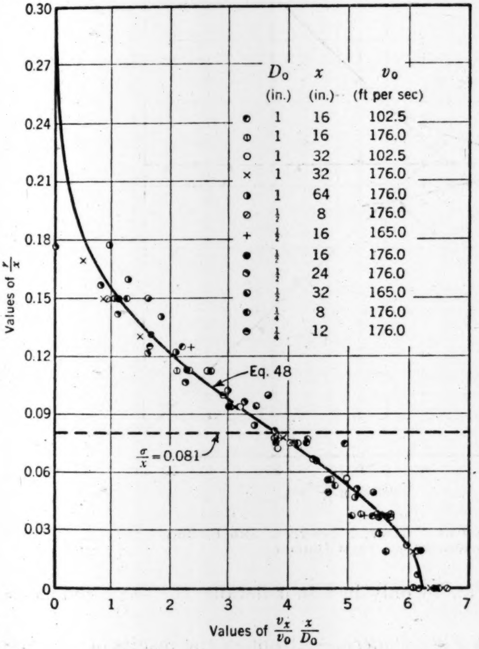

FIG. 17.—DISTRIBUTION OF LONGITUDINAL VELOCITY IN ZONE OF ESTABLISHED FLOW FROM ORIFICES

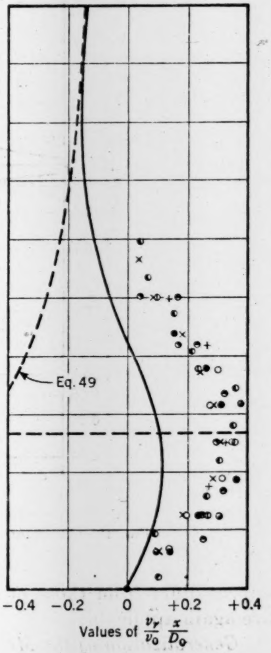

FIG. 18.—DISTRIBUTION OF RADIAL VELOCITY IN ZONE OF ESTABLISHED FLOW FROM ORIFICES

Eqs. 50, 51, 52, and 53 are plotted in Fig. 19, together with points representing the corresponding integrals of measured velocity-distribution curves for typical runs in which x was the only independent variable.

Except for the fact that the development leading to Eq. 35 must be revised according to the distinction between the cubic element of the two-dimensional case and the cylindrical element of the three-dimensional case, evaluation of the approximate characteristics of the turbulence proceeds in essentially the same manner. Solution of the corresponding differential equations results in the functions for τ, $\sqrt{\overline{(v'_r)^2}}$, l, and ϵ plotted in Fig. 20 in dimensionless form together with the function for v_x for purposes of orientation. Inasmuch as the curves differ from those of Fig. 14 only in minor details, the same comments are again applicable.

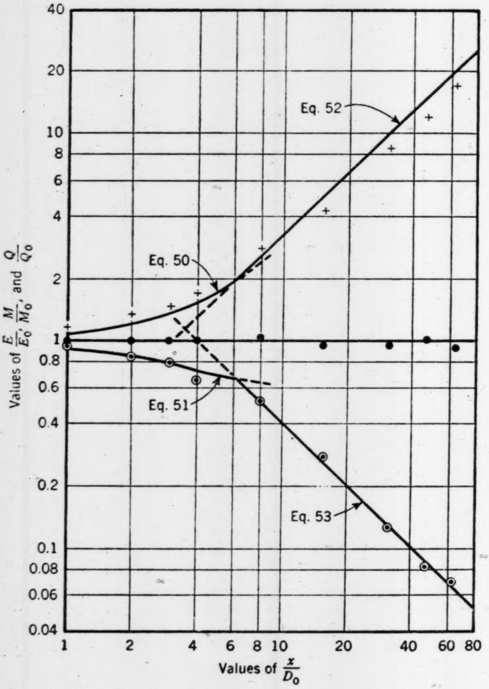

FIG. 19.—DISTRIBUTION OF VOLUME, MOMENTUM, AND ENERGY FLUX DOWNSTREAM FROM ORIFICE

Generalization of the Mean Flow Pattern.—Although the results of the combined analytical and experimental investigation presented in the foregoing must be considered as approximate rather than as rigorous, the series of specific equations and plotted curves are nevertheless sufficiently close to actual fact to be used with confidence in general problems of design. The equations or diagrams themselves indicate no limit of applicability. In fact, no systematic deviation from the respective functions was noted with change in either length or velocity scale. Two probable limits of applicability should therefore be mentioned. At great distances from the efflux section—however accurately the measured curves may be extrapolated—the velocities of flow become so low that extraneous effects rather than the dynamics of the jet itself will begin to govern the flow pattern. Although no significant deviation of the center-line velocities from the predicted function was found in experiments at distances from the efflux section equal to several hundred times the orifice diameter and to several thousand times the slot width, it must be emphasized that extreme care was taken to eliminate all possible disturbance of the jet. At low Reynolds numbers for the efflux section, moreover, it is to be expected that viscous influences will prevent the full development of turbulence in the region of diffusion. The lowest Reynolds number, $\mathbf{R} = v_0 L_0/\nu$, attained in the tests described herein was about 1,500. There is, of course, no reason to expect any

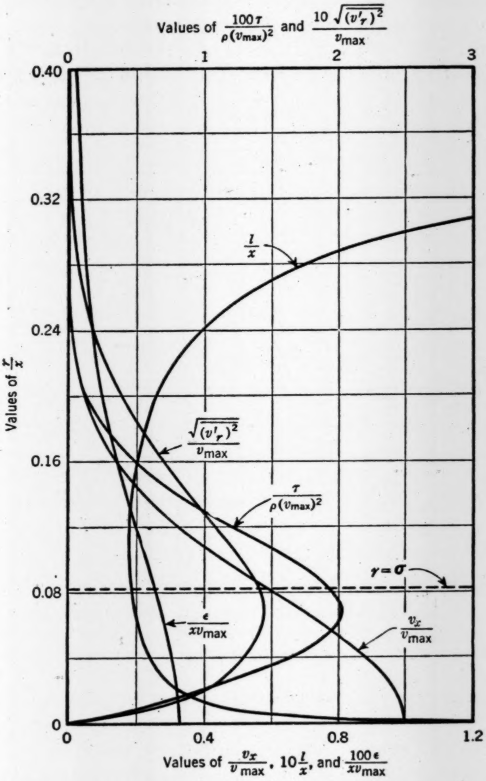

FIG. 20.—APPROXIMATE DISTRIBUTION OF TURBULENCE CHARACTERISTICS IN THREE-DIMENSIONAL ZONE OF ESTABLISHED FLOW

difference between the flow of a gas and the flow of a liquid at comparable Reynolds numbers. The results may be considered applicable, therefore, to the efflux of either type of fluid—provided only that gas velocities do not become so high that acoustic effects begin to influence the flow pattern and that liquid velocities do not become so high that appreciable cavitation occurs within the eddies.

If a streamline or stream surface of the mean motion is defined as a line or surface (depending on whether the flow is two-dimensional or three-dimensional) across which no flow takes place, it should be possible from the foregoing results to develop—either from the velocity-distribution curves or from the stream function corresponding to these curves—a systematic sequence of such lines or surfaces showing at a glance the pattern of mean motion in both the zone of establishment and the zone of established flow. Such a diagram is presented in Fig. 21, in which the upper half corresponds to efflux from a long slot and the lower half to that from a circular orifice. In the two-dimensional case, the spacing of the streamlines is inversely proportional to the local magnitude of the velocity, so that the speed as well as the direction of flow is readily apparent. In the three-dimensional case the spacing of the stream surfaces is inversely proportional to the velocity times the radial distance to the point in question, so that—although the direction of motion is immediately evident—comparison of velocity magnitudes by eye is not readily possible. Each of the diagrams contains the locus of the point of inflection (that is, the maximum slope) of the error curve for the longitudinal velocity distribution, which corre-

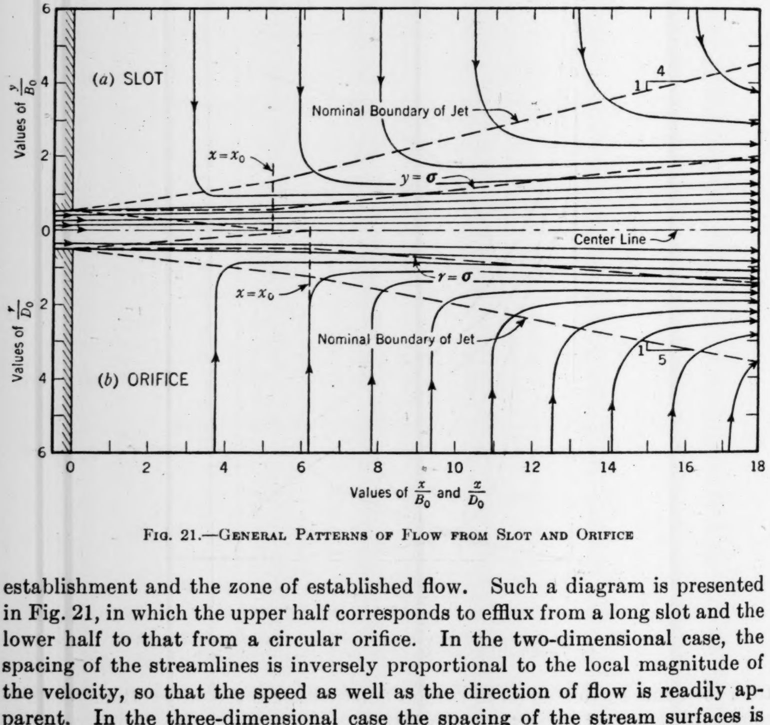

FIG. 21.—GENERAL PATTERNS OF FLOW FROM SLOT AND ORIFICE

sponds roughly to the region of maximum intensities of shear and turbulence. Also indicated is the nominal boundary of the jet—that is, the zone in which the longitudinal velocity is only a small percentage of that at the center line. The nominal boundary of the jet from a slot will thus have a slope of approximately 1:4, whereas that for an orifice is approximately 1:5. As previously indicated, this diagram should be characteristic of all jets under comparable flow conditions.

Not until after the completion of the present paper did the authors receive a copy of the very excellent and comprehensive treatise by H. Reichardt,[12] which discusses submerged jets as one of a series of related diffusion phenomena. Noteworthy in connection with the pattern of mean flow just presented is Mr. Reichardt's conclusion that the lateral streamlines of the two-dimensional jet—in contradistinction to the radial streamlines of the three-dimensional jet (and to the lateral streamlines as shown at the top of Fig. 21)—must be asymptotic to a family of parabolas having a common focus at the origin of the coordinate system, in order that the velocity of the lateral flow, like that of the radial flow, may properly be zero at infinity. This would accordingly modify the indicated outskirts of the two-dimensional pattern, but would affect the streamlines of the central flow to a wholly imperceptible degree.

CONCLUSIONS

Through assumptions of hydrostatic pressure distribution, dynamic similarity of the diffusion process, and normal-probability variation of the velocity, analytical expressions have been developed for the patterns of mean flow within submerged jets from both slots and orifices. Measurements conducted on the flow of air at different velocities from slots and orifices of different sizes have been shown to be in substantial agreement with the analytical relationships, and have provided the single experimental coefficient required to complete the analysis in the two-dimensional and in the three-dimensional case. Algebraic expressions and plotted curves have been presented in dimensionless form for the distribution of velocity, volume flux, and energy flux in the zone of flow establishment and the zone of established flow for each boundary condition. In addition, a qualitative treatment of the variation in mixing characteristics and a generalized diagram of the mean flow pattern have been given. All results are considered applicable for design purposes to the flow of any liquid or gas at moderate to high Reynolds numbers.

ACKNOWLEDGMENTS

This paper represents an original contribution of the Iowa Institute of Hydraulic Research of the State University of Iowa, all measurements having been conducted in the Institute laboratories and all analytical developments having been carried to completion by members of the Institute staff. The experiments on efflux from slots—with the exception of measurements in the zone of flow establishment contributed by D. E. Metzler—were performed by

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[12] "Gesetzmässigkeiten der freien Turbulenz," by H. Reichardt, *Forschungsheft 414*, Verein Deutscher Ingenieure, 1942.

Messrs. Albertson and Jensen as a seminar project in the Department of Mechanics and Hydraulics of the University. The experiments on efflux from circular orifices were performed by Mr. Dai as a thesis investigation in partial fulfilment of the requirements for the degree of Master of Science in the Graduate College of the University. Because of its direct application to certain studies being conducted by the Institute for the David Taylor Model Basin (Carderock, Md.), Bureau of Ships, United States Navy, part of the analysis presented herein was supported by Navy research funds. J. S. McNown, Assoc. M. ASCE, acted frequently as consultant during the preparation of the paper, and W. D. Baines, Jun. ASCE, evaluated many of the more complex integrals. All phases of the project were under the direct supervision of Professor Rouse.

DISCUSSION

James S. Holdhusen,[13] Jun. ASCE.—The experimental program on which this paper is based was planned and executed very carefully, and it seems to justify the assumptions made in deriving the formulas. To the reader of the paper, however, the final form of the equations defining the characteristics of the diffusion process appears to be dependent on the authors' third assumption —that the velocity distribution in the diffusion region follows the normal error curve. Since no semblance of an analytical proof is offered for this assumption, the reader may have some reservations concerning the generality of the subsequent analysis. The following discussion will show that only the first two assumptions are necessary in order to carry through the analysis and arrive at the same form of the final equations.

The first assumption made by the authors—that the pressure is hydrostatically distributed throughout the flow—is entirely justifiable. Theoretical considerations[5] indicate that the pressure differences arising as the consequence of the flow in the y-direction are negligibly small, being of the order of magnitude of a fraction of 1% of the velocity head. The second assumption—that the diffusion process is dynamically similar at all sections of the flow—is apparently justifiable as a consequence of the "free" nature of the diffusion process. This second assumption can be expressed mathematically as

$$v_x = v_{\max} f(\eta) \qquad (54)$$

in which v_x is the velocity at any point in the cross section of the flow that has $v_{\max}$ as a maximum velocity. The variable η is defined by $\eta = y/x$. In this discussion, y is defined as the distance, transverse to the flow direction, from the plane of symmetry in the two-dimensional case or from the axis of symmetry in the three-dimensional case in the zone of established flow. In the zone of flow establishment of both jets, y will be measured from the inner edge of the zone of diffusion, for the sake of convenience. The distance x is measured from the plane of the slot or orifice. The second assumption states that a single function of η defines the velocity distribution in the diffusion region for any value of x in a particular zone of flow. In the following development, all symbols and terminology will indicate the same quantities that they do in the paper, unless specifically noted otherwise.

In the zone of flow establishment of the two-dimensional jet (Fig. 3) evaluation of the momentum-flux integral (Eq. 4) shows directly that

$$\frac{M}{M_0} = 1 = \frac{\int_{-0.5B}^{0} (v_0)^2 \, dy + \int_{0}^{\infty} (v_0)^2 f^2(\eta) \, dy}{0.5 \, B_0 \, (v_0)^2} = \frac{0.5 \, B + Y_m \, x}{0.5 \, B_0} \qquad (55)$$

In the discussion concerning both zones of the two-dimensional jet, the following designations will be used, in order to simplify typography:

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[13] Research Fellow, St. Anthony Falls Hydr. Laboratory, Univ. of Minnesota, Minneapolis, Minn.

$$\left.\begin{aligned} Y_m &= \int_0^\infty f^2(\eta)\cdot d\eta \\ Y_q &= \int_0^\infty f(\eta)\, d\eta \\ Y_e &= \int_0^\infty f^3(\eta)\, d\eta \end{aligned}\right\}\ldots\ldots\ldots\ldots(56)$$

It will be noted that each of these integrals has a constant, dimensionless, value for an assigned value of $f(\eta)$—that is, for a given velocity distribution. Thus Eq. 55 shows immediately that

$$\frac{B}{2} = \frac{B_0}{2} - Y_m\, x \ldots\ldots\ldots\ldots(57)$$

The inner boundary of the diffusion region is thus shown to be a plane and

$$x_0 = \frac{B_0}{2\, Y_m} \ldots\ldots\ldots\ldots(58)$$

Similar evaluation of the volume-flux and energy-flux ratios (Eqs. 3 and 5) shows that

$$\frac{Q}{Q_0} = 1 + \frac{2\,x}{B_0}\,(Y_q - Y_m) \ldots\ldots\ldots\ldots(59a)$$

and

$$\frac{E}{E_0} = 1 + \frac{2\,x}{B_0}\,(Y_e - Y_m) \ldots\ldots\ldots\ldots(59b)$$

Differentiation of Eq. 59*a* will yield the following equations for the lateral velocity at a considerable distance from the jet:

$$\operatorname*{Limit}_{y\to\infty} \frac{v_y}{v_0} = -\,(Y_q - Y_m) \ldots\ldots\ldots\ldots(60)$$

Evaluation of the momentum-flux ratio in the zone of established flow of the two-dimensional jet shows that

$$\frac{\int_0^\infty (v_{\max})^2 f^2(\eta)\, dy}{0.5\, B_0\, (v_0)^2} = 1 = \frac{(v_{\max})^2}{(v_0)^2}\,\frac{Y_m\, x}{0.5\, B_0} \ldots\ldots\ldots\ldots(61)$$

Thus,

$$\frac{v_{\max}}{v_0} = \sqrt{\frac{B_0}{2\,x\,Y_m}} = \sqrt{\frac{x_0}{x}} \ldots\ldots\ldots\ldots(62)$$

which shows directly that the maximum velocity varies inversely as the square root of the distance from the orifice in the zone of established flow. The three remaining ratios are likewise determined as

$$\frac{Q}{Q_0} = \sqrt{\frac{2\,x\,(Y_q)^2}{B_0\,Y_m}} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (63a)$$

$$\frac{E}{E_0} = \sqrt{\frac{B_0\,(Y_e)^2}{2\,x\,(Y_m)^3}} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (63b)$$

and

$$\operatorname*{Limit}_{y\to\infty} \frac{v_y}{v_0} = -\sqrt{\frac{B_0\,(Y_q)^2}{8\,x\,Y_m}} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (63c)$$

In the zone of establishment of the circular jet, evaluation of the momentum-flux ratio will define the inner boundary of the diffusion zone as an ellipsoid of revolution, as was the case in the paper. This indicates that Eq. 54 is not satisfied, and consequently no analysis of the flow characteristics will be made here for this zone. As will be shown, theoretical considerations indicate that similarity of velocity distributions should not be expected in this zone.

In the zone of established flow of the three-dimensional (circular) jet, evaluation of the momentum-flux ratio will yield

$$\frac{M}{M_0} = 1 = \frac{\int_0^\infty (v_{\max})^2 f^2(\eta)\,dy\,2\,\pi\,y\cdot}{0.25\,\pi\,(D_0)^2\,(v_0)^2} = \frac{(v_{\max})^2}{(v_0)^2}\,\frac{2\,Y_m\,x^2}{0.25\,(D_0)^2} \quad\ldots\ldots (64a)$$

Thus,

$$\frac{v_{\max}}{v_0} = \frac{D_0}{x\sqrt{8\,Y_m}} = \frac{x_0}{x} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (64b)$$

which shows directly that the maximum velocity varies inversely as the distance from the orifice. In the discussion concerning the circular jet, the following designations will be used:

$$\left.\begin{aligned} Y_m &= \int_0^\infty f^2(\eta)\eta\,d\eta \\ Y_q &= \int_0^\infty f(\eta)\eta\,d\eta \\ Y_e &= \int_0^\infty f^3(\eta)\eta\,d\eta \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (65)$$

in which, as before, the integrals have a constant dimensionless value for a given value of $f(\eta)$. The three remaining ratios are then determined as

$$\frac{Q}{Q_0} = \frac{x}{D_0}\sqrt{\frac{8\,(Y_q)^2}{Y_m}} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (66a)$$

$$\frac{E}{E_0} = \frac{D_0}{x}\sqrt{\frac{(Y_e)^2}{8\,(Y_m)^3}} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (66b)$$

$$\operatorname*{Limit}_{y\to\infty} \frac{v_y\,y}{v_0\,D_0} = -\sqrt{\frac{Y_q^2}{8\,Y_m}} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (66c)$$

The same functional relationships as derived by the authors have been derived without the necessity of assuming any specific distribution of velocity in the diffusion region. The only limitation imposed on this velocity distribution thus far is that the defined integrals exist. One is thus free to choose one of the calculated theoretical distributions, the sources of which have been noted in the paper, and, by a simple graphical integration, the constants in the equation can be determined.

The authors' use of the normal error curve can be justified on the basis of its providing a convenient arithmetic determination of the constants in the final equations. However, in view of the simplicity of making the graphical integrations highly accurate, one is permitted to use an analytically derived form of the velocity distribution, which will allow the extension of the theory to boundary conditions other than those assumed in the present derivation. Also, the use of the error curve has the inherent weaknesses of specifying that the turbulence generated by the mixing extends to infinity, and of indicating a rather erratic variation in the mixing length (Figs. 14 and 20). This first discrepancy was pointed out by H. Reichardt[14] when he first postulated the use of the error curve to define the velocity distribution.

The writer has evaluated the constants in Eqs. 55 to 66 using the velocity distribution that has been derived on the basis of the momentum transfer theory, assuming that the mixing length is constant over any cross section.[5] The constants were evaluated for this distribution (Fig. 22) because, in general, it seems to follow experimentally determined distributions as well as, or better than, any other of the analytically determined curves. In addition, this distribution is such that there exists a finite breadth outside which there is no

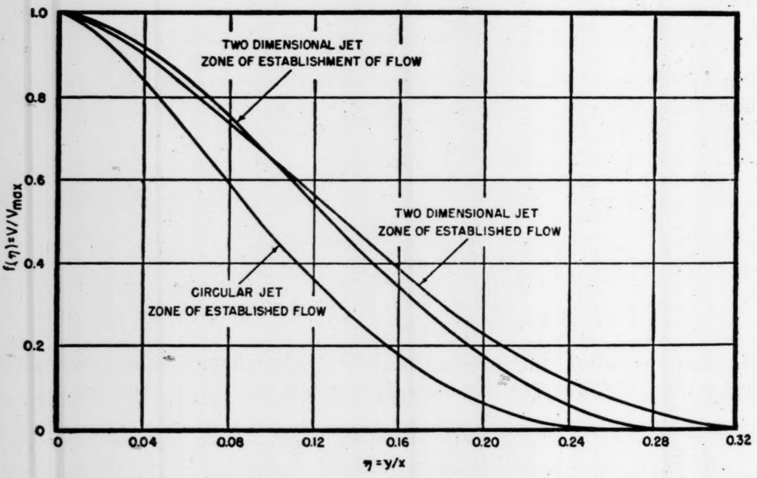

FIG. 22.—VELOCITY DISTRIBUTION IN ZONE OF DIFFUSION BY MOMENTUM TRANSFER THEORY

velocity in the x-direction, thus simplifying the evaluation of the integrals. The velocity distribution was analyzed completely by W. Tollmien with the exception of the determination of a single numerical constant in each case which determined the ratio between the mixing length and the distance x. This constant was ascertained by Mr. Tollmien in two cases from rather meager experimental data. However, the excellent data presented by the authors can be used to determine these constants, and the values of Y_m, Y_q, and Y_e have been determined from the values of $x_0/B_0 = 5.2$ and $x_0/D_0 = 6.2$ for the two-dimensional and three-dimensional cases, respectively. In Mr. Tollmein's paper,[5] it was pointed out that the distance x should not be measured from the plane of the orifice but rather from a plane about two diameters distant from the orifice in order that the profiles in the zone of established flow may be identical when plotted dimensionlessly. However, at large values of x the values of η would be practically identical for either definition so that it was assumed that $\eta = y/x$. The curves $f(\eta)$ were first drawn to a convenient scale of η and graphically integrated. The correct scale for η was then determined from Eqs. 57 and 64, and the corresponding values of Y_q and Y_e computed. In Table 1, the general form of the equations are listed with their numerical values computed from the velocity distributions plotted in Fig. 22 and from the formula:

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[14] "Über eine neue Theorie der Freien Turbulenz," by H. Reichardt, *Zeitschrift für angewandte Mathematik und Mechanik*, October, 1941, pp. 257–264.

$$f(\eta) = e^{-J} \qquad (67)$$

in which, to simplify typography,

$$J = \frac{\eta^2}{2\,(C_1)^2} \qquad (68)$$

Eq. 67 is obtained if it is assumed that the velocity distribution follows the normal error curve. It will be noticed that the differences between the two sets of equations (Cols. 3 and 4, Table 1) are not large and might well fall within normal experimental error.

To the writer, one of the most interesting aspects of free turbulence is the experimental verification, in a wide variety of problems, of the second assumption. However, it has been shown analytically by A. M. Keuthe[7] that the assumption is not valid in the zone of flow establishment of the circular jet. (Mr. Keuthe's excellent paper is a complete, but rather mathematically involved, theory concerning the zone of establishment of the circular jet.) It can be proved that at the edge of the orifice, the velocity distribution must be the same as that in the zone of establishment of a two-dimensional jet. From this distribution, there must be a gradual transition to the distribution of velocity in the zone of established flow from the orifice (Fig. 22). The equation of the mean motion in the circular jet is

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \frac{1}{\rho\, y} \frac{\partial(\tau\, y)}{\partial y} \qquad (69a)$$

The left side of Eq. 69a expresses the acceleration and the right side is the force per unit mass. At $x = 0$ and $y = 1$, designating the radius of the orifice as unity, the equation reduces to the equation of the mean motion in the two-

TABLE 1.—COMPARISON OF FORMULAS

Zone	Quantity	General form	Error curve	Momentum transfer theory
Zone of establishment; two-dimensional jet	x_0	$\frac{B_0}{2\,Y_m}$	$5.2\,B_0$	$5.2\,B_0$
	$\frac{v_{\max}}{v_0}$	1	1	1
	$\lim\limits_{y\to\infty}\frac{v_y}{v_0}$	$-(Y_q - Y_m)$	-0.040	-0.040
	$\frac{Q}{Q_0}$	$1 + (Y_q - Y_m)\frac{x}{B_0/2}$	$1 + 0.080\frac{x}{B_0}$	$1 + 0.080\frac{x}{B_0}$
	$\frac{E}{E_0}$	$1 + (Y_e - Y_m)\frac{x}{B_0/2}$	$1 - 0.035\frac{x}{B_0}$	$1 - 0.036\frac{x}{B_0}$
	Y_m	$\int_0^\infty f^2(\eta)\,d\eta$	$\frac{\sqrt{\pi}}{2}C_1$	0.096
	Y_q	$\int_0^\infty f(\eta)\,d\eta$	$\sqrt{\frac{\pi}{2}}C_1$	0.134
	Y_e	$\int_0^\infty f^3(\eta)\,d\eta$	$\sqrt{\frac{\pi}{6}}C_1$	0.078
Zone of established flow; two-dimensional jet	x_0	$\frac{B_0}{2\,Y_m}$	$5.2\,B_0$	$5.2\,B_0$
	$\frac{v_{\max}}{v_0}$	$\frac{B_0}{2\,Y_m}$	$2.28\sqrt{\frac{B_0}{x}}$	$2.28\sqrt{\frac{B_0}{x}}$
	$\lim\limits_{y\to\infty}\frac{v_y}{v_0}$	$-\sqrt{\frac{Y_q^2\,B_0}{8\,Y_m\,x}}$	$-0.155\sqrt{\frac{B_0}{x}}$	$-0.160\sqrt{\frac{B_0}{x}}$
	$\frac{Q}{Q_0}$	$\sqrt{\frac{2\,Y_q^2\,x}{B_0\,Y_m}}$	$0.62\sqrt{\frac{x}{B_0}}$	$0.64\sqrt{\frac{x}{B_0}}$
	$\frac{E}{E_0}$	$\sqrt{\frac{Y_e^2\,B_0}{2\,Y_m^3\,x}}$	$1.86\sqrt{\frac{B_0}{x}}$	$1.80\sqrt{\frac{B_0}{x}}$
	Y_m	$\int_0^\infty f^2(\eta)\,d\eta$	$\frac{\sqrt{\pi}}{2}C_1$	0.096
	Y_q	$\int_0^\infty f(\eta)\,d\eta$	$\sqrt{\frac{\pi}{2}}C_1$	0.140
	Y_e	$\int_0^\infty f^3(\eta)\,d\eta$	$\sqrt{\frac{\pi}{6}}C_1$	0.076
Zone of established flow; three-dimensional jet	x_0	$\frac{D_0}{\sqrt{8\,Y_m}}$	$6.2\,D_0$	$6.2\,D_0$
	$\frac{v_{\max}}{v_0}$	$\frac{D_0}{x\sqrt{8\,Y_m}}$	$6.2\frac{D_0}{x}$	$6.2\frac{D_0}{x}$
	$\lim\limits_{y\to\infty}\frac{v_y\,y}{v_0\,D_0}$	$\frac{-Y_q}{\sqrt{8\,Y_m}}$	-0.040	-0.042
	$\frac{Q}{Q_0}$	$\frac{x\sqrt{8\,Y_q^2}}{D_0\sqrt{Y_m}}$	$0.32\frac{x}{D_0}$	$0.34\frac{x}{D_0}$
	$\frac{E}{E_0}$	$\frac{D_0\,Y_e}{x\sqrt{8\,Y_m^3}}$	$4.1\frac{D_0}{x}$	$3.8\frac{D_0}{x}$
	Y_m	$\int_0^\infty f^2(\eta)\eta\,d\eta$	$\frac{C_2^2}{2}$	0.00325
	Y_q	$\int_0^\infty f(\eta)\eta\,d\eta$	C_2^2	0.00684
	Y_e	$\int_0^\infty f^3(\eta)\eta\,d\eta$	$\frac{C_2^2}{3}$	0.00201

dimensional jet:

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \frac{1}{\rho} \frac{\partial \tau}{\partial y} \qquad (69b)$$

If the same mechanism of turbulent shear exists in both cases, the distribution of velocity in both the circular and two-dimensional jets must be the same at the point where mixing begins. This argument also seems valid intuitively since at this point the breadth of the mixing region is infinitely small compared to the radius of curvature of the orifice. The experimental data presented in Fig. 16 also bear out this argument. Close examination of the plotted points will show that there is a definite tendency for the distribution to vary with x/D_0. Points for smaller values of x/D_0 have a less flat distribution and indeed follow closely the authors' equation for velocity distribution in the zone of establishment of the two-dimensional jet whereas those points for larger values of x/D_0 are approaching the curve for the zone of established flow of the circular jet.

The final forms of the equations presented in Table 1 are dependent on the values determined for x_0/B_0 and x_0/D_0 and as such should be used with some discretion in an application where the distribution of velocity at the slot or orifice is not uniform. A distribution of velocity in an orifice corresponding to a normal turbulent profile might cause a very significant shortening of the length x_0. Since the formulas for the zone of established flow from an orifice are linearly dependent on x_0/D_0, a change of 10% in x_0 would cause an equal change in the discharge-flux ratio. The variation in the constants with different inlet velocity distributions could be quite easily determined from measurements of velocity along the center line of the jet, using the principles as derived. Further experiments along this line would be of value in determining the importance of this factor.

Duilio Citrini.[15]—This research constitutes a remarkable experimental contribution to the knowledge of the diffusion process of both two-dimensional and three-dimensional jets from slots and orifices. Especially important is the fact that the devised equipment enabled the authors to make observations at cross sections at considerable distances from the efflux section, where the values of the velocity and kinetic energy of the fluid are only a small fraction of the initial values. In a strict sense, the simple laws relating to variations of the longitudinal velocity component, the volume flux, and the energy flux can be applied only to the flow from infinitely small slots and orifices. The measurements reported by the authors, however, seem to indicate the possibility that the laws can be extended to jets having a finite initial section. At least, the similarity between the two cases is reached asymptotically with increasing distance from the efflux section, because, in the region of the jets near the finite initial section (extending somewhat beyond the region which the authors call "of flow establishment"), the flow is certainly governed by more complicated laws.

The analytical presentation of the experimental results is interesting as an approach to the problem by synthesis. As the authors acknowledge, their conclusions must be considered to be only approximations, although in the writer's opinion they are reliable for practical purposes. They are based on substantially empirical hypotheses, as, for example, the assumed distribution of the longitudinal component of the velocity, according to the Gauss normal probability curve. H. Reichardt, who first proposed it in 1941,[16] tries to justify this distribution by an assumed analogy between the diffusion of jets and the diffusion of molecular agitation connected with heat transmission in solids, which is governed by laws that can be expressed through curves of the Gaussian type. Mr. Reichardt shows inductively that, as a consequence of the assumed distribution of the mean longitudinal component of velocity $\boldsymbol{v}_x$, the turbulent shearing stress can be expressed (in terms of the mean elements of the flow) by the following formula for circular jets (for two-dimensional jets a similar formula can be established):

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[15] Asst. Prof. of Hydraulics, Instituto di Idraulica del Politecnico, Milan, Italy.

$$\bar{\tau} = -\rho \left[\Lambda \frac{\partial \overline{(v_x)^2}}{\partial r} + \boldsymbol{v}_x \, \boldsymbol{v}_r \right] \ldots\ldots\ldots\ldots\ldots\ldots (70)$$

in which Λ is a linear dimension proportional to x. It does not seem easy however to justify such an expression theoretically.

W. Tollmien,[5] finds a different distribution of $\boldsymbol{v}_x$ by deduction from the well-known expression for $\bar{\tau}$ given by L. Prandtl, and also on the assumption that the turbulent mixing length l is a linear function of x only. The authors call attention to the fact that to make one or another assumption about the detailed characteristics of the turbulent flow pattern does not greatly affect the distribution of the mean velocity; and this is true for most turbulence problems, particularly for the one under discussion. This conclusion is also reached by later studies, such as those of H. Görtler,[17] or those of G. Moretti.[18] Mr. Görtler proposes a theory of the two-dimensional jet based on the hypothesis that the so-called kinematic eddy viscosity ϵ is constant on every cross section; and Mr. Moretti bases similar theories on different assumptions. On the other hand, even if all the velocity distributions proposed by the aforementioned investigations, and the distribution devised according to the normal probability curve, are equally useful for supplying an approximate interpretation of the experimental results, the fact cannot be disregarded that there exist between them some remarkable differences. It is enough to remember in this connection that, if the two curves (that by Mr. Tollmien and that of normal probability) identified with the same volume flux are compared for a circular jet, the first gives rise to a kinetic energy flux which is only 87% of that given by the second.[19]

Concerning the comparison of the different curves, it is interesting to consider the respective values of the Coriolis coefficient α—that is, the values of the ratio between the kinetic energy flux of the flow having a velocity distribution according to one of these curves, and that of an ideal flow having the same volume flux and cross section, but with a uniform velocity distribution. In Fig. 23 three curves are compared, with regard to a three-dimensional jet: The

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[16] "Über eine neue Theorie der Freien Turbulenz," by H. Reichardt, *Zeitschrift für angewandte Mathematik und Mechanik*, October, 1941.

[17] "Berechnung von Aufgaben der freien Turbulenz auf Grund eines neuen Näherungsansatzes," by H. Görtler, *ibid.*, 1942.

[18] "Scie e getti piani turbolenti," by G. Moretti, *Rendiconti*, Accademia dei Lincei, 1946.

[19] "Diffusione di una vena fluida effluente in un campo di fluido in quiete," by Duilio Citrini, *L'Energia Elettrica*, April, 1946.

Tollmien curve, the normal-probability curve, and an experimental curve obtained by the writer[20,21] as an average of nineteen measurements of mean velocity distribution. The latter observations were made on nineteen cross sections of six different water jets, flowing from the same orifice into still water, with different values of efflux velocity. For the first two curves the abscissa scale, at first, is indeterminate, as it depends (according to the theory of jets issuing from an infinitely small source) upon a constant, which must be established by experiment. This indeterminacy has been eliminated in Fig. 23 by the requirement that the three curves must meet at the place (abscissa $r = r_{0.5}$) where the velocity is reduced to half the axial velocity. The values of the Coriolis coefficient α have been calculated for the three curves, taking into account, for each one, only the part of the cross section covered by the experimental curve. This was done for the two theoretical curves, in an attempt to exclude the external region of the flow, which would have caused a great increase in the section area (infinite for the Gauss curve), whereas its effect on the volume flux and on the kinetic energy flux would have been negligible. The following values of α were obtained:

Curve	α
Tollmien	2.8
Normal probability	3.64
Experimental	3.5

This comparison demonstrates that the normal-probability curve is a good empirical representation of the velocity distribution in the diffusion zone of jets. The results presented in the paper lead to the same conclusion. Naturally this does not apply to the external region of the diffusion zone, where the velocity values really decrease rather quickly toward zero. It is the writer's opinion that this circumstance is responsible for the appreciable discrepancy between the experimental values of the volume flux plotted in Fig. 19 and the curve representing Eq. 52, which was obtained by the analytical integration of the Gauss curve. As to the kinetic energy flux, the agreement is rather good because that factor depends on the third power of the velocity, so that the external region contributes very little to it.

The following series of numerical values, defining the writer's experimental curve plotted in Fig. 23, is in good agreement with the results of his tests:

$\frac{r}{r_{0.5}}$	$\frac{v_x}{v_m}$
0.2	0.966
0.4	0.880
0.6	0.757
0.8	0.625
1.0	0.500
1.2	0.387
1.4	0.284
1.6	0.191
1.8	0.106
2.0	0.034

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[20] "Ricerca sperimentale sulla diffusione di una vena liquida effluente in un campo di liquido in quiete," by Duilio Citrini, *L'Energia Elettrica*, August, 1946.

[21] "Nuove ricerche sulla diffusione di una vena liquida in un campo di liquido in quiete," by Duilio Citrini, *ibid.*, June, 1947.

It is the writer's opinion (although lack of necessary numerical data prevents a direct comparison) that this curve also provides very good interpretation of tests presented in the paper.

As to the expansion of the diffusion zone and the law of decrease of the axial velocity (the variations of the volume flux and energy flux being a direct consequence of it), the analysis of the authors seems to have been guided by a criterion of practical convenience, rather than by theoretical exigence. Even accepting for the entire zone of established flow the hypotheses that define a diffusion along a dihedron or a cone (respectively, for the two types of jets studied), there does not seem to be any predictable reason why the edge of the dihedron or the vertex of the cone should lie in the efflux section. Possibly, this appreciable discrepancy between the analytical function and the experimental values (shown by Figs. 11 and 17) is due also to such an assumption, whereas each group of tests could be interpreted more logically by a more suitable Gauss function, correlated to a more suitable law of expansion of the diffusion zone. In order to compare the writer's tests on three-dimensional jets, it must be noted that the Gauss function yields

$$r_{0.5} = 1.178\,\sigma \ldots\ldots\ldots\ldots\ldots\ldots\ldots (71a)$$

The value, $C_2 = 0.081$, given by the authors for the empirical coefficient that enters their theory, corresponds therefore to the expansion law:

$$r_{0.5} = 0.0954\,x \ldots\ldots\ldots\ldots\ldots\ldots\ldots (71b)$$

On the other hand, for the region farthest downstream in the writer's tests, the law—

$$r_{0.5} = 0.1\,x - 0.21\,\mathrm{D}_0 \ldots\ldots\ldots\ldots\ldots\ldots (71c)$$

—has been found most suitable. This law does not differ greatly from the previous law (Eq. 71b), but it places the vertex of the diffusion cone somewhat farther than the distance 2 D_0 downstream from the efflux section. Upstream from this region (that is, for approximately $x < 15\,D_0$), and also for the entire region which the authors term "flow establishment," the writer has found that the real expansion process is much more exactly represented by a law of parabolic type—that is:

$$\frac{r_{0.5}}{D_0} = 0.55 + \left(0.08\,\frac{x}{D_0}\right)^2 \ldots\ldots\ldots\ldots\ldots (71d)$$

Undoubtedly the Iowa experiments lead to much greater values of x than the writer's experiments. This is due partly to the difficulty of measuring low velocities in the water, and partly to the limited dimensions of the stilling basin used by the writer in his tests. In this connection, it is the writer's opinion (differing somewhat from the authors' view) that, if the downstream region has a greater theoretical meaning, owing to its greater conformity with the assumptions of the theory, the upstream region, which has higher values of velocity and energy flux, is far more interesting for practical purposes. This is the reason that the writer has not derived a law of decrease of the axial velocity of the type of Eq. 47 (which undoubtedly is in much closer agreement with the theoretical assumptions) but has preferred to adopt the form:

$$\frac{v_m}{v_0} = 6.6\,\frac{D_0}{x} - 0.49 \ldots\ldots\ldots\ldots\ldots\ldots (72)$$

Eq. 72 certainly has an empirical basis; but it seems to follow better the trend of the real process, at least for penetrations of 40 diameters or less.

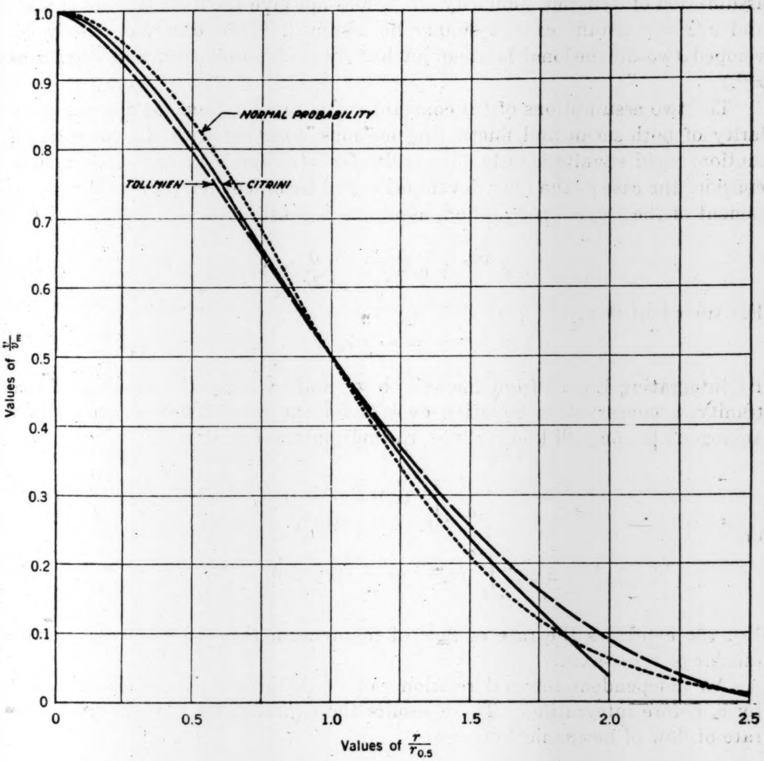

FIG. 23

These differences concern only details, however, for all practical purposes, the authors' and the writer's formulas can be considered as perfectly equivalent. The comparison proves therefore that the results of both groups of tests are reliable, and that all the formulas proposed in the paper can be applied in practical problems.

An interesting property, which should affect the stream surfaces for the circular jet, if all the theoretical assumptions were fulfilled, is that these surfaces should be mutually homothetical, the homothetic center being at the vertex of the diffusion cone.

STANLEY CORRSIN.[22]—This thorough experimental study of the mean flow fields in submerged jets is particularly interesting since the authors have extended the measurements to extremely large distances along the jet axis. In the introductory analytical part of the paper, however, the authors have not explained which assumptions lead to which results in the integral computations of width (σ) and $v_{\max}$ variation in the downstream direction. Specifically, the assumption of dynamic similarity alone does not give the linear spread of a jet, and σ/x = constant cannot simply be assumed. (For example, a fully developed two-dimensional laminar jet has dynamic similarity, and spreads as $x^{2/3}$.)

The two assumptions of (*a*) constant static pressure and (*b*) dynamic similarity of both mean and fluctuating motions, together with the equations of motion, yield simultaneously the results for $\sigma(x)$ and $v_{\max}(x)$. For example, consider the case of the fully developed round turbulent jet. From the x-component of the Reynolds equation, assuming p constant—

$$v_x \frac{\partial v_x}{\partial x} + v_r \frac{\partial v_x}{\partial r} = \frac{1}{\rho r}\frac{\partial}{\partial r}(r\tau) \dots\dots\dots\dots (73)$$

For turbulent flow,

$$\tau \doteq -\rho\, \overline{v_x\, v_r} \dots\dots\dots\dots (74)$$

By integrating Eq. 73 from the axis to ∞, and by using the equation of continuity, a conservation equation evolves for the rate of flow of momentum, analogous to the well known "von Kármán integral relation":

$$\frac{d}{dx}\int_0^\infty (v_x)^2\, r\, dr = 0 \dots\dots\dots\dots (75a)$$

or

$$2\,\pi\,\rho \int_0^\infty (v_x)^2\, r\, dr = \text{constant} = M_0 \dots\dots\dots\dots (75b)$$

Eq. 75*b* expresses the rate of flow of momentum through all planes perpendicular to the x-axis.

An independent integral relation can be deduced by multiplying Eq. 73 by v_x before integrating. There results the equation for the conservation of rate of flow of mechanical energy—

$$\frac{d}{dx}\int_0^\infty \left[\tfrac{1}{2}\,\rho\,(v_x)^3\right] r\, dr = -\int_0^\infty \tau\, \frac{\partial v_x}{\partial r}\, r\, dr \dots\dots\dots\dots (76)$$

The left-hand side of Eq. 76 (within the boundary layer type of approximation) is the x-rate of increase in the rate of flow of mean kinetic energy. The right-hand side is the total rate at which the turbulent shear stress does work on the mean motion over a cross section; thus, it is the rate at which mean flow kinetic energy is transformed to turbulent kinetic energy—a "less available" form of energy.

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[22] Asst. Prof. in Aeronautics, School of Eng., The Johns Hopkins Univ., Baltimore, Md.

The assumption of dynamic similarity requires that

$$\frac{v_x}{v_{\max}} = f(\eta) \qquad (77)$$

in which

$$\eta = \frac{r}{\sigma} \qquad (78a)$$

(Of course, any characteristic lateral jet dimension may be used as reference length) and

$$\tau = \rho\,(v_{\max})^2\,g(\eta) \qquad (78b)$$

Substitute the similarity assumptions (Eqs. 77 and 78) into Eqs. 75*b* and 76 to obtain simultaneous equations for the two unknowns, $\sigma(x)$ and $v_{\max}(x)$:

$$(v_{\max})^2\,\sigma^2 = \frac{M_0}{2\,\pi\,\rho\,I_1} \qquad (79a)$$

and

$$\frac{d}{dx}\left[(v_{\max})^3\,\sigma^2\right] = -\,2\,\frac{I_3}{I_2}\,(v_{\max})^2\,\sigma \qquad (79b)$$

In Eqs. 79, the symbols I are pure numbers depending only upon the assumed dimensionless profile shapes:

$$I_1 \equiv \int_0^\infty f^2\,\eta\,d\eta \qquad (80a)$$

$$I_2 \equiv \int_0^\infty f^3\,\eta\,d\eta \qquad (80b)$$

and

$$I_3 \equiv \int_0^\infty g\,\frac{df}{d\eta}\,\eta\,d\eta \qquad (80c)$$

Simultaneous solution of Eqs. 79 leads to the required results:

$$v_{\max} = \left(\frac{1}{2}\,\frac{I_2}{I_3\,\sqrt{I_1}}\sqrt{\frac{M_0}{2\,\pi\,\rho}}\right)\frac{1}{x} \qquad (81a)$$

and

$$\sigma = 2\,\frac{I_3}{I_2}\,x \qquad (81b)$$

The constants of integration have been made equal to zero by (arbitrarily) defining $x = 0$ where $\sigma = 0$ or $v_{\max} = \infty$.

W. Douglas Baines,[23] Jun. ASCE.—The simple, easily understood analysis of submerged jets, presented in this paper, should be of considerable value to the engineering profession. The writer assisted in the evaluation of some of the mathematical expressions and subsequently extended the study of axisymmetrical jets by investigating the Reynolds effect and the turbulence. On the basis of the material gained in this study, revaluation of certain results which the authors present appears to be warranted.

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[23] Research Associate, Iowa Inst. of Hydr. Research, State Univ. of Iowa, Iowa City, Iowa.

The authors' assumption that the Reynolds effect is negligible is unnecessary because the basic assumptions of the analysis (constant pressure throughout, similarity of velocity profiles, and constant spread of the jet) do not specifically involve the Reynolds number. The analysis should describe the flow at any given Reynolds number where the assumptions are satisfied. Thus, it would appear that the method of attack of the general problem would be to investigate the flow pattern at several different Reynolds numbers and to note in what ways the pattern changes as the Reynolds number changes. Of the three main assumptions, the first (constant pressure) has been checked theoretically and found satisfactory by W. Tollmien,[5] and the other two have

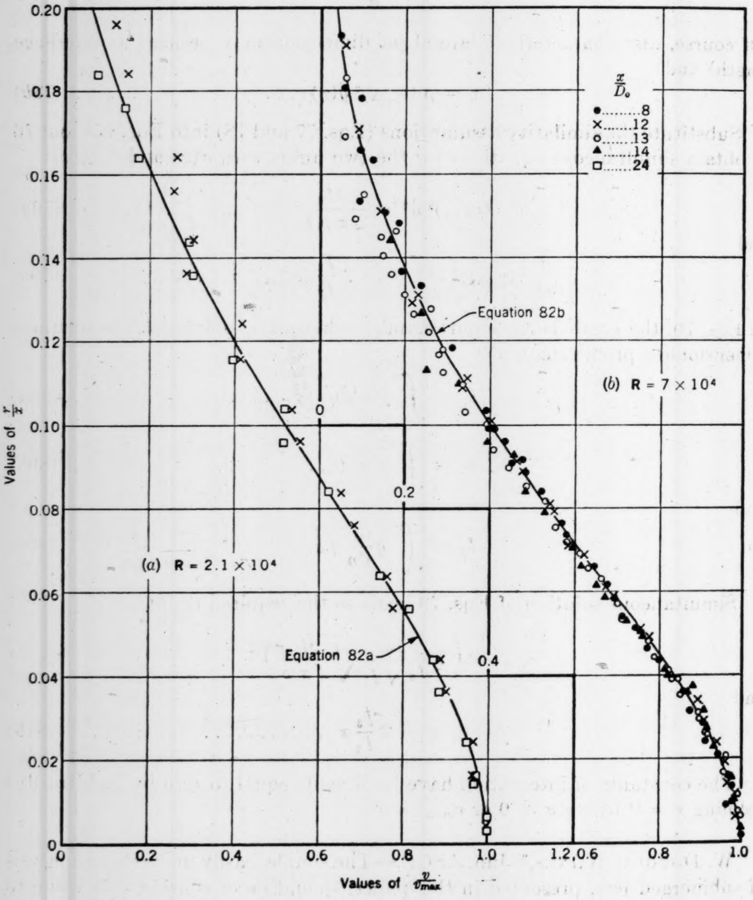

FIG. 24.—DISTRIBUTION OF LONGITUDINAL VELOCITY IN THE ZONE OF ESTABLISHED FLOW

been corroborated by the writer. The similarity of velocity profiles for a constant Reynolds number is shown in Fig. 24, which gives the velocity profile for a different Reynolds number.

The corresponding velocity distributions can be expressed empirically as follows: For $\mathbf{R} = 2.1 \times 10^4$—

$$\frac{v_x}{v_{\max}} = \exp\left[-43.3\left(\frac{r}{x}\right)^{1.82}\right] \ldots\ldots\ldots\ldots\ldots (82a)$$

and, for $\mathbf{R} = 7 \times 10^4$—

$$\frac{v_x}{v_{\max}} = \exp\left[-64.4\left(\frac{r}{x}\right)^{1.84}\right] \ldots\ldots\ldots\ldots\ldots (82b)$$

These curves, plotted through observed data, are shown in Figs. 24(a) and 24(b), respectively. The linear spread of the jet is substantiated in Fig. 25, a plot similar to Fig. 15. Again, for a constant Reynolds number the assumption appears to be satisfied.

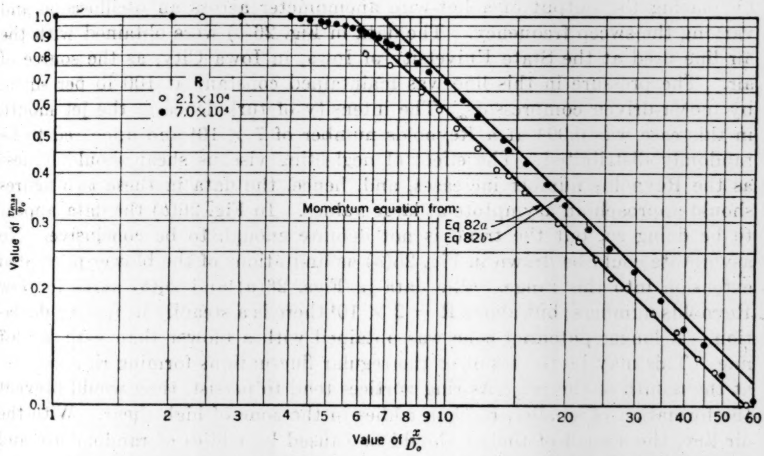

FIG. 25.—DISTRIBUTION OF CENTER-LINE VELOCITY FOR $\mathbf{R} = 2.1 \times 10^4$ AND 7.0×10^4

A more rigorous mathematical analysis, assuming a general velocity profile, also results in Eqs. 23, 25, 27, 28, 30, and 32, thus illustrating the fact that the probability function is not a necessary assumption in the analysis. However, in the general case, the constants C_1 and C_2 are more complicated and thus not so convenient in practical problems as those derived with the probability distribution. The main part of such an analysis is given by H. W. Liepmann and J. Laufer[11] for the two-dimensional jet and by the writer for the three-dimensional case.[24] This analysis shows that the bulk properties of the flow can be described by the constant x_0/B_0 (or x_0/D_0) as is shown in the special case by the authors. This constant, the length of the potential zone, is a physical property of the jet that is easy to visualize and is perhaps preferable to C_1 or C_2, the constant appearing in the equations. Thus, extending the reasoning that the analysis describes the jet at any given Reynolds number, the effect of changing the Reynolds number should be described by the variation of x_0/B_0 (or x_0/D_0).

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[24] "Investigations in the Diffusion of Submerged Jets," by W. D. Baines, thesis presented to the State Univ. of Iowa, at Iowa City, Iowa, in August, 1948, in partial fulfilment of the requirements for the degree of Master of Science.

Experimental verification of the foregoing reasoning is given by Fig. 26, in which the length of the potential zone is plotted as a function of the Reynolds number. The data in Fig. 26(a) were obtained using a blower to furnish the air stream, for which the intensity of turbulence, $\frac{\sqrt{\overline{[(v')_x]^2}}}{v_0}$ was 0.008 at a Reynolds number of 7×10^4. The fluctuations were not completely random, because the blades of the blower induced a dominant turbulent frequency of 360 cycles per sec. The regularity of these pulsations could easily be noted by placing the output of a hot-wire anemometer across an oscilloscope and varying the sweep frequency. The data in Fig. 26(b) were obtained with the air line used at the State University of Iowa, in Iowa City, as the source of air. The pressure in this line was maintained constant at 100 lb per sq in. by motor-driven compressors. The intensity of turbulence in the jet mouth in this case was 0.004 at a Reynolds number of 7×10^4 and appeared to be randomly distributed. The effect of neglecting viscous shear should be less as the Reynolds number increases, and, hence, the data in these two figures should approach an asymptote at large values. In Fig. 26(b) the data appear to be doing so, but the trend is not definite enough to be conclusive. No asymptote could be drawn in Fig. 26(a) as limitations of the blower prevented extension into this range. The data in Figs. 26(a) and 26(b) agree for low Reynolds numbers; but above $\mathbf{R} = 2 \times 10^4$ there is a steadily increasing deviation. A longer potential cone was obtained with a blower than with the air line. This may be the result of the regular fluctuations forming ring vortices at the mouth of the jet. As ring vortices tend to persist, they would prevent the formation of smaller, random eddies in the zone of high shear. With the air line, the spread of the jet should be caused by eddies of random size and occurrence forming in the high shear zone.

The effects of the Reynolds number and the turbulence of the issuing stream can be noted in the authors' results and those of other investigators. The results presented by the authors, by S. Corrsin,[10] D. Citrini,[21] Poul Becher,[25] and D. W. Appel,[26] and by tests of the velocity distribution downstream from a 6-in. bellmouthed opening mounted in the small wind tunnel of the Iowa Institute of Hydraulic Research, at Iowa City, are plotted in Fig. 26 for comparison. If regular fluctuations could be expected, the results would be plotted in Fig. 26(a); if not, they would be plotted in Fig. 26(b). It may be noted that Mr. Corrsin also observed the regular fluctuations in the velocity at the jet mouth, presenting an oscillograph,[10] similar to that which the writer observed. Considering the scatter of the writer's data and the unknown turbulence conditions in the other cases, the agreement between various investigations is surprisingly good. All the investigations show the definite lengthening of the potential cone with an increasing Reynolds number.

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[25] "Om beregning af Indblaesningsåbiniger," by Poul Becher, I Kommission Hos Jul. Gjellerups Forlag, Copenhagen, 1949.

[26] "The Development of a Turbulence Pitot for Use in Water," by D. W. Appel, thesis presented to the State Univ. of Iowa, at Iowa City, Iowa, in June, 1949, in partial fulfilment of the requirements for the degree of Master of Science.

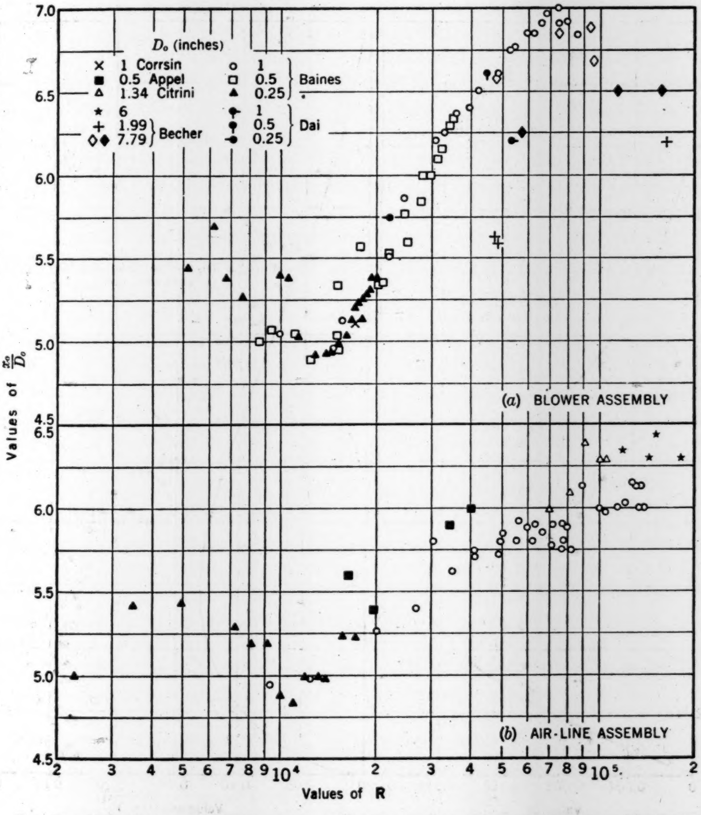

FIG. 26.—VARIATION IN THE LENGTH OF THE POTENTIAL CONE WITH THE REYNOLDS NUMBER

The authors' assumption of constant momentum flux was corroborated by the writer's measurements. This is illustrated by computing the length of the potential cone from the velocity profiles, Fig. 24, using Eq. 6, and comparing the result to that obtained by direct measurement. This computation gave values of 5.76 and 6.90 from the empirical curves in Figs. 24(*a*) and 24(*b*), respectively. These values are close to the values obtained from Fig. 25, which are 5.7 and 7.0 for corresponding Reynolds numbers. It should be noted that the scatter of the experimental points in Fig. 17 is a result of the

Reynolds effect. The lack of this scatter in the writer's data thus makes the corroboration of the momentum assumption more definite.

The turbulent properties of the flow are of great theoretical importance because they give an insight into the general turbulent flow of fluids. In the authors' simple analysis of the turbulence, the equations relating the shear to the mean flow pattern—Eq. 35 and the corresponding one for axisymmetric flow—are fundamental. It is anticipated, therefore, that good agreement between the properties derived by these equations and the experimental values should exist. In Fig. 27 the measurements of turbulent shear which the writer made with a hot-wire anemometer are presented. These consist of traverses at $x/D_0 = 12$ and $x/D_0 = 20$ and indicate that there is similarity of shear

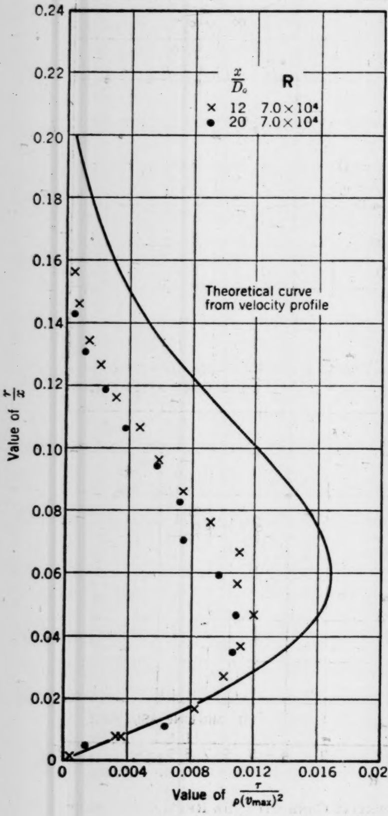

FIG. 27.—DISTRIBUTION OF TURBULENT SHEAR IN THE ZONE OF ESTABLISHED FLOW OF AN AXISYMMETRICAL JET

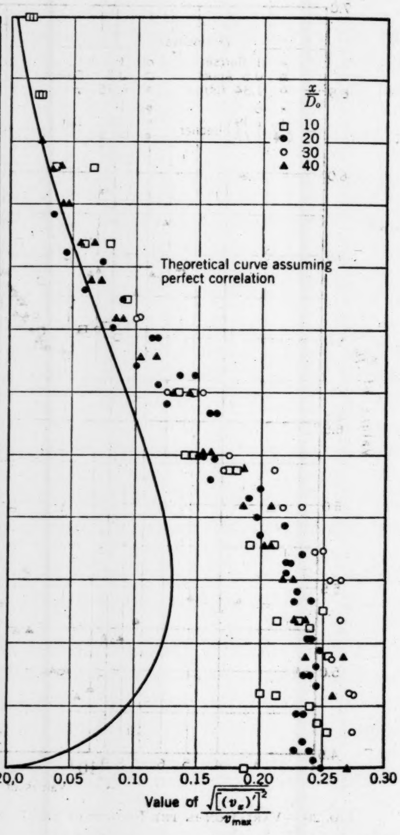

FIG. 28.—DISTRIBUTION OF TURBULENCE REFERRED TO THE CENTER-LINE VELOCITY IN THE ZONE OF ESTABLISHED FLOW FOR R = 7.0×10^4

profiles in this zone. The experimental points, however, do not follow the shear profile computed by the equation corresponding to Eq. 35. This large discrepancy, which also has been noted by Mr. Corrsin,[10] was probably the result of a combination of inaccuracies in the mechanical setup and in the evaluation of calibration constants in the writer's experiments.

A clearer understanding of the turbulence may be gained by considering it from the viewpoint of two of the turbulence theories. The authors' use of the Prandtl mixing-length concept is not of practical significance since the quantities derived cannot be measured directly. Consideration of the problem in the light of the statistical theory of turbulence is not too satisfactory since the properties of the turbulence in cases of nonisotropic flow do not naturally follow the same definitions as are being used in isotropic turbulence. The intensity and shear have the same definition in each case but have three nonequal components in nonisotropic flow. A mixing length defined as the area under the correlation curve is difficult to use for two reasons: (1) At present, correlation coefficients cannot be measured by the double-wire, hot-wire technique unless both wires are in the same mean velocity zone; and (2) the abscissa of the correlation curve should follow either the fluid particle in its motion or a path perpendicular to it. In the first case, this would mean following a streamline, which is usually a difficult procedure. Evaluation of the eddy viscosity, defined as the coefficient of diffusion at a point, suffers from the same lack of natural coordinates. A quantity that would be of great importance, and would retain the same definition, is λ, the size of the smallest eddies.[27] The size and distribution of λ would indicate the manner and rate of dissipation of turbulent energy. Unfortunately, a satisfactory method of measuring this quantity was not available to the writer.

The qualitative pattern of the turbulence can be deduced from oscillographs, however. Upon moving through the shear zone lengthwise down the jet, it was noted that the frequency of the turbulence was very high near the mouth, and that it decreased as the distance increased. A traverse taken radially outward from any point in the zone of established flow illustrated the transition of the flow from turbulent to laminar. This transition[10] progresses from pure turbulent flow in the central core of the jet, through an annular transition zone in which the flow is neither laminar nor turbulent, to a region of laminar flow that extends to infinity. The oscillograms in these zones are sharp-peaked waves of high frequency passing to more rounded wave forms of longer periods, finally becoming slow, easy undulation in the laminar regions.

The authors' assumption that v'_x and v'_r are proportional in deriving the curve of $\sqrt{\overline{(v')^2}}$ in Fig. 20 from the velocity profile does not seem valid for the following reasons: A direct result of this assumption is that there is similarity in the radial distribution of the intensity of turbulence. Inspection of Fig. 28 shows that similarity of intensity profiles is a fact only when $x/D_0 > 20$, whereas the velocity profiles are similar for $x/D_0 > 8$. Making use of the foregoing assumption and the shear profile of Fig. 27, the theoretical curve (similar to the curve in Fig. 20) plotted in Fig. 28 was obtained. In order to agree with the experimental points, v'_r would have to be about four times v'_x. Thus the assumption of perfect correlation does not seem valid since measurements[10] of these quantities have not borne out this proportionality.

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[27] "Statistical Theory of Turbulence," by G. I. Taylor, *Proceedings*, Royal Soc. of London, Series A, Vol. 151, 1935, pp. 421–478.

Problems requiring a detailed examination of submerged jets have confronted the civil engineer only recently. For many years a rough approximation of the characteristics of jets was sufficient for design purposes; but in recent times the design of stilling basins for large dams has necessitated a detailed study of the properties of jets. This has been accomplished mainly by hydraulic models, as has the companion problem of the scour caused by high-velocity flows. It should be noted that the latter is simply a study of the effects of submerged jets on sediment. Since a reasonable and workable design of a structure is possible only when the engineer is intimately acquainted with the phenomena involved, this paper should greatly assist the designer of the aforementioned types of structure. In the usual case, the Reynolds number is of the same order as those in the experimental investigations reported by the authors, so that the designer is free of a further assumption. Thus, it is the writer's opinion that this simple, direct analysis of submerged jets should be of great assistance to engineers in the future.

Acknowledgment.—The investigations reported herein were conducted at the Iowa Institute of Hydraulic Research under the auspices of the Bureau of Ships and the Office of Naval Research of the United States Navy, and the Department of Mechanics and Hydraulics of the State University of Iowa.

ABRAHAM STREIFF,[28] M. ASCE.—Submerged jets of water are of practical importance because of their use in underwater spilling through dams and powerhouses. For this reason large-scale tests were made at the Croton Dam on the Muskegon River in Michigan in 1915. Based on the results of these tests, the Mio, Alcona, Hodenpyl, and Hardy dams on the Ausable and Muskegon rivers in Michigan have been provided with underwater spilling tubes. The northern part of the southern peninsula of Michigan is covered with glacial drift, which causes fairly uniform runoff conditions. The application of spillway tubes through the powerhouse therefore enabled a reduction in cost by entirely or partly eliminating an overflow spillway.

Considering that the tests reported in this paper were made on a very small scale and with air, it is not surprising to find that tests with water jets, having a velocity of 45 ft per sec issuing from nozzles from 12 in. to 20½ in. in diameter and discharging with the top 17 in. under the water surface, would give results wholly different from the authors' findings. Instead of the Reynolds number $\frac{V\,a}{\nu}$ the Froude number $\frac{V^2}{a\,g}$ is used, because of the close proximity of a free surface, thus calling in the influence of gravity.[29,30]

The pattern of flow resulting was wholly different from that reported in the paper. The jet diffusion shown in Fig. 1 (rectilinear diffusion under a constant angle) was completely absent. Instead, a flow pattern appeared

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[28] Cons. Engr., Austin, Tex.

[29] "Applied Hydro and Aeromechanics," based on lectures of L. Prandtl, by O. G. Tietjens, translated by J. P. Den Hartog, McGraw-Hill Book Co., Inc., New York, N. Y., 1st Ed., 1934, p. 10.

[30] "Hydrodynamics," by Horace Lamb, Univ. Press, Cambridge, England, 6th Ed., 1932.

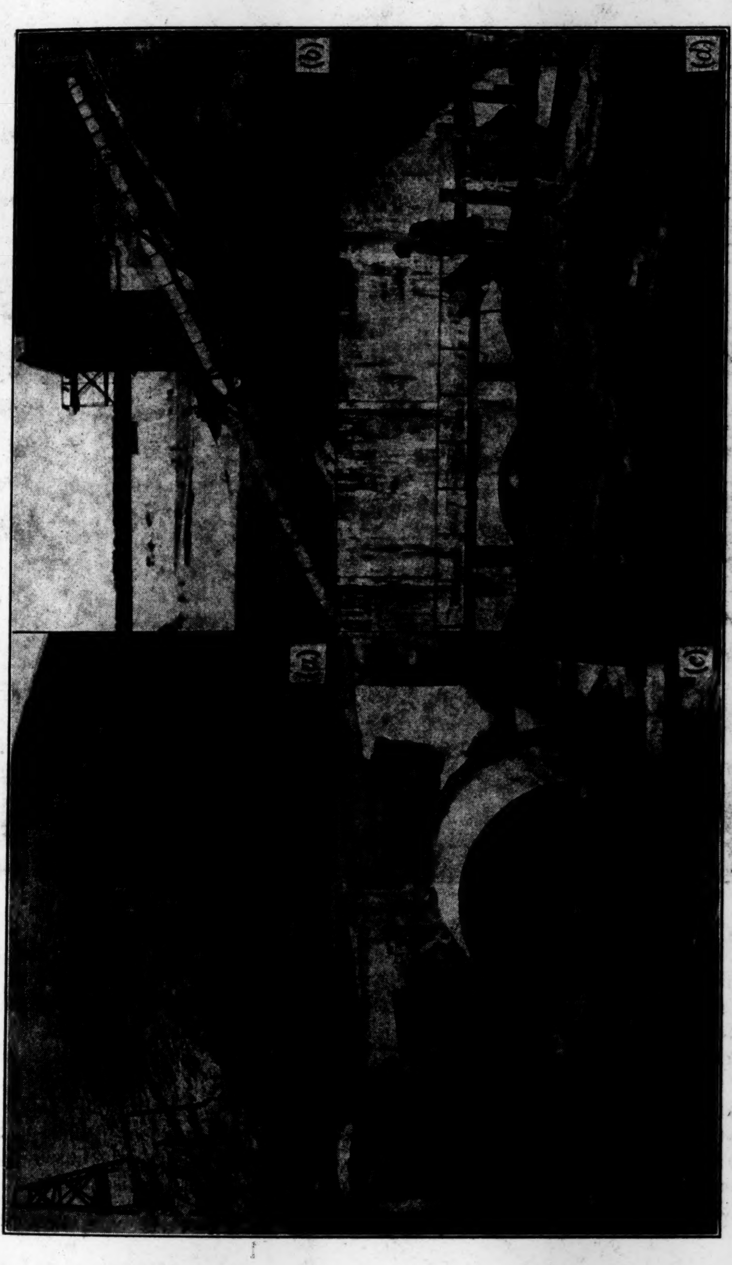

Fig. 29.—Large-Scale Tests at Croton Dam, on the Muskegon River, in Michigan, January 29-30, 1915

which was quite similar to the well-known streamline pattern into a Borda mouthpiece:

$$u = x + e^x \cos y \qquad (83a)$$

and

$$v = y + e^x \sin y \qquad (83b)$$

This is shown plainly in Fig. 29(*a*), which shows discharge (no baffle) from a 12-in. nozzle, with the gate fully opened. The water-surface elevation was 93.2; the elevation of the top of the nozzle, 91.8. The effect of the wind in producing ripples away from the nozzle is apparent.

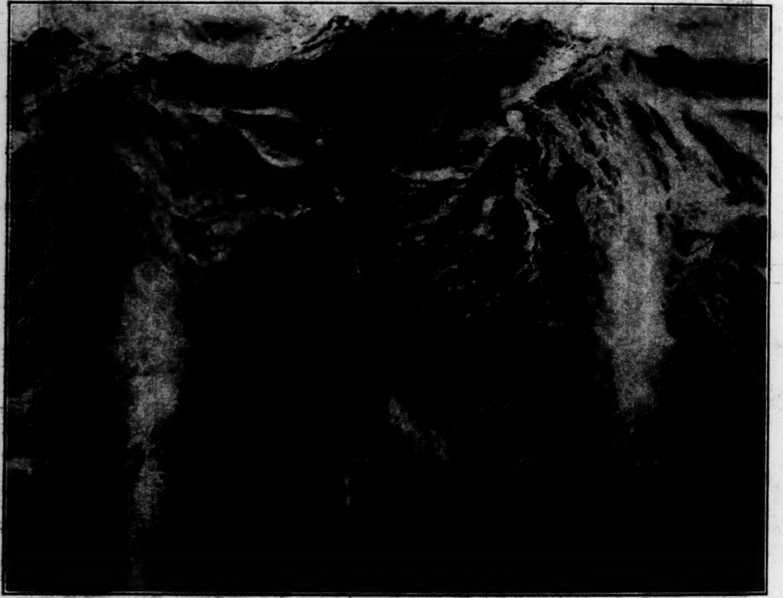

FIG. 30

The conclusion that "All results are considered applicable for design purposes to the flow of any liquid * * *" (under the heading, "Conclusions") is not confirmed. The small scale, the use of air instead of water with the resulting scale effect, and the absence of a free surface contribute to the complete dissimilarity of the results.

The large-scale tests were made by siphoning a 30-in. steel penstock over an earth dam adjoining the powerhouse of the Croton Dam. The nozzles varied in diameter from 12 in. to 20½ in. and operated under a gross head of 40 ft. This is shown in Figs 29(*b*) and 29(*c*). The elevation of the nozzle was such that the jet became submerged about 1 ft because of the rise in tailwater after the units in the powerhouse were opened wide.

The theory proposed by the authors, following previous investigators such as W. Tollmien,[5] H. Reichardt,[12] P. Ruden,[6] assumes (under the heading, "Statement of the Problem") that "* * * the sole effective force [of the jet] was the tangential shear." The large-scale tests with water on Croton Dam led to opposite conclusions. There is so much kinetic energy in the jet that the tangential shear around its circumference is minimal in comparison. The writer concluded that the decelerating force is the reaction and equal to twice the head. This assumption was in satisfactory agreement with the test results. The kinetic energy is finally consumed by the tangential shear in the resulting turbulence.

The submerged jet and the hydraulic jump are the same; with rising tailwater the jump is finally converted into the submerged jet. The theory of the jump does not consider at all the sole decelerating force of the tangential shear (under the heading, "Statement of the Problem"), whereas in appearance the turbulence of the jump is greater than that of the submerged jet. Test results show that in both cases the deceleration occurs at about the same distance. It will be necessary to bring both theories into satisfactory agreement with each other, inasmuch as the only difference between the two phenomena is the elevation of the tailwater.

Fig. 30 shows a 12-in. jet issuing at a velocity of 45 ft per sec at a depth of 12 in. under the water surface, looking down on the surface. This surface was entirely unperturbed, the jet "drilling" a hole through the surrounding water. Sitting in a rowboat, the bottom of which was only a few inches over the jet (Fig. 29(*d*)), the writer could not notice any evidence betraying the close proximity of a submerged jet issuing at 45 ft per sec.

The turbine discharge, shown in Fig. 29(*d*), as smooth water upstream from the jet, flows at right angles over the jet as if it were a solid log lying in the water. The jet acts like a weir, under and over which the turbine discharge flows at right angles. There was not the slightest indication of a straight-line diffusion cone in accordance with the theory propounded in the paper. On the contrary, the spreading is curvilinear and quite rapid, in accordance with deceleration by the reaction force; the jet suddenly breaks through and spreads in all directions following approximately the pattern of the Borda mouthpiece. The authors' dimensionless graphs, therefore, are not in agreement with these tests.

These large-scale experiments were used in the design of apron protection for four hydroelectric plants which were placed on soil foundations. The largest one, the Hardy Dam, operates under a head of 100 ft and has been in operation since 1938. The submerged jets discharge under the hydraucone plate of the turbine units.

Harold R. Henry,[31] **Jun. ASCE.**—The value of this paper lies not only in the specific results obtained, but in the method of attack, which may be applied with modification if necessary to submerged jets of a different nature and application. For instance, one of the specific examples referred to by the authors, efflux from a submerged sluice gate, is cited for comparison. The upper limit of submergence of a sluice gate (that is, infinite vertical extent of tailwater) may be represented by substituting a boundary for the plane of symmetry of the completely submerged two-dimensional jet shown in Figs. 1 and 21, whereas the lower limit is represented by a free hydraulic jump forming on this boundary downstream from the efflux section. By far the majority of cases involving flow under a sluice gate occur nearer the limiting case of the free jump: Fig. 31 represents, schematically, a typical intermediate condition.

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[31] Instr., Civ. Eng. Dept., Georgia Inst. of Technology, Atlanta, Ga.

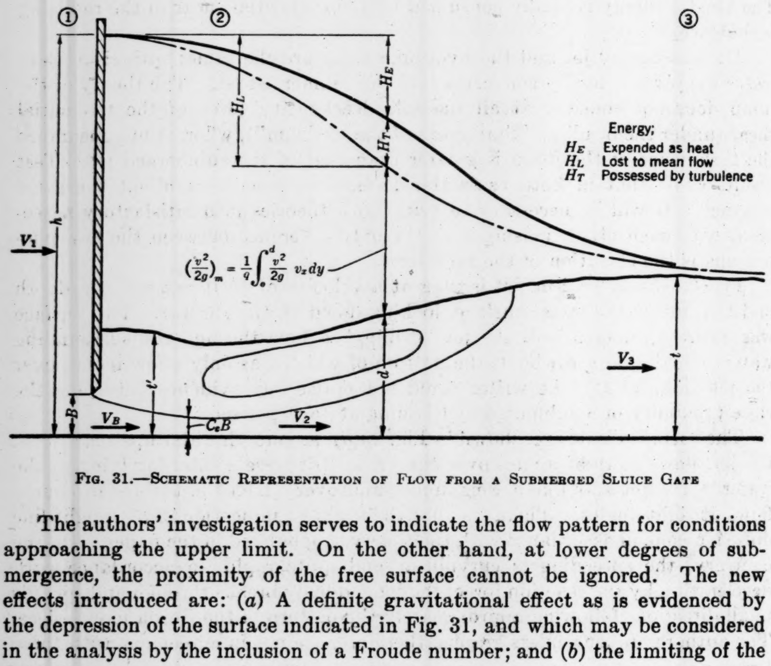

FIG. 31.—SCHEMATIC REPRESENTATION OF FLOW FROM A SUBMERGED SLUICE GATE

The authors' investigation serves to indicate the flow pattern for conditions approaching the upper limit. On the other hand, at lower degrees of submergence, the proximity of the free surface cannot be ignored. The new effects introduced are: (*a*) A definite gravitational effect as is evidenced by the depression of the surface indicated in Fig. 31, and which may be considered in the analysis by the inclusion of a Froude number; and (*b*) the limiting of the fluid downstream from the efflux section to a finite vertical extent which calls for a variable describing the relative depth of the tailwater. The foregoing variables may be $F_B = \frac{q}{B\sqrt{g\,B}}$ and $\frac{t}{B}$, in which B is the height of gate opening $(= B_0/2)$; t is the tailwater depth; q is the mean flow per unit width or gate opening; and g is the acceleration of gravity. Hence, when modified to apply to two-dimensional flow from a submerged sluice gate, the formula for relative velocity at any point in the flow (Eq. 1) would take the form:

$$\frac{v}{V_B} = f\left(F_B, \frac{t}{B}, \frac{x}{B}, \frac{y}{B}\right) \quad \text{(84)}$$

in which $V_B = q/B$, the mean velocity at the gate.

Eq. 84 and the reasoning leading to this equation indicate that an orderly method to determine experimentally the effect of proximity of the free surface would be to investigate the flow phenomena with systematic variation of F_B and t/B. The writer has conducted a series of tests to examine the mean flow pattern and the intensity of turbulence (v', the root-mean-square of the velocity fluctuation) for the three conditions holding $F_B = 2$—namely, $t/B = 3.35$, 4, and 8. The results are presented in Figs. 32, 33, and 34. The first value of t/B was necessary for the formation of a free jump at the vena contracta; the other values were chosen to indicate the effects of increased submergence.

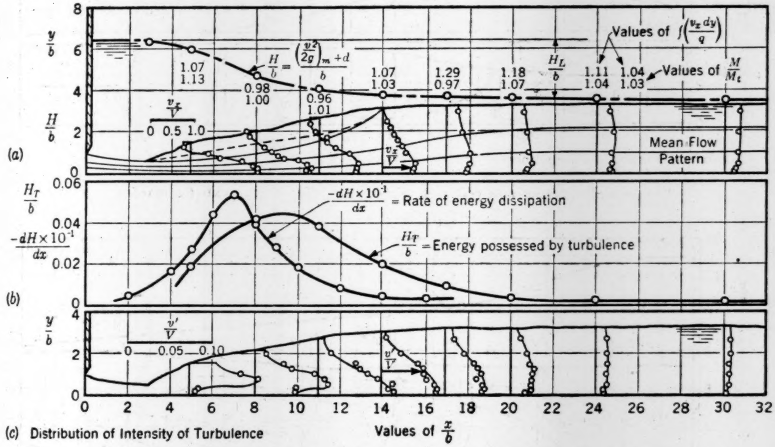

FIG. 32.—PATTERNS FOR FREE JUMP: $F_B = 2.0$; $F_1 = 4.30$; $t/B = 3.35$; $h/B = 6.35$; $C_c = 0.60$; AND $C_d = 0.561$

It is enlightening to consider the effect of submergence from the standpoint of the discharge characteristics of the gate. Simultaneous consideration of F_B and the headwater depth, h, leads to the expression:

$$F_B = \left(\sqrt{\frac{2\,h}{B}}\right)\frac{q}{\sqrt{2\,g\,h}} = \sqrt{\frac{2\,h}{B}}\,C_d \ldots\ldots\ldots\ldots (85)$$

in which $C_d = q/\sqrt{2\,g\,h}$ may be treated as a discharge coefficient. Thus, whenever F_B appears in any functional expression, as Eq. 84, it may be replaced by C_d or h/B. Hence, the variation of C_d and h/B at various degrees of submergence should be noted. A dimensionless expression for the experimental determination of this variation is

$$C_d = f\left(\frac{h}{B}, \frac{t}{B}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (86)$$

The discharge characteristics of a sharp-edged sluice gate, as determined experimentally by the writer, are shown as the solid curves in Fig. 35. The center-line curves, lines of constant F_B, were computed according to Eq. 85.

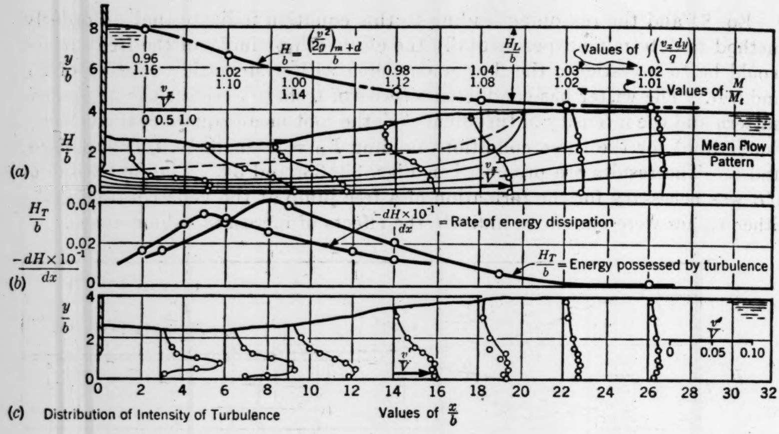

FIG. 33.—PATTERNS FOR $t/B = 4$; $F_B \doteq 2.0$; $h/B = 7.95$; AND $C_d = 0.501$

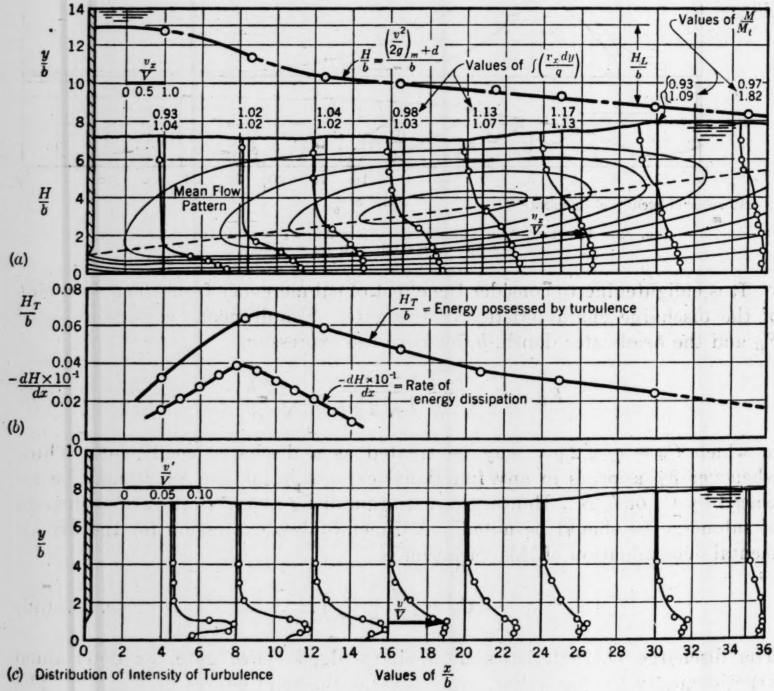

FIG. 34.—PATTERNS FOR $t/B = 8$; $F_B = 2.0$; $h/B = 13$; AND $C_d = 0.391$

The three solid points represent values at which the patterns of mean flow and turbulence were investigated, and serve to indicate clearly the change in degree of submergence.

The experimental trend of the curves, C_d versus h/B, in Fig. 35 was verified by an approximate theoretical approach, utilizing the authors' assumption of

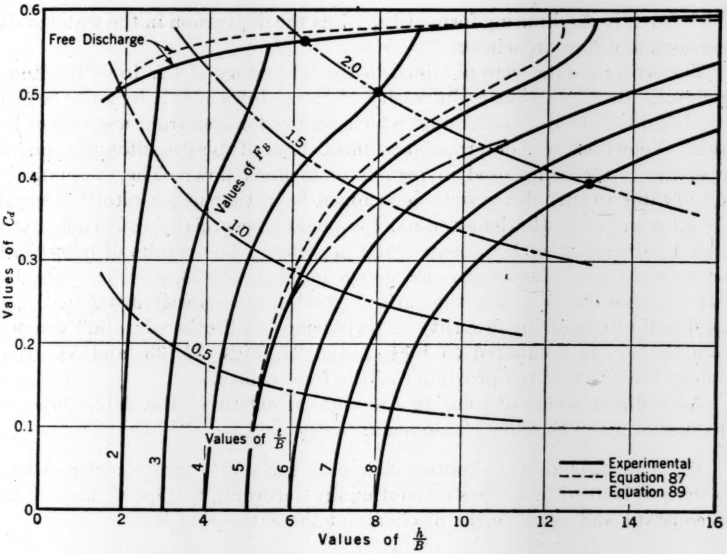

FIG. 35.—DISCHARGE CHARACTERISTICS OF A SUBMERGED SLUICE GATE

hydrostatic pressure distribution throughout the flow plus the assumptions (see Fig. 31): No energy loss between sections 1 and 2; uniform velocity distribution at sections 1 and 3; and, at the vena contracta (section 2), zero velocity in the roller and uniform velocity in the jet. The energy relation between sections 1 and 2 and the momentum relation between sections 2 and 3 allow the following equations to be written:

$$C_d = K\, C_c \sqrt{1 - \frac{t'}{h}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (87)$$

and

$$\frac{t'}{b} = \frac{K^2 \pm \sqrt{K^4 + 4\,A\left[A\left(\frac{t}{B}\right)^2 - \left(\frac{h}{B}\right)K^2\right]}}{2\,A} \ldots\ldots\ldots (88)$$

in which $K = \dfrac{1}{\sqrt{1 - \left(\dfrac{C_c\, B}{h}\right)^2}}$ and $A = \dfrac{1}{4\, C_c\left(1 - \dfrac{C_c\, B}{t}\right)}$. Values of C_d

given by the resulting forms of Eq. 87 for $t/B = 5$ and for free discharge (assuming $C_c = 0.6$) are plotted in Fig. 35 as the dotted curves. The dashed curve represents the results of applying, for $t/B = 5$, the common formula:

$$C_d = C_c \sqrt{1 - \frac{t}{h}} \qquad (89)$$

which is really the limiting form of Eq. 87 as the depression in the water surface decreases and t' approaches t.

The writer's data were obtained in the laboratory of the Iowa Institute of Hydraulic Research (State University of Iowa, Iowa City), by using a sharp-edged sluice gate in a 1-ft flume which received water from a constant-head tank. Velocities were determined by pitot tubes of the Prandtl and cylindrical type, the latter being used in regions of inclined flow. The fluctuations in velocity due to turbulence were determined by a turbulence pitot[26,32] (capable of indicating only the longitudinal component), especially constructed at the Iowa Institute for such measurements in water. The results of investigating the internal flow phenomena are shown in Figs. 32, 33, and 34. The basic data of these diagrams are the surface profiles, the velocity distributions, and the distributions of the intensity of turbulence. All other data are computed from them. As compared to Figs. 1 and 21, Figs. 32, 33, and 34 indicate vividly the effect of the proximity of the free surface.

As a direct result of limiting the vertical extent of the fluid, three basic changes occur in the flow phenomena:

(1) The discharge or volume flux past successive sections perpendicular to the jet, instead of increasing continually according to Eqs. 42 and 44, must be constant and equal to the discharge at the gate—that is,

$$\frac{\int_0^d v_x \, dy}{q} = 1 \qquad (90)$$

in which d is the depth at any section;

(2) As a direct consequence of item (1), fluid undergoing the induced motion because of the accelerative action of the jet must be replenished by a return flow both at and near the surface, forming what is termed the "roller"; and

(3) The vertical expansion of the jet does not continue indefinitely but is limited by the intersection of the uppermost streamline of the jet with the free surface. Inasmuch as the free surface is a line of constant pressure, this intersection must be at right angles, necessitating a curvilinear pattern of jet diffusion instead of the linear pattern indicated in Figs. 1 and 21. The "nominal boundary of the jet," as indicated by the authors, corresponds closely to the lines of zero horizontal velocity in Figs. 33, 34, and 35.

The depression and unevenness of the surface, due to the gravitational effect, make invalid the reasoning leading to Eq. 6. Rather they lead to the

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[32] "An Evaluation of Several Experimental Methods of Measuring Turbulence in Flowing Water," by P. G. Hubbard and J. S. Norton, *Report No. 2*, rept. prepared by the Iowa Inst. of Hydr. Research for the Office of Naval Research under contract N8onr-500, February, 1949.

constancy of the sum of pressure forces on any vertical section and the flux of the longitudinal component of momentum at the same section. Again, using the authors' assumption of hydrostatic pressure distribution (which is the only one of the three original assumptions that could reasonably apply to flow conditions with finite tailwater depth) and letting M represent the indicated sum, the following relation may be written:

$$\frac{M}{M_t} = \frac{\frac{1}{2}\,\gamma\, d^2 + \rho \int_0^d v^2_x\, dy}{\frac{1}{2}\,\gamma\, t^2 + \rho\, q\, V_t} = 1 \qquad (91)$$

in which γ is the specific weight of water. The subscript t in Eq. 91 applies to uniform tailwater conditions.

Actual values of the ratios presented in Eqs. 90 and 91 have been computed from the experimental data and appear above the mean flow patterns. The constant value of unity is indicated; the variations are no doubt due in part to air entrainment and to accelerative forces within the roller.

With the complex flow pattern exhibited downstream from a submerged sluice gate, one could not expect relations in the nature of Eqs. 43 and 45 to represent energy flux at any section. Instead, the writer has shown variation in total head, H, plotted above the mean flow patterns, computed from the experimental data as follows:

$$H = d + \frac{1}{q} \int_0^d \frac{v^2}{2\,g}\, v_x\, dy \qquad (92)$$

Here the authors' assumption of hydrostatic pressure distribution is again utilized.

At any section the energy lost to the mean flow, as referred to the energy at the efflux section, is the sum of the kinetic energy of turbulence and the energy expended as heat over the preceding distance. This is the relation indicated by the authors (under the heading, "Statement of the Problem")—

> "* * * the kinetic energy of the oncoming flow steadily [is] * * * converted into kinetic energy of turbulence, and the latter steadily * * * [decays] through viscous shear."

This may be expressed in the equation:

$$H_L = H_E + H_T \qquad (93)$$

in which H_L is the energy lost to the mean flow; H_T is the kinetic energy of turbulence; and H_E is the energy already expended as heat. This relation is represented schematically in Fig. 31.

The actual values of H_T were computed from the experimental data by the following equation:

$$H_T = 3\,\frac{1}{q} \int_0^d \frac{(v'_x)^2}{2\,g}\, v_x\, dy \qquad (94)$$

In Eq. 94 the assumption of equal turbulence components in the three coordinate directions has been made. The values of H_T so computed were too small to allow satisfactory representation as indicated in Fig. 31, but necessitated an expanded scale as plotted in Figs. 32, 33, and 34. The discovery that the energy of the turbulence, which is involved in so great a dissipation of the mean flow energy, is on the order of only 1% as great as the total energy lost was one of the most interesting results of the writer's experiments.

To allow rapid and vivid comparison, $-\frac{dH}{dx}$ (the rate at which energy is lost to the mean flow) was obtained graphically as the slope of the total head line and plotted on the same set of coordinates as H_T. Because of the smallness of H_T this rate represents essentially the rate that energy is converted into heat. If the scale of the smallest turbulent eddies were constant throughout the flow, $-\frac{dH}{dx}$ and H_T should attain maximum values at identical sections.[27] Since the writer had no way of determining the scale, such a statement cannot be made. The problem of energy dissipation due to turbulence will not be completely solved until the discovery of some method of evaluating the size of these smallest eddies.

The writer has concluded that the method of experimental attack as applied in the paper is of great value in investigating any type of submerged jet. It only behooves the investigator to recognize such factors as the dissimilarities resulting from boundary conditions, and to modify the approach accordingly.

Acknowledgment.—This discussion is based on a thesis by the writer[33] and part of the data for the mean flow patterns for $F_B = 2.00$, $t/B = 4$, and $t/B = 8$ were obtained from a thesis by H. K. Liu.[34]

M. L. ALBERTSON,[35] JUN. ASCE, Y. B. DAI,[36] R. A. JENSEN,[37] AND HUNTER ROUSE,[38] M. ASCE.—Among the six published discussions may be found essentially all the features that make such discussion worthwhile: Alternative methods of analysis, which either simplify or shed further light upon the problem; corroborative as well as supplementary experimental data; application of the general method to different boundary conditions; and—although in itself hardly contributory—the inevitable question as to the usefulness of the results.

Since, as was indicated by the references cited, the investigation of submerged jets was by no means originated by the writers, perhaps a rephrasing of the objective of the paper is in order. The manuscript was prepared in an effort to present to hydraulic engineers, in a simplified yet systematic manner, the essential mean flow characteristics of jet diffusion. The probability function, which represented the simplest analytic approximation to the writers' plotted data, was introduced for the sake of mathematical convenience. In no sense were the results—particularly with regard to details of the zone of establishment and of the derived turbulence functions—stated to be exact, since precision in such details was purposely sacrificed to emphasize the primary similarity of the over-all picture.

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[33] "A Study of Flow from a Submerged Sluice Gate," by Harold R. Henry, thesis presented to the State Univ. of Iowa, at Iowa City, Iowa, in February, 1950, in partial fulfilment of the requirements for the degree of Master of Science.

[34] "Diffusion of Flow from a Submerged Sluice Gate," by H. K. Liu, thesis presented to the State Univ. of Iowa, at Iowa City, Iowa, in February, 1949, in partial fulfilment of the requirements for the degree of Master of Science.

[35] Associate Prof., Civ. Eng. Dept., Colorado Agri. and Mech. College, Fort Collins, Colo.

[36] Associate Prof., National Hunan Univ., Changsha, Hunan, China.

[37] Research Engr., Fuels Div., Battelle Memorial Inst., Columbus, Ohio.

[38] Director, Iowa Inst. of Hydr. Research, State Univ. of Iowa, Iowa City, Iowa.

In only one respect do the writers fear that their objective was far from achieved. In the effort to familiarize the hydraulic engineer with the basic aspects of jet diffusion, insufficient emphasis appears to have been placed on the fact that the actual form of the flow pattern must be expected to vary considerably with the boundary geometry. The limiting case of the infinite fluid was chosen to illustrate the principles involved, and, hence, direct application of the results obtained requires that the boundaries in question be approximately comparable. On the other hand, although a pronounced departure from the assumed conditions—as represented, for example, by a pipe expansion—assuredly requires further experimental tests, the fact remains that essentially the same general method of analysis will apply.

Messrs. Holdhusen and Corrsin rightly contend that the writers' specific assumption of the Gaussian probability function is not essential to the development—provided, one must note, that some known relationship for the velocity distribution is at hand. Whether this relationship is a mean curve drawn through plotted data, or an algebraic approximation thereof, obviously makes little difference. Surely the experimental plots would have sufficed had no simple relationship been available, just as a simpler mathematical relationship providing a still better approximation would have been even more desirable. The probability curve is commonly used in turbulence investigations, however, and a simpler and closer approximation to the data presented is unknown to the writers. Furthermore, the point must be emphasized that the experimental scatter of the writers' data was due in considerable measure to their fully generalized method of plotting; that is, all measured velocities for each zone of flow were included in a single composite plot in terms of the independent quantities v_0, L_0, and x rather than in the more favorable (and customary) form of their ratio to the dependent variable v_{max}. The latter type of plot, presented by Mr. Baines in Fig. 24, shows beyond doubt that the Gaussian function is in itself a very close approximation to actuality.

Mr. Holdhusen's more general method of analysis and his tabulation of results from several alternative methods are very helpful, provided that the impression is not received that the momentum analysis leads any closer to final numerical values than the others. His plotted curves, like those of Mr. Citrini, permit a further over-all comparison of the various functions. Mr. Citrini's investigation of the Coriolis coefficient is an interesting sidelight on the problem; however, such an evaluation would appear more pertinent to the case of jet diffusion in a bounded fluid, since only then is the mean velocity a significant parameter.

Mr. Corrsin's alternative analysis of the diffusion mechanism is not only beautifully concise, but also discloses in Eq. 76 a very fundamental expression of the rate at which energy is transferred from the mean flow to the turbulence. Mr. Corrsin rightly contends that the assumption of dynamic similarity of the velocity profiles is not in itself sufficient reason for the linear spread of the jet, since the inherent constancy of the momentum flux is also requisite to this condition. This reveals a misstatement in the lines following Eq. 8 which should be revised to read: "but the constancy of the momentum flux together with the similarity of the velocity profiles at successive sections will be found to require that: $\sigma/x = C$ (Eq. 9). That is, the jet will spread at a linear rate defined by the constant C." With regard to the great discrepancy between the values of the lateral components determined from continuity requirements and from the angularity measurements, the writers are disappointed that neither Mr. Corrsin nor the other discussers suggested possible explanations. Since the writers themselves could find no further clue, the matter unfortunately remains unclarified.

Very essential contributions are made by Mr. Baines in his evaluation of the Reynolds-number effect and in his measurements of the intensity of turbulent shear and the turbulence itself. Although measurements of the latter types have been published elsewhere, those included in his discussion represent a supplementary phase of the original experimental project. As far as the writers know, investigation of the extent to which the Reynolds number enters the problem had not previously been made, and the influence of fluctuations in the oncoming flow had not been determined. These factors, without doubt, have been the cause of many discrepancies in experimental data.

The probability that the channel boundaries had some influence on Mr. Citrini's experimental indications at considerable distances from the efflux section is mentioned in his discussion, and for several other reasons he expresses greater interest in the intermediate region. As a matter of fact, it is this very factor of relative boundary effect which determines the limit of applicability of the results presented in the paper. If the fluid body into which a jet issues is large in all directions, the results will apply over a considerable distance; if the fluid body is limited in extent, the region of safe application will also be limited proportionately. Since the region of approximate applicability and the surrounding region of boundary influence may be considered roughly of the same order, it is obvious that a boundary spacing which is not many times the dimension of the efflux section will finally reduce the zone in which the boundary effect is negligible to the immediate vicinity of the issuing jet.

The type of jet which Mr. Streiff claims bears little resemblance to that investigated by the writers seems to be a typical case of mixed boundary geometry. On the one hand, the efflux section lies only a few diameters from a longitudinal boundary (the free surface), which of necessity changes the local diffusion and entrainment pattern. On the other hand, the jet appears to be directed against an opposite wall which is not sufficiently distant for the resulting deflection and return flow to be negligible. Nevertheless, the writers cannot agree that Mr. Streiff's observations indicate a complete dissimilarity between the two cases, since he appears to have made no observations in the more nearly comparable regions—that is, in the relatively unconfined zone below the jet.

Mr. Streiff's initial thesis that a small air jet in air and a large jet under water are inherently different is a contention of the type that has been disproved regularly in fluids research. The only known limits of similarity, for comparable fixed boundaries and Reynolds' numbers, are those of sonic effects in air and cavitation effects in water. His belief that it is the Froude number, rather than the Reynolds number, which governs the flow pattern in water is correct only in so far as the wave action in the vicinity of an outlet near or at the surface is concerned—and even then the wave pattern and the flow pattern must not be confused. There is, moreover, no physical basis for the statement: "There is so much kinetic energy in the [water] jet that the tangential shear around its circumference is minimal in comparison"—unless Mr. Streiff mistakenly presumes that the shear in question is that of purely laminar flow. In any type of turbulent diffusion the turbulent shear plays the dominant internal role (see Mr. Corrsin's Eqs. 73 to 76) and it is only when integrated beyond the zone of diffusion (as in the writers' analysis or in the customary momentum relationship for the hydraulic jump) that the shear terms become replaced by external characteristics upon which they depend. What Mr. Streiff means by "reaction" is not clear.

As shown by Mr. Henry's correct analysis of the diffusion below a partly submerged sluice gate, the fully submerged jet and the free hydraulic jump are (Mr. Streiff's implication to the contrary) by no means identical, representing as they do the two distinct limits of a continuous change in boundary geometry. A theory such as that proposed by Mr. Streiff for the submerged jet as a hydraulic jump cannot be brought "into satisfactory agreement" with the writers' analysis, since the limiting case of expansion from one depth to another depth under gravitational action is similar to that of the jet submerged in an infinite fluid only to the extent that both involve turbulent shear and both are subject to the same basic principles of continuity, momentum, and energy transformation.

Mr. Henry's illustration of the methods whereby these principles may be applied to boundary conditions other than those discussed by the writers is most gratifying. His contribution formed, in fact, still another phase of the original project, which is at the time of writing being extended to the case of abrupt conduit expansions as well.[39] Although, because of instrumental limitations, Mr. Henry's measurements of turbulence must be regarded as qualitative rather than quantitative, they nevertheless show the correct order of magnitude and as such represent the first published data known to the writers for the conditions in question. That the kinetic energy of the turbulence is small in comparison with the energy thereby dissipated is in full accord with measurements made in other studies of turbulent diffusion.[39,40]

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[39] "Characteristics of Mean Flow and Turbulence at an Abrupt Two-Dimensional Expansion," by H. C. Hsu, thesis presented to the State Univ. of Iowa, at Iowa City, Iowa, in February, 1949, in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

[40] "Conversion of Kinetic to Potential Energy in Flow Expansions," by A. A. Kalinske, *Transactions*, ASCE, Vol. 111, 1946.

AMERICAN SOCIETY OF CIVIL ENGINEERS
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TRANSACTIONS

Paper No. 2410

APPLIED COLUMN THEORY

By F. R. Shanley[1]

With Discussion by Messrs. William R. Osgood, J. V. du Plessis, Jacob Karol, E. P. Popov, Jack R. Benjamin, Herbert A. Sawyer, Jr., E. C. Hartmann, and F. R. Shanley.

Synopsis

The generalized Euler equation (involving the tangent modulus) is used as a basis for column strength analysis. The influence of the shape of the stress-strain diagram is shown by nondimensional column curves and an envelope formula is derived. The effects of initial eccentricity are investigated by interaction curves for combined bending and compression. Correction of test data, end fixity effects, and optimum design of columns are discussed briefly.

1. The Generalized Euler Theory

The failure of members under compressive loading has long been a subject of great interest to the structural engineer. In aircraft structures the phenomenon of instability is the basis for many strength calculations, because nearly all the structure is critical under some design condition that causes compressive loads. The demand for minimum structural weight has had two important effects—it has forced the aircraft designer to use special alloys and it has raised the level of design stresses well beyond the proportional limit, into the inelastic range. This combination of circumstances has made it necessary to abandon many of the formulas and methods that were satisfactory for heavier types of construction.

The search for more rational methods leads the investigator backward, chronologically, to the year 1757, when Leonhard Euler published his famous treatise on the strength of columns. The following quotation is taken from a translation[2] of the Euler paper by J. A. Van den Broek, M. ASCE:

> "To begin with I should indicate that this moment is not limited to elastic bodies, among which we have good reason to doubt that columns could be included. It concerns in essence a force by which any body resists a change in curvature, and it is totally immaterial whether such a body after

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Note.—Published in June, 1949, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Consultant, The Rand Corp., Santa Monica, Calif.; Lecturer, Univ. of California, Los Angeles, Calif.

[2] "On the Strength of Columns," by Leonhard Euler, translation by J. A. Van den Broek, *American Journal of Physics*, July–August, 1947, p. 315.

> flexure is endowed with a force to re-establish its original shape or not. For this reason we might preferably designate this moment as stiffness moment (*moment de roideur*) because it occurs in all bodies that resist flexure, whether they are elastic or not."

The "moment" to which Euler referred was designated by him as $E\,k\,k$, the product of the modulus of elasticity and the square of the shape parameter. Today the product of E and the moment of inertia I replaces $E\,k\,k$ in the Euler formula.

The foregoing quotation shows that Euler did not intend to restrict his formula to the elastic range. Because Young's modulus and the tangent modulus were unknown in Euler's time, he could not have been more specific; but he left the door wide open for a general interpretation of the term E.

Since E is usually considered to be a constant (up to a certain stress), Euler's formula is restricted to the elastic range. In 1889, F. Engesser published a paper[3] in which he suggested that E be considered as the slope of the compressive stress-strain diagram, commonly called the "tangent modulus." (In a letter to the writer, Theodor von Kármán, M. ASCE, noted that he had independently made a similar suggestion in an early paper written in Hungarian). Several investigators later pointed out that this theory would not account for the fact that, when the strains are decreasing on one side of the column, during bending, the stresses will decrease at a rate given by the elastic modulus, E. In 1909, Mr. von Kármán[4] made a thorough analysis of column action and performed some very careful experiments, by which the double-modulus (or reduced modulus) theory was apparently substantiated.

The writer has shown[5,6] that the tangent-modulus theory is actually correct for predicting the load at which buckling of a perfect column will begin and that the double-modulus theory gives the upper limit for the load as the bending increases. Mr. von Kármán's own interpretation is clearly indicated by the following excerpts from his discussion of one of the writer's papers:[7]

> "* * * the first equilibrium bifurcation from the straight equilibrium configuration occurs at a load given by the Euler formula when the Young's modulus is replaced by the tangent modulus. * * * The definition of the stability limit must be revised for nonreversible processes. * * * Although the Euler formula with the tangent modulus does not, in general, give the maximum axial load to which the column can be subjected without large deflection, it is conservative and therefore advisable to use this formula for practical computation of column loads. As I have shown in my paper of 1909, also the load deflection curve that starts from the upper limiting load in general soon assumes a negative slope. Consequently, it is difficult to determine the actual peak of the axial load. It will certainly be between the two values that correspond to the tangent and the reduced moduli. These two values can be respectively designated as the lower and upper limits of the critical load."

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[3] *Zeitschrift fur Architektur und Ingenieurwesen*, 1889.

[4] "Untersuchungen über Knickfestigkeit," by Theodor von Kármán, *Forschungsarbeiten No. 81*, Berlin, 1910.

[5] "The Column Paradox," by F. R. Shanley, *Journal of the Aeronautical Sciences*, December, 1946, p. 678.

[6] "Inelastic Column Theory," by F. R. Shanley, *ibid.*, May, 1947, p. 261.

[7] *Ibid.*, p. 267.

The generalized Euler formula (which should perhaps be designated the Euler-Engesser formula) may now be written as

$$P = \frac{\pi^2 E_t I}{L^2} \qquad (1)$$

in which P is an axial load on the column; E_t is the tangent modulus; and L is the length for a column with pinned ends. (The effects of end restraint are discussed in Section 8.)

To obtain the average axial stress f corresponding to the load P, Eq. 1 is divided by the cross-sectional area:

$$f = \frac{P}{A} = \frac{\pi^2 E_t I}{L^2 A} = \frac{\pi^2 E_t}{(L/\rho)^2} \qquad (2)$$

in which $\rho^2 = I/A$ defines the least radius of gyration.

The following statement may be used as a basis for further discussion: The generalized Euler equation gives the load at which a perfect column will begin to bend.

The word "perfect" is used to describe a hypothetical centrally loaded straight column of uniform cross section and homogeneous material properties. It should be noted, too, that only primary bending failure is involved, not the bending that would result from a local instability failure. The statement takes care of another point that has sometimes caused confusion. It is well known that, in the elastic range, the buckling load is not a constant but increases with increasing deflection. In fact, for very large deflections the load required to maintain equilibrium may be considerably greater than the Euler load. In the inelastic range, too, it is possible for the column load to exceed the load given by the tangent-modulus formula, as previously noted. In both these cases, however, the generalized Euler formula is correct for predicting the load at which bending (or buckling) begins.

There are now two questions to answer:

(*a*) Shall the theoretical load at which bending of a perfect column begins be taken as the upper limit for structural design?

(*b*) How is this load further reduced by various imperfections that may exist in the actual structure?

It should not be difficult to agree on the answer to the first question. In the elastic range the column load will not exceed the Euler value by any appreciable amount until the bending deflection becomes quite large.[8] In the inelastic range the increase beyond the Euler value will depend on the variation of the tangent modulus with increasing stress. In the highly curved region of the stress-strain diagram (sometimes termed the knee) the tangent modulus drops very rapidly with increasing stress (shown subsequently in Fig. 5); consequently very little increase in load would be obtained. In the so-called plastic range, beyond the knee of the diagram, the tangent modulus usually has a relatively low value and does not change rapidly with increasing strain. The

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[8] "An Introduction to the Theory of Elasticity," by R. V. Southwell, Oxford Univ. Press, Oxford, England, 2d. Ed., p. 434, Fig. 114.

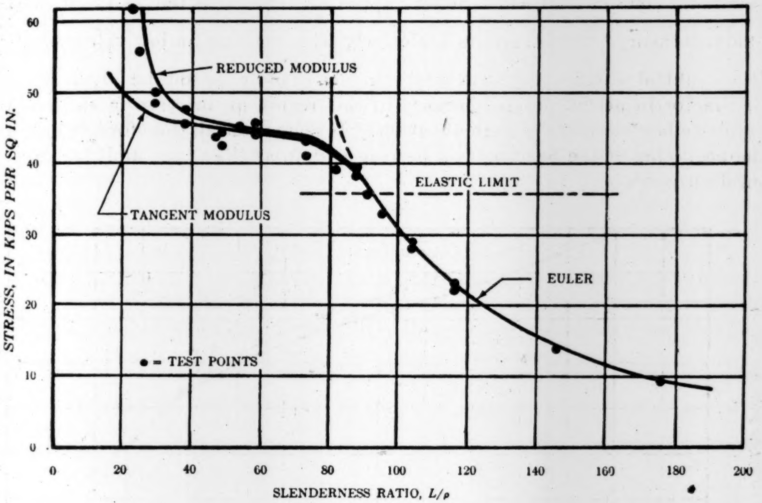

FIG. 1.—RECTANGULAR STEEL BAR WITH KNIFE-EDGE ENDS (THE ABBREVIATION "KSI" DENOTES KIPS PER SQUARE INCH—THAT IS, THOUSAND POUNDS PER SQUARE INCH)

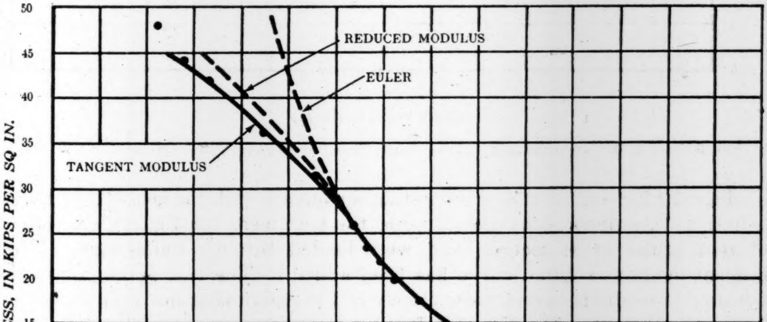

FIG. 2.—SQUARE ALUMINUM ALLOY COLUMN WITH SPHERICAL ENDS (24ST ½ IN.)

maximum column load will therefore approach that predicted by the double-modulus theory,[9] which permits a relatively large increase for low values of $\frac{E_t}{E}$. It is doubtful whether any structural engineer would wish to take advantage of this factor in actual design, because in this region an increase in stress will involve a large amount of permanent set. It seems logical, therefore, to regard the beginning of the bending in a perfect column as the upper limit for structural purposes.

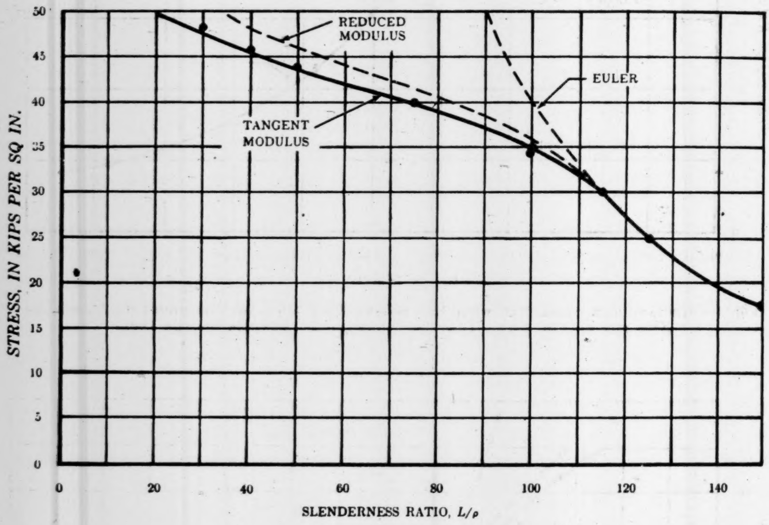

FIG. 3.—SOLID ROUND ALUMINUM (17ST) ALLOY ROD, WITH FLAT ENDS; DIAMETER 0.282 IN.

Figs. 1,[4] 2,[10] and 3[11] show how test data compare with the generalized Euler equation. As denoted by the caption, the specimens for Fig. 1[4] were steel, of rectangular cross section, and were loaded through knife-edges. (The tangent-modulus column curve has been added.) The test points approach the double-modulus curves more closely in the region that lies well above the proportional limit. Mr. Van den Broek's tests[10] were made with square bars of 24ST aluminum alloy (see Fig. 2), using spherical steel loading blocks. (The symbol 24ST denotes a trade designation by which the alloy is commonly recognized. In the drawings, stresses are abbreviated ksi for kips—or thousands of pounds—per square inch, and psi for pounds per square inch.) Fig. 3[11] shows the results of tests made by the Research Laboratories of the Aluminum

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[9] "The Double-Modulus Theory of Column Action," by William R. Osgood, *Civil Engineering*, March, 1935, p. 174, Fig. 2.

[10] "Column Formula for Materials of Variable Modulus," by J. A. Van den Broek, *The Engineering Journal*, December, 1945, p. 772.

[11] "Column Strength of Various Aluminum Alloys," by R. L. Templin, R. G. Sturm, E. C. Hartmann, and M. Holt, *Technical Paper No. 1*, Aluminum Research Laboratories, Pittsburgh, Pa., 1938.

Company of America, in Pittsburgh, Pa. The specimens were 17ST aluminum alloy of round cross section and were tested with flat ends (in these tests a fixity coefficient of 4 was developed).

It would be possible to devote the remainder of this paper to the reproduction of test data that show excellent agreement with the generalized Euler theory. Figs. 1, 2, and 3, however, should be sufficient evidence that the theory can be substantiated experimentally. In fact, it is doubtful whether any other structural theory could be verified so accurately by tests covering such a wide range of variables (materials, shape, method of loading, and end fixity). Even the departure of the test points from the tangent-modulus curve is predicted by the theory, as explained previously.

Although it may be agreed that the generalized Euler equation represents the upper limit for column strength, the engineer will very properly raise the second question, previously noted, regarding the reduction in strength caused by imperfections. The major imperfections may be classified as follows:

(*a*) Variations in material properties;
(*b*) Eccentricity of load application;
(*c*) Initial crookedness of the column; and
(*d*) Effects of end conditions:

Items (*b*) and (*c*) are usually classed together, since their effects are very similar. The effect of end conditions should probably be regarded as a separate problem, not as a type of imperfection. The difficulty in this case is the calculation of the degree of end restraint in actual structures.

Before these subjects are considered, it will be helpful to review some of the methods that are used in the engineering application of the generalized Euler theory.

2. ENGINEERING APPLICATION OF THE COLUMN THEORY

To one who has not actually tried to use the tangent modulus in the Euler equation the lack of a formula that can be solved on a slide rule may appear to be a disadvantage. Actually, however, the entire column curve for a given material can be obtained in a few minutes, directly from the stress-strain diagram. The slope at various stress values is obtained by a straightedge.[12] These values of the tangent modulus may then be used, together with the corresponding stresses, to solve Eq. 3 (presented subsequently) for the slenderness ratio. It is interesting to plot the values of the tangent modulus against stress. This method will aid in obtaining a smooth curve and will also reveal how the stiffness of the material decreases as the stress is increased. Mr. von Kármán followed this procedure[4] in obtaining the column curve shown in Fig. 1. The curve presented by Mr. von Kármán is reproduced as Fig. 4. The reduced modulus is also shown in Fig. 4. It is important to note that the difference between the tangent modulus and the reduced modulus does not cause a very large reduction in stress on the column curve (Fig. 1). The reason for this can be noted by writing Eq. 2 as follows:

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[12] "Compressive Stress-Strain Properties of Some Aircraft Materials," by P. E. Sandorff and R. K. Dillon, *Proceedings*, A.S.T.M., Vol. 46, 1946, pp. 1039–1052.

$$\frac{L}{\rho} = \pi \sqrt{\frac{E_t}{f}} \quad \text{(3)}$$

At a given stress, a change in the value of the modulus will simply displace the column curve to the left or right, and then only by the square root of the ratio involved. For stress-strain diagrams with a relatively sharp knee, the difference between the two theories is almost negligible, over the range of slenderness ratios used in design.

Fig. 5 shows the tangent-modulus curves for several materials. The column curves derived from these curves are shown in Fig. 6. It is quite evident that the shape of the column curve may vary widely between different materials. With such an array of curves to be fitted, it is no wonder that engineers have used so many different types of empirical formulas for the "short-column" range.

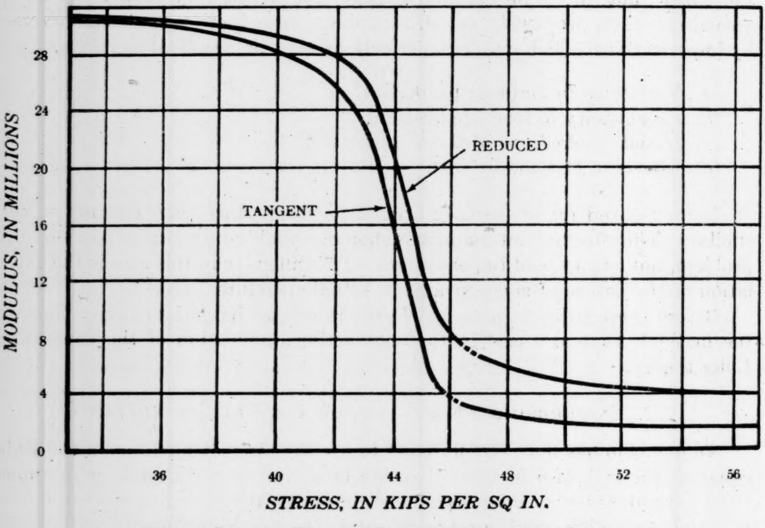

FIG. 4.—RELATION BETWEEN TANGENT MODULUS AND STRESS

One of the very practical advantages of the tangent-modulus method is that there is no need to define the "short column." One formula covers the entire range and it is impossible to make the mistake of using the wrong formula. It would be well to abandon the term "short column."

Some of the other engineering advantages of the generalized Euler equation will become apparent in the discussion of material variations and fixity effects. Before considering these questions, however, two very useful engineering tools will be discussed: The mathematical stress-strain diagram and the nondimensional column curve.

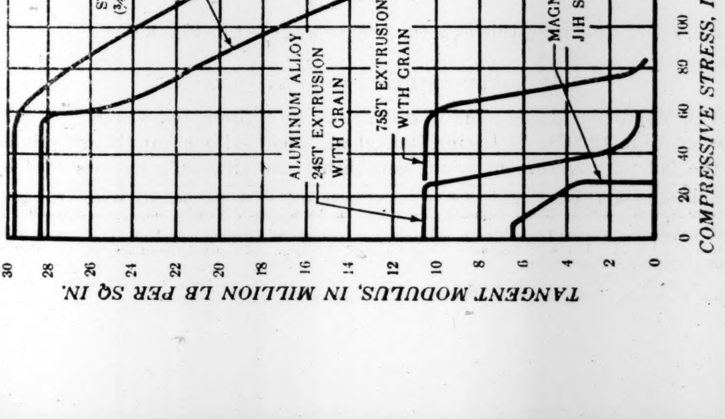

Fig. 5.—Tangent-Modulus Curves for Various Materials

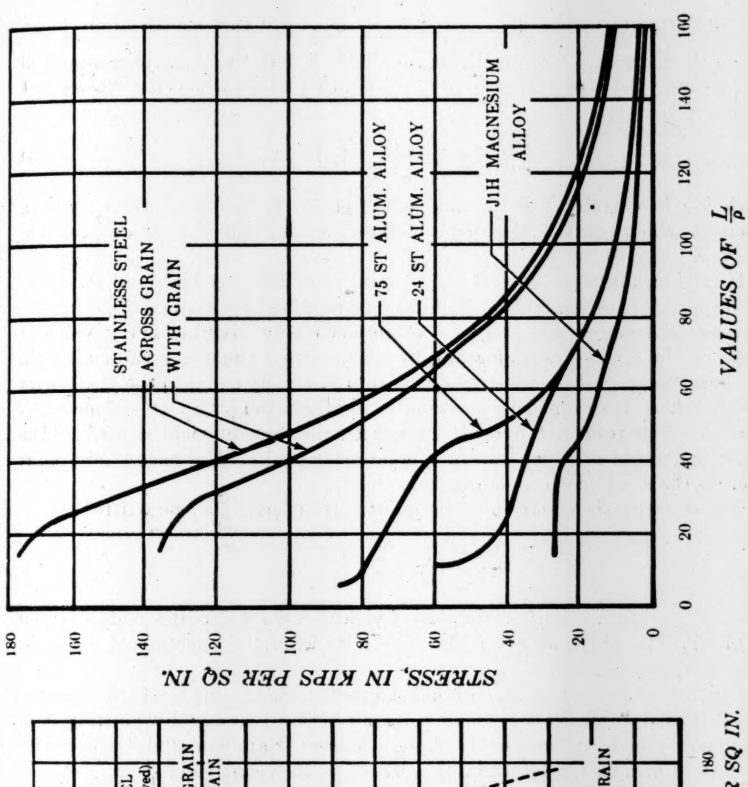

Fig. 6.—Generalized Euler Curves for Various Materials

3. Mathematical Determination of the Tangent Modulus

W. Ramberg and William R. Osgood,[13] M. ASCE, have provided a method of expressing the stress-strain diagram by an equation which has the general form:

$$\epsilon = \frac{f}{E} + K\left(\frac{f}{E}\right)^n \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

in which ϵ is a strain; E is the modulus of elasticity (slope at origin); n is an exponent which controls the shape of the diagram; and K is a constant (see note under Eq. 5).

If the constants are known, the tangent modulus may be found by differentiation. A more general solution may be obtained by writing Eq. 4 in nondimensional form—which may be accomplished by dividing every value of stress, on the stress-strain diagram, by a base stress and every value of strain by a base strain. The base stress, f_o, is approximately equal to the yield stress (0.002) but is determined by drawing a line from the origin at a slope equal to 0.7 E. The resultant stress is designated the "secant yield stress." (The secant method is used because it is nondimensional.) The base strain, ϵ_o, is taken as the strain corresponding to E and f_o.

The stress-strain equation, in nondimensional form, may be written as

$$\frac{\epsilon}{\epsilon_o} = \frac{f}{f_o} + \frac{3}{7}\left(\frac{f}{f_o}\right)^n \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

in which $\epsilon_o = f_o/E$; and f_o is the secant yield stress for a secant modulus equal to 0.7 E. The constant 3/7 in Eq. 5 results from the selection of 0.7 E for the secant modulus.

Fig. 7 shows[13] the nondimensional stress-strain diagrams for various values of n. It is possible to determine n for an actual diagram by plotting it in nondimensional form on such a chart. Messrs. Ramberg and Osgood also present[13] a formula for determining n by a second secant yield stress.

The equation for the tangent modulus, in nondimensional form, may be obtained from Eq. 5:

$$\frac{E_t}{E} = \frac{1}{1 + \frac{3}{7}n\left(\frac{f}{f_o}\right)^{n-1}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Curves representing this equation are reproduced in Fig. 8.[13]

4. Nondimensional Column Curves

The nondimensional column curve was first presented (to the writer's knowledge) in 1929.[14] Sometime later it was included in a handbook[15] printed by the federal government which establishes the allowable stresses used by

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[13] "Description of Stress-Strain Curves by Three Parameters," by W. Ramberg and William R. Osgood, *Technical Note No. 902*, National Advisory Committee for Aeronautics, Washington, D. C., July, 1943.

[14] "Strength of Tubing Under Combined Axial and Transverse Loading," by L. B. Tuckerman, S. N. Petrenko, and C. D. Johnson, *Technical Note No. 307*, National Advisory Committee for Aeronautics, Washington, D. C., June, 1929.

[15] "Strength of Metal Aircraft Elements," *ANC-5*, Army-Navy-Civil Committee on Aircraft Design Criteria, U. S. Govt. Printing Office, Washington, D. C., 1942.

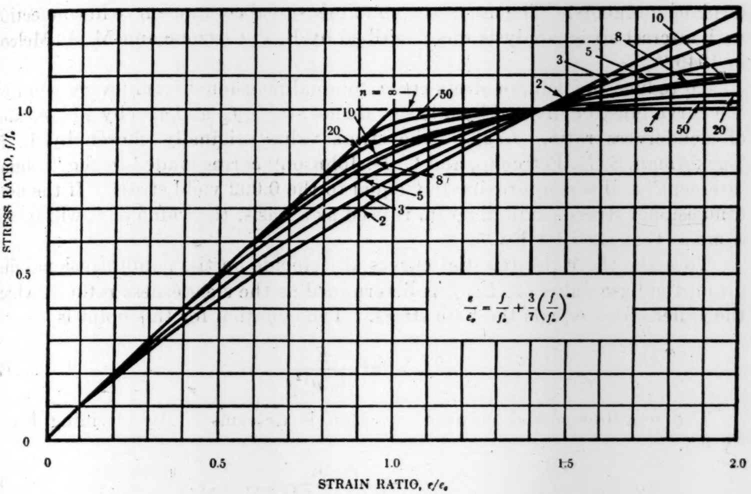

FIG. 7.—NONDIMENSIONAL STRESS-STRAIN DIAGRAMS

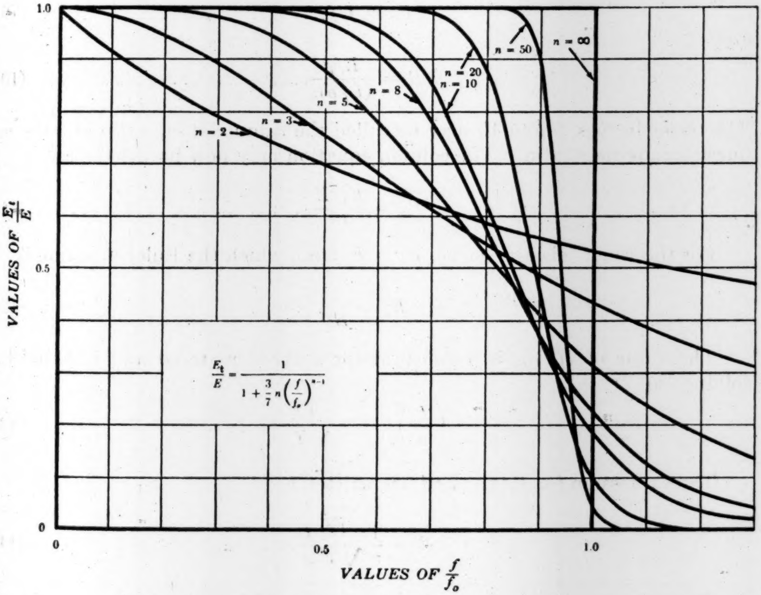

FIG. 8.—NONDIMENSIONAL TANGENT-MODULUS CURVES

airplane designers. The use of the nondimensional column curve in connection with aircraft stress analysis was described by F. P. Cozzone and M. A. Melcon in 1946.[16]

To reduce the column stress curve to nondimensional form, every point on the curve must be divided by a base value of stress, f_o, and also by a base value of slenderness ratio, $(L/\rho)_o$. The stress value originally chosen by L. B. Tuckerman, S. N. Petrenko, and C. D. Johnson[14] corresponded to the "column intercept" and was approximately equal to the 0.002 yield stress. If the nondimensional stress-strain diagram is used as a basis, the value of f_o will be the same as that used for Eq. 5.

To make the Euler (elastic) curves all coincide on the nondimensional diagram, the base value of $(L/\rho)_o$ is determined as the slenderness ratio at which the Euler stress equals the base stress. The equation for this point is

$$f_o = \pi^2 \frac{E}{(L/\rho)^2_o} \qquad (7)$$

The nondimensional column equation is determined by dividing Eq. 2 by Eq. 7:

$$\frac{f}{f_o} = \frac{E_t/E}{\left[\dfrac{L/\rho}{(L/\rho)_o}\right]^2} \qquad (8)$$

Let

$$R_f = f/f_o \qquad (9)$$

and

$$B = \frac{L/\rho}{(L/\rho)_o} \qquad (10)$$

The terms in Eqs. 9 and 10 may be called the column stress ratio and the reduced slenderness ratio. The column equation may now be written as

$$R_f = \frac{E_t/E}{B^2} \qquad (11)$$

For the Euler (elastic) curve, $E_t = E$, from which the Euler equation is

$$R_f = \frac{1}{B^2} \qquad (12)$$

The value of $(L/\rho)_o$ is a constant for a given material and is found by solving Eq. 7:

$$\left(\frac{L}{\rho}\right)_o = \pi \sqrt{\frac{E}{f_o}} \qquad (13)$$

The expression for B is sometimes written as

$$B = \frac{1}{\pi}\sqrt{\frac{f_o}{E}}\,\frac{L}{\rho} \qquad (14)$$

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[16] "Nondimensional Buckling Curves—Their Development and Application," by F. P. Cozzone and M. A. Melcon, *Journal of the Aeronautical Sciences*, October, 1946, p. 511.

This form, however, conceals the physical significance of B. It is probably best to determine the constant $(L/\rho)_o$ for the material being investigated and then to use that value in Eq. 10 to determine B for any given slenderness ratio.

Messrs. Cozzone and Melcon[16] show that the curves of Fig. 8 may also be used for any type of buckling in which the equation has the form:

$$f = K\frac{E_t}{(a/b)^2} \qquad (15)$$

in which K is a constant; and a and b are the two dimensions that affect the buckling stress. In this more general form,

$$B = \frac{a/b}{(a/b)_o} \qquad (16)$$

and

$$\left(\frac{a}{b}\right)_o = \sqrt{K\frac{E}{f_o}} \qquad (17)$$

The nondimensional column curve may be derived equally well from the load-length curves. Thus, test data can be plotted directly in nondimensional form, if desired. The column load, P, is divided by the base load, P_o, in which

$$P_o = f_o A \qquad (18)$$

The effective column length, L, is divided by the base length, L_o:

$$L_o = \pi\sqrt{\frac{EI}{P_o}} \qquad (19)$$

Then

$$R_f = \frac{P}{P_o} \qquad (20)$$

and

$$B = \frac{L}{L_o} \qquad (21)$$

Using Eqs. 18 to 21, any nondimensional column curve may be transformed directly into a design curve of load versus length, for a member of given material and cross section.

To express the nondimensional column curves in mathematical form it is necessary only to insert in Eq. 11 the expression for E_t/E from Eq. 6. The generalized column equation then becomes:

$$R_f + \frac{3}{7}n\,(R_f)^n = \frac{1}{B^2} \qquad (22)$$

Eq. 22 is plotted in Fig. 9,[16] revealing a number of interesting points. The effects of varying the shape of the stress-strain diagram (by varying n) are greatest in the region where $B = 1.0$. Thus, it is apparent that the shape of the stress-strain diagram near the knee will have a relatively large influence on the column stress in this region.

5. The Envelope Equation

Fig. 9 shows that there is an envelope curve representing the minimum values of column stress. The equation for this envelope may be obtained from Eq. 22:

$$R_f = e^{-1/n} \quad \text{(23)}$$

from which

$$n = -\frac{1}{\log_e R_f} \quad \text{(24)}$$

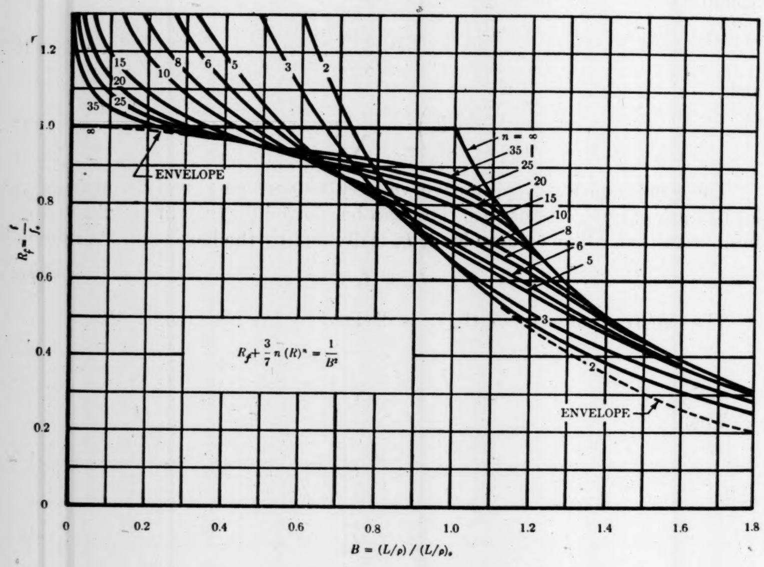

Fig. 9.—Nondimensional Tangent-Modulus Column Curves

Substituting the value of n in Eq. 24 into Eq. 22 gives the equation of the envelope curve:

$$R_f - \frac{3}{7} \frac{1}{e \log_e R_f} = \frac{1}{B^2} \quad \text{(25)}$$

Eq. 25 is based on the tangent-modulus theory, but it does not involve the tangent modulus; in fact, the only two material properties that enter into Eq. 25 are E (Young's modulus) and f_o (the secant yield stress). The physical significance of Eq. 25 is evident from Fig. 10. Together with the Euler curve, Eq. 25 bounds a region which encloses the column curves for all mathematical stress-strain diagrams (that is, all diagrams expressed by Eq. 22) that can be passed through the shaded area of Fig. 10(a); and it covers all positive values of n in Eq. 22.

Eq. 25 would be too conservative for most design purposes (at least for aircraft) since it does not permit the designer to take full advantage of available knowledge on the shape of the stress-strain diagram. Where weight saving is not too important, however, Eq. 25 could be used as a kind of over-all column curve for any material. It would then only be necessary to know the secant yield stress (the stress at which a line having a slope of 0.7 E cuts the stress-strain diagram) and the modulus of elasticity E.

The resemblance of the envelope curve to a parabola shows why parabolic curves have been found useful in representing the "short column" range. The relationship of various other column formulas is shown in Fig. 11. The equations of these formulas, in nondimensional form, are: Parabolic formula (exponent = 2):

$$R_f = 1 - \frac{B^2}{4} \qquad \text{(26)}$$

straight line formula (intercept at $R_f = 1.0$):

$$R_f = 1 - 0.385\,B \qquad \text{(27)}$$

Gordon-Rankine formula (with Ritter's constant):

$$R_f = \frac{1}{1 + B^2} \qquad \text{(28)}$$

and Van den Broek formula:[17]

$$R_f = \sqrt{1 + \left(\frac{B^2}{2}\right)^2} - \frac{B^2}{2} \qquad \text{(29)}$$

As a whole, the various empirical formulas form a "scatter band," as shown in Fig. 11.

6. EFFECTS OF MATERIAL VARIATIONS

From the discussion of nondimensional column formulas, it is obvious that the three primary material variables are the initial modulus of elasticity E, the secant yield stress f_o, and the shape of the knee of the stress-strain diagram (indicated by n).

It is unnecessary to consider E, as far as variations between lots of material are concerned. From an engineering point of view E can usually be regarded as constant for a given material or alloy and at a given temperature.

The secant yield stress based on 0.7 E is approximately equal to the conventional (0.002) yield stress.[13] For material-correction purposes either type of yield stress may be used. In general terms it can be stated that the compressive yield stress controls the height of the left end of the column curve. In the nondimensional form this stress controls the value by which all the ordinates of the diagram are multiplied. Therefore, it is the most important property, as far as material variations are concerned.

The specifications for structural materials to be used in compression should contain a minimum value for the compressive yield stress. The tensile yield

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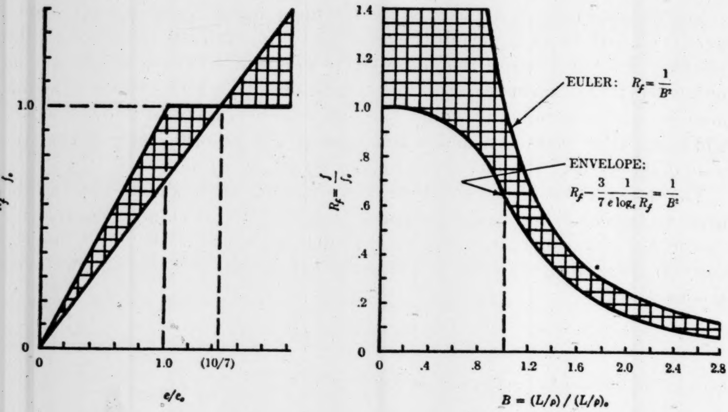
[17] "Theory of Limit Design," by J. A. Van den Broek, John Wiley & Sons, Inc., New York, N. Y., 1948, p. 103, formula IX.



(a)

(b)

FIG. 10.—THE ENVELOPE COLUMN CURVE

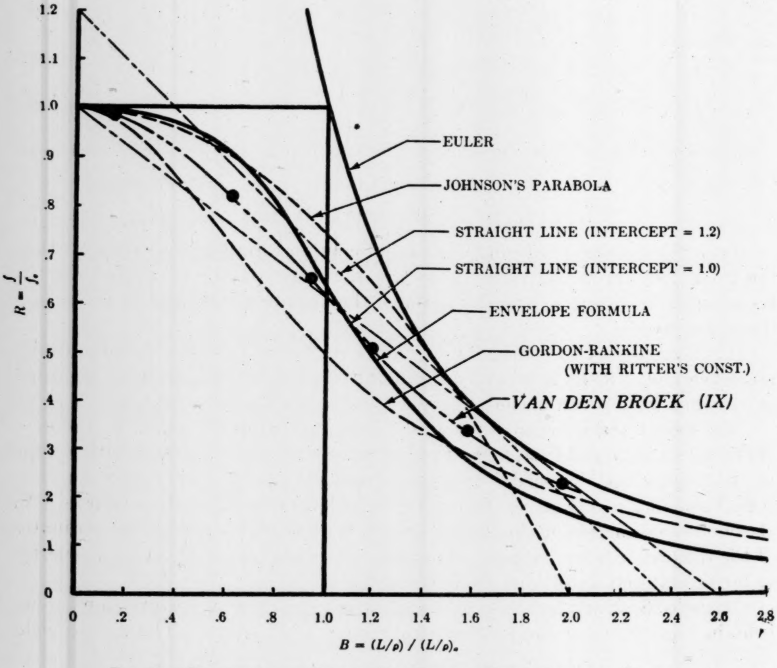


FIG. 11.—COMPARISON OF COLUMN FORMULAS IN NONDIMENSIONAL FORM

stress is sometimes a reasonably good substitute since the yield stresses in tension and compression are nearly the same, but this assumption can be misleading. Certain types of magnesium alloy show considerably lower yield stresses in compression than in tension. The "grain" (direction of rolling) may also have a pronounced influence (see Fig. 6). Most aluminum alloy sheet is straightened by stretching, which raises the tensile yield stress in one direction and raises the compressive yield stress in the other (cross grain).

To be entirely safe, it would be necessary to select, as a basis for the design column curve, the lowest probable value of compressive yield stress and the "flattest" type of diagram likely to be obtained (minimum values for f_o and n, when the stress-strain diagram is expressed mathematically). This procedure would take care of all material variations. (Theoretically the curve should be obtained from the two limiting values of n, with the envelope curve in between.)

In aircraft work many compression tests are performed to obtain allowable stresses for complicated types of construction. The results of such tests must be "reduced to standard," or to minimum specified values. Several methods are available, but the most logical is that using the nondimensional column curve. All test data may be divided by the constants corresponding to the test material, to obtain the nondimensional diagram. To convert back to "standard," the nondimensional values are multiplied by the constants for the standard material.

Fig. 12 shows a family of column curves for a high-strength aluminum alloy. The effects of varying the yield stress have been determined by the nondimensional system, assuming that the stress-strain diagrams remain affine (that is, that n does not vary).

Curves of this type may be used directly for "correcting" from one yield stress to another, as indicated in Fig. 12. Such curves may also be used directly for any type of buckling in which failure occurs by bending in one plane only, even though L/ρ is not the proper geometrical ratio. (This procedure is similar to the use of B for various types of buckling, as discussed in Section 4.)

7. THE EFFECTS OF ECCENTRICITY

The theories and formulas that have been discussed are all based on the conception of a perfectly straight, centrally loaded column. Since no actual column can fully meet this requirement, engineers have long been interested in the effects of eccentric loading. There is a definite need, also, for a sound method of design for columns that have design eccentricities of considerable magnitude. This problem therefore has two distinct phases: (*a*) The effects of small "incidental" eccentricities, and (*b*) the design of columns with known eccentricities.

For the design of columns that are nominally straight, it is customary and logical to consider small degrees of crookedness as equivalent to a certain initial eccentricity. There has been much discussion of the proper value to use for the equivalent eccentricity in actual design work, but obviously this problem comes under the heading of "tolerances" and must be controlled by the same processes that are used to control other tolerances in manufacture. It will be assumed, therefore, that the designer either controls this variable by suitable specifications, shop practices, etc., or estimates a maximum value that is compatible with the kind of structures with which he is dealing. The question then becomes, is it necessary to make a further reduction in the column strength to account for the maximum probable equivalent eccentricity?

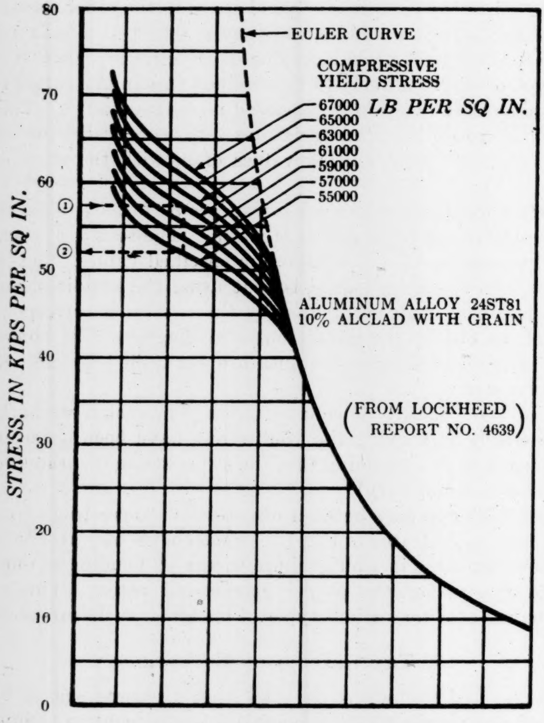


Fig. 12.—Method of Correcting Test Data for Yield Stress

Before this question can be answered it is necessary to have a method of determining the effects of eccentricity. For the design of elastic members the answer is simple—use the secant formula. This well-known formula gives the maximum fiber stress that will be produced by a definite load (or average stress) in the column, provided that the stresses stay within the elastic range. If certain values are chosen for the eccentricity and the maximum permissible stress, the equation becomes a column formula.

For a material which has a very pronounced yield point, and a "flat-topped" stress-strain diagram, the secant formula can be used by substituting the yield stress for the maximum permissible stress. (It is still necessary to specify the eccentricity, however.) For materials that do not exhibit this characteristic it is impossible to select a single value of maximum stress that will make the formula fit the test data. Furthermore, the shape of the secant column curve will have no relation to the shape determined by the tangent-modulus theory for zero eccentricity; yet it is obvious that the eccentric column curve must approach the basic column curve as the eccentricity approaches zero. Because of these shortcomings, the secant column formula cannot be considered as generally satisfactory.

The eccentric column was treated very thoroughly by Mr. von Kármán, in 1910.[4] He showed that, in the elastic range, initial eccentricity simply causes a constantly increasing lateral deflection as the axial load is increased. The curves are asymptotic to the Euler load, which therefore represents the maximum load for the eccentric elastic column. (This theory neglects the further increase in load at large deflections, previously discussed.)

For the inelastic case, Mr. von Kármán showed that increased eccentricity causes greater bending strains (for the same axial load), which force some of the material farther into the inelastic region of the stress-strain diagram. Thus, a lower value of effective modulus of elasticity is obtained. During loading, a point is finally reached where the internal resisting moment does not increase as rapidly as does the external moment. The external load must then decrease to maintain equilibrium. The maximum load reached in this process will depend on the slenderness ratio, the degree of initial eccentricity, the shape of the stress-strain curve, and the shape of the cross section.

This problem is extremely complex. Mr. von Kármán solved it for a particular material (steel) and cross section (rectangular) using the double-modulus theory to determine the internal resisting moment. Graphical integration methods were used. The resulting curves have been reproduced in many textbooks, but the exact solution has never been improved on, to the writer's knowledge. In fact, the recent discovery that the double-modulus theory (which Mr. von Kármán used) is not strictly accurate for small deflections further complicates the situation.

Mr. Van den Broek[10] has developed a formula for predicting the maximum load of the eccentric column. He assumes, in effect, that the stress-strain diagram becomes substantially flat at some known value of stress and that the column has bent to a point where opposite edges are working in these flat regions, one in tension and one in compression. By making some simplified assumptions as to the shape of the deflection curve, he obtains a formula for the maximum load, for a column of rectangular cross section. For materials having a curved stress-strain diagram, the Van den Broek method has the same limitation as the secant formula—there is no good definition for the "upper yield stress."

K. Ježek[18] has developed an algebraic method based on a simplified stress-strain diagram composed of two straight lines, one horizontal. He obtains factors by which the concentric column values are divided. Thus, the errors caused by the original assumptions are reduced, but for materials that have a gradually curving stress-strain diagram the error may still be considerable. (The writer is indebted to Friedrich Bleich, M. ASCE, for the information on the Ježek method.)

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[18] "Approximate Theory of Eccentrically Loaded Steel Columns," by K. Ježek, *Der Stahlbau*, Vol. 8, 1935.

These examples indicate that the eccentric column, in the inelastic range, cannot be treated adequately by any simple engineering formula or theory. The reason for this may be that there are in reality two problems—(1) stability under axial loading and (2) failure in bending. The stability problem is complicated even for zero eccentricity, but it can be handled by the tangent-modulus theory. Failure in bending is in itself a large subject, involving both the shape of the cross section and the shape of the stress-strain diagram.

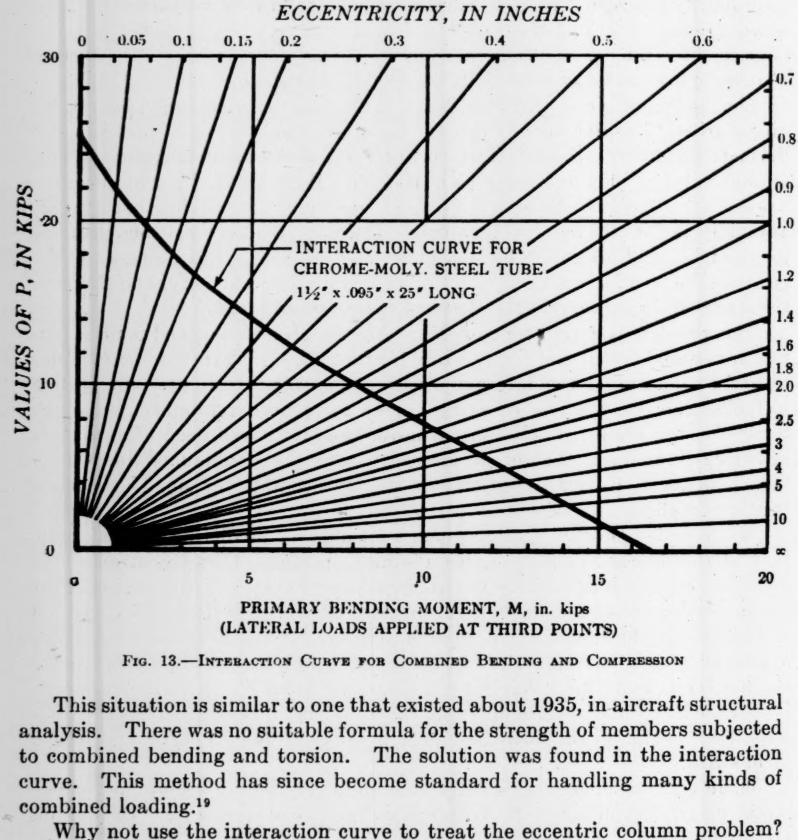

Fig. 13.—Interaction Curve for Combined Bending and Compression

This situation is similar to one that existed about 1935, in aircraft structural analysis. There was no suitable formula for the strength of members subjected to combined bending and torsion. The solution was found in the interaction curve. This method has since become standard for handling many kinds of combined loading.[19]

Why not use the interaction curve to treat the eccentric column problem? The member can be regarded as being subjected to combined compression and bending, with the bending moment equal to $P\ e$ (e denoting eccentricity).

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[19] "Basic Structures," by F. R. Shanley, John Wiley & Sons, Inc., New York, N. Y., 1944, Chapter 20 p. 340.

Interaction curves for combined bending and compression were obtained from tests of steel and aluminum alloy tubing by Messrs. Tuckerman, Petrenko, and Johnson.[14] In these tests the bending moment was applied by transverse loading at the third points, whereas in the eccentric column the bending moment must be considered as being applied at the ends of the member. Since the interaction curves for the two cases should be quite similar, the curves developed by Messrs. Tuckerman, Petrenko, and Johnson[14] will be used to illustrate the proposed method.

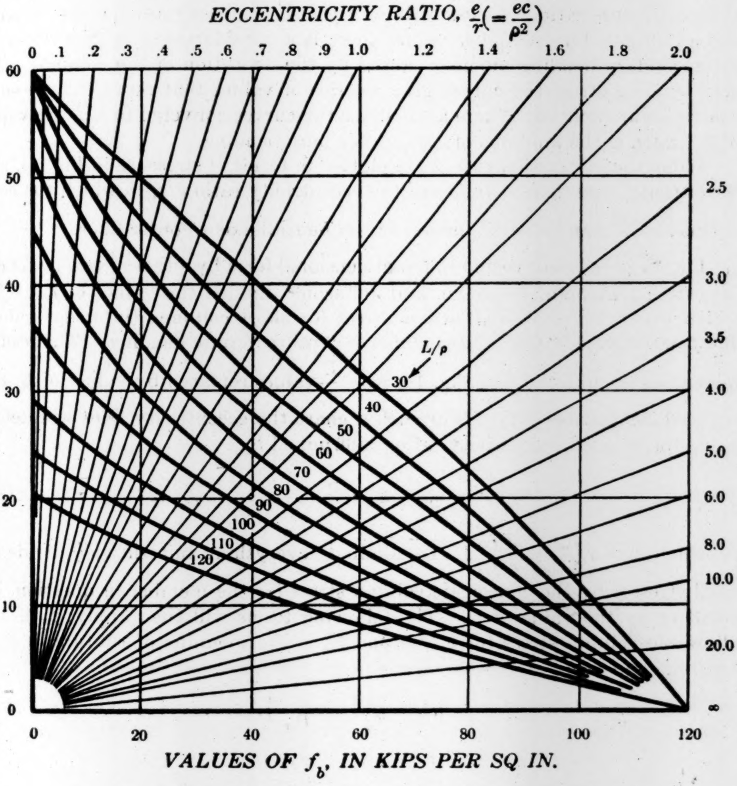

FIG. 14.—INTERACTION CURVES FOR CHROME-MOLYBDENUM STEEL TUBING

Fig. 13 shows a typical interaction curve for a single member under all possible combinations of axial compression and bending.[14] The two end points of the curve correspond to the column strength and to the ultimate bending moment, respectively. If it is assumed that the bending moment was applied at the ends of the member, as in an eccentric column test, the eccentricity (e)

is given by the ratio M/P. Straight lines of constant eccentricity therefore radiate from the origin, as shown in Fig. 13. The interaction curve may thus be used directly to determine the effects of eccentricity in axial loading.

Tests of different kinds and sizes of round tubing were made[14] and the results were plotted as interaction curves, both in terms of stress and in terms of stress ratios. Fig. 14 shows[14] that there is a family of curves for different values of the slenderness ratio. At low values of the slenderness ratio the curves tend to be straight, but at higher values there is a considerable sag (the result of the secondary bending moment caused by the deflection of the column). At zero bending stress the curves give a series of values that represent the concentric column curve; at zero axial stress the curves converge to a single value of f_b, which is the modulus of rupture for pure bending.

When the interaction curves are plotted in terms of stress, as in Fig. 14, the eccentricity lines denote values of the eccentricity ratio e/r, in which the core radius $r = \frac{\rho^2}{c}$; and c is the distance from neutral axis to outer fiber.

Fig. 14 may be converted to nondimensional form by either of two methods. Messrs. Tuckerman, Petrenko, and Johnson[14] divided the values of f_c by f_{co}, which was taken as the allowable stress for short columns (intercept value) and f_b was divided by f_{bo}, the modulus of rupture in pure bending. The resulting curves are like those of Fig. 14, with $\frac{L}{\rho}$-values replaced by values of B, the reduced slenderness ratio. For such a chart the eccentricity lines will define values of reduced eccentricity ratio, determined by

$$R_e = \frac{e}{e_o} \qquad (30)$$

in which $e_o = r\frac{f_{bo}}{f_{co}} = \frac{\rho^2}{c}\frac{f_{bo}}{f_{co}}$. To eliminate everything but the sag, divide all axial stresses by the concentric column stress for the member in question, instead of by the value of f_o. The following terms will then apply: f_{cr} is the allowable column stress for $e = 0$ and f_{bo} is the modulus of rupture in pure bending, and

$$R'_c = \frac{f_c}{f_{cr}}\left(= \frac{P}{P_{cr}}\right) \qquad (31)$$

$$R_b = \frac{f_b}{f_{bo}}\left(= \frac{M}{M_o}\right) \qquad (32)$$

$$R'_e = \frac{e}{e'_o} \qquad (33)$$

and

$$e'_o = \frac{\rho^2}{c}\frac{f_{bo}}{f_{cr}}\left(= \frac{M_o}{P_{cr}}\right) \qquad (34)$$

The terms in parentheses show that the ratios in Eqs. 31, 32, and 34 may all be expressed in terms of axial load and bending moment. Thus, test data may be converted directly into the nondimensional form. Conversely, the nondimensional design charts may be used directly to find the allowable eccentric column load, if the load for the concentric column, and its ultimate bending moment, are known. The curves of Fig. 14 have been replotted in Fig. 15, in this reduced form. The actual slenderness ratio (L/ρ) has been retained, instead of the reduced slenderness ratio B. If B is used, the curves will be similar in shape; the eccentricity lines will remain unchanged.

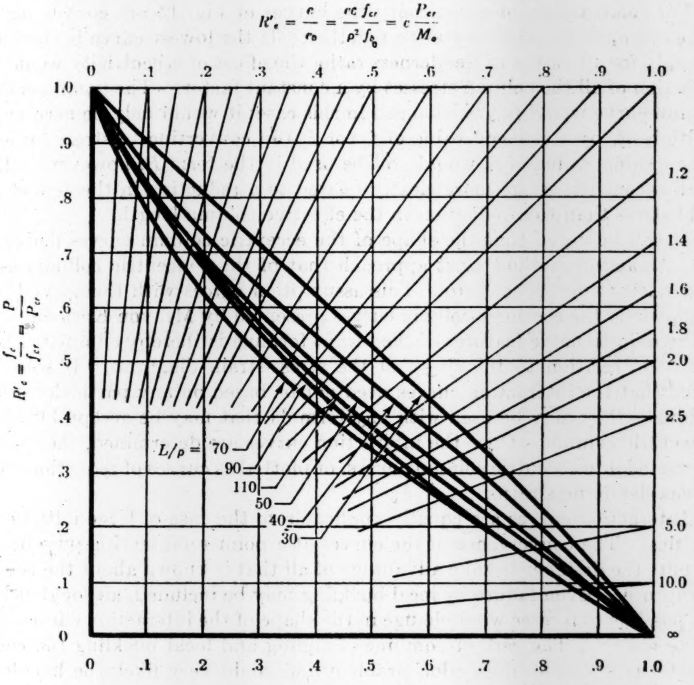

FIG. 15.—REDUCED INTERACTION CURVES FOR COMBINED BENDING AND AXIAL LOADING

Fig. 15 indicates that the nondimensional interaction curves tend to lie within a definite band and that the amount of sag is a maximum for slenderness ratios in the region corresponding roughly to $L/\rho = 70$ (or to $B = 1.0$ to 1.2). At higher values of slenderness ratio, the interaction curves tend to become straighter again. If the bending moment had been applied at the ends of the member, the amount of sag would no doubt be greater.

Since the tests on which Figs. 14 and 15 are based did not truly represent the eccentric column, it would be unsafe to use them for design purposes, but they can be used to draw some interesting conclusions. One of the most valuable features of interaction curves is that their shape seems to be primarily a function of the kind of loading conditions involved and is little affected by other variables. Further research may indicate that a few sets of interaction curves, such as those shown in Fig. 15, could be used for a wide variety of columns. By using the reduced slenderness ratio B, it may be possible to show that the shape of the curves depends on the value of the exponent n for the nondimensional stress-strain diagram.

For small values of eccentricity the curves of Fig. 15 are converging and therefore must lie relatively close together. If the lowest curve is considered to apply for all values of slenderness ratio, the effect of eccentricity would be a reduction of all the column stresses by a constant factor. The nondimensional column curve would not be changed in this case; it would only be necessary to multiply R_c by a reduced value of f_o (or P_o) in converting to stress (or load). (The original value of f_o would still be used in the term B, however.) Thus, small eccentricities are more closely related to a reduction in the compressive yield stress than to a reduction in the effective column length.

It can be noted that the shape of the eccentric column curves derived by the interaction method must approach that of the concentric column, as the eccentricity approaches zero. This assumption agrees with theory and tests, as shown by the eccentric column curves presented by Mr. von Kármán.[4] One of the objectionable features of the secant formula is thereby eliminated (since it has no relation to the shape of the stress-strain diagram). It should be noted that the interaction curves need not be based on any particular column formula—they may be used with any formula that may be acceptable for the concentric column. Once the interaction curves are determined, they may be converted into eccentric column curves, or plotted as curves of reduction factors versus slenderness ratio.

Interaction curves are equally applicable to the case of large initial eccentricities. The convergence of the curves at a point representing pure bending permits the designer to take advantage of all that is known about the modulus of rupture. Even failure by local buckling may be included, although it is not yet possible to foresee what change in the shape of the interaction curves would be necessary. The case of combined bending and local buckling (no column effects) is a simple interaction problem and could very likely be handled by using the interaction curve for low slenderness ratios and substituting the crushing strength for the column strength in determining f_o or P_o.

The effect of small eccentricities may now be determined approximately. Assume, for example, that the equivalent eccentricity, e, for a thin-walled tube of mean radius R lies within the tolerance $\pm 0.025\,R$ (for a column 20 in. in diameter this would be $\pm$ 0.25 in.). The value of the core radius r $(= \rho^2/c)$ for such a tube is $R/2$. The eccentricity ratio e/r is therefore 0.05. The corresponding reduction in the column stress may be determined from Fig. 14 or Fig. 15. (If Fig. 15 is used, the eccentricity must be further reduced by using Eqs. 33 and 34.) The maximum reduction so determined is in the order of 6%, at $L/\rho = 70$. Since the curves are probably only slightly on the unsafe side at such low eccentricities, this result should be a fairly accurate indication of the reduction involved for the assumed case. The magnitude of the reduction appears to be no greater than the usual variation in material properties and could be provided for (at least in part) in selecting a conservative value for the minimum compressive yield stress.

Mr. von Kármán[4] recommended that columns be designed on the basis of their concentric buckling strength and that the effects of initial eccentricities be taken care of in the safety factor. This recommendation is still sound and is followed in modern aircraft practice in the United States. (Although it is not specifically stated in the airworthiness requirements, the matter is provided for by a factor of safety to increase the applied loads, not to reduce the allowable stresses.) With the use of the tangent modulus in the column formula, together with proper control of manufacturing tolerances, the specification of a minimum compressive yield stress, and the reduction of all test data to minimum guaranteed properties, it would seem that the unknown factors in compressive loading are certainly no greater than those for any other type of loading and that no increase in the usual factor of safety is required.

8. EFFECTS OF END RESTRAINT

In theory, end restraint has a very simple effect: It merely changes the effective column length. In practice, however, the accurate determination of the effects of end restraint for any column except a pin-ended column is a difficult problem. The situation that exists in aircraft structural analysis has been very well described by Paul E. Sandorff,[20] and there is little more that can be added at this time. Experience has shown that a fixity coefficient of 1.5 has been generally satisfactory for the design of the continuous compression panel forming the top of the wing structure. Mr. Sandorff shows that very little of this fixity is supplied by the supporting ribs. Edge restraint, chordwise curvature, and other effects usually provide the increase corresponding to the assumed fixity coefficient.

The long-standing conception of a fixity coefficient as a multiplying factor has perhaps given rise to an exaggerated idea of the effects of end restraint. Mr. von Kármán has shown[4] that the fixity coefficient can be used as a multiplying factor, but only for the elastic case (Euler formula). In the inelastic case the coefficient must be used indirectly, to obtain the effective length:

$$L_{eff} = L/\sqrt{C} \qquad (35)$$

It is easy to overlook this important fact in dealing with the generalized Euler equation. (The writer made this mistake in his own textbook!) It would have been much better if the effects of fixity had been introduced by a length coefficient in the first place, because length is really what is directly affected, not the buckling load nor the stress.

It is important to realize that efficiently designed structures cannot be made proportionately stronger by increasing the amount of end restraint. This fact, too, was pointed out by Mr. von Kármán[4] when he declared, in effect, that end restraint causes a marked increase in buckling strength only for long slender columns, not for shorter ones. In aircraft structures, for example, the compression members are designed to develop ultimate stresses that approach the compressive yield stress.

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[20] "Notes on Columns," by Paul E. Sandorff, *Journal of the Aeronautical Sciences*, January, 1944, p. 1.

Fig. 16 illustrates the effects of the fixity coefficient by column curves (24ST aluminum alloy) plotted for different degrees of end fixity C. Assume, for example, that a designer has used $C = 1$ to design a structure for the column stress f_1. If he now goes to considerable trouble to restrain the ends he may be able to develop a fixity coefficient C of 3.0, but the increase in stress thus obtained is almost negligible, as shown by case A, Fig. 16. On the other hand, assume that he has selected too high a value for the fixity coefficient (say, $C = 3.0$) and that in the actual structure it proves to be only $C = 1.0$. The decrease in allowable stress is now quite large, as shown by case B. This situation is rather disturbing. An excess of actual fixity (over the assumed value) can do little good, but a lack of fixity (under the assumed value) can do much harm. Thus, conservatism should be used in selecting the fixity coefficient for use in design. Research on this subject cannot be expected to result in substantial weight savings for efficiently designed structures.

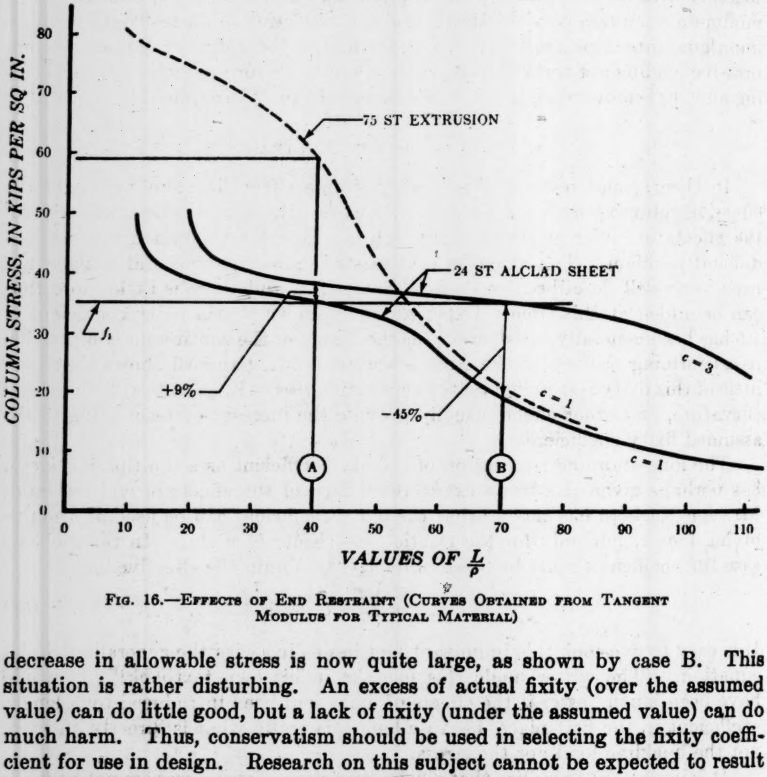

FIG. 16.—EFFECTS OF END RESTRAINT (CURVES OBTAINED FROM TANGENT MODULUS FOR TYPICAL MATERIAL)

Fixity effects should be thought of as "length" effects. In aircraft wing structures, for example, changes in end restraint primarily affect the rib or bulkhead spacing. A few extra ribs represent the price that would be paid for being quite conservative in selecting the fixity coefficient for design. Fig. 16 also shows how the use of a material having a higher yield stress pays dividends in the lower range of slenderness ratios, but such material would give little benefit at high slenderness ratios.

End restraint in an eccentric column reduces the effective eccentricity. An examination of the elastic case will show that the reduction increases as the column load is increased. In the inelastic case this effect should be even more pronounced. This condition is one more reason why, in actual structures, the effects of small eccentricities may prove to be relatively unimportant. The combined effect of large eccentricities and end restraint is an attractive subject for further research; the interaction curves previously described might be useful in this connection.

9. DESIGN FOR MINIMUM WEIGHT

The principles of structural design for minimum weight have been treated in some detail elsewhere.[21] In this paper the discussion will be confined to the procedure for determining the optimum design as applied to simple pin-ended columns of tubular cross section, using the generalized Euler theory. Since the weight of the column (for a given material) will be a minimum when the allowable stress is a maximum, the object is to find the maximum value of allowable stress in terms of the load P, the distance through which it must be transmitted L, and a "shape parameter" that controls the proportions of the cross section. If this is done for various materials, a direct comparison of weights may be made.

The generalized Euler stress formula, Eq. 2, contains one of the design variables, L, but it does not include the load P directly. The radius of gyration, being dimensional, cannot be used as a shape parameter. Since ρ^2 has the dimensions of an area, a ratio can be obtained by dividing it by A, giving the shape parameter:[22]

$$k = \frac{\rho^2}{A} \quad \text{(36)}$$

Eq. 2 may now be written

$$f = \frac{\pi^2 E_t}{L^2}\left(\frac{\rho^2}{A}\right) A \quad \text{(37)}$$

Substituting k for (ρ^2/A), (P/f) for A, and solving for f or P/L^2 gives the desired equations:

$$f = \pi \sqrt{E_t}\sqrt{k}\sqrt{\frac{P}{L^2}} \quad \text{(38)}$$

or

$$\frac{P}{L^2} = \frac{f^2}{\pi^2 E_t k} \quad \text{(39)}$$

The term P/L^2 is called the "structural index" or the "structural coefficient." (A nondimensional coefficient may be obtained by dividing by the elastic modulus, E.) Eq. 39 is more convenient to use than Eq. 38 since the value of

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[21] "Principles of Structural Design for Minimum Weight," by F. R. Shanley. *Journal of the Aeronautical Sciences*, March, 1949, p. 133.

[22] "Structures of Minimum Weight," by H. Cox and H. Smith. *Reports and Memoranda No. 1923*, H. M. Stationery Office, London, 1943.

E_t depends on f. From a curve of tangent modulus versus stress the value of $\frac{f^2}{E_t}$ may be determined. Eq. 39 can then be plotted as allowable stress f against P/L^2, for different values of k.

For a thin-walled tube of average diameter D, the value of the shape parameter is

$$k = \frac{\rho^2}{A} = \frac{1}{8\,\pi}\left(\frac{D}{t}\right) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(40)$$

For the round tube, Eq. 39 then becomes

$$\frac{P}{L^2} = \frac{8\,f^2}{\pi\,E_t\,\dfrac{D}{t}} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(41)$$

From Eq. 41, it can be noted that high values of the stress (f) for a given material will be obtained at high values of the structural index (P/L^2) for large values of D/t. There is a limit to the value of D/t that may be used, however, because the local buckling or crushing stress decreases as D/t is increased.

The optimum stress and the corresponding $\frac{D}{t}$-ratio may be obtained graphically by plotting Eq. 41 as a family of curves for different values of D/t, after which another curve of local buckling stress against D/t is drawn, by which the curves are "cut off." If an equation for the crushing stress can be written in terms of D/t and E_t, the problem may be solved mathematically.

Mr. Timoshenko[23] has suggested an equation of the following type for local buckling:

$$f_{cr} = K\,\frac{\sqrt{E\,E_t}}{D/t} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(42)$$

(Recent developments in the theory of plasticity indicate that this may be considerably on the safe side.) If this value is substituted for f in Eq. 41, the following equation will be obtained:

$$\left(\frac{D}{t}\right)_{\text{opt}} = 2\left[\frac{K^2\,E}{\pi\,(P/L^2)}\right]^{\frac{1}{3}} \quad\ldots\ldots\ldots\ldots\ldots\ldots(43)$$

It is interesting to note that the tangent modulus does not appear in Eq. 43 even though the equation applies to the inelastic range. Since E remains practically the same for any one class of alloys (steel, aluminum, etc.), the optimum value of D/t will be unaffected by changes in the shape or height of the stress-strain diagram: It depends only on the initial slope of the diagram.

By substituting the value from Eq. 43 in either Eq. 41 or Eq. 42, the optimum stress is obtained as a function of P/L^2:

$$\frac{P}{L^2} = \frac{8\,f^3}{\pi\,K\,E^{\frac{1}{2}}\,E^{\frac{3}{2}}{}_t} \quad\ldots\ldots\ldots\ldots\ldots\ldots(44)$$

Mr. Timoshenko[23] has shown that the theoretical value of K is about 1.2, but that much lower values are usually obtained in tests. For practical use in design, Eq. 44 can be plotted for various materials[21] as shown in Fig. 17, where $K = 0.40$. The procedure for optimum design consists in determining the structural index for the case in question and selecting the optimum stress from such a chart. The load P is divided by the optimum stress to obtain the cross-sectional area. Eq. 43, or curves derived from it, may be used to determine the optimum ratio D/t, from which the mean diameter may be determined by the following equation for thin-walled tubes:

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[23] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1st Ed., 1936, p. 442.

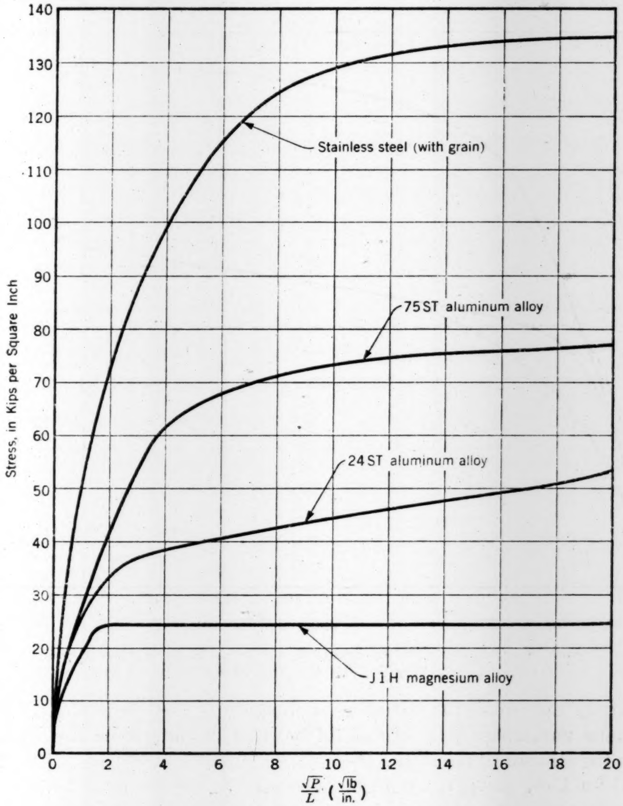

FIG. 17.—OPTIMUM COLUMN STRESS FOR ROUND TUBES

$$D = \sqrt{\frac{A}{\pi}\frac{D}{t}} \qquad (45)$$

To show how the weight of a column is affected by the material and the loading conditions, the optimum stress may be divided by the relative density. If aluminum alloy is used as a base (density 0.10 lb per cu in.), the curves will appear as in Fig. 18. (The square root of the structural index has been used in Figs. 17 and 18, to contract the scale.) It is important to note that curves of this type may be plotted directly from test data.[24] If all available data for a given type of loading were plotted on a single chart, the optimum conditions would be indicated by the envelope to the test points.[21] Application of the method, therefore, is not limited by the lack of a theory for optimum design.

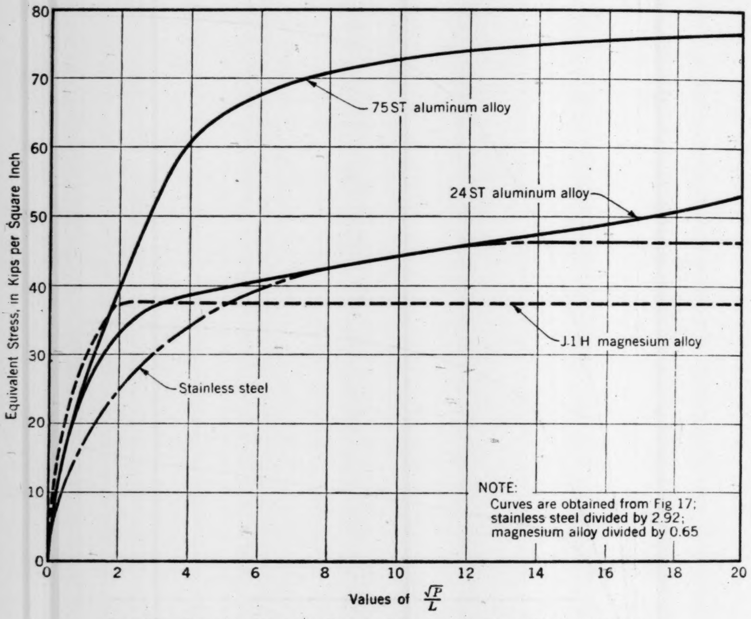

FIG. 18.—OPTIMUM COLUMN STRESS REDUCED TO SAME WEIGHT BASIS (ALUMINUM ALLOY USED AS BASE)

To apply the method to members such as those used in steel structures the same shape parameter (Eq. 36) could be used to determine the relationship between the structural index and the column buckling stress, as in Eq. 39. For the local buckling stress it would be necessary to use the ratio between one of the major cross-sectional dimensions (such as width) and the thickness of some part of the cross section. Since it is difficult to calculate the local buckling stress with accuracy, for such sections, a series of tests would be advisable in which the width/thickness ratio would be systematically varied. This procedure could be carried out for various basic shapes, such as I-beams, WF columns, and H columns. The results would be used in place of Eq. 42, to obtain curves similar to those in Fig. 17. A systematic program of this type would answer all questions regarding the efficiency of various cross-sectional shapes. It would show, for example, that beyond a certain value of the structural index any attempt to improve efficiency by revising the cross-sectional shape would be futile, at least for structural steel. More research of this type is needed, not only to supply valuable design information, but also to answer questions regarding the relative efficiency of various materials. For any structure that fails through buckling the term "strength/weight ratio" is misleading. The only way in which materials can be properly compared, in such cases, is through the use of optimum design procedures based on the structural index.

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[24] "Design Charts for Flat Compression Panels Having Longitudinal Extruded Y-Section Stiffeners and Comparison with Panels Having Formed Z-Section Stiffeners," by N. F. Dow and W. A. Hickman, *Technical Note No. 1389*, National Advisory Committee for Aeronautics, Washington, D. C., August, 1947.

The round tube, loaded as a simple column, has been used as an example of the general method of optimum design. H. Cox and H. Smith extended the method to various other types of construction and loading conditions.[22] The National Advisory Committee for Aeronautics has determined the optimum design for flat panels with various types of stiffeners and for several different materials.[24] Some information is also available for shear webs and for shells loaded in bending.[21]

10. Recommendations and Conclusions

Eight general conclusions supported by this paper are:

(*a*) The generalized Euler formula, Eq. 2 (using the tangent modulus), is recommended for the design of columns that fail by primary instability;

(*b*) Specification of a minimum guaranteed compressive yield stress provides for most of the effects of material variations;

(*c*) The effects of small eccentricities may be provided for by the factor of safety;

(*d*) The interaction curves for combined bending and compression may be applied to eccentrically loaded columns;

(*e*) The effects of end restraint should be conservatively estimated in design;

(*f*) The coefficient of fixity should be replaced by an effective length coefficient;

(*g*) Tests are needed to determine interaction curves for various cross sections and materials; and

(*h*) Optimum design methods should be used to evaluate the efficiency of various types of construction and materials.

DISCUSSION

William R. Osgood,[25] M. ASCE.—One aspect of the art of engineering consists in designing without adequate information. However much one may admire the artist, most engineers, nevertheless, would like the art reduced to a science. In what he calls "the generalized Euler formula," and modestly suggests calling the "Euler-Engesser formula," Mr. Shanley has made a signal contribution to column theory—that is, to science. The formula, in the writer's opinion, should be called the Euler-Shanley formula. It is true that Engesser first proposed it, but Engesser later gave it up in favor of the reduced modulus, or double-modulus, formula. It remained for Mr. Shanley to show that Engesser was right the first time.

Mr. Shanley has traced nondimensional column curves back to 1929. Actually they go back further than that, to May, 1918, when R. W. Hawken[26] introduced them.

In connection with the determination of the constants K and n in the stress-strain formula, Eq. 4, the preferable way (certainly for a new material) of determining these constants is the following: Write Eq. 4 as

$$e - \frac{f}{E} = K \left(\frac{f}{E}\right)^n \quad \text{.....................} (46)$$

and take the logarithm of both sides:

$$\log \left(e - \frac{f}{E} \right) = n \log \frac{f}{E} + \log K \quad \text{...............} (47)$$

Now plot the observed values of $\log \left(e - \frac{f}{E} \right)$ against $\log \frac{f}{E}$ and draw a straight line through the plotted points that seem to lie along a straight line. The slope of this line will be n and the intercept on the axis of $\log \left(e - \frac{f}{E} \right)$ will be $\log K$. Alternatively, of course, $\left(e - \frac{f}{E} \right)$ may be plotted against f/E on log-log paper.

In connection with the interaction curves it may be well to emphasize that a bending moment M arising from transverse loads and an equal moment $P\,e$ arising from eccentrically applied longitudinal loads are not equivalent. If the interaction curve is obtained by transverse tests, as the curve in Fig. 13 was, then in using the curve for the design of an eccentrically loaded column, for a given e the load P read from the curve will be on the unsafe side—that is, the eccentrically loaded column will actually fail at a lower value of P. This becomes evident by considering Fig. 19. The moment diagrams on which Fig. 13 is based are like ABCD in Fig. 19, in which M is the maximum bending moment arising from the transverse loads. The corresponding diagram for terminal couples equal to M is AEBCGD. The maximum deflection produced by the terminal couples will be greater than that produced by the transverse loads. If an axial load is now added, it will take a smaller axial load to produce failure of the member loaded by the terminal couples than to produce failure of the transversely loaded member. (The terminal couples plus the axial load are the same as an eccentrically applied load.) On the other hand, if the interaction curve is obtained from tests on eccentrically loaded columns and the abscissas are $M = P\,e$, then in using the curve for the design of a beam-column for a given maximum moment from the transverse load the axial load P read from the curve will be conservative.

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[25] Senior Structural Engr. (Acting), David Taylor Model Basin, Washington, D. C.

[26] "Column Analysis and Design," by R. W. Hawken, Univ. of Queensland and the Sydney Univ. Eng. Soc., Brisbane, Australia, 1918.

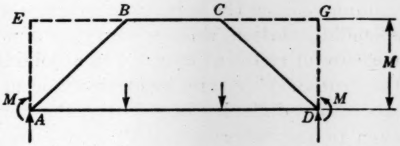

FIG. 19

In 1947 the writer suggested[27] a method of compromising between an unsafe design and an uneconomical one. The method assumes that the test data, in the form of an interaction curve, for example, are based on eccentrically loaded column tests. As intimated in the preceding paragraph, the eccentrically loaded column is a limiting case of a beam-column. The interaction curve for eccentric-load tests will be lower than the curve for any combination of axial and transverse loads, M being equal to $P\,e$ for the eccentric-load tests and equal to the maximum bending moment for transverse loads. As a practical matter it is easier to make accurate eccentrically loaded column tests than tests under combined axial and transverse load. The latter call for considerable art.

J. V. DU PLESSIS,[28] JUN. ASCE.—This comprehensive expression of the views of one who recently has shed further light on the fundamentals of column behavior is both refreshing and welcome.

It cannot be emphasized enough, however, that the "generalized Euler theory" is a "critical load" or "buckling load" theory and as such applies only to the "perfect" column or (the word used in this sense) "strut." Insisting, with the full rigor of Webster, that conditions existing are concentric loading, straightness, symmetricality, and homogeneity (even to postulating rigid monotonic rather than statistical location of atoms), it is possible to conceive a situation in which a column may be obtained in a straight loaded state, which, under some slight disturbance, will snap out of this position into the von Kármán nonstraight equilibrium bifurcation. In some cases, if the velocity of buckle is such as to induce large dynamic energy effects, the strut will surmount the barrier of increasing equilibrium loadings for larger deflections and then, behaving according to the negative slope of the deflection-end-load curve, rush to catastrophic collapse. If this last condition were in danger of occurring, the use of the buckling criterion would deserve some support as a logical design basis in order to guard against the occurrence of the "perfect" column. Since the column in practice, perfect or otherwise, seldom occurs in splendid isolation, as in some types of water towers, but generally with some measure of restraint afforded in adjoining framing, it is not reasonable to fear the "runaway" of any incidentally occurring perfect column. Apart from the laboratory difficulty of securing the perfect straight equilibrium bifurcation, even in the "short column" compression test, practice adequately indicates that column failure is a matter of bending rather than buckling.

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[27] "Beam-Columns," by William R. Osgood, *Journal of the Aeronautical Sciences*, March, 1947, p. 167.

[28] Research Student, Faculty of Eng., Bristol Univ., Bristol, England.

To cite a practical case, an old reinforced concrete frame building was subjected by new owners to warehouse type loading in the region of two to three times the design loading, and one of the columns under the loaded floor began to fail. Distortion was a gradual process (was not suddenly there) and proceeded so slowly as to allow adequate time for supporting timbers to be introduced. Whatever it was that happened, it certainly was not buckling. Apart from such records of column failures as may exist, the laboratory struggle to produce perfect pin-ended columns is sufficient testimony that column failure in civil engineering structures is a matter of bending rather than buckling, and that columns undergo bending deformation from the inception of loading.

On the basis of this observation it is impossible to agree with the author's statement (in Section 1) that:

> "It is doubtful whether any structural engineer would wish to take advantage of this factor [increasing equilibrium load with increasing deflections] in actual design, because in this region an increase in stress will involve a large amount of permanent set. It seems logical, therefore, to regard the beginning of the bending [that is, buckling] in a perfect column as the upper limit for structural purposes."

As a basis of discussion, accept the premise that a certain amount of permanent set is the maximum permitted for design purposes. If bending occurs from the beginning, due to some cause classified by buckling theorists under the term "imperfection" (as it undoubtedly will), the critical amount of permanent set will be induced at a lower load than in the case of the initially perfect but later buckling strut. The permissible load will therefore be different in practice from that derived from a buckling analysis. For other criterions of design, similar arguments will still apply. The greatest argument in support of buckling formulas is that there is no other theoretical analysis that does not presuppose knowledge either unobtainable or securable only by investigation or test, and, in any case, whose degree of variation has never been subject to control or even adequate determination. Until an adequate theoretical analysis not requiring unknown information exists, and until determined and specifiable systems of tolerances are used, it appears more practical to use empirical formulas and data, or practically proved formulas, rather than to rely on a formula based on a case unlikely to occur in practice. The interaction curves described by the author are a valuable step in this direction; but they still involve undecided tolerances and, by so doing, only serve further to underline the need for investigation as to the range of tolerance that actually occurs in practice.

Although buckling formulas are more of academic interest, their value as one set of known boundary conditions to a complex region of structural actions should not be forgotten, and in this connection the author's clarification of the stress situation that may occur across a section is of the greatest value.

It is worthwhile to stress the fact, however, that this clarification (which, in effect, is a statement that the ratio of stress change to strain change applicable to analytical consideration of any fiber is, in some cases, for all the fibers, the ratio corresponding to the loading stress strain relationship) applies in some (and probably in most) instances to columns which for some reason (such as eccentric loading) bend right from the inception of the loading program. Any load analysis or distortion analysis of columns in the elasto-plastic range must therefore be subject to "co-analysis" as to whether all the fibers are being loaded or whether some are being unloaded. Analysis of the condition for loading to occur over the entire section has been made by D. C. Drucker,[29] Jun. ASCE, for the case of buckling of the perfect column, and extension can be made on similar lines to the more general "non-perfect" column problem. The case in which (given increasing end load) all fibers continue to load is obviously more tractable analytically since it is not necessary to have a stress-strain relation sensitive to "direction." In this category may also be placed those cases in which, if a fiber is subject to unloading, it is a fiber that has been subject to a past history of only elastic strain. Only the perfect, or near perfect, columns are likely to fall outside this category before reaching the maximum load they will carry; and, in general, this will be more likely to occur with materials having relatively steep-sloping strain-hardening curves. Therefore, the arguments that led the author to support the tangent modulus Euler formula are of much wider application.

Jacob Karol,[30] M. ASCE.—This comprehensive paper by Mr. Shanley is an admirable presentation of the theory underlying the present methods of stress analysis of compression members in aircraft design. The mathematical development of the nondimensional stress-strain diagrams and column curves may seem somewhat "impractical" to the civil engineer who uses allowable stresses in the design of columns of steel, concrete, or timber that are fixed by the design specifications at values well within the elastic range. In aircraft structures, however, the design stresses are in the inelastic range. Since airplanes are generally static tested for full design loads, it was necessary for the stress analyst to develop a design theory which, in the ideal case, would result in a structure that failed at slightly beyond the design load. Therefore, the detailed study, by the author, of the several phases of applied column theory is undeniably "practical."

The writer will confine his discussion to Sections 6 and 7.

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[29] "The Plasticity of Metals—Mathematical Theory and Structural Applications," by D. C. Drucker, *Proceedings-Separate No. 27*, ASCE, August, 1950.

[30] Design Engr., Howard-Needles-Tammen & Bergendoff, Kansas City, Mo.

The method of correcting test data to "standard," as indicated in Fig. 12, is correct only for columns that fail by primary instability, since the affinely related column curves are derived from the tangent modulus. The horizontal scale marked "not used" is designated as "effective L/ρ" in the original Lockheed report (see Fig. 12). This value is actually the important variable in the general case, as will be evident subsequently.

It is supposed that a series of compression tests of sheet stiffener combinations have been made. Since the sheet and stiffener may each be subject to local buckling, the column curve plotted from the test data will not be affinely related to the tangent modulus column curve of the specimen material. It may be noted in passing that determination of f_c and L/ρ for a sheet stiffener combination is quite a problem in itself, but it is assumed that they have been determined somehow. It is desirable to plot the test data on a graph, which also has the Euler curve plotted thereon, to determine the point of intersection. The procedure for correcting the data for a test stress f_2 to standard is as follows:

Let: f_o be the yield strength for standard material; $K f_o$ be the yield strength for specimen material; f_1 be the stress for $(L/\rho)_1$, as reduced to standard; f_2 be the stress for $(L/\rho)_1$, read from the curve; and f_3 be the stress for $(L/\rho)_3 = (L/\rho)_1/\sqrt{K}$, read from the curve. Then, for the ratio $(L/\rho)_1$,

$$f_1 = \frac{f_3}{K} \qquad (48)$$

The foregoing procedure is merely a verbal statement of the mathematics expressing the affine relationship between the test column curve and the derived column curve for the standard material. The interested reader can readily prove this for himself by assuming that the test column curve is mathematically expressible in terms of f_o and $K f_o$, as indicated elsewhere,[15] and by calculating the stresses f_1, f_2, and f_3. Hence, in the general case, correction to standard is a function of the shape of the test column curve, the yield strength ratio, and L/ρ.

The author's extended treatment of the effects of eccentricity in pin-ended columns is undoubtedly an indication of the importance of the subject. After rejecting the secant formula as inapplicable to materials not having a pronounced yield point, he suggests using interaction curves to solve the problem, and presents in Fig. 15 such a series of curves for axial loading and bending due to third-point loading. He assumes correctly that, for an initial moment due to constant eccentricity, the sag would be greater. He further states that the shape of these curves may be primarily a function of the kind of loading involved, which is also true. Hence, a number of sets of interaction curves would be necessary in design, unless there is some other approach to the problem.

After following the author's lead and verifying the statements referred to in the preceding paragraph, it occurred to the writer that the problem could be generalized so that one interaction curve would apply for various loadings, and perhaps for any shape of cross section and any type of stress-strain diagram. The reasoning was extremely simple. For any column subject to combined bending and compression, the maximum stress on the section is the sum of the axial stress and the total bending stress. The total bending stress is itself the sum of the primary bending stress and the secondary bending stress due to lateral deflection. By using the total bending stress as a parameter, the type of loading is automatically cared for. If the total stress on the section is known, then, by deducting the axial stress, the total bending stress is obtained. The primary bending stress can then be determined by using the available theories of secondary bending.

The problem resolved itself into finding an expression which would equal the critical column stress for pure compression and the modulus of rupture for pure bending. The expression was originally determined to fit the test data in Fig. 14 for the entire range of R'_c and $L/\rho = 30$. It was later found that the same expression checked quite well any point taken at random from Fig. 14. This might be considered as an indication of its possible generality.

The total stress in combined compression and bending is given by the expression:

$$f_c + f_{bt} = f_{cr}\,[1 + (m - 1)\,(0.839 \tan \theta)] \qquad (49)$$

in which

$$m = \frac{f_{bo}}{f_{cr}} \qquad (50a)$$

and

$$\theta = \frac{5}{9} \operatorname{arc\,tan} \frac{f_{bt}}{f_c} \qquad (50b)$$

Dividing through both sides of Eq. 49 by f_{cr},

$$\frac{f_c}{f_{cr}} + \frac{f_{bt}}{f_{cr}} = 1 + (m - 1)\,(0.839 \tan \theta) \qquad (51)$$

The stress ratios are

$$R'_c + m\,R_{bt} = 1 + (m - 1)\,(0.839 \tan \theta) \qquad (52)$$

in which

$$\theta = \frac{5}{9} \operatorname{arc\,tan} \frac{m\,R_{bt}}{R'_c} \qquad (53)$$

Eq. 52 is plotted in Fig. 20(*a*) for various values of *m*. Since *m* is known for a particular problem—

$$R_{bt} = \frac{f_{bt}}{f_{bo}} = \frac{M_{bt}}{M_o} \qquad (54)$$

—can be determined for any value of

$$R'_c = \frac{f_c}{f_{cr}} = \frac{P}{P_{cr}} \qquad (55)$$

—and vice versa.

The remaining problem is to determine the primary bending stress f_b from the total bending stress f_{bt}. For any distributed loading (including the test third-point loading), as well as for initial curvature, the stress is given accurately enough by

$$f_b = (1 - \alpha)\,f_{bt} \qquad (56a)$$

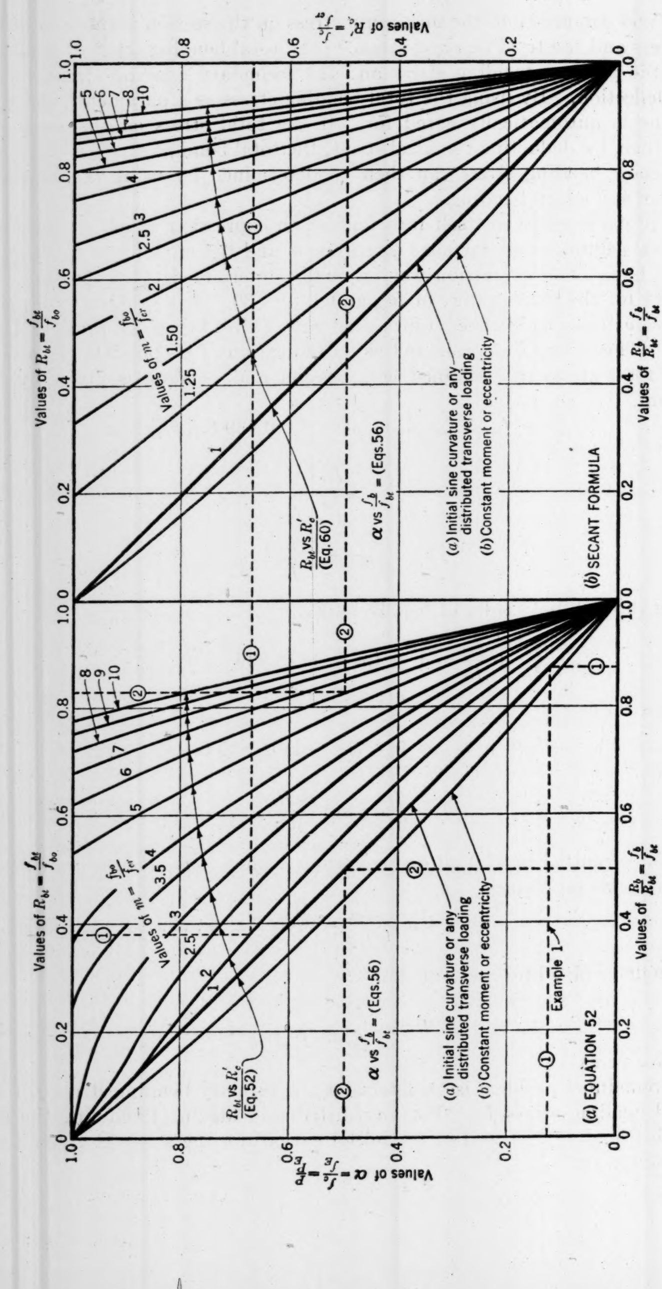

FIG. 20.—INTERACTION CURVES FOR COMBINED BENDING AND AXIAL LOADING

and, for constant eccentricity of axial load,

$$f_b = \frac{1 - \alpha}{1 + 0.272\,\alpha} f_{bt} \quad \text{.....................} (56b)$$

in which

$$\alpha = \frac{f_c}{f_E} = \frac{P}{P_E} = \frac{f_c}{\frac{\pi^2 E}{(L/\rho)^2}} \quad \text{.....................} (57)$$

Hence the ratios

$$\frac{R_b}{R_{bt}} = \frac{f_b}{f_{bt}} = \frac{M}{M_{bt}} \quad \text{.....................} (58)$$

can be determined for any value of α, and vice versa.

Eqs. 56 are also plotted in Fig. 20(*a*). Thus, the problem of combined axial load and bending due to eccentricity or transverse loading is solved from one chart.

Two examples of the use of Fig. 20(*a*) applied to the data in Fig. 14 follow:

(1) Given: $L/\rho = 30$; and, in kips per square inch, $f_{bo} = 120$, $f_{cr} = 60$, $f_c = 40$, and $E = 29{,}200$. The problem is to compute f_b. From the assumed data, $m = 120/60 = 2$; $f_E = 320$ kips per sq in; $\alpha = 40/320 = 0.125$; and $R'_c = 40/60 = 0.667$. From Fig. 20(*a*), $R_{bt} = 0.38$; hence, $f_{bt} = 0.38 \times 120 = 45.6$ kips per sq in. From Fig. 20(*a*), $f_b/f_{bt} = 0.875$; hence, $f_b = 0.875 \times 45.6 = 39.9$ kips per sq in. From Fig. 14, test $f_b = 40$ kips per sq in.

(2) Given: $L/\rho = 120$; and, in kips per square inch—$f_{bo} = 120$, $f_{cr} = f_E = 20$, and $f_b = 50$. It is required to compute f_c. From the given data, $m = 120/20 = 6$; $f_b/f_{bo} = 50/120 = 0.417$. Guess $R'_c = 0.498 = \alpha$. From Fig. 20(*a*), $f_b/f_{bt} = 0.502$; hence, $R_{bt} = 0.83$. From Fig. 20(*a*), $R'_c = 0.498$, which checks the guess. Hence, $f_c = 20 \times 0.498 = 9.96$ kips per sq in. From Fig. 14, test $f_c = 10$ kips per sq in.

It is recognized that the addition of stresses in the inelastic range is a convenient fiction; but, in the foregoing treatment, Eq. 49 has the merit of being a close approximation to the experimental truth and of indicating the important variables in a problem not solved hitherto in the general case. Since Eq. 49 assumes a gradual change in the total stress as the loading changes from pure compression to pure bending, it seems logical enough to be generally valid. Furthermore, if f_{bo} is considered to represent not only the ultimate modulus of rupture, but also a limiting bending stress in general, the usefulness of Fig. 20(*a*) is extended. Thus, in aircraft design, it can be used for limit loads as well as for ultimate loads. The writer was able to check the test data by J. A. Van den Broek,[31] M. ASCE, on 1-in. mild steel round bars for both initial yielding and ultimate capacity by using the yield point as the limiting bending stress in the first case and 1.7 times the yield point as the modulus of rupture in the second case.

The column theory, from which the secant formula is derived, is also presented for comparison. The total stress is given by the expression:

$$f_c + f_{bt} = m\, f_{cr} \quad \text{.....................} (59)$$

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[31] "Rational Column Analysis," by J. A. Van den Broek, *The Engineering Journal*, December, 1941.

Dividing through both sides of Eq. 59 by $m\ f_{cr}$,

$$\frac{f_c}{m\ f_{cr}} + \frac{f_{bt}}{m\ f_{cr}} = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (60)$$

Expressed as stress ratios, Eq. 60 becomes

$$\frac{R'_c}{m} + R_{bt} = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (61)$$

Eq. 61 is a family of straight lines, as represented in Fig. 20(*b*) for various values of m. It is obviously correct only for a value of $m = 1$. This is true in the short column range for structural steel, in which the limiting stress for both bending and compression is the yield point stress. For values of m greater than 1, it gives less conservative results than Eq. 52. The relation of primary to total bending stress is given by Eqs. 56, and use of the chart is similar to the method presented for Fig. 20(*a*).

E. P. Popov,[32] Assoc. M. ASCE.—An excellent comprehensive treatment of primary column failure has been presented in this paper. Emphasis on the tangent-modulus theory as a basis for column strength analysis is gratifying. In an earlier paper [6] Mr. Shanley capably justified the use of the tangent-modulus method for columns, theoretically. He showed that the bending of a column begins at the tangent-modulus load and that the consideration of a single critical section is sufficient to predict the capacity of a member. In this paper several graphs showing the agreement between this theory and tests are cited as supporting evidence for justification of the theory. However, these citations are limited to metals and it is noteworthy that even for reinforced concrete columns similar good agreement is found.[33]

Fig. 21 [33] shows the results of a series of tests on spirally reinforced concrete columns, in which ρ, the radius of gyration, is based on a transformed concrete section, and an average load-strain curve for a short column has been used to determine the tangent-modulus and reduced-modulus column curves. The agreement thus found is truly remarkable; and, although relatively few tests have been made on slender concrete columns, F. E. Richart and J. O. Draffin, Members, ASCE, T. A. Olson, and R. H. Heitman,[33] Assoc. M. ASCE, venture the following conclusion:[34]

> "It appears from the procedure followed that the strength of any long slender spiral column may be estimated from the properties of a short column of similar materials and arrangement of reinforcement."

However, in all these matters a word of caution must be given, as the tangent-modulus theory is adequate only if the material is stable with time.

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[32] Associate Prof. of Civ. Eng., Univ. of California, Berkeley, Calif.

[33] "The Effect of Eccentric Loading, Protective Shells, Slenderness Ratios, and Other Variables in Reinforced Concrete Columns," by F. E. Richart, J. O. Draffin, T. A. Olson, and R. H. Heitman, *Bulletin No. 368*, Univ. of Illinois Eng. Experiment Station, Urbana, Ill., November, 1947, p. 103.

[34] *Ibid.*, p. 86.

Gradual plastic deformation of concrete, or creep of the material used, may substantially lower the carrying capacity of a column. The generalized Euler equation is a criterion of column strengths for loads of short duration and for materials that do not creep.

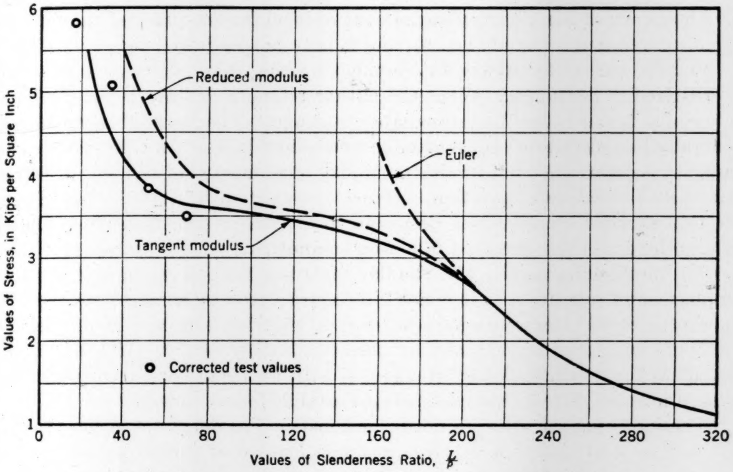

FIG. 21.—STRENGTH OF LONG REINFORCED CONCRETE COLUMNS (Average Over-All Diameter of Columns, 12.40 In.; Average Core Diameter, 12.15 In.; and $\frac{\text{Maximum Length}}{\text{Diameter}} = 19.7$)

The writer is in agreement with the general conclusions reached by the author; but one of the basic assumptions made in the paper should be clearly kept in mind, as it is significant. After justifying the use of the tangent modulus in Euler's formula, an algebraic expression for a tangent modulus, stemming from the paper by W. Ramberg and William R. Osgood,[13] M. ASCE, is introduced by Mr. Shanley. This expression for the tangent modulus (Eq. 6) is obtained from Eq. 5 by differentiation. The latter expression with proper parameters is supposedly capable of representing, reasonably well, the stress-strain curve for any material in compression. Unfortunately, ordinary steel (which hardens after some yielding strain) cannot be represented accurately by Eq. 6, in that particular form. As may be noted from Fig. 7, none of the non-dimensional stress-strain curves resemble the entire range for an ordinary steel. Perhaps there are other cases in which a mathematical three-parameter expression for a stress-strain diagram is not adequate. Moreover, the author does not actually use these curves but goes even further and works with the tangents to such curves. Also, he calls attention to the shape of the knee (mathematically indicated by n) of the stress-strain diagram as having a relatively large influence on the column stress curves as this affects the tangent moduli considerably, in an important zone.

Messrs. Ramberg and Osgood [13] show that good agreement of computed-to-experimental tangent moduli is found for several materials whose curves may be represented by a small value of parameter n. In such cases the value of n was always less than 20 and the knees of the corresponding stress-strain diagrams were not sharp. On the other hand, for the 1025 carbon steel which was also investigated, n is greater than 50 and a considerable discrepancy is indicated by the original data between computed and actual tangent moduli. To a large extent Mr. Shanley's argument rests on the adequacy of the empirical algebraic relation selected (Eq. 6), and it may not be fully adequate.

Finally, the writer wishes to comment on the term "short column" which is criticized in Section 2. What the author refers to as "short-column" range is usually termed the "intermediate"[35] range. The term "short column" generally is reserved for still shorter compression blocks. In the latter case the capacity is determined not by the instability phenomenon of an inelastic column but by an allowable deformation. Hence, retention of the term "short column" for this shortest range seems desirable to emphasize the transition point in column lengths from one kind of action to another.

The discussion given by Mr. Shanley regarding the end restraint is worthy of emphasis and repetition as it is not fully appreciated by some engineers. The development of interaction curves for combined bending and compression of members suggested by the author deserves further study. It is a practical expedient for accurate design of columns with any load eccentricity, although a large number of tests are necessary to establish such curves.

The foregoing remarks are not to be construed as detracting from the overall excellence of the paper. Even a single feature of the paper, such as the use of nondimensional forms for column formulas, is a valuable addition to civil engineering literature on columns. For completeness, in his concluding remarks, the author should comment on the limitation of the tangent-modulus column formula which occurs when $E_t = 0$; it even has a negative slope as in the case of an ordinary steel where it does not apply.

Jack R. Benjamin,[36] **Assoc. M. ASCE.**—The statement that the theoretical load at which bending of a perfect column begins is the upper limit for structural design leaves much to be desired. The design problem is not concerned basically with the upper limit for a theoretical column, but with the allowable design load on an actual column in a structure. The distance between the limit for such a column and this upper limit can be large and is difficult to analyze.

The "perfect" column does not exist. Every column is subject to variations in material and initial crookedness. These are items over which the designer has no control other than to set a limit for each. Eccentric load application always occurs and the eccentricity is normally variable about both principal axes. The end conditions in the usual structural column may be subject to analysis, but more often are not.

Taking the theoretical load at which bending of a perfect column begins as the upper limit for structural design and then modifying this limit in designing may be satisfactory for aircraft work, but it does not apply under existing codes and specifications to civil engineering structures. Two possible general design criteria exist. The normal criterion for civil engineering structures is to consider the upper limit of column strength to be reached when the unit stress at any point in the primary column material reaches the yield point. In recent years, the other criterion of ultimate strength has been given much attention. However, no matter how the design is made, under normal working loads the stresses are within the elastic limit and the column acts elastically. If the column is a part of a complex structure, as most columns are, the load on the structure which will cause failure of any single column is very difficult to determine. If the yield point criterion is used, an approximation can be made, but if ultimate collapse is the basis, only a pure guess is possible. The structural designer must first consider the stresses in the column under working loads with some arbitrary limit set by code or specification which will determine the factor of safety. If superposition applies to the system, the factor of safety thus found is reasonable. However, when ultimate strength is the basis of design, a radical change in approach must be made by comparing the column in the structure with an isolated theoretical column in only a simplified mathematical structure at best. If the two can be compared at all, it can be only an arbitrary guess, pure and simple.

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[35] "Mechanics of Materials," by P. G. Laurson and W. J. Cox, John Wiley & Sons, Inc., New York, N. Y., 1947, 2d Ed., p. 233.

[36] Asst. Prof. of Civ. Eng., Stanford Univ., Stanford University, Calif.

The tests of structural columns reported in the literature are open to the same kind of arbitrary decision as are designs by ultimate strengths. First, the experimental end conditions are unknown. Testing with flat ends certainly results in semielastic supports and the action of the column is not completely determined in either the elastic range or the plastic range. To accept test data with flat ends as a basis for design requires considerable imagination. The pin-ended testing is scarcely better since true pin ends are not obtained mechanically, and very small eccentricities can produce large stresses. Thus, tests of columns leave gaps in both experiment and application, the latter being virtually unbridged except by arbitrary opinion.

The design of columns in the elastic range by the secant formula is subject to question. First, only rarely are eccentricities constant throughout a column length. Eccentricities vary normally and exist about both principal axes. The further use of positive eccentricities combined with a reduced length coefficient based on a negative eccentricity places it in the same category with the parabolic and other empirical formulas.

The writer does not understand why eccentricity and end restraint are considered separately. End restraint is often the cause of eccentricity and then the two are inseparable. The same basic error contained in the secant formula is made in the paper. The idea of end restraint producing a reduced length is incompatible with positive eccentricities. Furthermore, eccentricities are not normally constant and occur about both principal axes.

The paper was found most challenging; it is a valuable contribution to the subject of columns.

Herbert A. Sawyer, Jr.,[37] **Jun. ASCE.**—This well organized and stimulating paper can be mainly criticized for its quick rejection of the secant formula. The secant formula is a special case of the general column formula:

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[37] Asst. Prof. of Civ. Eng., Univ. of Alabama, University, Ala.

$$f = \frac{P}{A} + \frac{M\,c}{I_e} \qquad \text{(62)}$$

in which M is the actual maximum moment in the column from all possible causes. This expression is true even in the plastic range if the term c/I_e is properly modified. Contrary to what the author states in Section 7, the secant curve from this general formula has a very distinct relationship to the Euler column curve—in fact, the Euler Engesser curve is only a special case of the secant curve. The general column formula also appears to agree very well with the experimental results[14] cited by the author in his treatment of eccentricity. Thus these results (and evidently the entire problem of eccentricity) seem to have, in the general column formula, a theoretical basis which should be recognized, and which serves as a direct tie between the two main parts of the paper—one on perfect columns, and the other on eccentric columns.

To verify these statements, the value of M in the general equation must first be found. For this discussion, it is sufficient to do this only for the cases of the paper—for any initial eccentricity, e, and any initial moment from transverse loads, M_b, both of which are symmetrically distributed with respect to the transverse center line of the column. Examples of these cases are shown in Figs. 22(a) and 22(b). Many investigators have evaluated M for various conditions of eccentricity, curvature, or transverse loading.[38,39] Prominent among early investigators was J. M. Moncrieff,[40] M. ASCE, and the author[41] himself has suggested an approach to the problem.

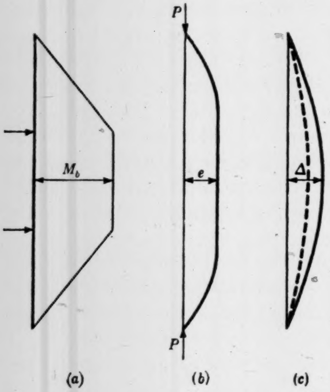

FIG. 22.—EXAMPLES OF INITIAL MOMENTS AND DEFLECTIONS

The writer's reason for adding another basically similar column formula to the existing superabundance is that a formula was needed for this discussion which, using the author's notation, would apply generally to the author's various examples of eccentrically and transversely loaded columns. Existing formulas have been developed, for the most part, for specific types of transverse loadings or eccentricities.

Successive approximations will be used, although the principle of superposition is true only in the plastic range as applied subsequently, and for infinitesimally small values of M_b and e. The moment M will be found at the midheight of the column, where it is a maximum.

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[38] "Buckling of Elastic Structures," by H. M. Westergaard, *Transactions*, ASCE, Vol. LXXXV, 1922, p. 586.

[39] "Numerical Procedure for Computing Deflections, Moments, and Buckling Loads," by N. M. Newmark, *ibid.*, Vol. 108, 1943, pp. 1161–1234.

[40] "The Practical Column Under Central or Eccentric Loads," by J. M. Moncrieff, *ibid.*, Vol. XLV, June, 1901, p. 341.

[41] "Basic Structures," by F. R. Shanley, John Wiley & Sons, Inc., New York, N. Y., 1944, Chapter 19, p. 309.

The primary moments at the midheight will be $M_b + P e$. These moments, distributed, for example, as shown in Figs. 22(*a*) and 22(*b*), each cause a transverse deflection, and, when added, would result in a deflection curve like Fig. 22(*c*). The total midheight deflection from these moments, Δ, calculated by any standard method, would be

$$\Delta = C_b \frac{M_b L^2}{\pi^2 E_t I} + C_e \frac{P e L^2}{\pi^2 E_t I} \qquad \text{(63)}$$

in which C_b and C_e are constants depending on the shape of the curves corresponding to Figs. 22(*a*) and 22(*b*). Values of these constants, as defined by Eq. 63, are as follows:

Distribution of moment	C_b or C_e
Triangular (apex at midheight)	0.882
Third point trapezoidal (Fig. 22(*a*))	1.050
Quarter point trapezoidal	1.131
Rectangular (M_b or $P e$ constant)	1.26
Parabolic (vertex at midheight)	1.027
Sinusoidal (vertex at midheight)	1.000

The tangent modulus is accurate here only for infinitesimally small values of M_b and e, as shown by the author for perfect columns. For larger values, although not accurate, it is certainly more accurate than the elastic modulus.

The additional midheight moment, M_1, from the deflection Δ will be

$$P\,\Delta = M_1 = (C_b M_b + C_e P e) \frac{P L^2}{\pi^2 E_t I} \qquad \text{(64)}$$

and the additional moment at any height will be P times the deflection at the height, producing a moment diagram geometrically similar to Fig. 22(*c*). This curve approximates a sine curve, with negligible error for the foregoing values of the constants, C_b and C_e, since its curvature usually approaches zero at either end and is maximum at the midheight. Assuming a sine curve, the additional midheight deflection from these moments will be $\frac{P\,\Delta\,L^2}{\pi^2 E_t I}$, and the additional midheight moment from this deflection will be

$$M_2 = P \left(\frac{P\,\Delta\,L^2}{\pi^2 E_t I} \right) = M_1 \frac{P L^2}{\pi^2 E_t I} \qquad \text{(65)}$$

This increment of moment (with sine curve distribution again assumed), in turn, causes another increment of midheight deflection. This deflection, multiplied by P, gives the next increment of moment:

$$M_3 = M_1 \frac{P L^2}{\pi^2 E_t I} \frac{P L^2}{\pi^2 E_t I} \qquad \text{(66)}$$

This process may be continued indefinitely, each successive increment of moment containing an additional term $\frac{P L^2}{\pi^2 E_t I}$. Totaling all these increments,

$$M = M_b + P e + (C_b M_b + C_e P e) \frac{P L^2}{\pi^2 E_t I} \times \left[1 + \frac{P L^2}{\pi^2 E_t I} + \left(\frac{P L^2}{\pi^2 E_t I} \right)^2 \cdots \right] \ldots\ldots\ldots\ldots\ldots (67)$$

Summing up the geometric progression, and substituting this value of M in Eq. 62, the general column formula becomes

$$f = f_c + f_b + f_e + \frac{f_b C_b + f_e C_e}{\frac{\pi^2 E_t \rho^2}{f_c L^2} - 1} \ldots\ldots\ldots\ldots\ldots (68)$$

in which $f_b = M_b c/I$; $f_e = P e c/I$; and $f_c = P/A$.

For the special case in which M_b and e are infinitesimal, when the term $\frac{\pi^2 E_t \rho^2}{f_c L^2}$ becomes unity, or $P = \frac{\pi^2 E_t I}{L^2}$, f will become infinite, indicating failure from instability. The curve of this failure will be the Euler-Engesser curve, the outermost curve of the infinite number of curves of f_c versus L/ρ, for the general formula, Eq. 68.

For the special case in which M_b and f_b are zero and e is constant throughout the length, C_e becomes 1.26. With this value the formula becomes practically identical to the secant formula, the series of Eq. 67 becoming practically identical to a secant series.

Many arguments can be advanced as to whether this approach and formula are useful in showing failure above the elastic limit for a material without a sharp knee on its stress-strain diagram. As indicated previously, it certainly becomes less accurate in these regions, except for very small values of M_b and e. In any case, the value of this approach will be similar to the value of the fictitious modulus of rupture in predicting pure bending failure.

The best answer to this question is to compare the general formula with the test results. This procedure has been used for the series of interaction curves in Fig. 14. In doing this, only the values for f_{bo} (the modulus of rupture), the value $f_{co} = 60$ kips per sq in. for average ultimate stress for a short perfect column, and a derived value of E_t for $f_c = 50$ kips per sq in., as given elsewhere,[14] were used; all other points were calculated, using a form of the general column formula, giving "stress" at ultimate loading:

$$f_c + \left(1 - \frac{f_c}{f_{co}} \right) f_{bo} = f = f_c + f_b + \frac{1.050 f_b}{\frac{\pi^2 E_t \rho^2}{f_c L^2} - 1} \ldots\ldots\ldots (69)$$

The left-hand term of Eq. 69 (the "stress" at ultimate loading) is based on the reasoning that, if, for example, f_c is seven twelfths of f_{co}, failure will occur when the total bending stress exceeds five twelfths of f_{bo}. This result is probably not provable theoretically. Simplifying Eq. 69,

$$f_b = \frac{f_{bo}\left(1 - \frac{f_c}{60}\right)\left[1 - \frac{f_c}{\pi^2 E_t}\left(\frac{L}{\rho}\right)^2\right]}{1 + 0.05\,\frac{f_c}{\pi^2 E_t}\left(\frac{L}{\rho}\right)^2} \quad \text{.............} (70)$$

Curves plotted from Eq. 70 for $f_{bo} = 120$ kips per sq in., as indicated in Fig. 13 (and also the original[14] Fig. 10), did not check the experimental curves, especially at high eccentricity ratios. Further investigation showed that the experimental curves were plotted incorrectly. From the original source,[42] it can be deduced that the actual modulus of rupture varied from roughly 115 kips per sq in. to 135 kips per sq in., generally increasing with decreasing slenderness ratio and decreasing wall thickness. However, the original investigators[42] concluded that until some definite relationship was found it would be best to ignore this variation and use a constant value of 120 kips per sq in., thus making the experimental failure stresses relatively too high with respect to f_{bo}, and making the experimental curves plotted against this artificial limit err on the unsafe side for low slenderness ratios. Thus, through no fault of the author, the curves in Fig. 14 and Fig. 15 are inaccurate, on the unsafe side, in representing failure of compression tubing loaded transversely at the third points.

Curves were then plotted from Eq. 70 on the basis of a sliding value of f_{bo}, starting at 134 kips per sq in. for $L/\rho = 30$, and decreasing by 2 kips per sq in. for each increase of 10 in L/ρ, making $f_{bo} = 116$ kips per sq in. for $L/\rho = 120$. This very arbitrary variation seemed to fit the small amount of information[42] available concerning f_{bo}. It can be seen that the results, shown in Fig. 23, agree with the experimental curves very well. Discrepancies for $f_b > 96$ should be neglected because there were no tests in this region. The other discrepancies can probably be accounted for by the uncertainty as to f_{bo} and by experimental errors. These are especially large at small eccentricities and high axial loads, where any imperfection of the specimen tends to make experimental results very conservative relative to theoretical results.

Eq. 70 in general nondimensional form would be

$$R_b = \frac{(1 - R_c)(1 - R^2_L)}{1 + (C - 1)R^2_L} \quad \text{.................} (71)$$

in which C is either C_b or C_c as defined previously;

$$R^2_L = \frac{f_c}{f_{cr}} = \frac{\left(\frac{L}{\rho}\right)^2}{\left(\frac{L}{\rho}\right)^2_{cr}} = \frac{\left(\frac{L}{\rho}\right)^2}{\frac{\pi^2 E_t}{f_c}} \quad \text{...............} (72)$$

$$R_c = \frac{f_c}{f_{co}} \quad \text{.........................} (73a)$$

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[42] "Strength of Tubing Under Combined Axial and Transverse Loading," by L. B. Tuckerman, S. N. Petrenko, and C. D. Johnson, *Technical Note No. 307*, National Advisory Committee for Aeronautics, Washington, D. C., June, 1929, pp. 6–7.

and

$$R_b = \frac{f_b}{f_{bo}} \text{ or } \frac{f_e}{f_{bo}} \quad \text{.....................(73}b\text{)}$$

Eq. 71 is plotted for constant *e*, a truly eccentric column, in Fig. 24. The author's radial lines showing eccentricity are very useful for this type of chart. For Eq. 71 and Fig. 24 they must be plotted for

$$R_e = \frac{R_b}{R_c} = \frac{f_{co}}{f_{bo}} \frac{c}{\rho^2} e \quad \text{.....................(74)}$$

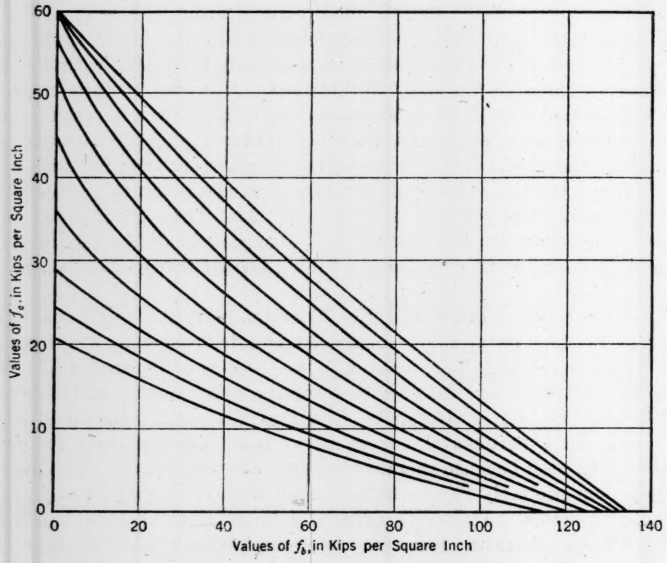

FIG. 23.—THEORETICAL INTERACTION CURVES FOR CHROME-MOLYBDENUM-STEEL TUBING

Similar diagrams could be plotted very quickly for any one of the values of the constants, C_b or C_e, by computing the ten or more points at $R_e = 0$ required to determine the straight lines for R_L.

Eqs. 71 and 72 support the author's prediction that, since R_L depends upon L/ρ and E_t, the lines of the nondimensional curve (for example, Fig. 24) would change with the value of the exponent *n* if drawn for L/ρ instead of R_L. Drawn for R_L, the lines are independent of *n*; and one set of nondimensional lines (such as those in Fig. 24) are good for any column of constant *E I* of any material with a certain type loading (such as the constant *e* of Fig. 24), within the limitations of the foregoing theory. These include, besides the limitations noted in the foregoing derivation, neglect of variation in tangent modulus (from size, shape, age, temperature, and direction and history of stress), failure in detail, and any imperfection not included in the terms M_b, C_b, *e*, and C_e.

To use Fig. 24 for a given material, f_{bo}, f_{co}, and the stress-strain curve must be known. Then, opposite each value of R_c should be placed the square of the corresponding maximum slenderness ratio for the material, $(L/\rho)^2_{cr}$, computed from the Euler-Engesser formula (using the stress-strain diagram to obtain E_t). Then, for a given member and load, the allowable e may be found directly. If the member and e are given, R_c must be found by trial and error, two trials usually being sufficient. If the designer uses the concept of the factor of safety traditional to civil engineers, the design must be made on the basis of an ultimate load equal to the working load multiplied by the factor of safety, since f does not vary linearly with P.

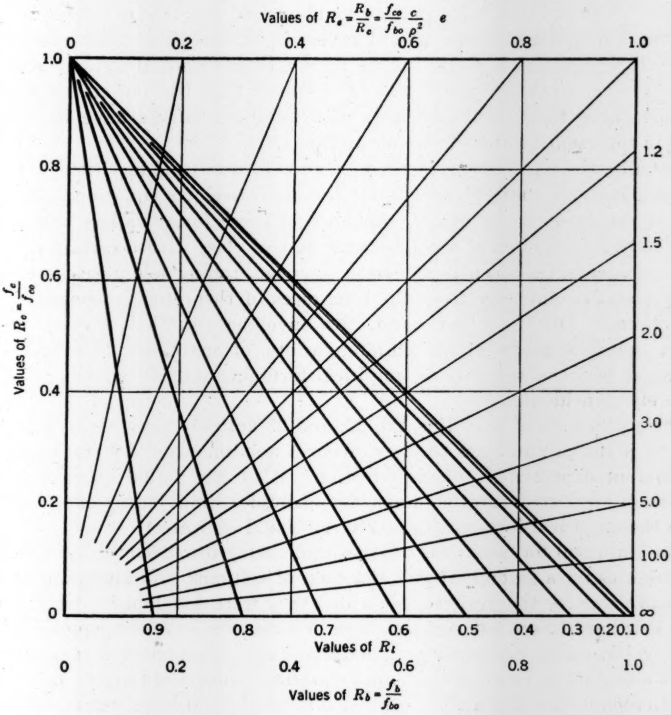

FIG. 24.—REDUCED INTERACTION LINES FOR ECCENTRIC LONGITUDINAL LOADING

Another check of Eq. 70 against experimental results was made, using the remaining data,[14] which was for duralumin tubing loaded in the same manner as the chrome-molybdenum-steel tubing. For these tests the value of the short perfect column strength had to be revised (for reasons similar to those for changing f_{bo} for the chrome-molybdenum-steel tests) to a value estimated at 38 kips per sq in. from the test data. The resulting theoretical and experimental curves did not check as well as the steel curves, the discrepancy being from -4 kips per sq in. to $+6$ kips per sq in. for values of f_b up to 55 kips per sq in. Even though this difference is well within experimental error,[43] this check of Eq. 70 against experiment is not very strong evidence of its usefulness for materials less elastic than chrome-molybdenum-steel. More evidence, pro or con, is needed.

A second criticism of the paper concerns the envelope equation of Section 5, in which it is stated that this equation could be used as a kind of over-all column curve for any material where weight saving is not too important. In Fig. 11 the envelope equation is plotted along with various standard column formulas, which form a "scatter band," and it is stated that the resemblance of the envelope curve to a parabola shows why parabolic curves have been found useful in representing the short column range.

Actually, the parabolic, straight-line, and Gordon-Rankine-Ritter formulas have applications entirely different from the envelope equation, also their applications are more practical. Any one of these formulas has been set up empirically for columns of a single material and class with a certain probable degree of accidental eccentricity (from warpage, rolling tolerances, accidental damage, and eccentricity from the tolerances of theoretically concentric end connections). On the other hand, the envelope equation applies only to perfect columns made of an infinite variety of materials. Certainly any agreement between the various empirical formulas and the envelope formula is entirely coincidental.

The coefficients of B in Eqs. 26, 27, and 28 actually vary widely from material to material and class to class, even in nondimensional form, to provide for different expectancies of imperfection. Also, the actual coefficients used for carbon steel structural columns, for example, show better agreement between the empirical formulas than Fig. 11 would indicate.[44]

The empirical formulas, easy to use, and taking into account unpredictable eccentricities for a given material and class of columns, will always have great practical value for the purposes for which they were developed. On the other hand, it is hard to see a design situation in which the envelope equation would be of value. For extraordinary materials and eccentricities, experimental results along with the general column equation (using the tangent modulus so ably advocated by the author, and maximum tolerable values of accidental eccentricity) should be used, the generalized Euler equation being a special case of this application.

The fundamental importance of accidental eccentricity to routine column design is illustrated by an example. A "perfect" 14 WF 202 column, 30 ft long, would take an ultimate load of more than 2,100 kips according to the generalized Euler formula. Taking into account chance eccentricities, the American Institute of Steel Construction (AISC)[45] gives a 3/8-in. "out-of-lineness" rolling tolerance for this column, and it would probably not be too conservative to add a constant e of 1/8 in. from all other causes. Using the general column formula with $f_{bo} = f_{co} = 40{,}000$ lb per sq in., the ultimate load is about 1,420 kips. The allowable load on this column, using the parabolic formula in the 1946 AISC specifications,[46] equals about 784 kips, which when multiplied by the usual factor of safety[47] of about 1.8 agrees roughly with the foregoing result of 1,420 kips.

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[43] "Strength of Tubing Under Combined Axial and Transverse Loading," by L. B. Tuckerman, S. N. Petrenko, and C. D. Johnson, *Technical Note No. 307*, National Advisory Committee for Aeronautics, Washington, D. C., June, 1929, p. 7.

[44] "Resistance of Materials," by Fred B. Seely, John Wiley & Sons, Inc., New York, N. Y., 3d Ed., 1947, p. 229.

[45] "Steel Construction," AISC, New York, N. Y., 5th Ed., 1947, p. 285.

E. C. HARTMANN,[48] M. ASCE.—No structurally-minded engineer can read Mr. Shanley's excellent paper without finding much to stimulate his thinking, especially with regard to the significant part of the stress-strain curve for metals which lies above the range of purely elastic action. In his usual, clear, concise, and nonmathematical manner Mr. Shanley has covered some rather complicated concepts in a way which makes them more understandable, thus setting a good example to other engineering authors who do not always qualify for words of praise along these lines.

In the section of his paper dealing with the effects of eccentricity, Mr. Shanley has confined his attention to those members which fail in the plane of the applied bending moment. It might be well to point out that there is another classification of members, such as I-beams and H-beams, which may tend to fail at right angles to the plane of the applied moment, thus complicating the problem beyond that which is covered in Mr. Shanley's treatment. If an I-beam, for example, is loaded to failure as a column with the end load applied in the plane of the web, but eccentric with regard to the centroid, it is highly probable (assuming no local buckling) that the I-beam will fail by buckling in a direction at right angles to the plane of the web.

The problem of the lateral buckling of eccentrically loaded I-shaped and H-shaped columns is not an academic one in civil engineering practice and deserves more attention than it has received in the past. Among those who have contributed are S. Timoshenko[49] in 1936; Bruce Johnston,[50] M. ASCE, in 1941; and H. N. Hill, Assoc. M. ASCE, and J. W. Clark,[51, 52] Jun. ASCE, in 1948.

F. R. SHANLEY.[53]—The discussions of this paper by a number of experts have been most interesting and have contributed greatly to the knowledge of the subject. Most of the comments were of a constructive nature and were clearly presented. The writer wishes to thank the discussers collectively.

During the time that has elapsed since the preparation of the paper in the summer of 1947 a number of significant developments has occurred. Some of these have a bearing on the questions raised in the discussions and will therefore be mentioned briefly.

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[46] "Steel Construction," AISC, New York, N. Y., 5th Ed., 1947, p. 64.

[47] "Standard Specifications for Highway Bridges," A.A.S.H.O., Washington, D. C., 5th Ed., 1949, p. 242.

[48] Chf., Eng. Design Div., Aluminum Research Laboratories, Aluminum Co. of America, New Kensington, Pa.

[49] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1936, p. 243.

[50] "Lateral Buckling of I-Section Column with Eccentric End Loads in Plane of the Web," by Bruce Johnston, *Transactions*, A.S.M.E., Vol. 63, 1941, p. A-176.

[51] "Alcoa Structural Handbook," Aluminum Co. of America, Pittsburgh, Pa., 1948, p. 57.

[52] "Lateral Buckling of Eccentrically Loaded I-Section Columns," by H. N. Hill and J. W. Clark, *Proceedings-Separate No. 34*, ASCE, October, 1950.

[53] Prof., Univ. of California, Los Angeles, Calif.; Consultant, The Rand Corp., Santa Monica, Calif.

Mr. du Plessis mentions the case of a reinforced concrete column which bent gradually under heavy load and had to be supported laterally. This was evidently a case of instability as a result of compressive creep, a phenomenon which has been of some concern to the writer. Several years ago some experiments were made (at Lockheed Aircraft) in which a plastic (lucite) column, at room temperature, was subjected to a steady load equal to approximately two thirds of the buckling load determined under ordinary testing machine conditions. Careful measurements of lateral bowing showed a small but gradual increase over a period of several months; sudden buckling then occurred, with practically no warning.

Investigation of this phenomenon has since been under way by the writer and his associates at the Rand Corporation. Thomas Higgins has developed a method of predicting the deflection-time curve, using a procedure similar to that by which Theodor von Kármán,[4] Hon. M. ASCE, treated the eccentric column in the inelastic range. Creep curves were used as a basis, instead of the usual stress-strain diagram. Tests made at Battelle Institute, in Columbus, Ohio, show that column failure may occur under steady load at some time after initial application, the length of time depending on the average compressive stress, slenderness ratio, and temperature. It is of interest to note that complete collapse usually occurs before the bowing exceeds the core radius by any appreciable amount.

Mr. Karol's discussion of the interaction curves for combined bending illustrates how these curves can be derived by applying well known principles, and how they are related to the secant-modulus formula. This type of analysis may be useful in reducing the number of tests needed to establish the curves; however, the writer recommended the interaction curve method primarily because its simplicity permits the investigator to determine, with relatively few tests, the actual curves for various cross sections and materials. This seems to be important for members of complex cross section in which local buckling or other forms of instability may play a part.

It should be noted that an interaction formula is used for combined bending and compression in the manual[54] of the American Institute of Steel Construction. If secondary bending is not included in the computation of the bending stress, the formula would be represented by a straight line in Fig. 15, indicating unsafe values for slenderness ratios much greater than 40.

Mr. Popov states that the Ramberg-Osgood formula does not adequately represent the stress-strain diagram for some materials (such as ordinary steel), and that the writer's argument, to a large extent, "* * * rests on the adequacy of the empirical algebraic relation selected * * *." On the contrary, the mathematical formulation of the stress-strain diagram should be regarded only as a convenience to be used when applicable. It was, perhaps, misleading to use the mathematical curves in presenting the nondimensional column curves. Any column curve or formula may be presented in this manner. In actual practice the tangent-modulus column curve is usually determined directly from the stress-strain diagram for the material.

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[54] "Steel Construction," AISC, New York, N. Y., 5th Ed., 1947, p. 284.

Mr. Sawyer's remarks on the original data from which Figs. 14 and 15 were derived are important in connection with any attempt to apply a theoretical analysis to the interaction curves. As previously stated, however, these curves were presented only to illustrate the method and were labeled unsafe (see Section 7) for general use.

The writer certainly did not mean to add another column equation to the long list of empirical and semi-empirical equations. The remarks on the possible use of the envelope equation were intended to emphasize the point that all types of stress-strain curves would be safely covered by such an equation, if the value of f_o were defined as indicated.

Mr. Hartmann's remarks reveal another class of eccentric columns in which an additional interaction phenomenon must be considered—namely, the interaction between the eccentric bending moment in one principal plane and the induced or secondary bending moment in the other principal plane. Perhaps some of the analytical methods presented in the other discussions could be extended to include this unsymmetrical bending phenomenon, but one solution would be to find some interaction curves by actual tests. It might then be possible to convert the eccentric bending moment in one plane to an equivalent (reduced) bending moment in the other. From a practical point of view, however, the use of the straight-line interaction curve previously mentioned[54] might suffice, since the "sag" of the curves would probably be reduced. The lowest values of the allowable axial loads or bending moments thus obtained would of course be used.

In closing this discussion the writer would like to emphasize the importance of defining the level at which the analysis or investigation takes place. The most frequent criticism of the tangent-modulus theory is a statement to the effect that there is no such thing as a perfect column. Such a criticism can be applied to any scientific theory, since all such theories are based on idealizations or abstractions.[55] The only thing that can be discussed intelligently is the degree of abstraction involved and its relation to the actual phenomenon, which can never be completely described by words or equations.

In this paper the emphasis was on the application of the column theory in structural engineering. It was shown that an accurate estimate of the buckling load can easily be obtained from the compressive stress-strain diagram. A simple engineering method for reducing this load to account for eccentric loading was developed. Simple methods of reducing test data to standard values were presented. It is possible to move in either direction from this level. On the one hand, empirical equations may be fitted to the resulting curves, perhaps for handbook or specification purposes. On the other hand, one may attempt to analyze the action of the eccentric inelastic column in a detailed scientific manner.[56] Confusion of these levels of abstraction can lead only to arguments that have no end and delay real progress. The writer hopes that this paper will help to eliminate some of the confusion which seems to have surrounded the column problem for so many years.

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[55] "Science and Sanity," by Alfred Korzybski, Science Press, Lancaster, Pa., 3d Ed., 1948, p. 434.

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[56] "Inelastic Column Buckling," by Tung-Hua Lin, *Journal of the Aeronautical Sciences*, March, 1950, p. 159.

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TRANSACTIONS

Paper No. 2411

FLOW AROUND BENDS IN AN OPEN FLUME

By Ahmed Shukry,[1] Assoc. M. ASCE

With Discussion by Messrs. Gilbert W. Outland, Edward Silberman and Alvin G. Anderson, L. O. Croes, and Ahmed Shukry.

Synopsis

In this paper are presented the results of an investigation of the spiral motion around bends by direct measurements taken with a specially designed pitot sphere, which records the velocity components in the three planes. The characteristics of this spiral motion, as well as other pertinent aspects of the bend action, were studied experimentally. The investigations show the effect of varying the Reynolds number, the depth-breadth ratio, the radius-breadth ratio, and the deflection angle of the bend.

Introduction

Notation.—The letter symbols in this paper are defined where they first appear, in the text or by illustrations, and are assembled for convenience of reference in the Appendix.

Research on flow around bends has engaged the attention of hydraulic engineers for more than a hundred and fifty years. Most of the investigations were on closed bends and were mainly concerned with the energy lost as the result of bend resistance. Although considerable data have been published,[2,3,4,5,6,7,8] the results have not been satisfactory because of their

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Note.—Published in June, 1949, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Asst. Prof. of Irrig., Faculty of Eng., Farouk First Univ., Alexandria, Egypt.

[2] "Report Upon the Physics and Hydraulics of Mississippi River," by A. A. Humphreys and H. L. Abbot, Lippincott, Philadelphia, Pa., 1861, p. 315.

[3] "Loss of Pressure in Water Flowing Through Straight and Curved Pipes," by A. W. Brightmore, *Minutes of Proceedings*, Inst. C. E., London, Vol. CLXIX, 1906–1907, p. 315.

[4] "Der Verlust in 90°-rohrkrümmern mit gleichbleibendem kreisquerschnitt," by A. Hofmann, München, Oldenbourg, 1929.

[5] "The Flow of Water in Flumes," by Fred C. Scobey, *Technical Bulletin No. 393*, U.S.D.A., Washington, D. C., 1933.

[6] "Resistance to Flow in Curved Open Channels," by S. P. Raju, in Abridged Translations of Hydraulic Papers, *Proceedings*, ASCE, November, 1937, p. 49 after p. 1834.

[7] "Effects of Channel Shape on Losses in a Canal Bend," by C. H. Yen and J. W. Howe, *Civil Engineering*, January, 1942, p. 28.

[8] "Theorestiche Grundlagen der Fluss-und Wildbachverbauungen," by R. Müller, Leeman & Co., Zürich, 1943.

wide disagreement, which may be attributed to the absence of a standard basis for their correlation.

If the energy lost due to bend resistance is obtained dimensionally, as in the case of problems dealing with pipe resistance, the application of the π-theorem[9] to any bend would show that its coefficient of resistance ξ $\left(\text{which expresses the head lost } h_b \text{ in terms of the mean velocity head } \frac{V^2}{2g}\right)$ is a function of

(*a*) The Reynolds number **R**, expressed as

$$\mathbf{R} = \frac{V R}{\nu} \quad \text{.........................} (1)$$

in which V is the mean velocity; R is the hydraulic radius; and ν is the kinematic viscosity;

(*b*) The roughness parameter R/n_l in which n_l is a linear dimension of the wall roughness and the bed roughness; and

(*c*) The form parameter, which depends on the geometrical shape of the bend.

In a bend of an open, stable, rectangular channel the form parameter (item (*c*)) may be expressed in terms of the three ratios r_c/b, y/b, and $\theta/180°$ (r_c is the radius of the bend at the center of the stream; b is the width of the channel bed; y is the depth of flow; and θ is the angle of bend in degrees).

The general relationship between ξ and its parameters may be expressed as

$$\xi = C\,\phi\left(\mathbf{R}, \frac{R}{n_l}, \frac{r_c}{b}, \frac{y}{b}, \frac{\theta}{180°}\right) \quad \text{................} (2)$$

in which C is a constant and ϕ denotes a function.

Eq. 2 shows the importance of specifying the range of any experiment with respect to each of the five parameters. Furthermore, since the additional energy lost is but one of the phases in which the bend action manifests itself, other phenomena resulting from this action may reasonably be expected to vary with these parameters.

The lateral currents induced in bends were first observed by J. Thompson in 1876[10] in a series of experiments conducted in an open channel having an 180° bend, the spiral motion being observed by seeds and drops of anilin dyes. Later, many other investigators, following almost the same procedure as Mr. Thompson, explored the spiral flow phenomena in closed and open bends.[11,12,13,14] In most of these researches only the forward-velocity components were recorded because of the limitation of the then existing hydraulic instruments; thus, it was not possible to obtain a direct, clear image of the flow pattern. Moreover, the use of floats, threads, dyes, seeds, wet paints, and transparent models could show only the general features of the spiral flow.

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[9] "Fluid Mechanics for Hydraulic Engineers," by Hunter Rouse, McGraw-Hill Book Co., Inc., New York and London, 1938.

[10] "On the Origin and Winding of Rivers in Alluvial Plains, with Remarks on Flow Around Bends in Pipes," by James Thompson, *Proceedings*, Royal Soc. of London, March, 1876, p. 5.

[11] "Nebenströmungen in gekrümmten Kanälen," by A. Hinderks, *Zeitschrift*, Vereines Deutscher Ingenieure, Vol. 71, 1927, p. 1779.

[12] "Flow Around a River Bend Investigated," by F. L. Blue, Jr., J. K. Herbert, and R. L. Lancefield, *Civil Engineering*, May, 1934, p. 258.

[13] "Flow of Water Around 180°-Bend," by D. L. Yarnell, *Technical Bulletin No. 526*, U.S.D.A., Washington, D. C., 1936.

[14] "Flow Around Bends in Stable Channels," by C. A. Mockmore, *Transactions*, ASCE, Vol. 109, 1944, p. 593.

Spiral flow cannot be studied properly unless direct measurements of the velocity vectors have been made. As the flow is distinctly three dimensional, a pitot sphere was built especially for the work described herein to measure the velocity components.

THE PITOT SPHERE

The pitot sphere (Fig. 1) has five brass tubes a, b, c, d, and e (Fig. 1(*b*)), enclosed in a casing stem, A, which is 9.0 mm in external diameter (see Fig. 1(*a*)).

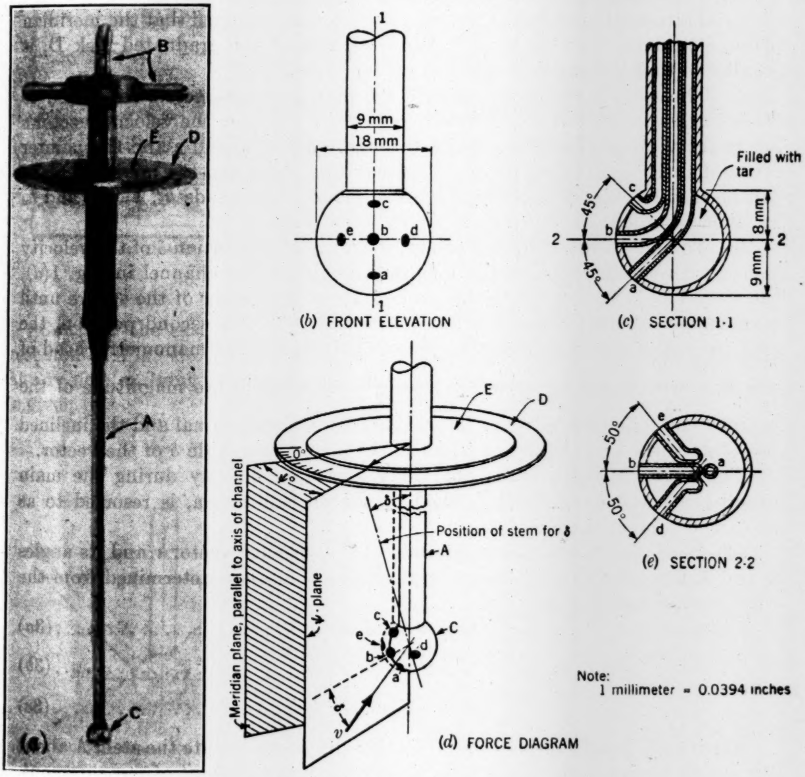
B
E
D
A
C
(a)
1
9 mm
18 mm
c
e
b
d
a
1
(*b*) FRONT ELEVATION
Filled with tar
c
45°
b
2
45°
a
8 mm
9 mm
2
(*c*) SECTION 1-1
E
D
0°
ψ°
δ
Position of stem for δ
A
Meridian plane, parallel to axis of channel
ψ-plane
c
e
b
d
a
C
δ
v
(*d*) FORCE DIAGRAM
e
50°
b
a
50°
d
(*e*) SECTION 2-2
Note:
1 millimeter = 0.0394 inches

FIG. 1.—THE PITOT SPHERE

Each tube has a small brass tapping at the top end to permit a rubber tubing connection with a pressure manometer. At the bottom end the tube is bent inside a sphere, C, 18.0 mm in diameter (3/4 in.), and then terminates at a certain hole in the shell of the sphere which thus contains five holes. The center of hole b coincides with the horizontal axis of the sphere; holes a and c lie on its vertical great circle and have central angles of 45°, whereas holes e and d lie on the horizontal great circle and have central angles of 50°.

Two circular disks D and E (Fig. 1(*a*)) are mounted on the stem A. Disk D is graduated in degrees and disk E is fixed to the stem and contains a pointer that lies in the plane of holes a, b, and c. The stem A passes through a hole in disk D so that when the disk is fixed at a certain position and the stem is rotated, the angle of deviation in degrees can be readily obtained.

To use the pitot sphere, the procedure is as follows:

(1) The instrument is held with the stem A perpendicular to the direction of the longitudinal axis of the channel. It is then so adjusted that the meridian plane, containing holes a, b, and c and the zero of the graduated disk D, is parallel to that longitudinal axis.

(2) The stem is then rotated until the pressure manometers connected to holes d and e indicate the same reading. In this position, the velocity vector v lies in the vertical plane that passes through holes a, b, and c and the pointer of the disk E (Fig. 1(*d*)). Angle ψ between this plane and the meridian plane can be readily observed on disk D. The manometric heads h_a, h_b, h_c, and h_d are then recorded.

(3) To find, by direct measurement, the angle of inclination δ of the velocity vector with the direction of the longitudinal axis of the channel in Fig. 1(*d*), the stem should be rotated in the plane ψ about the center of the sphere until manometers a and c indicate the same reading. In this second position, the velocity vector would be directly opposite hole b, and the manometric head of this hole would give the velocity head $\frac{v^2}{2\,g}$ from which the magnitude of the vector v could be determined. The angle between the normal and the inclined positions of the stem would be equal to the inclination angle δ of the vector.

The angle, δ, however, is not easily measured directly during the main experiments and an indirect method, based on calibration, is resorted to as explained subsequently.

(4) Having obtained the magnitude of the velocity vector v and its angles ψ and δ, the velocity components in the three planes are determined from the relationships (see Fig. 2):

$$v_x = v \cos \delta \sin \psi \quad\text{.....................}(3a)$$

$$v_y = v \sin \delta \quad\text{.............................}(3b)$$

and

$$v_z = v \cos \delta \cos \psi \quad\text{.....................}(3c)$$

During the main experiments it is rather difficult to rotate the stem A about the sphere whose center must remain stationary at the point of measurement. For this reason the pitot sphere has been calibrated beforehand in such a way that the manometric readings h_a, h_b, h_c, and h_d, taken during step (2), may be used directly to give values of the velocity head $\frac{v^2}{2\,g}$ of the vector and its inclination angle δ.

Calibration of the Pitot Sphere.—In this calibration, the instrument is given different angles of inclination δ, varying from $\mp 60°$ with a direction of a parallel flow whose velocity, v_o, and static pressure, p_o, are known (see Fig. 3). For any angle δ within the specified range, manometers e and d are first adjusted to give the same reading, and the manometric heads h'_a, h'_b, h'_c, and h'_d are then recorded.

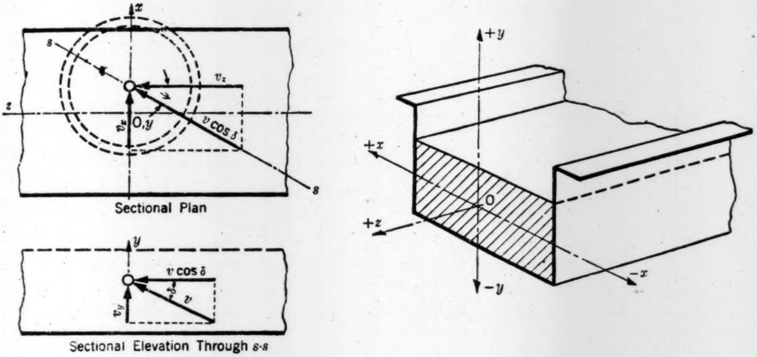
x
s
v_z
z
v_x
O,y
$v\cos\delta$
s
Sectional Plan
y
$v\cos\delta$
δ
v_y
v
Sectional Elevation Through s-s
+y
+x
O
+z
−x
−y

FIG. 2.—FLOW COORDINATES AND VELOCITY COMPONENTS

Consider the case of a potential flow around a frictionless sphere, as represented in Fig. 4. From the theory of fluid mechanics,[15] the pressure head at any hole m, lying at a central angle β_m in the direction of the parallel flow, is given by

$$h'_m = \frac{p_o}{\gamma} + \frac{v^2_o}{2g}\left(1 - \frac{9}{4}\sin^2\beta_m\right) = \frac{p_o}{\gamma} + k_m\frac{v^2_o}{2g} \qquad (4)$$

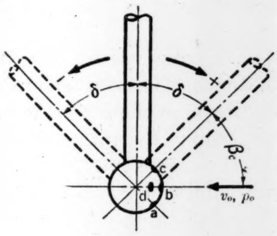
δ
δ
β_c
c
d
b
a
v_o, p_o

FIG. 3.—ANGLE OF INCLINATION δ

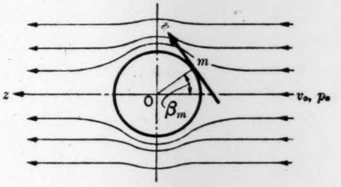
e
m
z
O
β_m
v_o, p_o

FIG. 4.—PATTERN OF STEADY FLOW AROUND A SPHERE

The symbol k_m is termed the "dynamic coefficient" of the hole m. As demonstrated by Eq. 4, it depends only on the central angle β_m. Substituting in Eq. 4 the manometric heads recorded in the calibration test (h'_a, h'_b, h'_c, and h'_d), the corresponding dynamic coefficients (k_a, k_b, k_c, and k_d), for any angle δ, are determined. If different values for δ are taken, the functions of k_a, k_b, k_c, and k_d, with respect to angle δ, can now be obtained from Fig. 5.

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[15] "Aerodynamic Theory," William Durand, editor, Julius Springer, Berlin, 1934, p. 213.

Furthermore, there is a functional relationship between any angle δ and its corresponding manometric heads h'_a, h'_b, h'_c, and h'_d. Calling this function k_δ, an "inclination factor," the relationship may be expressed as follows:

$$k_\delta = \frac{h'_c - h'_a}{h'_b - h'_d} = \frac{k_c - k_a}{k_b - k_d} \qquad \text{(5)}$$

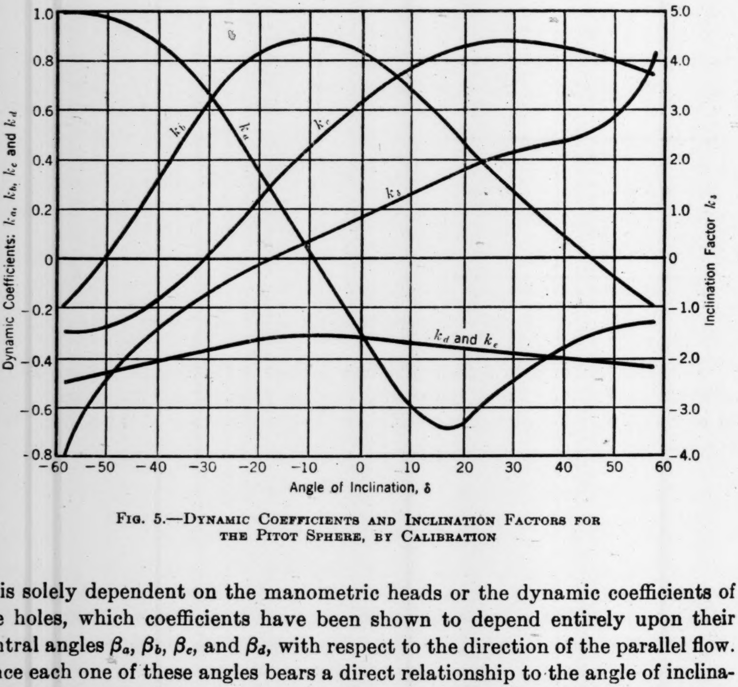


FIG. 5.—DYNAMIC COEFFICIENTS AND INCLINATION FACTORS FOR THE PITOT SPHERE, BY CALIBRATION

It is solely dependent on the manometric heads or the dynamic coefficients of the holes, which coefficients have been shown to depend entirely upon their central angles β_a, β_b, β_c, and β_d, with respect to the direction of the parallel flow. Since each one of these angles bears a direct relationship to the angle of inclination δ of the stem, as may be seen in Fig. 3, then k_δ depends only on angle δ. If k_δ is computed from Eq. 5 for the different values of δ lying between $\delta = \mp 60°$, then the relationship between k_δ and angle δ can be obtained from Fig. 5.

If the calibration curves of the pitot sphere have been predetermined, as in Fig. 5, step (3) of the foregoing measuring procedure may be replaced by the following step (3*a*), noting that v_o, p_o, and h', for a pitot sphere inclined in a parallel flow, are equal to v, p_s, and h, respectively, when the pitot sphere is held normal to the axis of the channel and the velocity vector is inclined at any angle δ.

(3*a*) Substituting the values of the manometric heads recorded in step (2) (namely, h_a, h_b, h_c, and h_d) in Eq. 4,

$$h_a = \frac{p_s}{\gamma} + k_a \frac{v^2}{2\,g} \qquad (6a)$$

$$h_b = \frac{p_s}{\gamma} + k_b \frac{v^2}{2\,g} \qquad (6b)$$

$$h_c = \frac{p_s}{\gamma} + k_c \frac{v^2}{2\,g} \qquad (6c)$$

and

$$h_d = \frac{p_s}{\gamma} + k_d \frac{v^2}{2\,g} \qquad (6d)$$

The velocity head of the vector may be computed from the following relationship:

$$\frac{v^2}{2\,g} = \frac{h_a - h_b}{k_a - k_b} = \frac{h_b - h_c}{k_b - k_c} = \text{(etc.)} \qquad (7)$$

The static pressure head at the measuring point will be

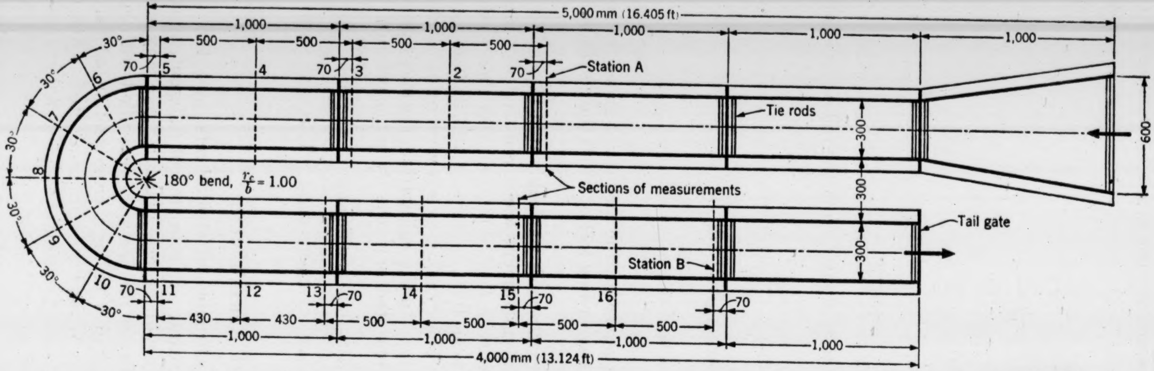
$$\frac{p_s}{\gamma} = h_a - k_a \frac{v^2}{2\,g} = h_b - k_b \frac{v^2}{2\,g} = \text{(etc.)} \qquad (8)$$

To find the angle of inclination δ of the vector, the inclination factor k_δ is first computed from the manometric heads according to Eq. 5 and the corresponding angle δ is then obtained from Fig. 5.

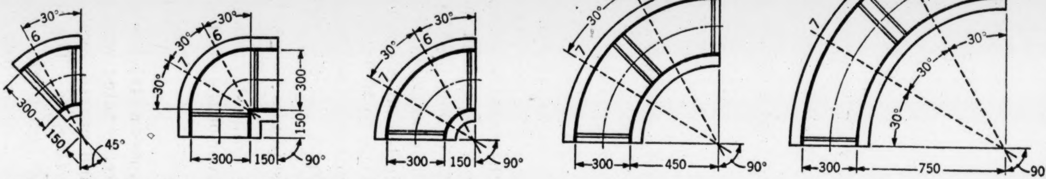
The pitot sphere used in this research work was calibrated in a still-water channel, 120 m long (393.70 ft). The instrument was given angles of inclination δ to the channel axis ranging between $\delta = \mp 60°$ and was then towed at various selected speeds ranging between 0.50 m per sec and 3.00 m per sec (9.8 ft per min to 59.0 ft per min). The dynamic heads were recorded during the uniform motion of the pitot sphere. In this calibration, the static pressure head in Eq. 4 was zero since h represented only the dynamic pressure head $k \frac{v_o^2}{2\,g}$. The results showed that the dynamic coefficients of the holes, for any angle δ, were almost constant in the range of speeds used. This is an important point in favor of the pitot sphere.

LABORATORY ARRANGEMENT

The experimental flume was constructed of steel sheets, and the inside surfaces were coated with smooth paint. It consisted of different, separate, straight and curved sections that could be assembled to give any desired alinement (Fig. 6). The joints between the sections were filled with rubber gaskets 5.0 mm thick. The flume had a uniform cross section 30 cm wide (11.8 in.) and 40 cm deep (15.75 in.). When tests were conducted on any given bend, the bend was preceded and followed by straight tangents 5.0 m and 4.0 m (16.40 ft and 13.12 ft) long, respectively; there were no other bends in the system. A stilling basin, floats, and a curved masonry entrance helped to provide a smooth flow in the flume; a tailgate was used for adjusting the depth


5,000 mm (16.405 ft)
1,000
500
70
Station A
Tie rods
300
600
180° bend, $\frac{r_c}{b}$ = 1.00
Sections of measurements
Tail gate
Station B
430
4,000 mm (13.124 ft)
30°
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16

(*a*) PLAN


30°
6
7
300
150
45°
90°
450
750

(*b*) 45° BEND, $\frac{r_c}{b} = 1.00$ (*c*) 90° BEND, $\frac{r_c}{b} = 0.50$ (*d*) 90° BEND, $\frac{r_c}{b} = 1.00$ (*e*) 90° BEND, $\frac{r_c}{b} = 2.00$ (*f*) 90° BEND, $\frac{r_c}{b} = 3.00$

FIG. 6.—GENERAL PLAN OF FLUME AND BENDS

of flow. The broken lines in Fig. 6 show the positions of the sections at which measurements were taken. In the bed of both the first section (Station A, Fig. 6(*a*)), and the last section (Station B), three holes were drilled which were connected to a common tube so that the observer was able to measure the drop in the water surface between the two stations. This was done by two glass containers, mounted on a common board, each fitted with a point gage. The rate of discharge was measured by two previously calibrated sills.

In all the experiments the bed of the flume was kept horizontal and the flow approached the bend in a streaming state. The experimental study covered the effect of the partial variations of the following parameters: $\mathbf{R}$, r_c/b, y/b, and $\theta/180°$. The mean value of Kutter's coefficient of roughness (as computed from the Manning formula) was found, for the range of experiments in the present research, to be 0.01012.

Velocity and Energy Distribution Around Bends

The Spiral Motion in a Straight Flume.—Fig. 7 presents the contour lines of the velocity components v_x, v_y, and v_z at the midsection of a straight flume 9.0 m long, the flow coordinates being as shown in Fig. 2. The magnitudes and directions of the lateral currents are represented by the dimensioned arrows at the measuring points. To visualize the lateral motion, "direction lines" were drawn by interpolation. These lines should not be confused with the "streamlines" since they do not obey the known equation of continuity. The zero contours of v_x and v_y intersect at point Z at which the flow is parallel to the longitudinal axis of the flume. Point Z may be considered as the intersection of the axis of the spiral motion with the plane of the cross section. Although the rotary motion about this axis is not uniform, a single spiral motion, nevertheless, is easily discernible. This differs from the spiral motion in straight channels, as discussed by A. H. Gibson,[16] M. Möller,[17] and others, in which the pattern included one spiral on each side of the center line where the maximum superelevation existed. It is quite probable that such a case of double spiral motion does not occur in a straight channel unless it has a long and uniform reach, sufficiently removed from the inlet, to permit the equalization of shear stresses on both sides. The highest water level would then be at the center of the stream. In short laboratory flumes, the small disturbance at the entrance, being usually an inevitable condition, is sufficient to cause the zone of highest water level to shift to one side, thus giving rise to a single spiral motion.

The Spiral Motion Around an 180° Bend.—Fig. 8 shows the contour lines of the velocity vector v and the lateral currents v_{xy} at eight sections around an 180° bend. The forward-velocity contours, v_z, are also given for some of these sections. The conditions of flow are the same as those for the straight flume discussed in the previous paragraph. Fig. 8 shows that the general distribution of the flow around the bend passes through three distinct stages.

In the first stage (sections 2 to 7, Fig. 6), the v-contours and the v_z-contours keep adjusting themselves until they become almost flattened out vertically

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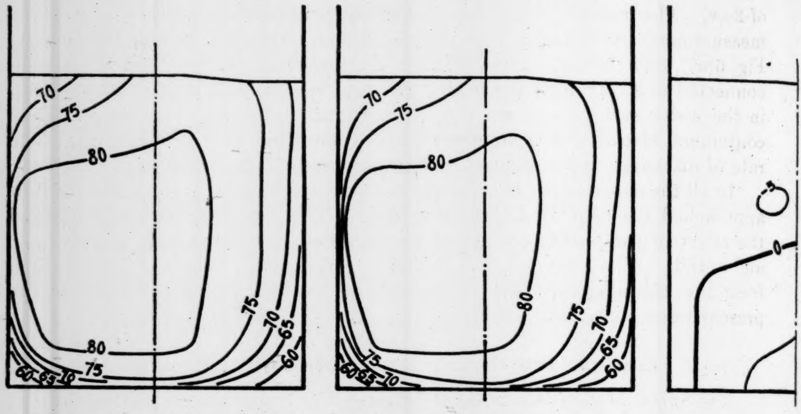
[16] "Hydraulics and Its Applications," by A. H. Gibson, Constable & Co., Ltd., London, 1934, 4th Ed., p. 332.

[17] "Hydraulic Laboratory Practice," by J. R. Freeman, A.S.M.E., 1929, p. 70.



(*a*) Contour Lines of Equal Vector (v) (*b*) Contour Lines of Equal Component (v_z) (*c*) Contour Lines of

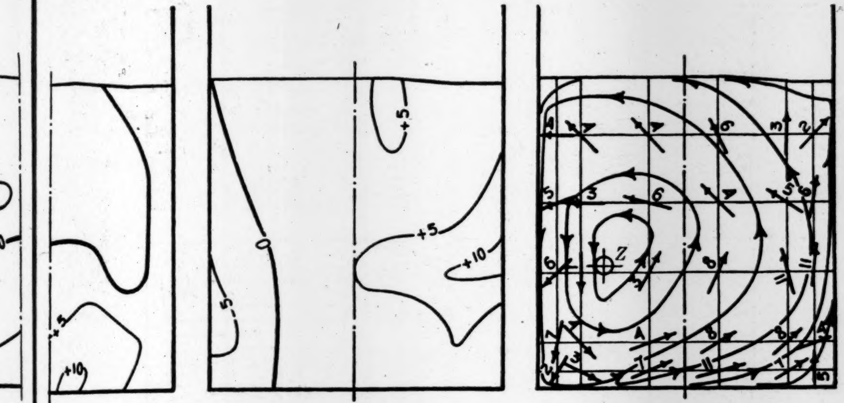
FIG. 7.—DISTRIBUTION OF THE VELOCITY COMPONENTS, FACING
(Velocities Are in Centimeters per Second (=1.9685 Ft per Min); $y/b = 1.0$;

and parallel to one another; and velocity contours of much higher values occur near the inside wall. This stage ends at section 7 (at 60°, Fig. 8(*c*)), where the superelevation of the water surface, the maximum velocity vectors, and the maximum forward-velocity components have values greater than those at any other section of the bend.

During the second stage (sections 7 to 11, Fig. 6), the contours of maximum velocity gradually move from the inside wall, causing such great deceleration there that a discontinuity region is formed at the exit of the bend adjacent to the inside wall (section 11). In spite of this discontinuity, however, the flow in the outer half of the section is fairly evenly distributed.

In the third stage (sections 11 to 16, Fig. 6), the distribution of the flow is readjusted until it reverts to the conditions in a straight flume.

By virtue of the centrifugal force, the lateral currents along the bend show a continuous tendency to deviate from the original pattern of the straight approach flume and to form a new one. The new pattern, with axis Z_2, will be called the "centrifugal spiral motion" to distinguish it from the "original spiral motion" of a straight flume whose pattern has an axis Z_1. The centrifugal motion becomes perceptible near the bed at section 7, and then expands gradually as the flow moves around the bend. As a result of this action, the original pattern is pushed to the upper outside corner of the section until it ultimately disappears near the exit of the bend, at section 11, Fig. 6. At this section the centrifugal motion almost occupies the full cross-sectional area, with the exception of the region close to the inside wall which is occupied by the wake of separation. As the flow leaves the bend and advances downstream, the original motion reappears in the upper regions (section 12). It then expands gradually,



es of Equal Component (v_x) (*d*) Contour Lines of Equal Component (v_y) (*e*) Direction Lines and Magnitudes of the Lateral Currents (v_{xy})

+ − +

ACING DOWNSTREAM AT THE MIDSECTION OF A STRAIGHT FLUME

= 1.0; R = 73,500; and Q = 70 Liters per Sec (=8,899.24 Cu Ft per Hr))

pushing down the centrifugal pattern, which, however, does not totally disappear until section 16 is reached, where the lateral motion becomes similar again to that of a straight flume.

It will be noted from the foregoing that the assumption that there is a single centrifugal spiral motion moving uniformly along the bend is erroneous. As a matter of fact, a complicated flow pattern is produced by the interference of the spiral motion (originally taking place in the straight approach flume) with that caused by the bend action. In the present work the centrifugal pattern was fully developed only for a short length of flume in the vicinity of section 11.

The effect of the spiral motion on the specific energy diagram of the curved flume is shown in Fig. 9, in which the five lines, 1, 2, 3, 4, and 5, may be defined as follows:

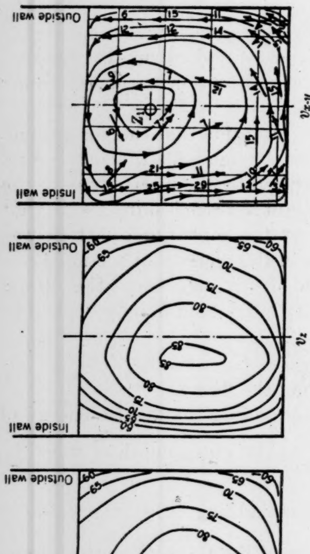
Curve Definition

1........The line of mean depth of flow along the flume

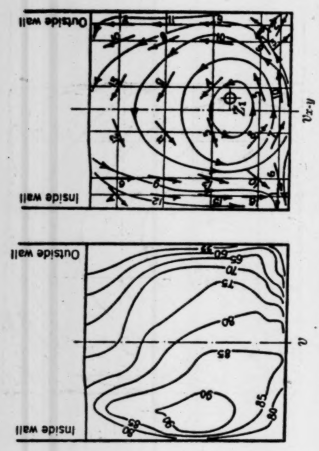
2........The specific energy line obtained by adding the kinetic energy of the mean forward velocity, $\frac{V^2_z}{2\,g}$, to the mean depth of flow, y_m, at any section

3........The specific energy line obtained by adding to y_m the value of $\left(\frac{V^2_z}{2\,g}\right)_m$, the mean kinetic energy of the forward motion at any section, as given by

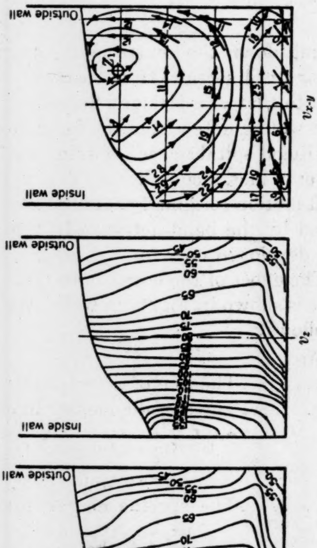
$$\left(\frac{V^2_z}{2\,g}\right)_m = \frac{1}{V_z\,A}\int_0^A \frac{v^3_z}{2\,g}\,dA \ldots\ldots\ldots\ldots (9a)$$



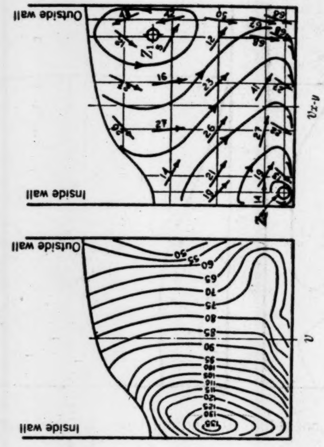
(a) Section 2 at 157 cm (5.2 ft) upstream from bend



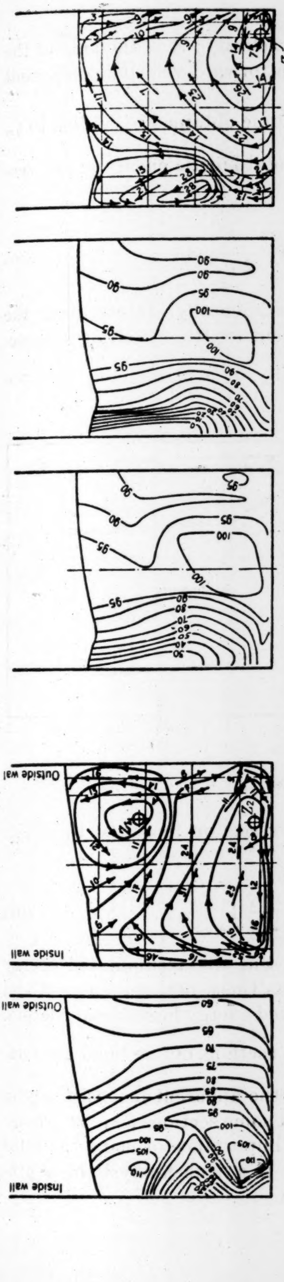
(b) Section 5 at 7 cm (2.75 in.) upstream from bend



(c) Section 7 at 60°

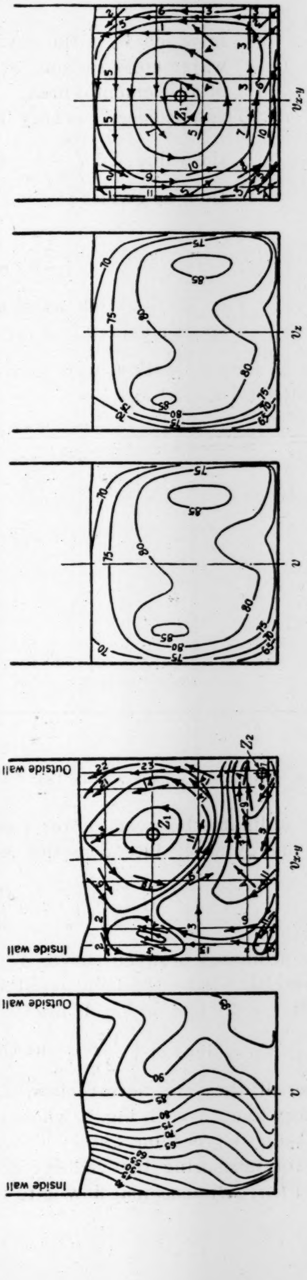


(d) Section 8 at 90°



(e) Section 10 at 150°

(f) Section 11 at 7 cm (2.75 in.) downstream from bend



(g) Section 12 at 50 cm (1.6 ft) downstream from bend

(h) Section 16 at 247 cm (8.1 ft) downstream from bend

FIG. 8.—DISTRIBUTION OF THE VELOCITY VECTORS, v, THE FORWARD-VELOCITY COMPONENTS, v_z, AND THE LATERAL CURRENTS v_{x-y}

Curve Definition

in which V_s is the mean forward velocity; A is the area of the water cross section; and v_s is the forward-velocity component of an elemental area

4........The true specific energy line of the flume obtained by adding to y_m the value of $\left(\frac{V^2}{2\,g}\right)_m$, the mean total kinetic energy at any section, as given by

$$\left(\frac{V^2}{2\,g}\right)_m = \frac{1}{V_s\,A}\int_0^A \frac{v^2}{2\,g}\,v_s\,dA \qquad (9b)$$

5........The friction line as obtained experimentally, representing the specific energy line for an equivalent length of a straight flume.

Eqs. 9, for any section, were solved graphically by a planimeter, from the v-contours and v_s-contours.

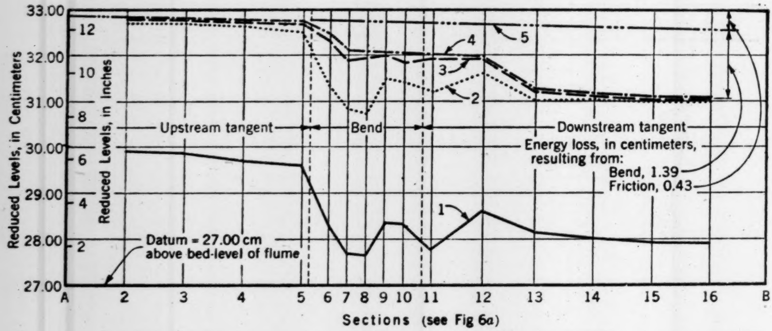
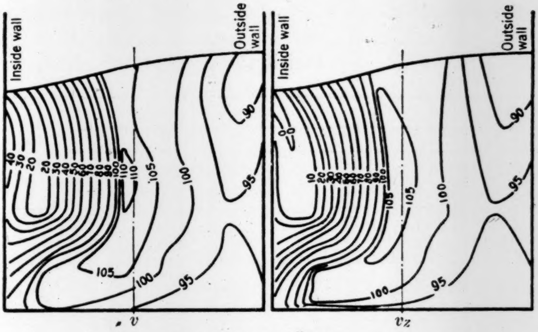


FIG. 9.—ENERGY DIAGRAMS FOR AN 180° BEND ($r_c/b = 1.00$; $y_A/b = 1.00$; and $R_A = 73{,}500$)

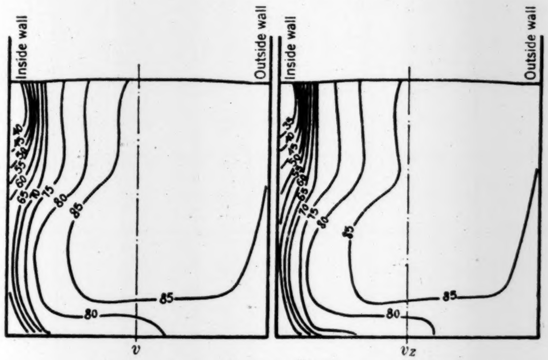
The ordinates between curves 4 and 3 at any section represent the mean kinetic energy of the lateral motion as given by

$$\left(\frac{V^2_{xy}}{2\,g}\right)_m = \left(\frac{V^2}{2\,g}\right)_m - \left(\frac{V^2_s}{2\,g}\right)_m \qquad (9c)$$

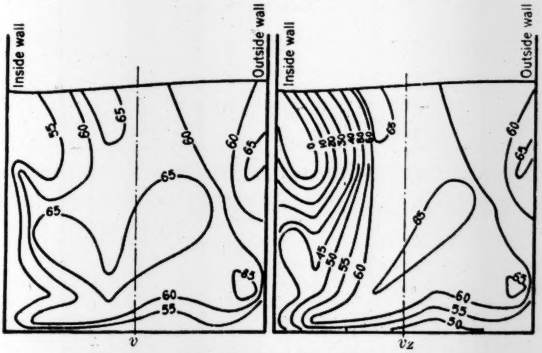
The ordinates between curves 4 and 5 represent the accumulated energy losses resulting from the bend resistance, whereas those between curve 5 and the static energy line, in Fig. 9, are the accumulated energy losses from friction. Although the values of $\left(\frac{V^2_{xy}}{2\,g}\right)_m$ at the different sections of the bend are relatively small, they are, nevertheless, the cause of the apparent inconsistency in the energy line, curve 3, Fig. 9, whose ordinates at some sections acquire greater values than at preceding ones. This shows the importance of using the pitot sphere for measuring the velocity components, if a true analysis of a pronounced three-dimensional flow is to be obtained.



(*a*) $\mathbf{R}_A = 73{,}500$, $\frac{r_c}{b} = 1.00$, $\frac{y_A}{b} = 1.00$, AND $\frac{\theta}{180^\circ} = 0.50$



(*b*) $\mathbf{R}_A = 73{,}500$, $\frac{r_c}{b} = 3.00$, $\frac{y_A}{b} = 1.00$, AND $\frac{\theta}{180^\circ} = 0.50$



(*c*) $\mathbf{R}_A = 48{,}100$, $\frac{r_c}{b} = 1.00$, $\frac{y_A}{b} = 1.00$, AND $\frac{\theta}{180^\circ} = 0.50$

FIG. 10.—EFFECT OF VARYING THE REYNOLDS NUMBER, AND THE RADIUS-BREADTH RATIO, ON THE DISTRIBUTION OF THE VELOCITY VECTORS AND THE FORWARD-VELOCITY COMPONENTS, v_z, AT THE DOWNSTREAM SECTION OF A BEND; SEE SECTION 11, FIG. 6
(Velocities Are in Centimeters per Second (=1.9685 Ft per Min))

Strength of a Spiral Motion.—No definition of the strength of a three-dimensional flow has come to the writer's notice. Such a definition, however, is essential in order to be able to visualize the effect of the spiral flow in different

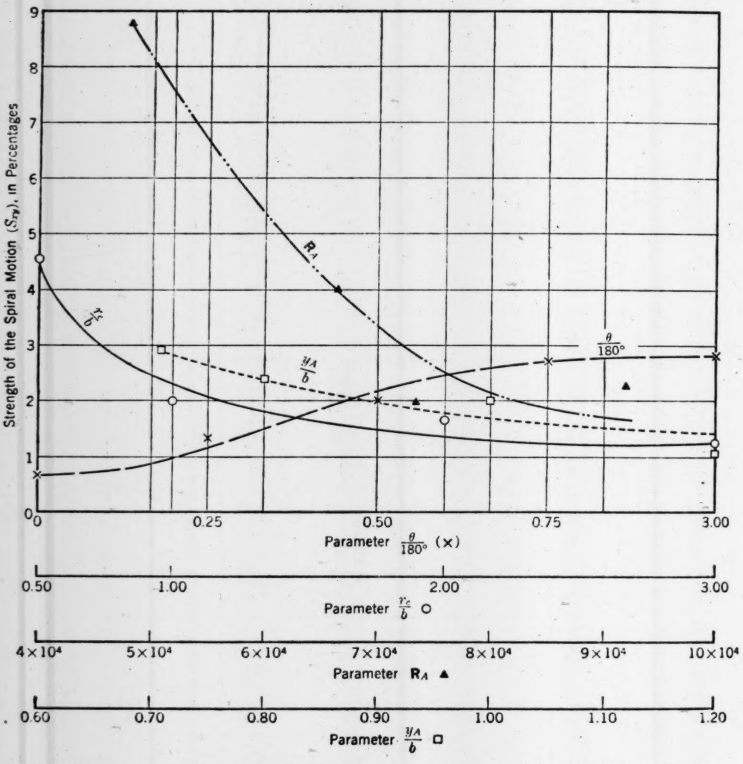


FIG. 11.—EFFECT OF THE DIFFERENT PARAMETERS ON THE STRENGTH OF THE SPIRAL MOTION

bends under variable conditions of flow. In this paper the strength of a spiral motion at any cross section is taken as the percentage ratio of the mean kinetic energy of the lateral motion to the total kinetic energy of flow, and is called S_{xy}:

$$S_{xy} = \frac{\left(\frac{V^2_{xy}}{2\,g}\right)_m}{\left(\frac{V^2}{2\,g}\right)_m} 100 \quad \ldots\ldots (10)$$

In the case of a flow with streamlines parallel to the axis of the conduit, $S_{xy} = 0$. The magnitude of S_{xy} is determined at any cross section from the contour lines of the velocity vector v and the forward-velocity components v_z, Eqs. 9.

TABLE 1.—MAGNITUDES OF CONSTANT PARAMETERS

Case	R_A	r_e/b	y_A/b	$\theta/180°$
R_A.......	..	1.0	1.0	0.5
r_e/b.......	73,500	..	1.0	0.5
y_A/b.......	68,500	1.0	..	0.5
$\theta/180°$....	73,500	1.0	1.0	..

The strength of the spiral motion was studied for each of the four parameters $\mathbf{R}_A$, r_c/b, y_A/b, and $\theta/180°$. (Subscript A in y_A and $\mathbf{R}_A$ refers to conditions at Station A, Fig. 6(a)). A particular section was chosen for comparing the results of all the bends, section 11, Fig. 6. Four groups of tests were run. In each group, three parameters were kept at constant values while the fourth parameter was given at least four different values. The v-contours and v_s-contours were plotted for each case. Fig. 10 shows the v-contours and the v_s-contours for only two values of each of the parameters $\mathbf{R}_A$ and r_c/b. Fig. 11 gives the four parametric functions of S_{xy} plotted on the same vertical axis but with a horizontal axis for each of the four parameters. The magnitudes of the three constant parameters for each case are shown in Table 1.

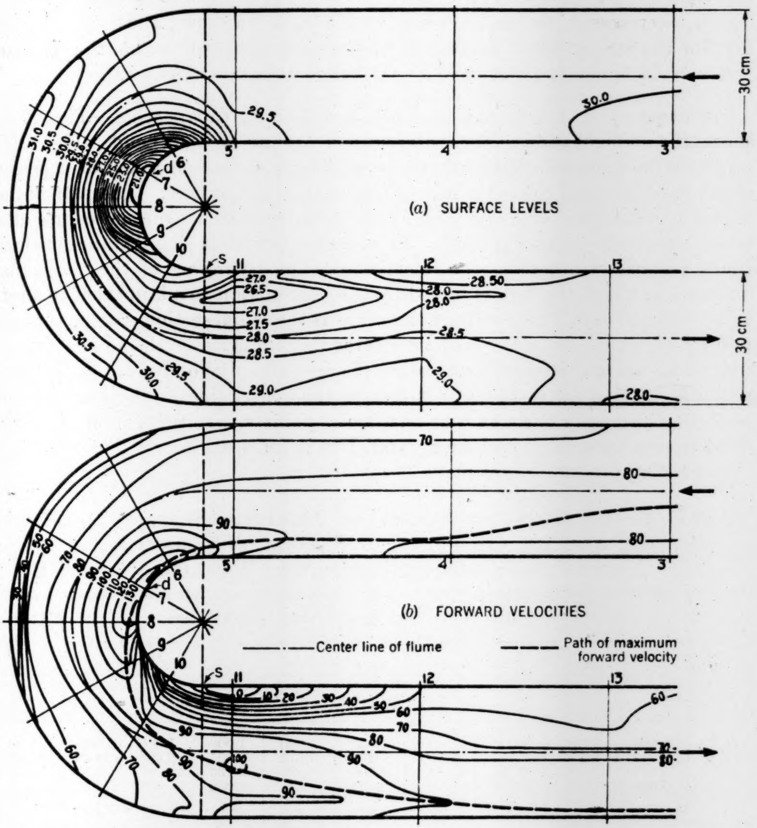


FIG. 12.—CONTOUR LINES OF EQUAL SURFACE LEVELS—IN CENTIMETERS (=0.3937 IN.) ABOVE THE HORIZONTAL BED OF THE FLUME—AND FORWARD-VELOCITY DISTRIBUTION IN AN 180° BEND—IN CENTIMETERS PER SECOND (=1.9685 FT PER MIN)

($r_c/b = 1.00$; $y_A/b = 1.00$; and, with $V_A = 77.8$ Cm per Sec (=152.95 Ft per Min), $\mathbf{R}_A = 73{,}500$)

From Fig. 11 the following is noted:

(*a*) At low values of $\mathbf{R}_A$, the corresponding S_{xy}-values are comparatively high (by increasing $\mathbf{R}_A$, the gradient of the curve decreases);
(*b*) S_{xy} does not decrease appreciably in value beyond the ratio $r_c/b = 3.0$;
(*c*) S_{xy} decreases as the depth-breadth ratio y_A/b increases; and
(*d*) For the range $\theta/180° = 0$ (straight channel) to $\theta/180° = 0.5$, the increase in S_{xy} is nearly twice that for the range $\theta/180° = 0.5$ to 1.0.

Behavior of Water Surface and Forward-Velocity Components.—Fig. 12(*a*) shows the configuration of the water surface around an 180° bend. Fig. 12(*b*) gives the forward-velocity contours at a horizontal section taken at a height above the bed equal to four tenths of the mean depth of flow. The path of the thread of maximum velocity, Fig. 12(*b*), deviates from its normal course at a section upstream from the bend. This section (which is at a distance of three times the bed width from the inlet) marks the beginning of the "adjusting process" in which the flow is forced by the bend action to adapt itself to a certain distribution. This distribution occurs at the radial section passing through point d at which the water surface attains its minimum level and the path of maximum velocity almost touches the inside wall. Beyond point d, the path gradually moves outward until it crosses the center line of the flume at the section that passes through the separation point s. It then approaches the downstream outer wall and runs parallel to it for a distance 8 *b* in which the "readjusting process" takes place.

TABLE 2.—LOCATIONS OF POINTS OF MAXIMUM SURFACE DEPRESSION (POINT d, FIG. 12) AND POINTS OF SEPARATION (POINTS s, FIG. 12) FOR VARYING PARAMETERS (THE PARAMETER $y/b = 1.00$, A CONSTANT)

Parameter	R = 10,500		R = 21,000		R = 31,500		R = 52,500		R = 73,500	
	θ_d/θ	θ_s/θ	θ_d/θ	θ_s/θ	θ_d/θ	θ_s/θ	θ_d/θ	θ_s/θ	θ_d/θ	θ_s/θ
r_c/b	(*a*) $\theta/180° = 0.50$, A CONSTANT									
0.50	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.00	0.111	0.250	0.189	0.500	0.250	0.611	0.306	0.889	0.667	1.000
2.00	0.033	..[a]	0.067	..[a]	0.100	..[a]	0.133	..[a]	0.333	..[a]
3.00	0.000	..[a]	0.000	..[a]	0.044	..[a]	0.056	..[a]	0.167	..[a]
$\theta/180°$	(*b*) $r_c/b = 1.00$, A CONSTANT									
0.25	0.111	0.222	0.333	0.667	0.333	0.667	0.400	0.710	0.556	0.889
0.50	0.111	0.250	0.189	0.500	0.250	0.611	0.306	0.889	0.667	1.000
0.75	0.148	0.333	0.185	0.445	0.296	0.556	0.371	0.926	0.445	1.000
1.00	0.056	0.778	0.111	0.778	0.167	0.805	0.278	0.833	0.333	1.000

[a] No separation.

The point of maximum surface depression d and the point of separation s were located for different bends under widely different conditions of flow, as cited in Table 2. The positions of the two points are given in terms of the ratios of their angles θ_d and θ_s, measured from the inlet of the bend to the total angle θ of the bend.

The ratios θ_d/θ and θ_s/θ were found to be only slightly affected by varying the parameter y/b. On the other hand, they decrease very quickly as the values of r_c/b increase. At $r_c/b = 3.0$, point d almost lies at the beginning of the bend and no separation in the flow occurs.

Fig. 12 shows that both the contour lines of the water surface and the forward velocity are more or less parallel to the walls of the bend in the vicinity of the radial section which passes through point d, indicating the existence of a functional relationship between the radius of curvature, the depth of flow, and the forward velocity. Investigations on different bends showed that the flow conditions at this radial section are very similar to those in a free or irrotational vortex, as long as the depth of flow at point d is greater than the critical depth of the streaming state. At radial sections on either side of point d, the similarity diminishes.

This characteristic of flow at section d may be applied to determine the forward-velocity distribution and surface profile curves. For example, in a free-vortex motion,

$$v_z = \frac{C}{r} \qquad (11a)$$

and

$$y = H - \frac{v^2_z}{2\,g} \qquad (11b)$$

also,

$$V_z = \frac{\int_{r_i}^{r_o} \frac{C}{r}\,dr}{r_o - r_i} = \frac{C}{r_o - r_i} \log_e \frac{r_o}{r_i} \qquad (12a)$$

$$y_m = \frac{\int_{r_i}^{r_o} y\,dr}{r_o - r_i} = \frac{\int_{r_i}^{r_o} \left(H - \frac{C^2}{2\,g\,r^2} \right) dr}{r_o - r_i} = H - \frac{C^2}{2\,g\,r_o\,r_i} \qquad (12b)$$

and

$$Q = V_z\,y_m\,(r_o - r_i) = C \log_e \frac{r_o}{r_i} \left(H - \frac{C^2}{2\,g\,r_o\,r_i} \right) \qquad (12c)$$

In Eqs. 12, r_i and r_o are the inner and outer radii of the bend, respectively; C is the circulation constant in a free-vortex motion; and H is the elevation of the specific energy line at any section.

If Q and H are known, the constant C can be determined from Eq. 12c. The velocity and the depth at any radius r are then obtained according to Eqs. 11. The superelevation Δh of the water surface will then be given by

$$\Delta h = \frac{C^2}{2\,g\,r^2_o\,r^2_i} (r^2_o - r^2_i) \qquad (13)$$

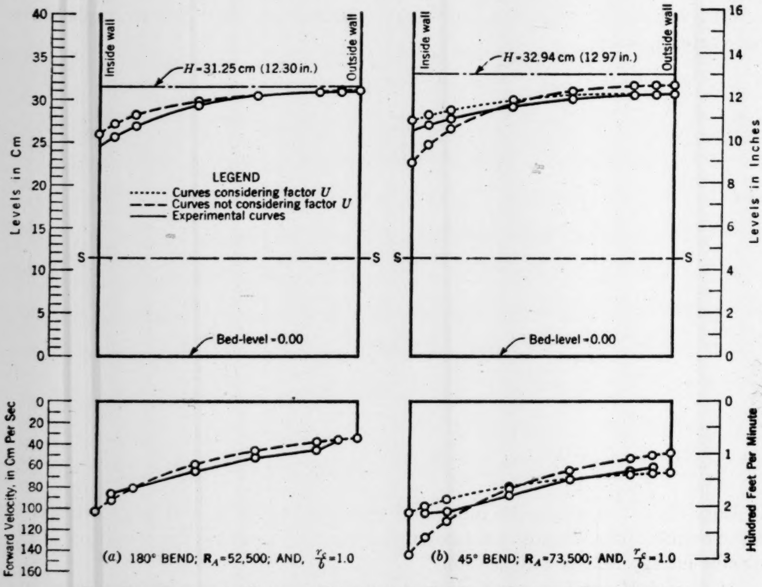
For the practical application of Eqs. 11a, 12c, and 13 to any bend, Table 2 is used to give the position of point d. The height of the specific energy line is first determined for any section A in the straight approach channel (far removed from the inlet of the bend) from the formula:

$$H_A = y_A + \alpha \frac{V^2{}_s A}{2\,g} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

in which the velocity head corrective factor α may be assumed equal to unity without appreciable error. The elevation H is then obtained by subtracting from H_A the energy lost in normal friction and the additional energy lost from bend resistance up to the radial section passing through point d. If ρ represents the ratio of the latter loss to its total value in the bend, and L is the distance between the two sections, then,

$$H = H_A - L\,h'_f - \rho\,h_b \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (15)$$

in which h'_f is the energy lost in friction per unit length of a straight channel and may be either determined experimentally or calculated from the Manning formula, and h_b is the energy lost due to bend resistance which may be obtained from Fig. 16 as explained subsequently. The coefficient ρ was found to be more or less constant for any bend and to have a value of 0.4.



(a) 180° BEND; R_A=52,500; AND, $\frac{r_c}{b}$=1.0

(b) 45° BEND; R_A=73,500; AND, $\frac{r_c}{b}$=1.0

FIG. 13.—CURVES OF SURFACE PROFILE AND FORWARD-VELOCITY DISTRIBUTION

The predetermined curves of the forward-velocity distribution and surface profile for any bend (obtained by assuming the existence of a theoretical free vortex) were found to be reasonably accurate as long as the angle of the bend was greater than 90°. The theoretical and experimental curves were obtained for several bends under different conditions of flow. An 180° bend is taken as an example and the pertinent curves are shown in Fig. 13(*a*) in broken and full lines, respectively. The experimental curves of the forward-velocity distribution are for the horizontal section S–S.

When the angle of bend is less than 90°, the differences between the theoretical and the experimental values gradually increase. These differences, however, could be reduced by introducing in the free-vortex formula a coefficient U (which is a function of the angle of bend) so that the formula becomes

$$v_z = U \frac{C}{r} \quad \text{(16a)}$$

The coefficient U is obviously also a function of the radius r, since v_z must be independent of r as the angle of the bend becomes zero (straight channel). Assuming that U varies linearly between $\theta = 0$ and $\theta = 90°$, and that its values at these two extremities are $\frac{r}{C\,V_{zs}}$ and 1.0, respectively (V_{zs} being the mean forward velocity in a straight channel), then, for any angle θ, the coefficient U will be given by

$$U = \frac{r}{C} V_{zs} + \left(1.0 - \frac{r}{C} V_{zs}\right) \frac{\theta}{90°} \quad \text{(16b)}$$

The experimental curves of the surface profile and forward-velocity distribution are compared in Fig. 13(*b*) with the theoretical curves as calculated with and without the introduction of the coefficient U.

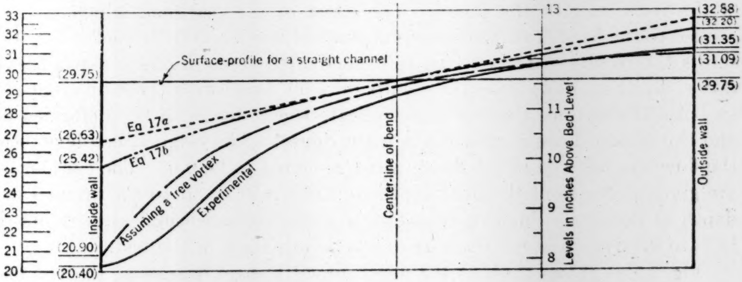


FIG. 14.—COMPARISON BETWEEN ACTUAL AND THEORETICAL SHAPES OF THE WATER-SURFACE PROFILE IN AN 180° BEND ($r_e/b = 100$; $y_A/b = 1.00$; AND $R_A = 73{,}500$)

Fig. 14 shows the actual water-surface profile in an 180° bend as compared with the theoretical profiles obtained by the foregoing assumption of a free vortex, and by the application of each of the following two formulas which are generally used in practice—

$$\Delta h = \frac{V_z^2\, b}{g\, r_e} \quad \text{(17a)}$$

and

$$\Delta h = \frac{V_z^2}{g} \log_e \frac{r}{r_i} \quad \text{(17b)}$$

In Eq. 17a, the surface profile is a straight line,[18] whereas in Eq. 17b (derived by F. Grashof[19]) it is a logarithmic curve. In both formulas the forward velocity is assumed constant throughout the entire cross section. In spite of the fact that the superelevation is attributed to the centrifugal force, the effect of this force on the velocity distribution, however, is not taken into consideration in either formula.

Effect of the Bend Action on Movable Beds.—When the bed of a curved channel is of an erodible material, a stable configuration in that bed is developed by virtue of the centrifugal spiral motion. It is noticeable in Fig. 8 that the strongest lateral currents occur near the outside wall of the midsection of the bend (section 8), with a downward movement. Beyond this section, the currents move diagonally along the bed toward the inside wall until they become close to it near the exit of the bend, section 11, with an upward inclination. These currents are not uniformly deflected toward the inside wall, however, but are grouped in separate zones which develop separate scour regions (pools), as shown in Fig. 15. The scoured material from these regions crosses the bend diagonally toward the inside wall near the exit of the bend where it settles because of the upward inclination of the bed currents. A sand bar is thus formed close to the inside wall at the downstream straight channel.

The different effects of the four parameters $\mathbf{R}_A$, r_c/b, y_A/b, and $\theta/180°$ were studied in four groups of tests. The bed level of the bend as well as that of an adjacent downstream section was lowered below the general bed level of the flume. The depression was then filled with sand of uniform grain, 1 mm to 2 mm in diameter. The surface of the sand was then given the same level as that of the remainder of the flume. The effect of the bed load which might be carried to the bend from the upstream channel was thus eliminated. Fig. 15(a) shows the results of tests on an 180° bend with three different depth-breadth ratios, and Fig. 15(b) gives the results on a 90° bend with three different radii. In both illustrations the full lines indicate the contours of the deformed bed, and the broken lines represent the main ridges. The original bed level before the flow was allowed in the flume was considered to be 0.0. The contour lines are given in terms of the final depths of deposit or scour to the original mean depth of flow y_A. Such representation produces dimensionless contours that help to illustrate the relative size of the deformation in the bed.

Fig. 15(a) shows that, as the depth-breadth ratio decreases, the volume of the sand bar at the inside wall of the downstream straight channel increases and the scour at the outside wall becomes deeper. In natural waterways, which generally have low depth-breadth ratios, this phenomenon might lead to a change in the watercourse downstream from the bend if the sides were also of an erodible material.

Fig. 15(b) demonstrates that an increase in r_c/b causes a considerable reduction in the size of bed deformation. For the present series of experiments no deformation occurred at $r_s/b = 3.0$.

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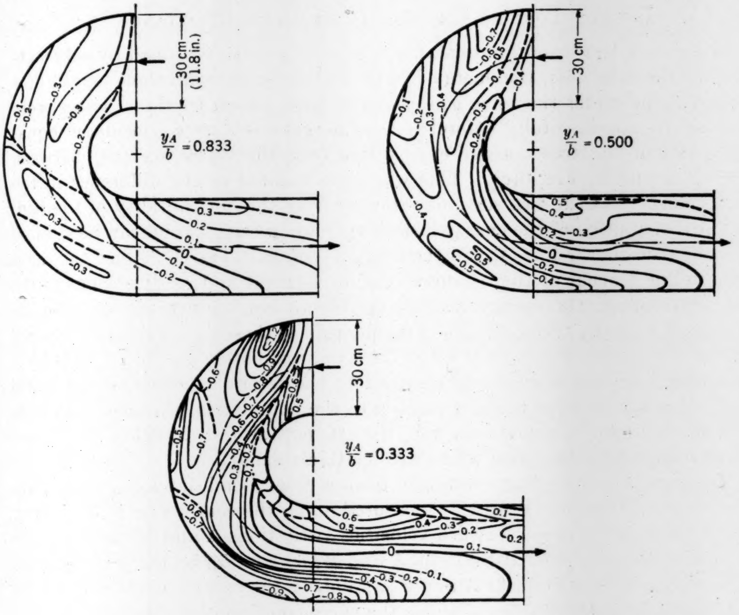
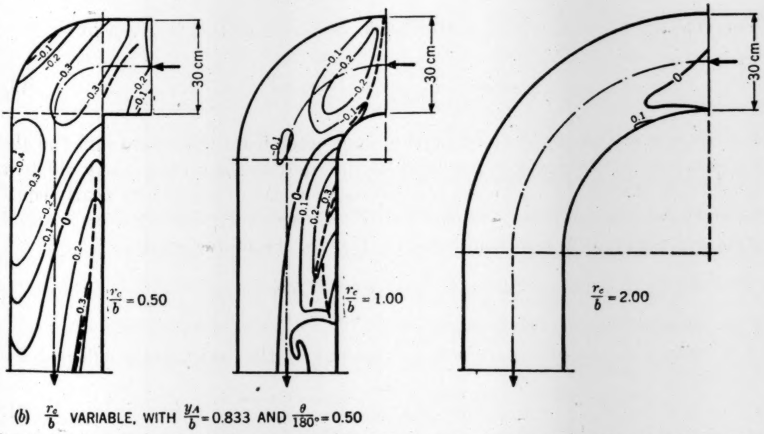
[18] "Hydraulics of Steady Flow in Open Channels," by S. M. Woodward and C. J. Posey, John Wiley & Sons, Inc., New York, N. Y., 1941, p. 112.

[19] "Hydraulic Structures," by A. Schoklitsch, A.S.M.E., 1937, Vol. I, p. 151.


30 cm (11.8 in.)
$\frac{y_A}{b}$ = 0.833
-0.1
-0.2
-0.3
0.3
0.2
0.1
0
30 cm
$\frac{y_A}{b}$ = 0.500
-0.4
-0.5
0.5
0.4
30 cm
$\frac{y_A}{b}$ = 0.333
-0.6
-0.7
-0.8
-0.9
0.6
0.5

(a) $\frac{y_A}{b}$ VARIABLE, WITH $\frac{r_c}{b} = 1.00$ AND $\frac{\theta}{180°} = 1.00$


30 cm
$\frac{r_c}{b}$ = 0.50
$\frac{r_c}{b}$ = 1.00
$\frac{r_c}{b}$ = 2.00
-0.1
-0.2
-0.3
-0.4
0
0.1
0.2
0.3

(b) $\frac{r_c}{b}$ VARIABLE, WITH $\frac{y_A}{b} = 0.833$ AND $\frac{\theta}{180°} = 0.50$

FIG. 15.—EFFECT OF THE BEND ACTION ON THE DEFORMATION OF MOVABLE BEDS (CONTOUR LINES ARE INDICATED BY THE RATIOS OF THE FINAL DEPTH OF DEPOSIT OR SCOUR TO THE ORIGINAL MEAN DEPTH OF FLOW)

(V_A = 40 Cm per Sec (=78.74 Ft per Min))

Energy Lost as the Result of Bend Resistance

The energy lost from the resistance of a bend, h_b, was obtained by subtracting from the total loss, the energy lost in friction in an equivalent length of a straight flume under the same conditions of flow. Such frictional losses were determined experimentally beforehand. Energy losses were considered equal to the fall of the true specific energy line (not the water surface) between Stations A and B, Fig. 6(*a*). This was done because of the difference in the mean velocities at the two stations resulting from the horizontality of the bed.

The distribution of the energy lost along a flume from a bend can be studied from the true energy line, 4, and the friction line, 5, in Fig. 9. Since the main drops in line 4 occur at the regions of the adjustment and readjustment of the flow distribution, the energy loss chargeable to bend resistance may be attributed principally to the change of momentum at these two regions. In other parts of the curved flume, lines 4 and 5 are approximately parallel. This indicates that both the energy loss ascribed to the lateral currents of the spiral motion and the increase in shear stresses at the boundaries (the latter resulting from the increase in the lateral velocity gradient as the thread of maximum velocity approaches the side walls) are relatively small.

To determine the energy resulting from a given bend, let L_{A-B} be the distance between Stations A and B, measured along the center line of the flume; V_{zA} and V_{zB} be the mean forward velocities at Stations A and B; y_A and y_B be the mean depths of flow at Stations A and B; α_A and α_B be the velocity head corrective factors at Stations A and B; and h'_f be the energy lost in friction per unit length in a straight flume under the same conditions of flow.

The energy loss chargeable to the bend will then be given by

$$h_b = \left(y_A + \alpha_A \frac{V^2_{zA}}{2\,g} \right) - \left(y_B + \alpha_B \frac{V^2_{zB}}{2\,g} \right) - (L_{A-B}\, h'_f) \ldots\ldots (18)$$

The velocity head corrective factor α for any cross section is given by

$$\alpha = \frac{1}{V^3_z\, A} \int_0^A v^2\, v_z\, dA \ldots\ldots\ldots\ldots\ldots\ldots\ldots (19)$$

Eq. 19 was solved graphically by the planimeter from the v-contours and the v_z-contours at both Stations A and B for different bends under widely different conditions of flow. In all cases α_A and α_B were each found to be nearly equal to unity, with a maximum value of 1.025. This showed that Stations A and B were not affected by the bend action. Eq. 18, therefore, becomes

$$h_b = (y_A - y_B) - \left(\frac{V^2_{zB} - V^2_{zA}}{2\,g} \right) - (L_{A-B}\, h'_f) \ldots\ldots\ldots (20)$$

The results of the four parametric functions of the resistance coefficient ξ—in which

$$\xi = h_b \frac{2\,g}{V^2_{zm}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (21)$$

—are presented in Fig. 16. For the parameter **R**, the three families of curves

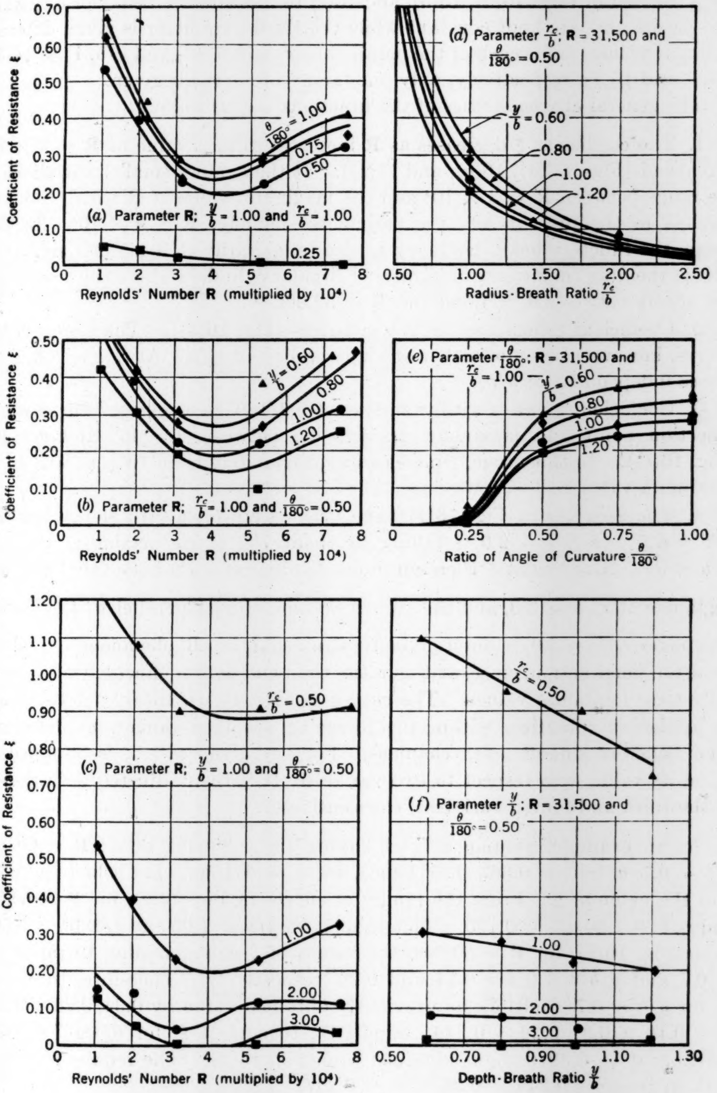


FIG. 16.—EXPERIMENTAL PARAMETRIC FUNCTIONS OF THE COEFFICIENT OF ENERGY RESULTING FROM BEND RESISTANCE (EQ. 19)

are presented in Figs. 16(*a*), 16(*b*), and 16(*c*) to indicate the behavior of ξ when two parameters are kept constant while the third parameter is given different constant values. For each of the parameters r_c/b, $\theta/180°$, and y/b, Figs. 16(*d*), 16(*e*), and 16(*f*), respectively, only one family of curves is given.

The general characteristics of the functions are, as follows:

1. The coefficient ξ decreases as **R** increases until a value of $\mathbf{R} = 3 \times 10^4$ is reached (Figs. 16(*a*), 16(*b*), and 16(*c*)). It then shows smaller variations in the range between $\mathbf{R} = 3 \times 10^4$ and the maximum value of **R** which could be reached in the present set of experiments—namely, $\mathbf{R} = 8 \times 10^4$. In this range, the function has a tendency to rise temporarily in a manner similar to that of the drag coefficients of immersed circular cylinders where the temporary rise occurs between $\mathbf{R} = 1{,}800$ and $\mathbf{R} = 32{,}000$.[20]

2. Coefficient ξ decreases as y/b increases (Fig. 16(*b*)). The slopes of the curves, however, become smaller at higher values of r_c/b. At $r_c/b = 3.0$, ξ becomes independent of y/b.

3. Coefficient ξ varies with the parameter $\theta/180°$ in a reversed curve whose inflection point and maximum gradient lie between $\theta = 45°$ and $\theta = 90°$ (Fig. 16(*e*)). In this range, ξ increases approximately from 10% to 80% of its maximum value at $\theta = 180°$.

4. The ξ-curves in Fig. 16(*f*) have very steep downward slopes between $r_c/b = 0.5$ and $r_c/b = 2.0$, ξ falling to about 7% of its maximum value at $r_c/b = 0.5$. Coefficient ξ then continues to diminish until its value becomes negligible at $r_c/b = 3.0$, and the curves become almost parallel to the $\frac{r_c}{b}$-axis. The ratio $r_c/b = 3.0$ is outstandingly significant in all phenomena resulting from the bend action, since, at this ratio, the bend action effects attain practically their minimum values. The curves in Figs. 16(*a*), 16(*b*), and 16(*c*) may be applied in practice, within the foregoing specified ranges, to determine ξ for bends of smooth open channels. For any given case, ξ is obtained by fixing its value with respect to two variables. It is then adjusted with respect to the third and fourth variables, consecutively.

As an example, assume a bend having the following data: $\mathbf{R} = 60{,}000$, $y/b = 0.7$, $\theta/180° = 0.667$ (120° bend), and $r_c/b = 1.50$. Let it be required to find the value of ξ. From the family of curves in Fig. 16(*c*), for $\mathbf{R} = 60{,}000$ and $r_c/b = 1.50$, $\xi'' = 0.190$. This value is for $y/b = 1.00$ and $\theta/180° = 0.500$. From Fig. 16(*b*), for $\mathbf{R} = 60{,}000$, the values of ξ corresponding to ratios $y/b = 0.7$ and $y/b = 1.0$ are 0.34 and 0.26, respectively. Therefore, to correct ξ'' for $y/b = 0.7$, multiply its previously obtained value by 0.34/0.26. Thus, $\xi' = 0.19 \times 0.34/0.26 = 0.249$. Similarly, to correct ξ' for $\theta/180° = 0.667$, multiply it by 0.30/0.26, as obtained from Fig. 16(*a*). The required value of ξ, thus, becomes $0.249 \times 0.30/0.26 = 0.287$.

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[20] "Applied Fluid Mechanics," by M. P. O'Brien and G. H. Hickox, McGraw-Hill Book Co., Inc., New York, N. Y., 1937, p. 186.

General Conclusions

(1) Spiral motion exists in straight flumes as well as in curved flumes. This fact is proved definitely by direct measurements taken by the pitot sphere. In a curved flume, the spiral motion induced by the centrifugal force is not uniform along the bend. There is a complicated pattern caused by the interference of the spiral motion (originating in the straight approach flume) with that produced by the bend. The strongest lateral currents appear close to the outer wall at the midsection of the bend, where they have a general direction toward the bed. Their direction and position gradually change through the second half of the bend until the exit is reached, where they approach the inside wall with upward inclination.

(2) The strength of the spiral motion S_{xy} in a bend (defined as the ratio of the kinetic energy of the lateral currents to the total kinetic energy of the flow at a particular section) decreases considerably by increasing either the Reynolds number **R** or the radius-breadth ratio r_c/b. At $r_c/b = 3.0$, S_{xy} attains practically its minimum value. The spiral motion S_{xy} also decreases (but in a lesser degree) as the depth-breadth ratio y/b increases. On the other hand, it increases as the deviation angle θ of the bend becomes greater. From $\theta = 0$ to $\theta = 90°$, the increase in S_{xy} is nearly double that of the range from $\theta = 90°$ to $\theta = 180°$. The kinetic energy of the lateral currents in a bend is relatively small, however, and, consequently, plays a minor part in producing the additional energy loss which is mainly the result of the change of momentum upstream and downstream from the bend.

(3) The coefficient ξ (of the energy lost due to bend resistance) varies considerably with each of the parameters **R**, r_c/b, y/b, and $\theta/180°$. This variation explains the enormous differences in the results if they are not properly correlated. Coefficient ξ varies with the parameter $\theta/180°$ in a reversed curve whose inflection point and maximum gradient lie between 45° and 90°. In this range, ξ increases approximately from 10% to 80% of its maximum value at 180°. Between ratios $r_c/b = 0.5$ and $r_c/b = 2.0$, ξ falls very rapidly. At $r_c/b = 3.0$, ξ attains practically its minimum value. Parametric families of curves are given here for practical use in determining ξ in smooth open bends.

(4) When the channel bed is formed of an erodible material, the bend action causes a stable configuration which diminishes considerably by increasing r_c/b or y/b. The practical minimum size of this configuration seems to be at $r_c/b = 3.0$.

(5) The forward-velocity distribution and the water-surface profile at the section passing through the point of maximum surface depression can be predetermined, with a reasonably good degree of accuracy, by the assumption of a theoretical free-vortex distribution. This assumption holds good as long as the depth of flow does not fall below the critical depth of the streaming state.

(6) For the proper proportioning of bends the ratio $r_c/b = 3.0$ is recommended as it gives the minimum radius at which all the bend action effects are negligible.

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APPENDIX. NOTATION

The following letter symbols adopted for use in this paper, and in its discussion, conform essentially with American Standard Letter Symbols for Hydraulics (ASA-Z10.2-1942) prepared by a Committee of the American Standards Association, with Society participation and approved by the Association in 1942.

A = area of the water cross section;
b = breadth of a water stream in a rectangular flume; width of the channel bed;
C = a constant; circulation constant in a free-vortex motion;
g = acceleration of gravity;
H = specific energy; elevation of the specific energy line at any section;
h = head, with subscripts generally denoting the point of measurement:
  h_a, h_b, etc. = manometric heads;
  h_b = energy head loss in bend from resistance only;
  h' = calibrated head;
  h'_f = energy head lost by friction per unit length of straight channel;
  Δh = difference in water levels at the outer and inner walls of the flume; superelevation of the water surface;
k = a dynamic coefficient, with subscript such as m denoting point of reference; k_δ = inclination factor;
L = length; distance between two sections;
n = roughness coefficient; n_l = a linear dimension of wall and bed roughness;
p = unit pressure:
  p_o = static pressure;
  p_s = pressure when pitot sphere is normal to the axis of the channel;

Q = quantity of flow; rate of discharge;
R = hydraulic radius;
$\mathbf{R}$ = Reynolds' number;
r = radius of bend; radius of curvature at any point:
  r_c = radius at the center of the stream;
  r_i = radius of the inner wall of the flume;
  r_o = radius of the outer wall of the flume;
S = strength of a spiral motion at any cross section, as S_{xy};
U = a coefficient that is a function of the angle of bend (Eq. 16*a*);
V = mean velocity in a given section; V_s = mean forward velocity;
v = velocity vector at any point (Fig. 1(*d*)):
  v_x = velocity component in the lateral direction;
  v_y = velocity component in the vertical direction;
  v_z = velocity component in the forward direction;
  v_{xy} = velocity vector projected on the plane of the cross section;
y = depth of flow; y_m = mean depth of flow;
α = velocity head correction factor;
β = a central vertical angle (Fig. 3) between the direction of the velocity vector and a given manometer, as indicated by subscript;
γ = weight of 1 cu ft of water; specific weight;
δ = angle of inclination of the velocity vector to the longitudinal axis of the channel;
θ = angle of deviation in a bend;
ν = kinematic viscosity;
ξ = bend coefficient of resistance; coefficient of energy loss from bend resistance;
ρ = ratio of energy lost from bend resistance to the total loss in the bends;
ϕ = a function; and
ψ = a horizontal angle of axial rotation (Fig. 1(*d*)) between the meridian plane and the plane of the velocity vector.

DISCUSSION

Gilbert W. Outland,[21] Assoc. M. ASCE.—In introducing Eqs. 17, the author states that they are generally used in practice. These equations are based upon free-vortex motion with the assumption that the velocity is constant across a radial section in the bend. In addition, it is also generally assumed in practice, when dealing with high velocities and low values of r_c/b, that the flow around a bend is based upon free-vortex motion with the specific energy constant across a radial section and with the pressure gradient or depth varying inversely as the radius of curvature. That these assumptions lead to Eqs. 11*a* and 13 will be shown.

The assumption that the pressure gradient or depth varies inversely as the radius of curvature may be written as

$$\frac{dp}{dr} = \frac{v_z^2\,\gamma}{g\,r} \qquad (22)$$

For the assumption of constant specific energy,

$$\frac{v_z^2}{2\,g} + \frac{p}{\gamma} = C \qquad (23)$$

Differentiating with respect to r,

$$\frac{v_z\,dv_z}{g\,dr} + \frac{dp}{\gamma\,dr} = 0 \qquad (24)$$

By substituting the value of $\frac{dp}{dr}$ from Eq. 22,

$$\frac{v_z\,dv_z}{g\,dr} + \frac{v_z^2}{g\,r} = 0 \qquad (25)$$

from which

$$\frac{dv_z}{v_z} + \frac{dr}{r} = 0 \qquad (26)$$

The solution of Eq. 26 is $v_z\,r = C$; or

$$v_z = \frac{C}{r} \qquad (27)$$

Eq. 27 is identical with Eq. 11*a*. It is seen therefore that the foregoing assumptions lead to the fact that the velocity varies inversely as the radius of curvature.

The value of v_z as given in Eq. 11*a* or Eq. 27, substituted into Eq. 22, yields

$$\frac{dp}{\gamma} = \frac{C^2\,dr}{g\,r^3} \qquad (28)$$

Also,

$$\int_{p_i}^{p_o} \frac{dp}{\gamma} = \frac{C^2}{g}\int_{r_i}^{r_o} \frac{dr}{r^3} \qquad (29)$$

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[21] Asst. Chf. Design Engr., Los Angeles County Flood Control Dist., Los Angeles, Calif.

from which

$$\frac{p_o - p_i}{\gamma} = \frac{C^2}{2\,g}\left(\frac{1}{r^2_i} - \frac{1}{r^2_o}\right) \qquad (30)$$

or

$$\Delta h = \frac{C^2}{2\,g\,r^2_o\,r^2_i}\,(r^2_o - r^2_i) \qquad (31)$$

which is identical with Eq. 13.

Since C is constant across any section, its value can be obtained from Eq. 11*a* or Eq. 27 provided that the velocity and its corresponding radius at some point are known. In practice these values are usually taken at the center line of the channel immediately upstream from the bend.

It is sometimes convenient to plot the water-surface curve using the specific energy grade line across a radial line as a base. Since the change in depth of water is equal to the change in velocity head, Eq. 28 may be rewritten as follows:

$$dh_v = \frac{C^2\,dr}{g\,r^3} \qquad (32)$$

from which

$$h_v = \frac{-\,C^2}{2\,g\,r^2} \qquad (33)$$

The position of the specific energy grade line may be determined from a known depth and velocity head, after which the values of h_v for various values of r may be plotted downward from the specific energy line.

Particular attention should be directed to paragraph (5) under the heading, "General Conclusions," in which it is stated that the theoretical assumptions hold only as long as the depth of flow does not fall below critical. It should be added that the theoretical assumptions will still be valid for a depth of flow below the critical, provided that suitable spiral or easement curves are placed in the channel.

EDWARD SILBERMAN[22] AND ALVIN G. ANDERSON,[23] ASSOC. MEMBERS, ASCE.—The Saint Anthony Falls Hydraulic Laboratory in Minneapolis, Minn., has made some studies of flow through radius bends, particularly in closed ducts. The studies have been both bibliographical[24] and experimental[25] and have been partly summarized in a bulletin of the laboratory.[26] The findings are in general agreement with those of the author. Actually, the character of the flow and the variation of spiral strength and bend loss with the flow parameters are much the same in closed and open bends.

There are two additional parameters that should be included in Eq. 2—(1) a parameter describing the flow approaching the bend and (2) a term of the Froude number type, $\frac{V^2}{g\,R}$, to introduce the effect of the free surface on the flow in an open channel. The late D. L. Yarnell,[27] M. ASCE, showed that the approaching flow (whether it be symmetrical or distorted toward the inside or outside of the bend) is quite important in determining head loss in closed bends. It is reasonable to assume that it is also important in open bends. Not only the velocity distribution but also the lateral currents in the approaching flow are important. An approaching spiral may either enhance or retard the normal development of a spiral in a bend. To obtain a truly quantitative understanding of the parameters studied in this paper, it is believed that a uniform, nonspiraling approach flow should have been provided even though such a flow may not exist naturally.

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[22] Asst. Prof., Saint Anthony Falls Hydr. Laboratory, Univ. of Minnesota, Minneapolis, Minn.

[23] Instr., Saint Anthony Falls Hydr. Laboratory, Univ. of Minnesota, Minneapolis, Minn.

[24] "Fluid Flow Diversion: A Summary and Bibliography of Literature," by Alvin G. Anderson, *Project Report No. 1*, Saint Anthony Falls Hydr. Laboratory, Univ. of Minnesota, Minneapolis, Minn., August, 1947.

[25] "The Nature of Flow in an Elbow," by Edward Silberman, *Project Report No. 5*, Saint Anthony Falls Hydr. Laboratory, Univ. of Minnesota, Minneapolis, Minn., December, 1947.

[26] "Hydraulics of Conduit Bends," by Alvin G. Anderson, *Bulletin No. 1*, Saint Anthony Falls Hydr. Laboratory, Univ. of Minnesota, Minneapolis, Minn., December, 1948.

Theoretically, the Froude number of the flow should be considered as a factor in the development of spiral currents. It seems unlikely that the character of the secondary currents would be similar, for a constant Reynolds number, if the Froude number were near the energy critical or if it were well below the critical or well above the critical. Most experiments on the flow in open channel bends have been conducted with the flow either definitely subcritical or definitely supercritical. To separate the influence of the Froude number from that of the Reynolds number would require experiments in channels of many different sizes, including those used in practice or a wide variation of the kinematic viscosity in a single channel. However, since the spiral currents are essentially a frictional phenomenon, there is reason to expect that for the Froude numbers well below the critical the development of the spiral currents would be more or less independent of the Froude number. In so far as the spiral currents can be considered independent of the Froude number when the flow is definitely subcritical, their development will be analogous to the development of spiral currents in bends in closed conduits. To this extent the results of the more numerous experiments on the flow in conduit bends will aid materially in understanding the similar flow phenomena in open channels.

The author has not discussed the physical factors producing the bend spiral, perhaps purposely. Neverthless, a knowledge of these factors is important for an understanding of the spiral flow. An explanation used by Mr. Silberman in 1947 is supported by data from closed bends,[25] but the data presented in Fig. 8 also fit the explanation if allowance is made for the approaching spiral. The spiral flow begins as a lateral boundary current near the point where streamline curvature begins and at the bottom, inside corner of the channel. The lateral current forms as a result of pressure differences created between the bulk of the fluid, which tends to be rapidly accelerated near the inside wall in accordance with a potential flow pattern, and the fluid near the bottom boundary which is retarded by wall friction. Initially, the lateral current is along the inside wall away from the bottom, and along the bottom toward the inside wall. These currents are recognizable in Fig. 8(*b*). Lateral currents are generated at the bottom, inside corner as long as the streamlines continue to curve with the bend. At first, the lateral currents are confined to a thin boundary layer and are very intense in that layer. In a 90° closed bend, with $r_e/b = 2$, the intense lateral currents were confined to a thickness of less than $b/12$ for the first 45° of the bend.[25]

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[27] "Flow of Water Through 6-Inch Pipe Bends," by D. L. Yarnell, *Technical Bulletin No. 577*, U.S.D.A., Washington, D. C., 1937, p. 12.

Downstream from the inception of curvature, the lateral currents spread along the bottom and up the inside wall by the requirements of continuity. Eventually, motion toward the inside can be detected along the entire bottom boundary; but along the inside wall motion toward the top is limited to a height equal to about two channel breadths, no matter how deep the water. These lateral boundary motions are diffused to the interior by viscous action so that the internal fluid is gradually given a rotary motion which, in combination with the through flow, produces the spiral. With uniform approach conditions, the lateral currents are always most intense near the inside wall where they are generated and least intense near the outside wall. A. Hinderks,[11] among others, calls attention to this characteristic.

The flow toward the surface near the inside wall finally results in a return flow to the interior near the surface and this, in turn, contributes toward a separation effect at the surface at the inside wall. Where the lateral currents have not yet penetrated, the flow is nearly potential. Frank L. Wattendorf[28] demonstrated the latter point in a very deep bend (large y_A/b) of 300° curvature; the tendency may also be seen in Figs. 8(*c*) and 8(*d*).

Under the foregoing assumed conditions of uniform, nonspiraling, approach flow, there is a single spiral flow in an open bend. In order that the spiral motion may become uniform, however, the bend curvature would have to continue farther than in any practical bend—probably through more than a full circle for most values of the parameters y_A/b and r_e/b.

The dip in the energy diagram near the beginning of the bend (Fig. 9, curve 4) may be explained by the initial formation of the lateral currents. At first, these currents are so close to the boundary that they are not included in the measurements, and there is an apparent loss of energy. As more lateral currents are generated along the bend, the original currents are diffused to the interior and are included in the measurements; thus, the apparent loss of energy is not so great. Actually, energy is being lost continually as the lateral currents are damped by viscosity. The energy content of the lateral currents near the boundary is quite large and a very large part of this energy is lost in viscous action. The energy loss from this cause is a major part of the bend loss, especially when the parameter r_e/b is not too small. The second dip in the energy diagram, beyond the bend, might be attributed to form loss (vortices resulting from separation of the flow). The magnitude of this loss should depend principally on the parameter r_e/b. (The author's statement (under the heading, "General Conclusions," paragraph (2)) to the effect that the additional energy loss in a bend is the result of change of momentum upstream and downstream from the bend is not understood by the writers.) If the ratio y_A/b is large enough so that the lateral currents occupy only a relatively small part of the cross section near the bottom and if form loss near the end of the bend is ignored, the bend loss scarcely exists as shown by Mr. Wattendorf's data.[28]

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[28] "A Study of the Effect of Curvature on Fully Developed Turbulent Flow," by Frank L. Wattendorf, *Proceedings*, Royal Soc. of London, Vol. 148, Series A, 1935, pp. 565–598.

The author's method of expressing spiral strength is novel and appears to be satisfactory as a basis of comparison, even though the total energy in the lateral currents is not being measured. One of the writers tried, unsuccessfully, to obtain a quantitative basis of comparison and appreciates the author's contribution toward this problem. When time permits, the author's method will be applied to the writer's data[25] in an attempt to correlate spiral strength and bend loss. (A complete study would have to consider the effect of separation loss as well as spiral strength.)

It is suggested that the data presented in Fig. 11 might have been extended to smaller values of the parameter y_A/b (which would have required a larger channel because of the Froude number effects). By comparison with closed bends (where a depth-breadth ratio of unity corresponds to $y_A/b = 0.5$ by symmetry) the spiral currents become very intense at smaller values of y_A/b. An idea of the spiral strength and associated energy loss at these smaller values would be useful. On the other hand, as y_A/b becomes large, the spiral strength should approach zero asymptotically.[28] The trends shown in Fig. 11 may be predicted qualitatively on the basis of the foregoing explanations of spiral origin. In addition to the effects shown, increasing roughness would be accompanied by more pronounced boundary retardation, stronger lateral currents, and a stronger spiral at the end of the bend.

It is believed that the spiral strengths shown in Fig. 11 and the bend energy loss values shown in Fig. 16 are both too small because of the spiral flow approaching the bend. Had the approaching spiral been in the opposite direction, the values would have been too large. The trends of the curves, however, are undoubtedly correct. Furthermore, the distance downstream from the bend at which the flow returned to its "pre-bend" characteristics has probably been reduced by the approaching spiral. Normally, a distance of twenty or more channel breadths should be required but only about eight were needed in the data represented by Fig. 9.

The author's experiments were performed in a channel with a single degree of roughness; consequently, the roughness parameter, $\mathbf{R}/n_l$, in Eq. 2 remains a constant and has no influence on the results. A discussion of the effect of relative roughness on the coefficient of resistance of a bend in a closed conduit will throw some light on the influence of relative roughness in the open channel bend.

K. Hilding Beij[29] has suggested that the relative roughness k/r for conduits might be estimated from

$$\frac{k}{r} = 280 f^3_s \qquad (34)$$

in which f_s is the friction coefficient of the straight pipe of the same character as the bend. A study[26] of the published literature indicates that, for the region where the coefficient of bend resistance was independent of the Reynolds number, the coefficient of bend resistance could be expressed as a linear function of the relative roughness for each value of relative radius. For those bends for which the value of f_s and the Reynolds number indicated that the bend was hydrodynamically smooth, the relative roughness was assumed to be zero. By this method the measured coefficient of bend resistance could be corrected and much of the scatter of points relating the coefficient to the relative radius was eliminated. For values of relative radius between 6 and 10, however, some anomalous results appeared which indicated that a factor other than relative roughness influenced the coefficient of bend resistance in this region. One might expect that for low Froude numbers the effect of roughness on the open bend resistance would be similar.

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[29] "Pressure Losses for Fluid Flow in 90° Pipe Bends," by K. Hilding Beij, *Journal of Research*, U. S. Bureau of Standards, Washington, D. C., Vol. 21, 1938, p. 1.

The author should be complimented on the use of the pitot sphere to delineate the complex flow pattern in a bend. However, it should be borne in mind that, when it is used near the boundary, it must be somewhat limited by wall effects. For measurements in this region the pitot cylinder might be more satisfactory for measuring the intense lateral currents. The latter instrument, together with yarns, dye bubbles, and wall coatings, was used in the 1947 work by Mr. Silberman.[25]

L. O. CROES.[30]—The characteristics of flow around bends in an open flume have been made measurably clearer by the data presented in this paper. In this connection, mention should properly be made of the studies by H. Wittmann and P. Böss as reported at the first meeting of the International Association for Hydraulic Structures Research, held in Berlin, October, 1937.[31] Working in the Technical University of Karlsruhe, Germany, they based their computations on the potential theory, and produced results comparable to those obtained by Mr. Shukry.

In Europe, the investigation of flow around bends dates from 1868, when L. Fargue applied empirical laws to the regulation of the Garonne River in France. Subsequently, in 1902, Mr. Fargue developed the theory supporting his laws.

In Holland, where the rivers are occupied by an unusually intensive shipping industry, the subject of Mr. Shukry's paper is particularly applicable to the design of groins to guide the flow in both the upper and lower reaches. However, the problem is extensive in the lower reaches of the river, where the reverse action of salt tidewater disturbs the normal flow around bends in the river channel,[32] because normal spiral motion is possible only in the part of a cross section that is filled with water of the same specific gravity.

AHMED SHUKRY,[33] ASSOC. M. ASCE.—All the discussions of the paper were interesting and appreciated by the writer. Mr. Outland states that Eqs. 17 are based upon free-vortex motion with the assumption that the velocity is constant across a radial section in the bend. In practice, unless these equations are applied only to very narrow bends, the writer believes that the pressure and velocity at any point cannot be dissociated. That the pressure gradient varies inversely as the radius of curvature of the streamline is a fact which was established by Leonhard Euler[9] in the following general form for irrotational, unsteady flow:

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[30] Civ. Engr., Rijkswaterstaat, The Hague, Holland.

[31] "Wasser und Geschiebebewegung in gekrümmten Fluszstrecken," by H. Wittmann and P. Böss, Julius Springer, Berlin, 1938.

[32] "Eenige Beschouwingen over De Waterbeweging in Beneden Rivieren," by J. J. Canter Cremers, *De Ingenieur*, September 17, 1921.

[33] Asst. Prof. of Irrig., Faculty of Eng., Farouk First Univ., Alexandria, Egypt.

$$\frac{\partial v_n}{\partial t} + \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial}{\partial n}(p + \gamma\, h) \dots\dots\dots\dots\dots (35)$$

Eq. 35 is given in the natural coordinate system, v representing the velocity of a fluid particle and v_n, the velocity component in direction n which is normal to the streamline at the origin of the coordinate system. Considering a steady flow and subtracting the quantity $\frac{\partial\left(\frac{v^2}{2}\right)}{\partial n}$ from both sides,

$$\frac{v^2}{r} - \frac{\partial\left(\frac{v^2}{2}\right)}{\partial n} = -\frac{1}{\rho}\frac{\partial}{\partial n}\left(\frac{\rho\, v^2}{2} + p + \gamma\, h\right) \dots\dots\dots (36a)$$

Assuming that the energy is constant across a radial section, then, $\frac{v^2}{r} = v\,\frac{\partial v}{\partial n}$ or,

$$\frac{\partial v}{v} = \frac{\partial n}{r} \dots\dots\dots\dots\dots\dots\dots (36b)$$

Eq. 36*b* expresses the general relationship between the velocities and the radii of curvature of the streamlines; but it does not give the free-vortex distribution of velocities and pressures except for the case in which the fluid moves in concentric layers around a common axis, where $\partial n = \partial r$ and

$$v\, r = C \dots\dots\dots\dots\dots\dots\dots\dots (37)$$

Eq. 37 is for a theoretical free-vortex motion. The magnitude of C is constant in the entire motion. In a bend, the circulatory motion being disturbed by the straight tangential channels, the value of C at the different radial sections across the bend should be variable. Therefore, Eq. 36*b* is generally used in the form:

$$v = C''\, e^{\int dn/r} \dots\dots\dots\dots\dots\dots\dots (38)$$

The results of the experiments reported in the paper justify the foregoing treatment. The similarity of the actual flow with the free-vortex motion diminishes by decreasing the angle of curvature of the bend. Moreover, this similarity exists only at the part of the curve that has the maximum superelevation. The value of the circulation constant C in Eqs. 11*a*, 27, or 37 cannot, therefore, be determined at the center line of the channel immediately upstream from the bend as stated by Mr. Outland. The writer appreciates the addition given by Mr. Outland to paragraph (5) of the "General Conclusions" that

> "* * * the theoretical assumptions will still be valid for a depth of flow below the critical, provided that suitable spiral or easement curves are placed in the channel."

The discussion by Messrs. Silberman and Anderson is constructive and mainly positive in its conclusions. The authors should be complimented for their exhaustive study of the problem. The writer agrees that the characteristics of flow around closed and open bends are much the same. The reason that the Froude number is ignored in the experiments is explained by Messrs. Silberman and Anderson:

> "* * * since the spiral currents are essentially a frictional phenomenon, there is reason to expect that for Froude numbers well below the critical the development of the spiral currents would be more or less independent of the Froude number."

The parameter describing the flow approaching the bend is really of a great importance especially for quantitative studies.

The explanation used by Mr. Silberman for the physical factors producing the spiral motion through a bend is interesting. For a uniform, nonspiraling, approach flow, there is a single spiral flow in an open bend. However, such a flow does not actually exist due to the effect of the wall friction which, when fully developed in a uniform, long, straight channel, would cause two opposite spiral motions on the sides of the center line.[16,17] On the other hand, since bed friction is the main cause of the spiral motion in the bend, the complicated pattern in the bend which results from the interference of the approaching spirals with the centrifugal spiral motion cannot generally be avoided. Changing the degree of the wall and bed roughness in the straight approach channel and in the bend might modify this pattern, but the small disturbance at the entrance of the experimental flume, which is an inevitable condition, will still give rise to a spiraling approach flow. In this connection it has to be stated that in a theoretical flow of an ideal fluid around a frictionless bend there will be no spiral motion, the lateral forces on both sides of the moving mass being completely in equilibrium along the full depth of bend.

The writer attributes the energy loss in a bend to the momentum changes in the adjusting and readjusting processes of the flow distribution upstream and downstream from the bend, respectively. This conclusion is obtained from results of the energy diagrams of different bends, an example of which is given in Fig. 9. In all these diagrams, lines 4 and 5, Fig. 9, are very nearly parallel, which indicates that no other losses (in addition to those of the normal friction) are taking place through the bend. Indeed, the explanation given by Messrs. Silberman and Anderson of the diffusion effect of the initial lateral currents illustrates the reason why the apparent loss of energy through the bend is not so great.

The writer is pleased to know from Mr. Croes that the subject of the paper has particular application to the design of groins in Holland. From the viewpoint of practical engineering, most of the bend effects are objectionable, for, in case of river regulation, water supply, irrigation canals, and high-head waterpower installations, additional energy losses due to bends in channels increase the total cost. On the other hand, engineers concerned with river and canal training require a clear understanding of what is taking place in the bend so that they may be properly guided in laying out their protective works. The paper clarifies only the different effects of a simple bend. Researches on open bends must continue to give fruitful results to the practical engineering profession.

In closing, the writer wishes to express his gratitude to all the discussers of the paper for their interesting and valuable contributions.

AMERICAN SOCIETY OF CIVIL ENGINEERS
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Paper No. 2412

MULTIPLE-PURPOSE RESERVOIRS
A SYMPOSIUM

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Note.—Published in March, 1949, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

INTRODUCTION

In September, 1945, a Joint Committee on Design and Operation of Multiple-Purpose Reservoirs was sponsored by the Hydraulics Division of the Society to study and report on the problems involved in the planning, design, and operation of multiple-purpose reservoir systems for optimum watershed development and utilization. The chairman of this joint committee is Raymond A. Hill, M. ASCE. Six Technical Divisions are represented on the committee; these with their representatives, all corporate members of the Society, are:

Division	Representative
Engineering Economics	Victor H. Cochrane
Hydraulics	Edward H. Sargent
Irrigation	W. G. Sloan
Power	C. N. Phillips
Structural	A. A. Brielmaier
Waterways	Emil P. Schuleen

(Mr. Cochrane died on September 25, 1948.)

It was very quickly found by the committee that there were no fixed principles upon which the design and operation of multiple-purpose reservoir systems could be based; nor was there even any agreement as to what constituted a multiple-purpose reservoir. Some engineers hold that any reservoir used or susceptible of use for more than one purpose should be classified as a multiple-purpose reservoir; others restrict the use of the term, at least for legislative purposes, to such major developments as those constructed by the federal government in the Tennessee Valley and those to be constructed in the Missouri Valley. In general, it has been the thought of the committee that the term "multiple-purpose reservoirs" should include all reservoirs actually designed and operated to serve more than one function and that it should exclude those whose design and operation are controlled by a single function, even though other benefits accrue as by-products.

So marked were the differences between the inchoate philosophies of those responsible for design and operation of multiple-purpose reservoirs that the committee felt it should obtain public expression and discussion of such divergent views. A number of papers were therefore solicited from representatives of most of the federal agencies concerned and from others affected by such projects. Ten of these papers were presented at the Ninety-fourth ASCE Annual Meeting; the remaining three were read at the ASCE Summer Convention in Duluth, Minn., the following summer (1947).

Papers Nos. 1 to 3 of this Symposium concern the development of the underlying principles of design and operation of multiple-purpose reservoirs by the Corps of Engineers, United States Army, by the United States Bureau of Reclamation, and by the Tennessee Valley Authority. Papers Nos. 4 to 7 are directed to the application of such principles and their effect on flood control, on navigation, on irrigation agriculture, and on electric power distribution.

Papers Nos. 8 to 11 describe the activities of certain federal agencies that influence materially the design and operation and evaluation of benefits of multiple-purpose systems. Paper No. 12 is a description of the relationships between the federal government and state governments. Paper No. 13 is a summary and review of the conflicting principles and views outlined in the other twelve.

This Symposium is tendered to the membership in order that there may be appreciation of the lack of agreement among the philosophies of the federal agencies primarily responsible for multiple-purpose reservoirs and of the conflicts between the several functions to be served by such reservoirs. It is the hope and expectation of the Joint Committee on Design and Operation of Multiple-Purpose Reservoirs that discussion of these papers will result in clarification of the issues involved and lead to a better understanding of the interrelated physical and economic factors which determine the optimum design of any multiple-purpose reservoir.

THEIR RELATION TO FLOOD CONTROL AND NAVIGATION

By Malcolm Elliott,[1] M. ASCE

Introduction

For practically all uses except flood control, reservoirs have as their main purpose retention of excess water for future systematic delivery for some beneficial use, generally over a prolonged period. For flood control, however, the main purpose of the reservoir is to keep surplus water out of the rivers during flood periods without regard to any future beneficial use of the water. The storage capacity having been used in decapitating one flood, the reservoir must be drawn down as soon as possible if security against future floods is to be assured. As long as this necessity exists, it is impracticable to schedule releases of the temporarily retained floodwaters over a prolonged period for future use because to do so would incur the risk of another flood arriving while the reservoir does not have capacity to retain it.

On the other hand, reservoirs which are depended on to furnish water for navigation schedule releases over a considerable period. The objective in such cases is not to dispose of the water expeditiously but to conserve it so that all deficiencies during the low-water navigation period will be supplied. Thus, there is a conflict in the use of a reservoir for flood control and the use of a reservoir for navigation. To understand the problem, the reader should first review the roles to be performed by reservoirs for each of the purposes to be considered.

Flood Control Reservoirs

The use of reservoirs as one of several methods of flood control is gaining in popularity and acceptance in the engineering profession. Arguments on levees versus reservoirs are no longer heard. Modern thought accepts the thesis that all methods of flood control—levees, channel enlargement and straightening, floodways, and reservoirs—are important in the solution of over-all flood problems.

Reservoirs are applicable to flood control, or at least are worthy of consideration, where:

1. Adequate reservoir capacity can be obtained;
2. Land or property values within the reservoir area are not excessive in proportion to the prospective benefits;
3. Suitable dam sites exist; and
4. Suitable sites are available, at localities where a substantial proportion of the flood runoff can be collected—if the reservoirs are too far upstream the bulk of the storm causing floods may occur downstream from the reservoir in which case the reservoir will be ineffective; and if they are too far downstream, important areas upstream will be left without protection.

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[1] Col., U. S. Army (*Retired*); Cons. Engr., Elliott and Porter, St. Louis, Mo.

Assuming that these conditions have been met adequately, the next important problem connected with flood control is the question of reservoir operation.

The complexities of this problem will probably never be fully comprehended until more experience has been obtained in actual operations. Unless a flood control reservoir has a large surplus capacity, the first concern of the operator should be to disperse the water as soon after it has been collected as possible. Floods can occur in the same basin in rapid succession. If a reservoir is filled by one flood and another flood comes along before the waters collected during the first flood are disposed of, the reservoir not only will not furnish protection from the second flood, but might make it worse. Furthermore, when there are reservoirs on several of the tributaries of a basin, unless the releases are carefully coordinated, there is the possibility that a combination of the releases might create a more serious flood in the main stream than nature could produce without reservoirs.

NAVIGATION RESERVOIRS

Reservoirs for navigation are of two different types: (1) Headwater or upstream reservoirs whose only function is to retain surplus waters during wet seasons for discharge into the navigable stream below during dry periods or seasons; and (2) reservoirs in the navigation channel itself which form navigable lakes or pools. Fort Peck Reservoir on the Missouri River in Montana and the headwater reservoirs in Minnesota which supply water for the Upper Mississippi River are examples of type (1). The Tennessee Valley Authority (TVA) reservoirs in the main stem of the Tennsesee River which constitute a continuous series of artificial lakes from the mouth to Knoxville, Tenn., are examples of type (2). These two types of navigation reservoirs may be conveniently designated as upstream reservoirs and channel reservoirs, respectively.

Upstream navigation reservoirs are filled with water during the season of surplus runoff, and the water is carefully conserved until the main stream needs supplementary water for the maintenance of navigable depths. Reservoirs are not generally a part of lock and dam navigation projects, because such improvements do not usually need any increase over the natural low-water flow of the river. Only when natural low-water flow is very small (probably less than 1,000 cu ft per sec), or where the waterway crosses a divide from one watershed to another, would water from reservoirs be needed for such projects.

Open-channel developments (that is, those in which navigable depths are obtained by regulation not involving any impoundment of the water in the channel), however, usually require a substantial minimum discharge of water if a navigable channel of suitable width and depth is to be maintained. If the natural discharge is insufficient for this purpose, the deficiency could be made up only by drawing on storage from reservoirs.

These reservoirs would be filled during the season of excessive runoff, of course, and their contents doled out during the low-water season in accordance with deficiencies in natural flow. No attention need be given to retaining enough vacant capacity to catch possible future floods, as is the case with flood control reservoirs.

Upstream navigation reservoirs should not be too far upstream from the navigable section of the river. In general, the closer the reservoir is to where the water is to be delivered to the channel the more efficiently will the water be used. To cite a specific example, Fort Peck Reservoir is not ideally situated for assisting navigation in the lower reaches of the Missouri River and the Mississippi River below the mouth of the Missouri, because of the distance the water must travel before it can be used. The dam is more than 400 miles above the head of the navigation project and 1,200 miles above the mouth of the river. A considerable part of the water released from Fort Peck is used up in valley storage before it reaches the navigable part of the river. Furthermore, in the case of the Mississippi River, the need for additional water must be anticipated by about 3 weeks to allow time for the water to travel from Fort Peck to the river. As a result, the supplementary water may not arrive soon enough; or, if rainfall should occur close to the Mississippi River during the 3-week lag period, the additional water already released might not be required. Both of these conditions tend to extravagance in the expenditure of stored water.

Channel reservoirs or pools vary in character from the low-lift navigation pools in the Ohio and Upper Mississippi rivers to the high-lift lakes in the Tennessee River. The low-lift pools of the Ohio and Mississippi rivers have no flood control significance and further discussion of these devices in a paper dealing with dual-purpose reservoirs is therefore unnecessary. In the case of the Tennessee River, the reservoirs themselves form the navigable waterway. As far as navigation on the Tennessee River is concerned, the reservoirs should contain only enough storage to reach the minimum level necessary to provide navigable depths throughout the length of the reservoirs. The actual storage capacity in these reservoirs over and above this minimum is utilized for flood control and power, and possibly could be diverted for the benefit of navigation in the Ohio and Mississippi rivers below the mouth of the Tennessee River. In regard to the latter possible use, the Tennessee River reservoirs would be classed as upstream reservoirs.

JOINT USE OF RESERVOIRS FOR NAVIGATION AND FLOOD CONTROL

Because of the conflict between uses for navigation and flood control, described previously, it will not generally be possible to build a reservoir primarily for one purpose and to expect that it will automatically and fully serve the other. Deliberate measures must be taken both in design and operation to make sure that both purposes will be served.

Instances may be found in which the aforementioned conflict does not occur. It is understood that in the case of TVA reservoirs the behavior and the time of occurrence of floods are believed to be sufficiently known and consistent enough to warrant filling the reservoirs toward the end of the usual season of excessive runoff and thereafter utilizing the flood control storage space for other purposes.

Thus, dual or multiple use of the same storage space is attained. This procedure would be safe in the Tennessee Valley or elsewhere only if it is certain that no flood can occur after the reservoirs have been filled.

It does not appear safe, however, to assume in every case (or even in the majority of cases) that the habits and the behavior of a river can be predicted with sufficient accuracy to insure that any particular flood is the last one of the season and that flood storage capacity can thereafter be dedicated to some other purpose. This view is consistent with the principles observed by the Corps of Engineers, Department of the Army, in the design of multiple-purpose reservoirs:[2]

> "In the case of a reservoir where storage is to be used jointly for flood control and for other purposes, it is essential that the storage capacity required for flood control be available when needed. Such a condition can exist only where floods always occur during a definite season of the year and where holding the required flood storage capacity empty during the flood season will not affect its availability for other purposes during the seasons when floods do not occur."

Any temptation to encroach on the capacity allocated to one purpose in favor of the other should be vigorously opposed. Floods unfortunately are soon forgotten. Lands may be cultivated, buildings erected, and transportation lines and industry located in the flood plains by people who either have forgotten the preceding major flood or have never experienced a major flood; and when a flood does come enormous damages occur in unprotected areas because of these encroachments. In the period between major floods—years, decades, or generations—it might be very difficult to retain flood storage capacities in reservoirs unused when it is plainly evident that the storage capacity might be used for assistance to navigation, the production of power, or other purposes. This problem is one of the most serious connected with the use of flood control reservoirs for other purposes. It is sufficiently serious to warrant examination and estimates in many, if not all, cases to determine whether the desired dual purposes cannot better and more economically, and reliably, be attained by separate reservoirs allocated to each purpose.

SINGLE-PURPOSE RESERVOIRS WITH INCIDENTAL COLLATERAL PURPOSES

The discussion up to this point has referred to truly dual-purpose reservoirs—that is, to reservoirs that are intended to serve both purposes without emphasis on either one to the disadvantage of the other. There is another type of reservoir development, however, in which the primary purpose might be either flood control or navigation with incidental benefits to the secondary purpose. A reservoir designed solely for the purpose of conserving water for future delivery to a navigable stream will automatically reduce many of the peak flood discharges. Even though such a reservoir may not always produce such incidental benefits—and might even make some floods worse—it is believed that a reservoir built for the primary purpose of furnishing water for navigation, in many instances, will automatically furnish a measure of flood control benefits. Also, single-purpose flood control reservoirs may produce incidental navigation benefits. The limitations on such use of single-purpose reservoirs should be thoroughly known, however, so that the public will not expect too much in the way of incidental benefits. It is particularly important that the public not be assured flood control benefits from a reservoir unless the storage capacity for that purpose is kept available.

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[2] "Orders and Regulations," Corps of Engrs., U. S. Army, Washington, D. C., May 1, 1940 (revised to February 5, 1945), paragraph 283.19(*b*).

CONCLUSIONS

Conclusions from the foregoing comments may be stated in four parts:

(1) Dual use of reservoirs for flood control and navigation is practicable under the following conditions—

a. When there are suitable reservoir sites far enough upstream to furnish flood protection to valley land, far enough downstream to control the desired proportion of the valley runoff, and near enough to the navigable waterway so that the water can be economically and promptly delivered thereto; and

b. When there is sufficient storage capacity to serve both purposes.

(2) Unless floods always occur during a definite season of the year, it is unsafe to allocate the same storage space to both navigation and flood control; separate storage zones should be established for each purpose.

(3) Although single-purpose navigation reservoirs may incidentally give some measure of flood control, and vice versa, they cannot be expected to serve both purposes fully. If dual-purpose reservoirs are essential, they must be deliberately provided for in the designs.

(4) Storage space for flood control may remain vacant and unused for many years. The temptation to use it for other purposes may be very strong, but it cannot be so diverted and still remain available for protection against floods.

DEVELOPMENT OF POLICY BY THE BUREAU OF RECLAMATION

By E. B. Debler,[3] M. ASCE

Introduction

In 1902, upon passage of the Reclamation Act, the Reclamation Service was organized as a branch of the United States Geologic Survey and so continued until 1904, when the Reclamation Service was established as an independent branch of the United States Department of the Interior. The name of the agency was changed to Bureau of Reclamation in 1923. The operations of the bureau have been guided by the Reclamation Act of 1902, together with numerous acts amending or supplementing it, of which the Reclamation Project Act of 1939 is probably the most important.

Irrigation

The Reclamation Act of 1902, comprised of ten short sections (of which one has since been repealed), was a relatively simple piece of legislation aimed only to bring federal aid into irrigation development which had been stalled by a combination of poor engineering, inadequate financing, and failure to recognize the problems of converting sagebrush to a going farm. That act nowhere makes reference to any use or control of water other than for irrigation. It contained one important provision, however, and that is the authorization for the withdrawal of public lands from entry for reservoir purposes. Since the western states in which the Bureau was authorized to operate by the Reclamation Act of 1902 still contain much public land, the power of withdrawal has been very useful in setting aside and holding for present and future use a large number of reservoir sites which are or will be useful for power production, flood control, and other purposes, as well as for irrigation.

The paramount purpose of irrigation has been reiterated in public statements of policy by the Department, the Bureau, and their responsible officials, time and again. Supporting legislation is found in such acts as those of April 16, 1906, February 24, 1911, and September 18, 1922. The Boulder Canyon Project Act of December 21, 1928, placed river regulation, improvement of navigation, and flood control ahead of irrigation on that project; but the irrigation purpose was tied to the compact which, with the approval of Congress, states that navigation has ceased to exist. The Central Valley Project in California has a like authorization by the Act of August 28, 1937, but there navigation has a real importance. When the maximum ultimate development of irrigation was threatened by navigation planning and authorizations, the Flood Control Act of December 22, 1944, and the companion River and Harbors Act of the following January effectively established the principle that irrigation is a paramount use throughout western United States, where it is essential to successful agriculture.

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The first instance of the recognition of power in Bureau legislation appears in the Act of April 16, 1906, Section V of which authorizes the secretary, where power is necessary for the irrigation of lands, to sell the surplus power produced, for periods not exceeding 10 years. Preference in such sales is given for municipal purposes; and the income is to be credited to the project from which the power is derived. That act and subsequent acts relating to power provide that the production of power must not impair the irrigation efficiency of the project.

In 1910 the enabling acts for the states of New Mexico and Arizona provided for reservation of lands actually or prospectively valuable for the development of water power and transmission of power, and later in the same year Congress authorized the withdrawal of lands generally for water power sites.

On February 24, 1911, Congress amended Section V of the original reclamation act, authorizing power development on all projects with power to be contracted for periods up to 10 years, except that on the Rio Grande Project authority was given for power contracts up to 50 years, and a similar authorization with respect to the length of the power contract was extended to the Salt River Project (Arizona) in the Act of September 18, 1922.

Nowhere in the Bureau legislation is power production accorded recognition as a function superior to the use of water for any other purpose. Section 8 of the Reclamation Act requires a procedure in conformity with state and territorial laws in establishing water rights as a result of which power rights have been established; and these power rights are in some cases in apparent conflict with subsequently established irrigation rights—yet the policy in power operations to date has been strictly one of noninterference with any irrigation.

Power

Although power development was not mentioned in the original reclamation act, and the transmission of power over material distances was still in its infancy in the earlier years of the twentieth century, it is noteworthy that, even before 1910, provision for future power development was made in the designs for practically every dam of real height, among these being the Shoshone (Wyoming), Arrowrock (Idaho), Elephant Butte (New Mexico), and Pathfinder (Wyoming) dams. Irrigation pumping required a power plant as at Minidoka Dam (Idaho).

All the earlier Bureau power plants were built either because they were required for irrigation pumping on Bureau projects, with their capacity wholly or largely so used, or because this power output was needed for project construction purposes. Among the first type were the Minidoka and Black Canyon plants in Idaho and the Kennewick plant in Washington. Among the second type were the Roosevelt (Arizona), Lahontan (Nevada), Boise (Idaho), Pilot Butte (Wyoming), Shoshone (Wyoming), and Lingle (Wyoming) plants, some of which, in anticipation of but a few years of operation, were not adequately planned for permanent use. Markets promptly opened up for the surplus output of all these plants.

Subsection I of the Fact Finder's Act (December 5, 1924) provided that profits from the operation of power projects be utilized in meeting the construction charges of the projects on which the power is produced, with the revenues ultimately to go to the irrigators themselves. This legislation expressed the policies that had been followed previously. However, during the succeeding few years there was a growing appreciation that such a policy was leading to inequalities in irrigators' obligations as between projects that did, and projects that did not, have power plants to assist in paying for the projects, and that furthermore the irrigators' resulting interest in power rates and power revenues tended to hold power rates at unduly high levels.

Following the passage of the 1924 Fact Finder's Act, it soon became evident that, because of the changing character of power developments, and in the interest of the best utilization of power production, power developments should be segregated from irrigation. In the late 1920's and the early 1930's this segregation was accomplished in a number of instances by special laws, and finally by the general legislation of the 1939 Reclamation Project Act which in Section IX provided for allocation of construction costs to power and the return of such costs to the United States from power revenues.

By 1920 the coming importance of power was clearly recognized and long-distance transmission was accepted as wholly feasible. Hydro power production became more attractive and Bureau river planning began to give due attention to the inclusion of power, with such joining of irrigation and power plans as to secure as much power ultimately as possible without material sacrifice in irrigation.

Flood Control

The projects initially selected for construction contained much public land and, on the whole, were located in areas where flood damage was relatively light (because of lack of widespread improvements) along the stream channels. Then, too, the settlement of the stream valleys was still new and there had been but little damage by floods. There was thus little incentive and no demand for flood control, as a result of which the reservoirs were designed, in the main, without provision for flood control; and the spillways were designed to pass floods with a minimum of cost.

There was one notable departure from this policy and that was on the Rio Grande Project where some 400,000 acre-ft of reservoir capacity was officially indicated to be available for flood control. The known floods of the preceding years, together with a very limited channel capacity below the reservoir, had already done much damage over the period of settlement of the valley below the reservoir—dating back fully 200 years.

Although construction of irrigation projects was started in 1904, no main stream reservoirs were completed before 1909. A considerable number was completed between 1909 and 1915. In the initial project operations the major interest of both the farmers and the project personnel was to assure a full reservoir; but it was not long until the people along the streams demanded that the constructed reservoirs be operated for flood control also, and the most vociferous among them usually had no obligation for repayment of costs.

The insistent demands for flood control operations at the important reservoirs led to developments. Agitation was begun for recognition by Congress of the need for providing more adequately for flood control by legislation which would relieve the irrigators from paying for flood control that would not benefit them. In Congress the feeling was that the reclamation program was already providing material subsidies and that desired arrangements providing flood control would be merely a further subsidy. Concurrently, it became a practice to adjust spillway plans so as to dampen flood peaks and to increase outlet capacities below the spillway level to the greatest extent practicable without materially adding to reservoir costs, thus accomplishing a measure of flood control.

The Boulder Canyon Project Act of December 21, 1928, was the first legislation to recognize flood control as a specific function of a reservoir but it failed to authorize an allocation of cost for that purpose, instead providing for the return of the entire cost, with interest, of the reservoir and power development from water and power revenues. On July 19, 1940, Congress designated $25,000,000 of the construction cost to be considered an allocation to flood control, repayable after 1987 without interest.

The Reclamation Project Act of August 4, 1939, in Section IX, laid down a general law applicable to all future projects with provisions for suitable nonreimbursable allocations to flood control, thus extending to projects of the Bureau of Reclamation the policy that had long been in force with projects of the Corps of Engineers—of federal responsibility for flood control. Later the Central Valley Project of California with its Shasta Reservoir and the Columbia Basin Project with its Grand Coulee reservoir, authorized by special laws prior to 1939, were brought under that general 1939 law.

The 1939 general law required consultation by the Bureau with the Corps of Engineers on flood control matters and this arrangement was strengthened by the Flood Control Act of December 22, 1944.

Navigation

In its operations, the Bureau first encountered navigation in an important degree on the Central Valley Project where the operations of the Shasta Reservoir are designed to assist navigation on the Sacramento River materially. In a lesser degree Marshall Ford Dam on the Colorado River in Texas and the Grand Coulee Reservoir were recognized as affecting navigation. The Reclamation Project Act of 1939 authorized suitable nonreimbursable allocations of costs for navigation in the same way as for flood control.

In more recent years, as Bureau activities have extended from the headwaters down onto the main streams, the need for consideration of navigation has increased and the stream operating plans, as well as the justified allocations for navigation, are receiving major consideration. In such work there must of necessity be close cooperation with the Corps of Engineers.

Silt Control

Prior to the settlement of the west most stream valleys were in substantial balance between the silt-carrying capacity of the streams and the silt loads brought to them. This balance was first disturbed by overgrazing with resultant increased erosion; later the disturbance was augmented by gullying along roads, railroads, and cattle trails. The southwestern streams then began building up their channels. The main stream irrigation reservoirs have deprived the streams of a goodly part of their silt-carrying capacity which rested largely in their flood flows. Irrigation of the river valleys has resulted in valuable lands and improvements, private and public, entitled to adequate flood control protection, which in turn will still further reduce the silt-carrying capacity of the stream. Some valleys which were substantially in balance before settlement are seriously endangered by rising river beds; the Middle Rio Grande Valley of New Mexico is probably one of the worst in this respect. With streams of relatively light silt loads, the impending silt menace becomes clear only as irrigation regulation enters its final stages. The North Platte Valley in Nebraska is of this type.

In a way, silt control is associated with flood control but suitable legislation giving complete recognition is needed in order that all situations requiring silt control can be assured of adequate attention without reliance on some other objective to meet the cost properly chargeable to silt control.

Fish and Wildlife

The early project plans and dam designs made no provisions for fish and wildlife. As a rule the reservoirs added measurably to the values of both—particularly so when power development assured the maintenance of adequate dead storage and thus avoided the complete drainage of reservoirs with consequent loss of fish life. Where such dead storage was not assured by power development, the justified criticism occasioned by the big loss of fish life when reservoirs were emptied soon led to the adoption of the policy with regard to design and operation that adequate pools would be retained even when the worst irrigation shortages existed. Reservoirs present major opportunities for an increase of fish and wildlife resources, but not until August 14, 1946, was legislation enacted enabling a nonreimbursable allocation of project costs for fish and wildlife benefits where such benefits exceed the damages by the same project. Aside from maintenance of dead storage pools, the initial demand by fish interests was for the fine screening of outlets. Further study resulted in discarding this requirement with deep outlets. Quite recently the Bureau has been requested in some instances to arrange for waters to be released from the surface instead of from deep outlets, to provide warmer waters and consequently better food for fish below the reservoirs. In other cases cool waters are wanted, and always there is a demand for elimination of extremely high and low discharges. More recently rearing ponds have been suggested around reservoirs to enable small fish to reach a size where they do not fall prey to the large fish and also to facilitate destruction of undesired species. Federal and state agencies are engaged in broad programs of needed studies to enable full realization of the opportunities for improvement of fishing values in reservoirs built by the Bureau of Reclamation. There is a prospect that such values, in some cases, will reach major proportions.

RECREATION

The recreational possibilities of reservoirs are only now beginning to be recognized, this delay being due to lack of access roads and lack of conveniences for use by the public around the shores. There are insistent and increasing demands for public acquisition and improvement of shore lands for public use. The recreational facilities so far provided are largely the result of work by the Civilian Conservation Corps (CCC) and the Works Progress Administration (WPA) camps. The Bureau is not authorized to make expenditures for this purpose and its part in the program has been to supply such work as could be done incidental to other authorized purposes. There is urgent need for legislation which will enable direct expenditures for the acquisition of shore lands for public use and for the improvement of recreational opportunities, such expenditures to be nonreimbursable by the users of water for power in recognition of their public uses. There is also need for authority to condemn land for such purposes, as shore line owners are reluctant to part with shore lands at reasonable prices.

SUMMARY

The Bureau of Reclamation was organized in 1902, pursuant to the Reclamation Act of 1902, with authority to construct irrigation projects. Irrigation alone was authorized by the original act. Possibly the authors of the Reclamation Act did not envision the multiple uses of reservoirs which are in effect and contemplated. The immediate need was the enactment of a law to give urgent assistance on irrigation. The Bureau took advantage of most of its opportunities in providing for power development, flood control, navigation, protection of fish, and recreational uses. As long as the entire construction costs had to be repaid by the users of water and power, expenditures for purposes other than the production of water and power of necessity had to be limited to negligible amounts. Special and general authorization was repeatedly, but often unsuccessfully, sought for adequate expenditures for other purposes; but there is always considerable opposition to nonreimbursable expenditures, and properly so. A further factor making for delays in such procedures is the relationship with other governmental agencies which have been assigned responsibilities in their respective fields, similar to those of the Bureau in the field of irrigation and power. The bureau recognizes the need for comprehensive multiple-use planning on any and all reservoirs yet to be built, and for adjustment in the operation of existing reservoirs to accomplish all purposes in so far as that may be attained without injury to vested rights. The accomplishment of these ends will require additional authorizing legislation and, where the circumstances so justify, nonreimbursable appropriations to the end that purchasers of water and power will not be required to pay for services provided for the general public.

GENERAL PROBLEMS OF DESIGN AND OPERATION

BY NICHOLLS W. BOWDEN,[4] M. ASCE

Beginning about 1928, the increasing recognition by the engineering profession of the advantages of multiple-purpose dam and reservoir projects for river control—particularly for irrigation, flood control, navigation, and power uses—resulted in the construction of many such structures in the United States and the proposed construction of many more in the near future. It is scarcely necessary to substantiate this statement in the light of projects completed during the period such as: (1) The Sacandaga Reservoir in the Hudson River Basin, a pioneer multiple-purpose project; (2) the development on the Colorado River consisting of Hoover, Parker, Imperial, and Davis dams built or building; (3) Grand Coulee and Bonneville dams on the Columbia River in Washington and Oregon; (4) the Central Valley Project in California; (5) the dams in the Allegheny and Monongahela rivers above Pittsburgh, Pa.; and (6) the Tennessee River system of multiple-purpose dams and reservoirs, shown in Fig. 1.

Previously (mostly since the turn of the century), the rivers of the United States had been developed for single purposes, in whole or in part by public funds, for navigation, irrigation, and flood control; and by private capital, for power. Some of these improvements involved large dam and reservoir projects.

The rapid shift to multiple-purpose projects presents a challenge to civil engineers in the planning, or general design, of the projects and, even more, in their operation. This is a new field of applied hydraulics, one largely unexplored and uncharted, and the engineer engaged in this field has had to depend largely on his own ingenuity in planning and in operating these projects. He should remedy this situation as soon as possible by giving to the profession the benefit of his experience. The planning of such reservoirs is so closely related to the operation that the two should not be separated. Because the size of a reservoir will more or less determine how it must be operated, the proposed method of operation should be critically considered during the planning stage. The primary operating purpose also should be clearly defined.

Multiple-purpose reservoirs should generally be larger than single-purpose reservoirs, how much larger depending on the uses to be served and economic considerations. The ideal size, often unattainable because of excessive flowage costs or topography, would afford ample allocation of storage space for each purpose. Ample storage space under one method of operation may be quite different from that under another. For example, the space required for flood control would be much larger if the operating method provided for storing all inflow during a flood or during a season instead of discharging part of that inflow. Also, in a large river basin where reservoir control obtains on all the principal tributaries, as well as on the main stream, thus affording the oppor-

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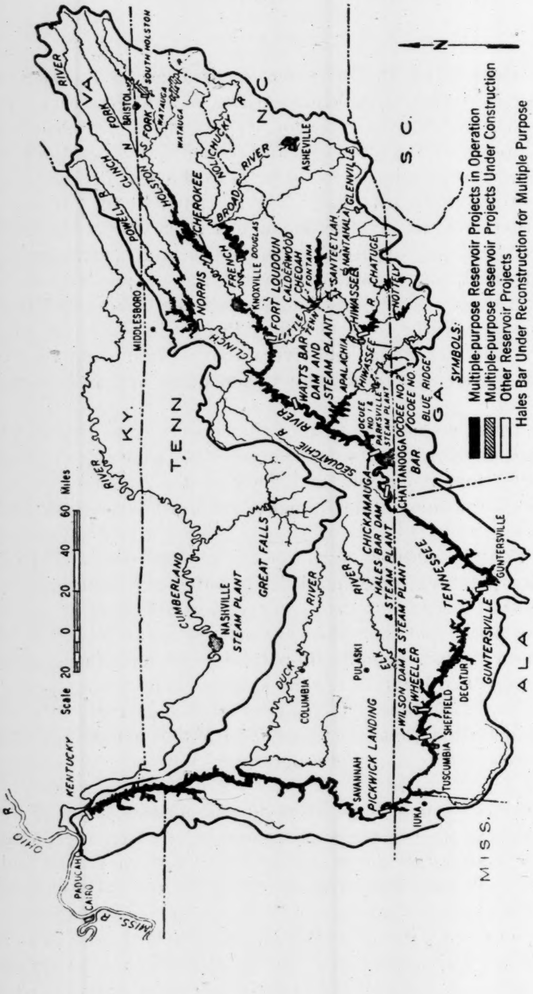
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DUCK
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COLUMBIA
PULASKI
ELK
RIVER
SAVANNAH
PICKWICK LANDING
IUKA
TUSCUMBIA
SHEFFIELD
WILSON DAM & STEAM PLANT
WHEELER
DECATUR
GUNTERSVILLE
TENNESSEE
HALES BAR DAM & STEAM PLANT
CHICKAMAUGA
CHATTANOOGA
HALES BAR
SEQUATCHIE R
RIVER
WATTS BAR DAM AND STEAM PLANT
APALACHIA
HIWASSEE
OCOEE NO. 1 & PARKSVILLE STEAM PLANT
OCOEE NO. 2
OCOEE NO. 3
BLUE RIDGE
NOTTELY
CHATUGE
HIWASSEE R.
NANTAHALA
SANTEETLAH
GLENVILLE
FONTANA
CHEOAH
CALDERWOOD
LITTLE TENN
FORT LOUDOUN
KNOXVILLE
DOUGLAS
FRENCH BROAD RIVER
NOLICHUCKY R
ASHEVILLE
CLINCH
NORRIS
MIDDLESBORO
POWELL R
CLINCH
HOLSTON
N FORK
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CHEROKEE
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WATAUGA
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SYMBOLS:
Multiple-purpose Reservoir Projects in Operation
Multiple-purpose Reservoir Projects Under Construction
Other Reservoir Projects
Hales Bar Under Reconstruction for Multiple Purpose

Fig. 1.—The Tennessee River System as of 1946

tunity for reregulating flows, the allocation of storage space for the several purposes may vary from that for single reservoirs or that for a less completely controlled basin.

These conditions do not signify that the sound principle of constructing reservoirs of adequate size should be abandoned; but instead that, under certain conditions, reservoirs of smaller size may be used effectively, as has already been demonstrated in practice. It is fortunate that smaller reservoirs are practicable, since, because of the development of river valleys with industries, transportation lines, etc., and sometimes because of topographic limitations, sites for large reservoirs are often nonexistent. A striking illustration of this condition is offered by the Allegheny and Monongahela river basins, where a system of reservoirs is being built for the purpose, among others, of regulating floods in the Pittsburgh section under considerable restrictions in flood storage possibilities. In this connection, Emil P. Schuleen,[5] Assoc. M. ASCE, has presented a table showing the inches of runoff control afforded by the existing and proposed reservoirs. If the engineers had insisted on greater control of runoff in the drainage areas of those reservoirs, the much needed regulation of floods in this important region (where one flood alone, in 1936, caused damages estimated to exceed $175,000,000) might not have been realized.

Since the principles involved in planning and operating reservoirs for power are well known, as are those pertaining to navigation improvements, this paper will deal mostly with the flood control features of multiple-purpose reservoirs, about which too little is known. Various criteria have been advanced as desirable for the size of reservoirs for flood control. Perhaps more applicable to the humid eastern section of the United States, they range from half the average annual runoff to 8 in., or somewhat less, and offer to the engineer a wide range within which to study the characteristics of his own particular problem, and on which to base his plans. Wherever feasible, reservoirs may be planned in accordance with these desirable requirements with great assurance of success. Some of the existing multiple-purpose reservoirs that meet the foregoing specifications are: Lake Mead (Arizona-Nevada), planned to store about 80% of the average annual runoff; Sacandaga Reservoir (Hudson River Basin), whose capacity is nearly 14 in. of runoff; and Norris, Fontana, Hiwassee, Chatuge, and Nottely reservoirs (Tennessee Basin), which have a storage capacity of about 10 in. of runoff at the beginning of the flood season on January 1, about 8 in. on March 15, and about 6 in. on April 1, near the end of the major, valley-wide flood season. Because it is often impracticable to locate reservoir sites that afford such good control of runoff, the engineer must either forego using reservoirs or be satisfied with less control. The latter will be his decision if he can devise operating methods adequate to meet the situation; and, thus, in a major sense, planning merges with operation.

What then should be the criterion for planning the size of such reservoirs? The writer is unable to give a specific answer, as that must come from a study of each situation, but hopes that the suggestions in this paper will be helpful. Personally, he believes that, when it becomes necessary in the future, engineers will find it feasible to reduce reservoir storage requirements considerably. The extent of control deemed essential will undoubtedly depend largely on operating methods to be followed, improved operating technique, and the reliability of precipitation predictions. It will also depend on weather characteristics and other features peculiar to the region.

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[5] "Control of Floods at Pittsburgh Planned," by Emil P. Schuleen, *Civil Engineering*, October, 1945, p. 454.

In the eastern United States, where, over large basins, precipitation averages, say, from 40 in. to 55 in. annually and a great flood producing storm may yield 20% of that amount, reservoirs capable of controlling 8 in. of runoff should be adequate. Reservoirs with less control can be operated (although it is more difficult) to save storage space for only a relatively short time at the crest of the flood, after which they can be drawn quickly to flood control level by spillage. This procedure is necessary for the main stream reservoirs on the Tennessee River, although in lesser degree for Kentucky Reservoir (see Fig. 1).

The guide curve governing the multiple-purpose operation of Douglas Reservoir, which does not have a high degree of runoff control in relation to the drainage area of 4,541 sq miles, and the actual operation of the reservoir in 1946 are shown in Fig. 2(*a*). The maintenance of multiple-purpose features is assured through the prompt restoration of required flood storage space by spilling water stored during the January flood (which, without regulation, would have been the fifth largest of record at Chattanooga, Tenn.) and also during the February flood. The deficiency in desirable filling of the reservoir after the end of the flood season in April would not have resulted if encroachment on flood storage space had been permitted by holding in storage some of the water from the January and February floods.

Where flood control and power are combined in the same reservoir, with or without other uses (not an infrequent condition), perhaps more uncertainty exists as to the feasibility of successful operation for flood control than in any other combination of uses. Experience with the Tennessee River system of reservoirs indicates that, with proper operation, there is no need for such uncertainty. Moreover, the increasing number of reservoirs throughout the United States combining these two purposes shows a growing confidence in their effectiveness. In such reservoirs, the superimposed storage space assigned to flood control should be adequate to control, properly, the runoff from the contributing drainage area under the plan of operation contemplated. The multiple-purpose reservoir will be just as efficacious for flood control as a single-purpose reservoir containing the same volume of flood storage space, provided that encroachment on the space reserved for flood control is not permitted in operation. In a properly planned multiple-purpose reservoir, particularly in a well integrated system having good interconnections and suitable capacity of stand-by power, the energy gained by keeping the reservoir full is of questionable temporary value, as it may be lost later by spillage.

The Tennessee River system of multiple-purpose reservoirs will be used largely as the subject for further discussion, as the writer is most familiar with that system. Clarence E. Blee,[6] M. ASCE, has presented a most complete

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[6] "Multiple-Purpose Reservoir Operation of Tennessee River System," by Clarence E. Blee, *Civil Engineering*, May, 1945, p. 219, and June, 1945, p. 263.

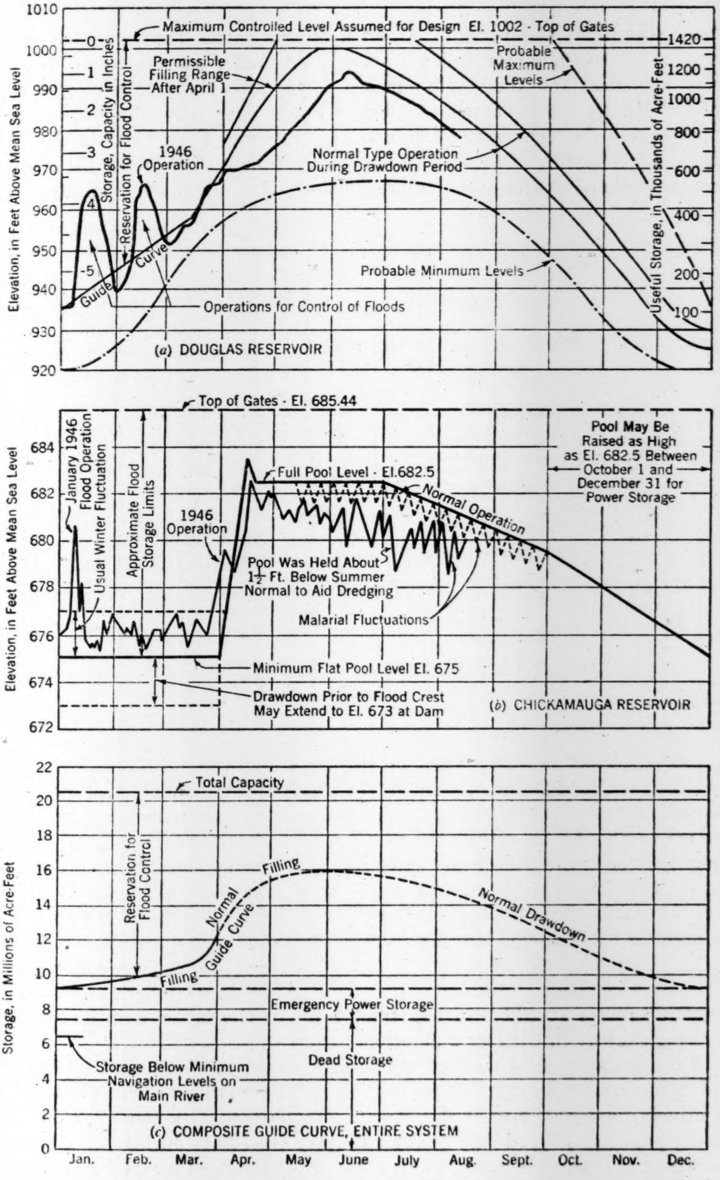

FIG. 2.—MULTIPLE-PURPOSE GUIDE CURVES AND CURVES OF 1946 OPERATION; RESERVOIRS OF THE TENNESSEE VALLEY AUTHORITY

exposition of this system. Including the five single-purpose power projects of the Aluminum Company of America on the Little Tennessee River (the operation of which is controlled by the TVA under an agreement between the two parties) and the single-purpose power projects acquired some years ago by TVA from the Tennessee Electric Power Company, there are twenty-six projects in the Tennessee Basin and one on Caney Fork River in the Cumberland River Basin. Of these, fifteen are multiple-purpose reservoir projects, some important features of which are shown in Table 1 and in Fig. 2(*c*).

TABLE 1.—MULTIPLE-PURPOSE RE

No.	Reservoir	River	Local drainage area[a] (sq miles)	FLOOD STOR		
				JANUARY 1		
				El.	Storage (acre-ft)	Runoff control (in.)
1	Kentucky........	Tennessee	7,380	354–375	4,010.800	10.2
2	Pickwick........	Tennessee	2,070	408–418	418.400	3.8
3	Wilson..........	Tennessee	1,160	504.5–507.9	52,500	0.8
4	Wheeler.........	Tennessee	5,140	550–556.3	348,900	1.3
5	Guntersville......	Tennessee	3,660	593–595.4	162.900	0.8
6	Chickamauga.....	Tennessee	2.512	675–685.4	329.400	2.5
7	Watts Bar.......	Tennessee	3,277	735–745	377,600	2.2
8	Fort Loudoun....	Tennessee	1,580	807–815	109,300	1.3
9	Cherokee........	Holston	3,429	1,020–1,075	1,145,900	6.3
10	Douglas.........	French Broad	4,541	935–1,002	1,311,200	5.4
11	Norris..........	Clinch	2,912	978–1,034	1,635,000	10.5
12	Fontana.........	Little Tennessee	1,571	1,615–1,710	771,160	9.2[c]
13	Hiwassee........	Hiwassee	565	1,455–1,526.5	291,100	9.7
14	Chatuge.........	Hiwassee	189	1,910–1,928	105.400	10.4
15	Nottely.........	Nottely	214	1,743–1,780	110,050	9.7
16	Total.........	....	....	....	11,179,610	....

[a] Below next upstream dam in case of Tennessee River reservoirs and Hiwassee reservoir. [b] Could use sto and Glenville areas.

The basin has an average annual precipitation of about 52 in., from which the runoff is about 23 in. Although the precipitation is fairly well distributed throughout the year (the monthly average varying from about 3 in. in the fall months to about 5 in. during the first 3 months in the year), natural runoff is much greater in the winter and early spring. During January, February, and March, it amounts to approximately 10 in. at Chattanooga, or nearly half the annual runoff. In wet years, it is relatively higher. This high runoff is accounted for by the heavy storms, usually several days in duration, which occur during the flood season, and by the high percentage of surface runoff caused by dormant vegetation. Subsequently, from April to October, inclusive, the runoff decreases steadily, to less than 1 in. in the latter month, before recovering slightly in the last 2 months of the year.

The annual cyclical method of operating reservoirs indicated in Figs. 2(*a*) and 2(*b*) (which are typical for tributary and main stream reservoirs, respectively) is designed to fit this stream flow pattern. In general, these guide curves provide for gradually filling tributary reservoirs during the first 4 or 5 months of the year, for maintaining water near its highest level for 2 or 3 months in the summer, and for gradually drawing the level down during late summer and fall to minimum flood control level by January 1 to complete the annual cycle. Somewhat similarly, main stream reservoirs are held at minimum flood control level, except during flood control operations, from January 1 until about April 1, then filled to pool-full level by about April 15, held at that level until July 1, and finally gradually drawn down to minimum flood control level by January 1. Although reservoirs are at high level for 2 or 3 months, storage space is still available, as shown in Table 1 and Fig. 2(*c*), for regulating such flash floods as may occur in that season.

SERVOIRS, TENNESSEE RIVER SYSTEM

AGE SPACE			STORAGE SPACE AVAILABLE DURING SUMMER PERIOD			No.
MARCH 15						
El.	Storage (acre-ft)	Runoff control (in.)	El.	Storage (acre-ft)	Runoff control (in.)	
354-375	4,010,800	10.2	359-365[b]	1,044,200[b]	2.7	1
408-418	418,400	3.8	414-418	179,200	1.6	2
504.5-507.9	52,500	0.8	....	....	....	3
550-556.3	348,900	1.3	556-556.3	20,800	0.1	4
593-595.4	162,900	0.8	595-595.4	31,200	0.2	5
675-685.4	329,400	2.5	682.5-685.4	108,500	0.8	6
735-745	377,600	2.2	741-745	163,500	0.9	7
807-815	109,300	1.3	813-815	29,990	0.4	8
1,042-1,075	807,200	4.4	1,073-1,075	61,400	0.3	9
958-1,002	1,019,800	4.2	1,000-1,002	62,100	0.3	10
990-1,034	1,377,000	8.9	1,020-1,034	520,000	3.3	11
1,644-1,710	581,770	6.9[c]	1,708-1,710	21,180	0.3[c]	12
1,472-1,526.5	245,100	8.1	1,524.5-1,526.5	12,400	0.4	13
1,916-1,928	75,100	7.4	1,926-1,928	13,980	1.4	14
1,755-1,780	83,500	7.3	1,778-1,780	8,420	0.7	15
....	9,999,270	....	....	2,276,870	....	16

rage space up to El. 375 if necessary (3,290,000 acre-ft). [c] On total area above Fontana, including Nantahala

The foregoing plan not only conforms to stream flow characteristics but also affords optimum benefits to the statutory purposes of navigation, flood control, and power production, and, in so far as consistent with those major uses, serves other useful purposes. Quite important among such other purposes, because the public health is involved, are operations for malaria control conducted on main stream reservoirs, as shown in Fig. 2(*b*). They include a 1-ft surcharge in April to strand flotage and a weekly, cyclical fluctuation about 1 ft deep between May 15 and October 1, combined with several feet of recession in pool level between July 1 and October 1. These operations and other water level management operations, such as the gradual drawdown of Kentucky Reservoir and tributary reservoirs during the mosquito breeding season, are effective in maintaining a clean shore line and in stranding and helping to destroy the larvae of the malaria-carrying mosquito.

Other incidental although important purposes served are (1) recreation, secured by maintaining suitable water levels and discharges, when practicable, for the propagation of fish during the spawning period and for such sports as boating and fishing; (2) sanitation and water supply, by making special discharges when needed; and (3) numerous others such as refloating stranded boats and ferries and aiding farmers in fording or ferrying to tend or harvest crops on islands, accomplished also by making special discharges.

The navigation purpose of providing for a minimum 9-ft draft in the main river is obtained by nine dams and locks, supplemented by dredging in the upper end of the pools. By regulating flows throughout the year, the pools are held at levels which provide the required or greater depth. Minimum pool level for this purpose is also the minimum allowable for power. The reduction of flood flows in the pools, through the operation of the system, is helpful to power by maintaining greater main river heads, and to navigation by reducing current velocities and increasing headroom under bridges. The storage of water permitted by the gradually rising guide curve in the winter and the spring flood season for controlling floods benefits both power and navigation through its later use in the low-flow season. Likewise, the low-flow season draft of this water from reservoirs in advance of the flood season benefits flood control by providing storage space for that purpose. This regulation has increased the minimum daily flow at the mouth of the river from less then 10,000 cu ft per sec to about 25,000 cu ft per sec. Gerard H. Matthes,[7] Hon. M. ASCE, chairman of the ASCE Joint Division Committee on Floods, in his 1943 Progress Report on "Floods," stated, in part:

> "* * * Two additional reservoirs, the Kentucky Reservoir near the mouth of the river and the Fontana Reservoir on the Little Tennessee River, with prospective storage capacities of 4,600,000 acre-ft and 1,200,000 acre-ft, respectively, are nearing completion and will make the Tennessee River and its tributaries the most perfectly regulated river system of its size in the world. * * *"

It must be remembered, in appraising the results of the first decade of operations of the TVA system, that many vicissitudes were encountered not hereafter to be expected, including: (1) The pioneering nature of multiple-purpose operations; (2) the frequent need of abnormal flow regulation, usually unsuited to best multiple-purpose results, to protect construction work in progress; (3) the temporary operation of several reservoirs during the World War II emergency primarily for power; (4) the greatest flood of record on the Mississippi River at Cairo, Ill., in 1937, when the system was too small to have much effect; and (5) the record-breaking dry period in 1939–1941 at a time of rapidly increasing power load. These factors imposed a severe load on the system, and on those responsible for its operation, while it was growing and changing and consequently less susceptible of efficient operation.

Storms severe enough to cause basin-wide floods usually travel across the Tennessee Basin in a northeasterly direction, causing heavy precipitation and increased runoff in the western section some 12 hr to 24 hr earlier than in the eastern section. This meteorological characteristic is a favorable one, because it permits the initiation of regulatory measures progressively upstream in the main river and the discharge of runoff from lower river areas ahead of upstream peaks. Quantitative predictions of precipitation, from 36 hr to 48 hr in advance, are furnished for seven basin subdivisions three times daily, and the general weather outlook up to 5 days ahead is supplied once daily by the United

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[7] *Proceedings*, ASCE, February, 1944, p. 176.

States Weather Bureau during the flood season under a cooperative arrangement initiated in 1939. These data are of inestimable value in day-to-day (and sometimes even more frequent) adjustment of discharges throughout the system. Although rainfall cannot yet be forecast with great accuracy, much improvement has been made in recent years, particularly under the stimulus of World War II. Without help of this nature, multiple-purpose reservoir operation would suffer a severe handicap.

Under natural flow conditions, peak flood discharges from the Tennessee River ordinarily reach the confluence of the Ohio and Mississippi rivers at Cairo in advance of the peaks from those two rivers. The continuous main river dam system on the Tennessee River makes it feasible to accelerate greatly the movement downstream, and offers the opportunity of passing the Tennessee peak into the Ohio and Mississippi rivers still farther ahead of the crest from those rivers. In the large flood of January, 1946, the acceleration amounted to about 5 days, as indicated in Fig. 3(*a*). Thus, space is available in the reservoirs for storing during the crest period at Cairo or Paducah, Ky., the control points, if the flood reaches dangerous proportions. In the event of successive crests, such as those in March and April, 1945, when a triple crest flood occurred which would have reached rather threatening stages ranging from 54.5 to 55.6 at Cairo, reservoir operations become a sustained effort of timing discharges to effect crest reductions, if stages are high enough to warrant this action. Fig. 3(*b*) shows the regulation of the three crests at Cairo and one at Paducah, the reductions at the former city varying from 0.7 ft to 1.7 ft, the largest reduction being from the highest crest, and 2 ft at the latter city. Because of construction operations, storage space in Kentucky Reservoir was not available for use in these operations except to a very limited degree.

In 1937, when the highest flood of record at Cairo occurred, the stage reached 59.5 ft on February 3, about a week after the New Madrid (Mo.) Floodway was opened at around stage 58. Cairo was seriously threatened, mud boxes placed on top of levees being nearly overtopped. At that time only Norris, Wheeler, and Wilson reservoirs were available, and the storage space of the latter was too small to be of consequence. Computations made subsequent to the flood indicated that a reduction of from 0.3 ft to 0.5 ft was effected. Although it was small, that regulation under the circumstances may have been of great importance.

Since that flood much raising and strengthening of levees has been done by The United States Engineers and probably a flood as great or greater now could safely be carried at and below Cairo. Protection works at Paducah will have a top elevation about 3 ft higher than the flood crest in 1937. It would seem, then, that henceforth no stage of less than, say, 55 at Cairo or Paducah should be considered dangerous and worthy of regulation. If higher stages are predicted, regulation should be based on the predicted crest, or on an even higher one in case of subsequent rainfall sufficient to affect the crest. The foregoing suggestion applies to the major flood season. In the summer, when damages to crops may accrue between the levees and in backwater areas, it may be found desirable to regulate floods reaching flood stage (40 ft) at Cairo.

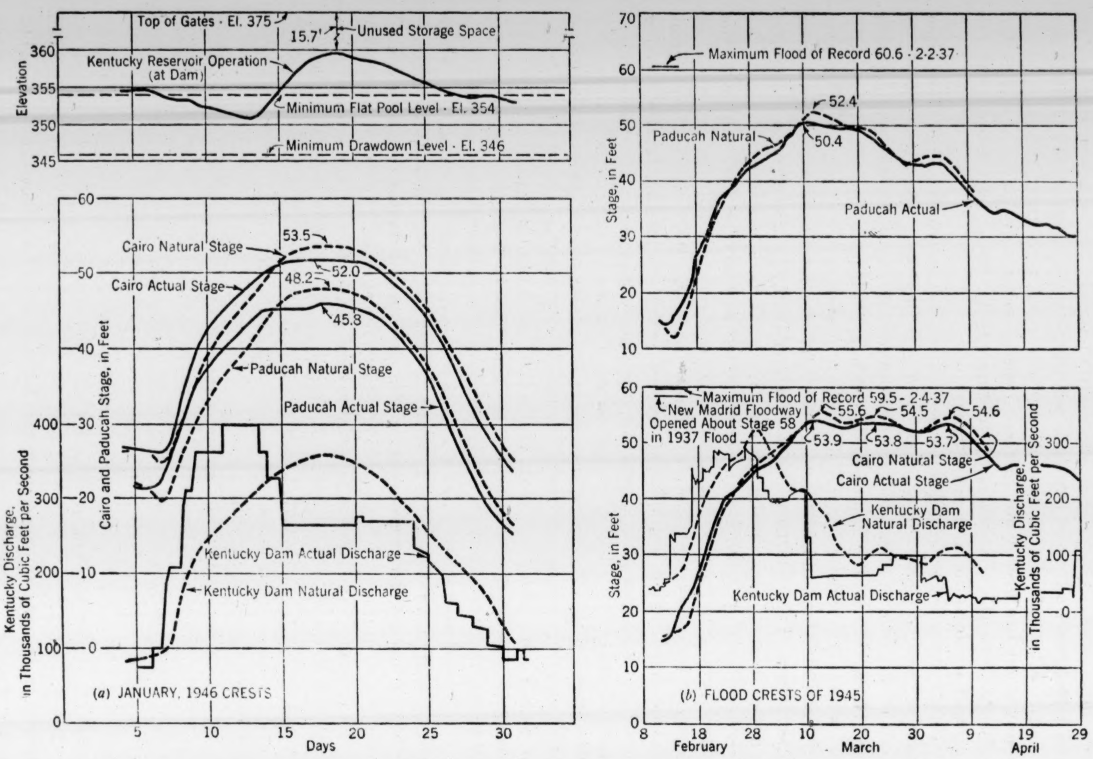

Fig. 3.—Regulation of Flood Crests at Paducah, Ky., and Cairo, Ill., by TVA Reservoir System

To control the contribution from the Tennessee to the Ohio and Mississippi rivers, so as to be most beneficial, frequent contact has been maintained during floods with the U. S. Engineer offices in charge of those rivers, as well as with the U. S. Weather Bureau at Cairo. Weather and river conditions and predictions obtained from these sources have made it possible to follow the course of floods down those rivers and to operate the Tennessee River reservoir system accordingly. Under Section 7 of the Flood Control Act of December 22, 1944, the TVA is directed to regulate the release of water from the Tennessee River into the Ohio River in accordance with such instructions as may be issued by the War Department in case of danger from floods on the lower Ohio and Mississippi rivers. This directive serves the purpose of integrating the flood control operations of the TVA system with those on the Ohio and Mississippi rivers under the control of the U. S. Engineers.

The 1946 flood season afforded the first opportunity for this new plan to function. Methods previously developed and put into effect for exchanging data provide that when the Cairo stage is 35 ft, with 40 ft or higher predicted, the TVA shall report by teletype, early each day, the observed data for main river reservoirs from Chickamauga to Kentucky dams, inclusive, giving rainfall, stages, and discharges. Later each day predicted schedules of stages and discharges from 3 days to 5 days in advance are reported. The Ohio River Division Office of the U. S. Engineers at Cincinnati (formerly at Columbus), Ohio, and the Upper Mississippi River Division Office at St. Louis, Mo., reciprocate by similarly furnishing the TVA observed data and predicted discharges at a number of key points on the Ohio and Upper Mississippi rivers and on certain tributaries. In addition, when the Cairo stage is 50 ft or higher, the Mississippi River Commission, at Vicksburg, Miss., furnishes the TVA daily with current stages and predicted stages on the Mississippi River from Cairo to New Orleans, La., 7 days in advance, together with predictions of date and height of crest at all stations.

The foregoing procedure was followed throughout the January, 1946, flood. This flood was the eighth highest of record in the lower Tennessee Valley but was of rather moderate height at Cairo and Paducah, where computed natural crest stages of 53.5 and 48.2, respectively, were reduced to 52.0 and 45.8 by reservoir action, as shown in Fig. 3(*a*). Since dangerously high stages were not reached or in prospect, the U. S. Engineers did not issue any instructions as to the regulation of releases from the Tennessee River into the Ohio River. In Fig. 3(*a*), it may be noted that:

(1) The modified hydrographs at Cairo and Paducah have exceptionally smooth, flat crests;

(2) A little more than 3 ft of the permissible 8-ft drawdown at the lower end of Kentucky Reservoir was effected in seeking to maintain constant volume in that reservoir before the flood peak arrived and in speeding the movement of the Tennessee peak at Kentucky Dam into the Ohio and Mississippi rivers ahead of the crests there; and

(3) The lower end of the reservoir was filled only to about El. 359, thus conserving storage space for use in case the flood should subsequently develop into one of dangerous height.

The method used at Kentucky Dam is a good example of reservoir operation planned in advance, as the result of several years of study. If the Kentucky Reservoir had been operated to obtain maximum effect, the Cairo and Paducah crests could have been reduced about 6 ft.

Because of the reservoir's large size (capacity equivalent to more than 10 in. of runoff on its local drainage area of 7,380 sq miles) and its strategic location (22 miles from the Ohio River and 67 miles from the Mississippi River, only about a day away in time of water travel), it serves admirably for reregulating discharges from the entire system of upriver reservoirs and for fixing and timing the output of water from the Tennessee River.

Although the method of operating the other main stream reservoirs is basically similar to that at Kentucky Dam, it is absolutely essential, because of their smaller size, to conserve as much as possible of their flood storage space for occupancy during a comparatively short period of flood crest regulation, if the reservoirs are to be effectual. Thus, delicate timing is required, usually complicated by weather changes and the necessity for prompt and accurate runoff computations and stream flow routings, both current and prospective.

By withholding water in tributary reservoirs until the peak from the main river contributory drainage has passed Chattanooga (as is the usual procedure, and as was done in the major flood of January, 1946, the fifth highest of record at Chattanooga in about 75 years), the burden on the main stream reservoirs is lightened. To accomplish this control, tributary reservoirs must be operated in coordination with main stream reservoirs, in both of which the required flood storage space must be held in reserve during the entire flood season except when used for regulating floods. Thus, operations during the late summer and fall drawdown period, as well as during the flood season, have an important bearing on subsequent flood control. The fall of 1945 was wet and, in order to provide the flood storage space required by the end of the year, a total of more than 1,000,000 acre-ft of water in excess of turbine requirements was spilled, in December, from the five major tributary reservoirs and through main stream reservoirs, augmented by additional spill from the latter.

During a great flood, such as those of 1867 and 1886, or some still larger possible flood, rather large releases from tributary reservoirs will be necessary. Nevertheless, a proper appraisal of the situation, promptly made, will permit maintaining smaller outflow than inflow from these reservoirs, so regulated as to effect a material crest reduction at Chattanooga, the key control point on the Tennessee River. However, because of the low elevations of important sections of that city, complete protection cannot be provided without supplemental levees. Usually, tributary contribution to the Chattanooga peak can be limited to bank-full stage on the several tributaries, or even less. In the January, 1946, flood, tributary discharge was limited to turbine discharges, varying from about 4,000 cu ft per sec to 7,000 cu ft per sec, until the peak passed Chattanooga, and thereafter to tributary bank-full stage, amounting in each tributary to 20,000 cu ft per sec or less. These discharges were timed to flatten the receding hydrograph at Chattanooga at around the flood stage where little or no damage occurs (see Fig. 4). In contrast to these discharges, peak inflows into the tributary reservoirs varied from about 15,000 cu ft per sec in the drainage above Hiwassee Dam to about 90,000 cu ft per sec in Douglas Reservoir.

This operation of the tributary reservoirs, together with the reregulation of flows in the three main stream reservoirs above Chattanooga, resulted in reducing not only the peak flow at Chattanooga from 320,000 cu ft per sec to 225,000 cu ft per sec and the peak stage from 45.8 ft to 35.7 ft, as shown in Fig. 4, but also the floods on the tributary streams below the dams to from 20% to 50% of their natural peak rates. The estimated savings in flood damages at Chattanooga were almost $10,000,000.

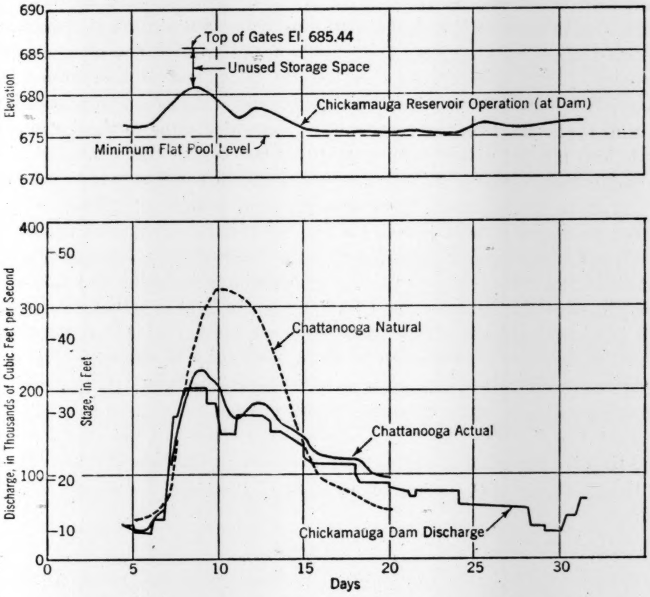

FIG. 4.—REGULATION OF JANUARY, 1946, FLOOD AT CHATTANOOGA, TENN., BY TVA RESERVOIR SYSTEM

In flood control operations of this kind, it is essential that a key point, or points, be selected as the objectives toward which such operations will be directed, as, for example, Chattanooga on the Tennessee River and Cairo on the Mississippi River in the case of the TVA system of reservoirs. Usually the objective is perfectly clear since its protection is the purpose for which the reservoir or reservoirs were built. In any event, where a big system of reservoirs is to be operated, such as that above Pittsburgh or that in the Tennessee Valley, selections should be made and operations planned accordingly. Control designed either for near-by objectives or for distant points may not always be dependable for the latter, because of the possible effect of flood runoff from other uncontrolled areas on the flood crest downstream, although such control should prove beneficial in the long term.

Handling the system of multiple-purpose reservoirs in the Tennessee Valley requires constant attention and at least daily scheduling of flows throughout the entire system for several days ahead. During floods, schedules must sometimes be revised two or three times daily to take into account changes in rainfall and runoff as they occur and effect flows and stages, with due regard for weather predictions, which are available three times daily. This scheduling is accomplished through centralized control under the chief engineer and the issuance of instructions as often an necessary to the Department of Power Operations, which is responsible for seeing that the required discharges are made at all the dams. At the central office, hydrologic data are received, assembled, and interpreted; runoff is estimated; flows are routed; and operations are decided upon by engineers thoroughly familiar with the characteristics of the system. Frequent reports received by teletype and telephone from the Operating Department at Chattanooga furnish checks on operations at the dams, and their effect on the reservoirs. Since decisions in time of flood must be made and put into effect promptly to be successful, this centralized control takes cognizance of the fact that no two storms, or the floods they produce, are alike and that each situation requires detailed treatment all its own, based, of course, on fundamental principles and procedures governing system operations. That success has been achieved under this method of control is evident from the aforementioned results. As to consistency, since the first reservoir (Norris) became available above Chattanooga in March, 1936, fourteen floods (which in their natural state would have reached or exceeded flood stage of 30 ft and caused varying amounts of damage) have occurred. The crests of all have been reduced, by amounts varying from less than 1 ft to more than 10 ft, with a total saving in damages at that point of well above $12,000,000. As of December, 1948, the number of floods reduced at Chattanooga had increased to seventeen and the total estimated savings in damages at that point to more than $40,000,000, based on estimated property values at the time of each flood.

SUMMARY

This paper being of limited length is inadequate to cover such an important subject. If it has not always seemed to present the conventional view (which is certainly easier and usually more popular), no apology is made. The unconventional of today may be commonplace tomorrow. Whatever the reaction, it is sincerely hoped that the paper may at least stimulate thought and study and thus be helpful in the planning and the operation of the great number of large reservoir projects being built and certain to be built soon, in the United States. In the interest of clarity, the following summary of the more important contents and of some of the writer's views is presented:

(1) Multiple-purpose reservoir design and operation are real and practical and are being successfully accomplished in several major drainage basins in the United States.

(2) In planning reservoirs for flood control, whether single purpose or multiple purpose, wherever practicable, and consistent with costs and benefits, large storage space should be provided for flood control (as much as half or

more of the average annual runoff on the drainage area); but, if economic or other considerations prevent, smaller space, providing 8 in. of runoff or even less, may be ample in certain locations under appropriate operating methods.

(3) A given volume of flood control storage space should be equally as effective in a multiple-purpose reservoir as in a single-purpose reservoir, if strict adherence is given to operating rules, and encroachment on flood space is not permitted. There is no more justification for intentionally "overloading" a reservoir than for overloading any other engineering structure. Careful, intelligent, honest operation will avoid this.

(4) If encroachment on flood space is allowed, the multiple-purpose reservoir, instead of being a beneficial instrument, becomes a serious threat of flood disaster. The primary purpose should be unmistakably fixed in advance, both for the safety of the project and for the guidance of those charged with its operation.

(5) Since a reservoir or system of reservoirs cannot be operated with complete assurance of results for distant points downstream, it is usually desirable to select and to operate for relatively near-by key points and rely on benefits accruing over a period of time at greater distances.

(6) Much has been gained and much more can yet be gained in operating reservoirs by the help of quantitative predictions of precipitation.

(7) Being a highly specialized activity, often requiring prompt decisions and quick execution, the control of water in multiple-purpose reservoirs, particularly in systems, should be entrusted only to engineers who are competent and resourceful and in whom confidence, authority, and responsibility can be placed.

THEIR USE FOR FLOOD CONTROL

BY ALBERT L. COCHRAN,[8] M. ASCE

INTRODUCTION

The subject of this paper raises the question: "How does the use of a 'multiple-purpose' reservoir for flood control differ from the use of any reservoir for that purpose?" It is apparent that there is very little difference, except in so far as reconciliation of the multiple interests is involved. In any case, the function of flood control storage is to impound water during periods of excess runoff in order that it may be released at nondamaging rates later. However, in certain river basins, meteorological and topographic influences limit major floods to well-defined seasons, or at least establish very consistent trends in flood magnitudes by seasons. Accordingly, part of the maximum flood control capacity may be "loaned" for other uses during the seasons of lower flood runoff.

Flood control capacity in a multiple-purpose reservoir includes only the volume that is specifically reserved for storage of floodwaters under undisputed jurisdiction of the flood control operator. Some of the flood control capacity may be utilized for other purposes during certain seasons of the year, but only at the discretion of the flood control operator. Storage of floodwaters to refill reservoir capacity allocated to some purpose other than flood control, after drawdown attending normal operation, may result in incidental flood control benefit, but such storage is not classified as "flood control," inasmuch as its availability when floods occur cannot be relied on.

Multiple-purpose reservoirs involving flood control may be placed in two principal groups:

(*a*) Reservoirs with constant allocations of storage capacity for flood control, irrespective of season of year; and

(*b*) Reservoirs with seasonal variations in flood control capacities (that is, reservoirs in which the storage capacity assigned to flood control is varied in accordance with seasonal trends in flood runoff volumes in the river basin involved).

MULTIPLE-PURPOSE RESERVOIRS WITH CONSTANT ALLOCATIONS OF FLOOD CONTROL CAPACITY

In many cases, floods capable of filling all available flood control capacity in a particular reservoir may be expected with a reasonable frequency during all seasons of the year. Accordingly, seasonal use of any part of the flood control capacity for other purposes cannot be permitted without loss of flood protection. Flood control capacity is incorporated in this class of multiple-purpose reservoirs either because such provision is more economical than construction of a separate flood control reservoir of comparable effectiveness, or because construction of a separate flood control reservoir is impracticable under prevailing circumstances. Operation of the flood control capacity in these reservoirs is essentially the same as that in straight flood control reservoirs.

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[8] Hydr. Engr., Chf. of Hydrology and Hydraulics Branch, Eng. Div., Civ. Works, Chf. of Engrs., Washington, D. C.

MULTIPLE-PURPOSE RESERVOIRS WITH SEASONAL VARIATIONS IN FLOOD CONTROL CAPACITIES

Two important objectives govern the operation of reservoirs in which the flood control capacity is varied by seasons:

(1) Sufficient capacity should be available in the reservoir on all dates to provide effective control of floods that might reasonably be expected to occur on those dates; and

(2) The reservoir storage should reach the highest level consistent with objective (1) by the end of the primary flood season, in order to provide as much water as possible for other functional uses of the reservoir—that is, the operator must strive to have just enough flood control capacity at all times, without wasting water needed for other purposes.

The best examples of this type of reservoir are found in the northwestern United States where maximum floods in large drainage basins are confined rather definitely to the winter and spring months. In that region, the greatest floods from rainfall generally precede the maximum snow-melt season. Accordingly, adequate reservoir capacity can be reserved for flood control during the winter, with the assurance that the part of the capacity not required for summer floods can be refilled during the spring snow-melt season to provide additional water for irrigation, power development, or other purposes during the summer and fall months.

In river basins in which snow melt is a consistent source of flood runoff, it is feasible to estimate approximately the volume of runoff to be expected from snow melt and incidental rainfall during the spring flood season from information regarding the water content and the extent of the snow cover within the watershed at the beginning of the melting season, with corrections and adjustments at appropriate intervals (usually monthly) during the season. In such cases, a fixed minimum storage capacity is usually reserved for control of floods that might reasonably be expected from above-normal rainfall, and the capacity between this minimum reserve and the maximum flood control capacity is varied in accordance with estimates of probable runoff from snow melt, combined with normal seasonal rainfall. Forecasts of probable runoff a month or more in advance are based, preferably, on information obtained from comprehensive snow surveys and analyses (in which checks of the water content, areal distribution, and conditions of the snow cover are made), correlated with records of observed runoff in previous years. Fig. 5 shows two sets of diagrams illustrating correlations of seasonal runoff, estimated from snow survey reports as of April 10 and May 10, with required reservoir storage capacity and regulated outflows. In the absence of adequate snow surveys and laboratory studies, various procedures have been used as expedients in estimating probable seasonal runoff in advance, with varying degrees of success.

In many river basins, snow melt constitutes an important influence on flood runoff only when combined with heavy rainfall; it cannot be considered as a governing limitation. The correlation between snow-cover indexes and flood control capacity requirements in downstream reservoirs is less definite than stated in the preceding paragraph. However, in recognition of the greater flood potentialities during the certain seasons, as indicated by runoff records from previous years, greater flood control capacity may be reserved during these seasons than for the remainder of the year. Correlations of runoff with some "basin mean wetness index" may provide a reasonable basis for varying the flood control capacity reservations during the respective flood seasons. Fig. 6 illustrates such a correlation, which was worked out by the Sacramento District Office of the Corps of Engineers for Folsom Reservoir on American River in California. Storage requirements are based on the control of flow to a maximum of 150,000 cu ft per sec at Fair Oaks, Calif. In this case, the "wetness index" parameters are in terms of inches of total precipitation during the preceding 60 days. For snow cover below El. 5000, the wetness index from Fig. 6 is increased by twice the water content in terms of inches over the entire

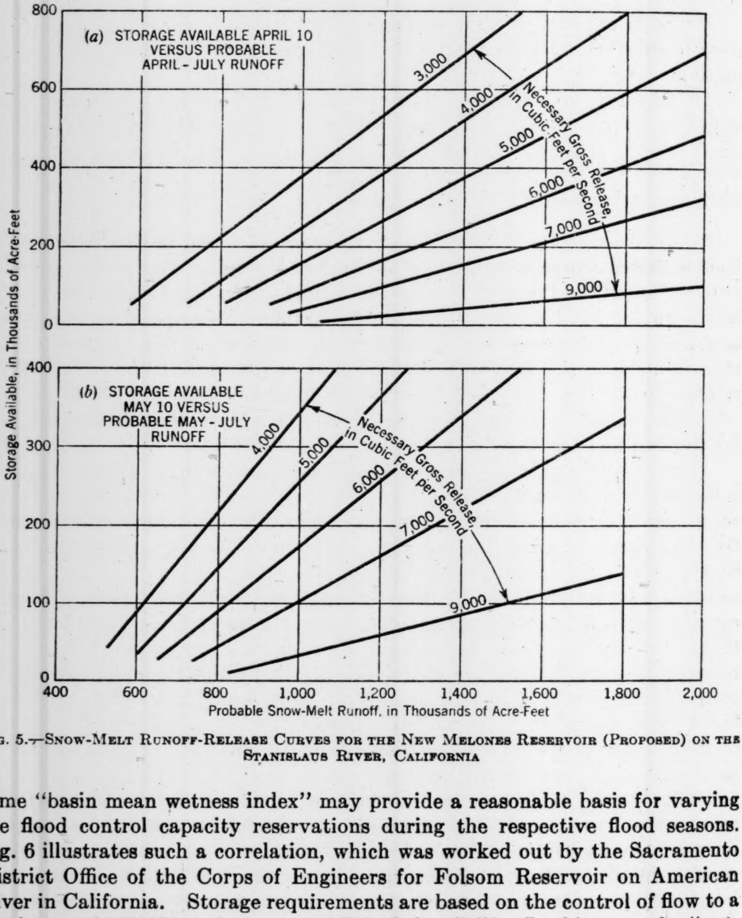

FIG. 5.—SNOW-MELT RUNOFF-RELEASE CURVES FOR THE NEW MELONES RESERVOIR (PROPOSED) ON THE STANISLAUS RIVER, CALIFORNIA

drainage area. In some basins, a "ground-water index" may serve a similar purpose. The best correlation must be determined from hydrologic studies for each project.

To understand the principles and the criteria that should determine the allocation of storage capacity for flood control in multiple-purpose reservoirs, it is necessary to consider certain aspects of flood control in general. There are two primary methods of providing flood control, and as a rule both are essential for satisfactory results. One is to increase the nondamaging capacity of downstream floodways by channel improvements and by the construction of levees or flood walls; the second is to reduce flood magnitudes by temporarily storing part of the runoff during flood periods in reservoirs. In many cases, appropriate land-use practices will serve to increase the absorption of rainfall by the soil; and, to the extent that this absorption is accomplished during important floods, such practices will help in reducing floods just as has the storage of water in reservoirs.

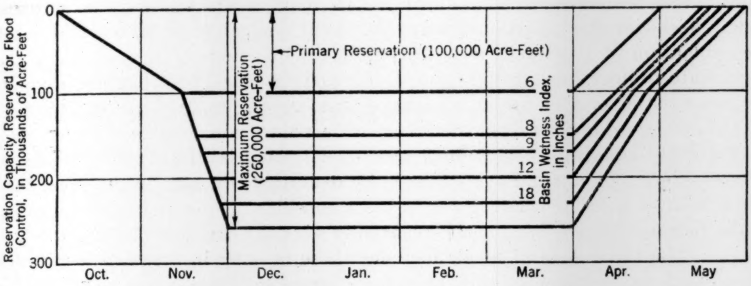

FIG. 6.—FLOOD CONTROL OPERATION DIAGRAM FOR FOLSOM RESERVOIR, AMERICAN RIVER, CALIFORNIA

It is seldom that adequate flood control can be provided by reservoirs alone, with the possible exception of the protection of limited areas very near the reservoir. As a rule it is profitable (and to some extent essential) to provide sufficient channel capacity downstream to carry at least moderate floods, reserving reservoir storage capacities for impounding water during relatively short periods of high flood discharges. It should be observed that the flood control storage requirements in any reservoir vary directly with the permissible rates of outflow during critical flood periods (that is, storage = inflow −outflow). Consideration of downstream channel capacity is inseparable from the determination of flood control storage requirements in any reservoir.

There are three general classes of floods to be considered in determining the degree of protection that should be adopted for a system of channel improvements and for a flood control reservoir:

1. The first class may be designated as "ordinary floods of record" in the given drainage basin—namely, floods that are reasonably certain to be equaled in magnitude one or more times within the estimated life of the project.

2. The second class includes floods exceeding the "ordinary floods," and equal to, or less than, the "standard project flood" for the particular drainage area involved. The "standard project flood" corresponds to a flood that would be exceeded in magnitude only on very rare occasions, and thus constitutes a standard for the design of structures that would provide a high degree of flood protection, as determined by flood potentialities of the drainage system involved without regard for localized economic or other practical limitations of individual projects. The standard project flood would normally be substantially less than the "maximum probable" flood that might occur with the worst combination of meteorologic and hydrologic conditions. In most cases, the standard project flood should equal or approximate the flood that would result, under existing or specified conditions of basin development, if the most critical storm of record in the region should occur over the drainage area involved when hydrologic conditions (including snow-melt potentialities) were reasonably favorable for flood runoff. Although, by definition, the standard project flood is a flood of such high magnitude as to be considered exceptionally infrequent, it is reasonably certain to occur eventually over the drainage basin concerned.

3. The third class of floods includes those ranging in magnitude from the standard project flood to the "maximum probable" (sometimes referred to as "maximum possible"), or the greatest flood that would occur if the most severe flood producing combinations of meterorologic and hydrologic conditions to which the basin appears to be reasonably susceptible should arise. The maximum probable flood is a "potential" flood that could occur in this given basin, but is extremely unlikely to occur.

Flood control works, like other public works, must be justified to warrant expenditure of the public funds required. It is not the purpose of this paper to deal with the evaluation of flood control benefits; but, since such evaluations are essential elements in the determination of appropriate storage capacities to be provided in reservoirs for flood control purposes, it is desired to call attention to certain primary considerations.

If it were economically feasible and practical to provide flood control works adequate for complete protection against all floods up to the maximum probable event, the problem would be greatly simplified. Since this is seldom practicable, it is necessary to decide how much money can be properly invested for flood protection. In making this decision it is necessary to consider both tangible and intangible benefits. The terms "tangible benefits" and "intangible benefits" are used in this paper in a limited sense, referring only to susceptibility to evaluation in monetary terms. For example, "tangible benefits" can be evaluated on a monetary basis, subject to uncertainties somewhat comparable to those attending other types of engineering estimates; thus:

(1) Reduction of damages to structures, lands, communication facilities, utilities, and other stationary property within the flood plain;

(2) Reduction of damages to movable stocks and supplies stored within the flood plain;

(3) Reduction of crop losses in rural areas (sale value, or investments if replantable);

(4) Reduction of losses from interruptions in normal business and industrial activities in the community that are attributable to floods;

(5) Reduction of interruptions in communications that are incident to floods;

(6) Enhancement of property values in protected areas (estimates of these "tangible" benefits usually being limited to apparent increases in property values corresponding to higher-class utilization permitted by protection from floods); and

(7) Reduction in flood fighting and relief costs.

The "intangible benefits" cannot be assigned definite dollar values, although they are recognized to be of great significance, in some cases outweighing tangible benefits in importance. Intangible benefits are:

(*a*) Protection of life and health (that is, reduction of hazards to occupants of protected area, flood fighting and relief personnel, users of transportation facilities subject to flood dangers, crews of rivercraft, etc.);

(*b*) Prevention of social dislocations and distress incident to major flood disasters;

(*c*) Improved general welfare of community resulting from higher standards of development in protected area, with incidental increase in production, higher earning capacities, and improved living conditions in the community at large;

(*d*) National benefits resulting from reduction of interruptions in interstate communications and general benefits derived from increased production, and improved welfare of populace directly affected by protection; and

(*e*) Increased national security resulting from protection of industries and communications against interruption by floods during times of war, and greater national strength incident to increased industrial development and improved general welfare of the people.

In determining the storage capacity to be allocated to flood control in any reservoir, the tangible and intangible benefits to be gained from various degrees of control of the three classes of floods previously discussed must be considered. In estimating these benefits, it is usually found that by eliminating the frequently recurring damages caused by "ordinary floods of record" the major part of the average annual "tangible" benefits that can be attributed to the protection project can be obtained. These benefits result in spite of the fact that a flood comparable to, or exceeding, the standard project flood may cause vastly greater damage than any single event of record in the particular river basin. However, provision of a partial degree of protection for an area suitable for industrial or residential use may serve to encourage intensive development in the protected area under a false sense of security, with the result that extensive loss of life and property may result in the event of a flood exceeding the design capacity of the project. In many cases, without protection against a flood reasonably comparable to the standard project flood, a supposedly "protected area" may be in imminent peril.

It is apparent that the several intangible benefits that are dependent on "security" warrant a substantially greater expenditure for dependable flood control in some instances than could be justified by measurable "tangible" benefits alone. In this connection, it is observed that annually large sums of money are spent in the United States for fire insurance on buildings and their contents. The security received by each property owner is considered justification for payment of insurance premiums, even if his building never burns down. In a similar sense, a substantial premium for a high degree of security against floods is usually justified in residential and industrial areas, even though there is some possibility that a flood approaching or exceeding the standard project flood might not occur in the particular basin under study within the estimated life of the project. Intangible benefits dependent on "security" against great floods must be given proper credit (in addition to the tangible benefits from control of the more frequent floods), when deciding on capacities to be selected for flood control purposes.

THEIR USE FOR NAVIGATION

BY ROBERT J. PAFFORD, JR.[9]

INTRODUCTION

The Missouri River Basin is the scene of a widespread, multiple-purpose, water use and control program. The technical phases of the basic program, commonly referred to as the Pick-Sloan plan, are being developed under the leadership of the Missouri Basin Inter Agency Committee, with the able and enthusiastic cooperation of many federal, state, and local agencies.

Although many other features are incorporated in the Missouri Basin program, reservoirs and storage are fundamental to its objectives. Present plans include some one hundred and five new federal reservoirs, in addition to the existing Fort Peck Reservoir built and operated by the Corps of Engineers and several smaller existing Bureau of Reclamation reservoirs in the basin. Total storage capacity of the entire approved system of federal reservoirs in the Missouri Basin will amount to some 100,000,000 acre-ft. Practically all this capacity will be obtained in multiple-purpose reservoirs.

The extent of control of the Missouri River itself which will result from completion of this reservoir plan is indicated by Fig. 7. For simplicity, Fig. 7 shows only those reservoirs which are at sites on the main stem, or on lower basin tributaries at sites close to the main stem. The total area of the Missouri Basin is 530,000 sq miles, divided as follows:

	Sq miles
To be controlled by reservoirs in plan	330,000
Not controlled by reservoirs	200,000

These reservoirs are a part of a comprehensive water use program for the control of destructive floods, expanded irrigation, improvement of navigation on inland waterways, hydroelectric power generation, and other collateral and incidental beneficial purposes such as stream sanitation, water supply, recreation, and wildlife uses. The primary emphasis in this paper is placed on how the requirements of navigation are to be served by the multiple-purpose reservoir system being constructed in the Missouri Basin.

FLOW REGULATION REQUIREMENTS FOR NAVIGATION

Reservoir regulation is needed for navigation on the Missouri River. Also, because of the Missouri River's large contributions to the flow of the Mississippi River, water stored in the Missouri Basin is of importance to navigation on the Mississippi River, particularly in the middle reaches between the mouth of the Missouri River, a few miles above St. Louis, and the mouth of the Ohio River, at Cairo.

The existing navigation channel project on the Missouri River provides for a channel 9 ft deep between Sioux City, Iowa, and the mouth to be obtained by revetment of banks, construction of permeable dikes to contract and stabilize

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[9] Hydr. Engr., Missouri River Div., Corps of Engrs., Omaha, Nebr.

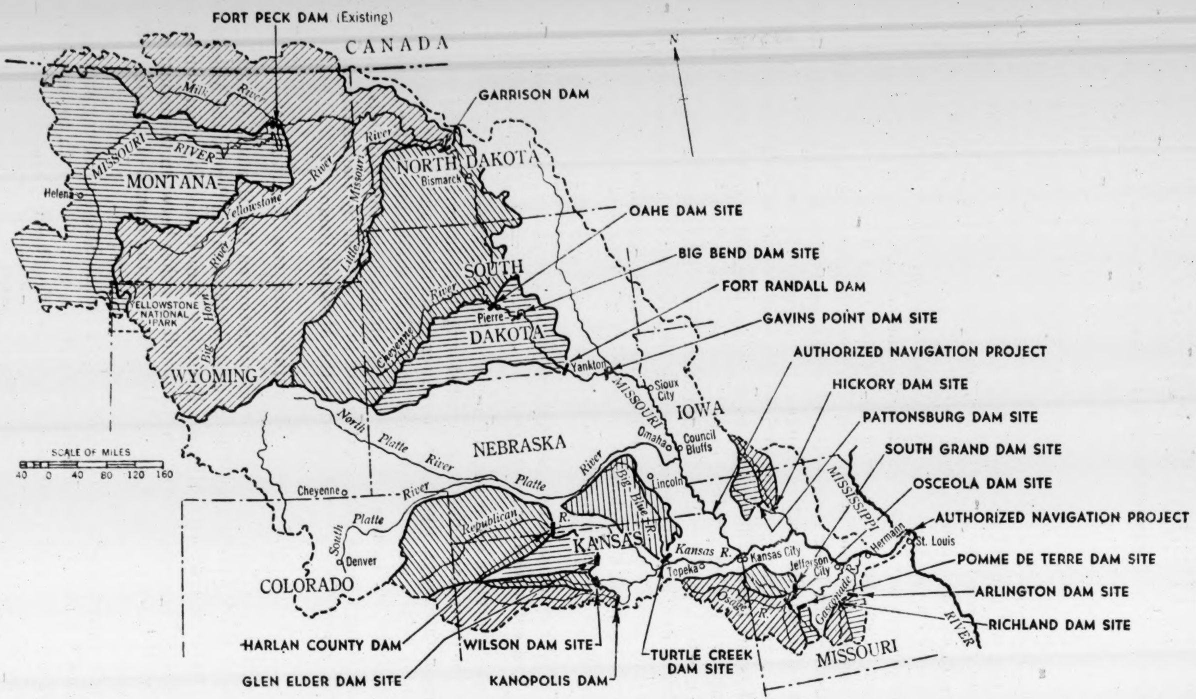
FORT PECK DAM (Existing)
CANADA
GARRISON DAM
OAHE DAM SITE
BIG BEND DAM SITE
FORT RANDALL DAM
GAVINS POINT DAM SITE
AUTHORIZED NAVIGATION PROJECT
HICKORY DAM SITE
PATTONSBURG DAM SITE
SOUTH GRAND DAM SITE
OSCEOLA DAM SITE
AUTHORIZED NAVIGATION PROJECT
POMME DE TERRE DAM SITE
ARLINGTON DAM SITE
RICHLAND DAM SITE
HARLAN COUNTY DAM
GLEN ELDER DAM SITE
WILSON DAM SITE
KANOPOLIS DAM
TURTLE CREEK DAM SITE
N
MONTANA
MISSOURI RIVER
Milk River
Helena
Yellowstone River
Little Missouri River
Big Horn River
YELLOWSTONE NATIONAL PARK
WYOMING
NORTH DAKOTA
Bismarck
SOUTH DAKOTA
Pierre
Cheyenne River
Yankton
MISSOURI
Sioux City
IOWA
Omaha
Council Bluffs
NEBRASKA
North Platte River
Platte River
Lincoln
Big Blue R.
Republican R.
KANSAS
Kansas R.
Topeka
Kansas City
Jefferson City
Osage R.
Gasconade R.
MISSISSIPPI
Hermann
St. Louis
RIVER
MISSOURI
Cheyenne
South Platte River
Denver
COLORADO
SCALE OF MILES
40 0 40 80 120 160

FIG. 7.—FEDERAL RESERVOIRS DIRECTLY AFFECTING THE MISSOURI RIVER

the waterway, cutoffs to improve alinement, closing of minor channels, removal of snags, and dredging as required. The Mississippi River navigation improvement downstream from the mouth of the Missouri River also is an "open-channel" project. With these improvements, substantial minimum river discharges are required to maintain minimum project depths, even with extensive dredging. Increased discharges above the minimum flow requirements decrease dredging requirements and costs, and are of additional benefit to river commerce through improvement of navigation conditions.

Adequate flows for the maintenance of minimum project depths are needed throughout the navigation season. On the Missouri River, because of severe winter ice conditions, the regular navigation season extends only from about April 1 to November 15 at Sioux City and Omaha, Nebr., and from about March 15 to November 30 at Kansas City, Mo., and at points downstream. On the Mississippi River downstream from the Missouri River, navigation continues throughout the year. During the autumn months of normal years (see Fig. 8), natural discharges of the Missouri River ordinarily are quite low,

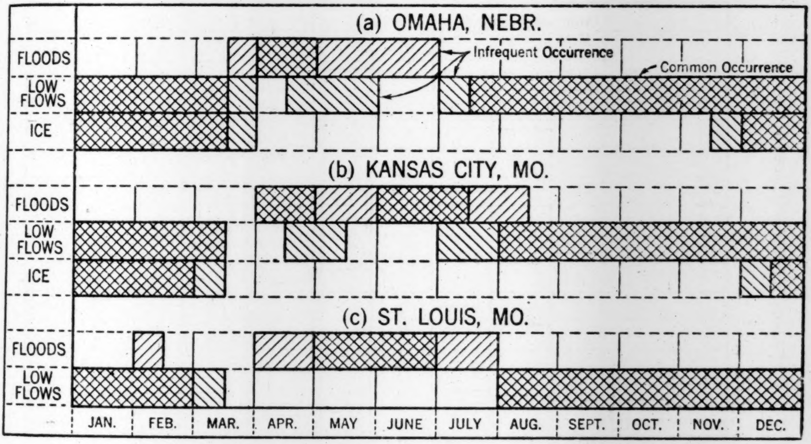

FIG. 8.—FLOOD AND LOW-FLOW MONTHS ON THE MISSOURI AND MISSISSIPPI RIVERS

and often are not sufficient for the Missouri River navigation project. During extremely critical low-flow years (not indicated in Fig. 8) adequate discharges may be available only for a few weeks. On the middle Mississippi River, natural river discharges are inadequate for navigation during the winter low-flow periods of ordinary years, and during the summer and autumn months of low-flow years. On the other hand, adequate discharges are available for these navigation projects during other periods—too much water is available during the flood seasons, and hinders river traffic and damages navigation improvement structures.

The periods of water deficiencies, under natural conditions, result from a great variation in the discharge of the Missouri River both within individual years and from year to year. There is a characteristic seasonal runoff pattern, as indicated by Fig. 9, with low winter discharges followed by a flood season in spring and early summer when river flows are seasonally high after which discharges rapidly fall off to much lower rates through late summer and autumn. This seasonal runoff pattern is superimposed on a quite variable annual pattern of normal years mixed with wet years, dry years, and droughts, as shown by Fig. 10. The Mississippi River at St. Louis reflects many of the same characteristics as the Missouri River, although variations are not so extreme. The variations in the natural flow of the Missouri River result in a twofold requirement for water supply for navigation: An annual regulation designed to adjust the greatly varying seasonal flow pattern to a more uniform and constant supply throughout the navigation season; and a long-range operation pattern planned to accumulate excess natural stream flows during wet years and to carry this excess water over for use during critical low-flow years and droughts.

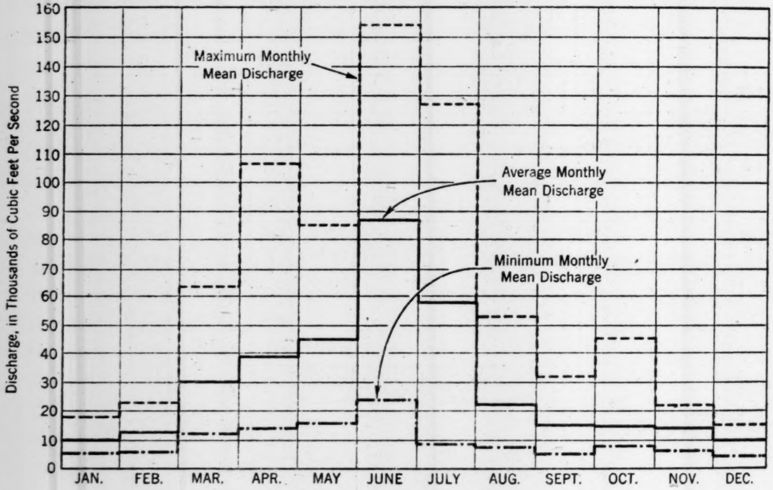

Fig. 9.—Normal and Extreme Monthly Runoff, Missouri River at Fort Randall (S. Dak.) Dam Site

Joint Use of Storage Capacity to Serve Navigation and Other Purposes

The need for annual and long-range storage regulation in the Missouri Basin for navigation is similar to, and consistent with, storage requirements for other purposes. The manner in which the various needs fit together, so that multiple-purpose reservoirs offer the ideal solution, is illustrated by the situation at the large main stem reservoir sites upstream from Sioux City.

Basic requirements at the main stem sites for annual flow regulation for navigation are quite similar to basic requirements for flood control. To obtain effective flood control along the Missouri River between Sioux City and the mouth, a large amount of flood control storage capacity is necessary at these main stem sites upstream from Sioux City. The flood control requirement is not simply one of providing sufficient storage to reduce excessive natural discharges at the dam sites to safe channel discharge capacities immediately downstream. Large drainage areas in the lower basin which will remain uncontrolled are themselves capable of producing serious floods on the Missouri River. The addition of large releases from the large main stem reservoirs, even though of less than channel-full magnitude at the dam sites, to flood flows from these uncontrolled tributaries, could result in a very ineffective form of flood control; yet, because of the long distances involved, river travel times are too long to permit forecasting of floods from uncontrolled lower basin tributaries in time to permit compensating reductions in releases from the main stem dams. (Water from the lowermost main stem dam will take at least 5 days to reach Kansas City, and 8 days to reach the mouth of the Missouri River.) Therefore for fully effective flood control all along the main stem of the Missouri River, floodwaters impounded in the main stem reservoirs, during the Missouri River flood season, should not be released until the low-water period, following the end of the flood season.

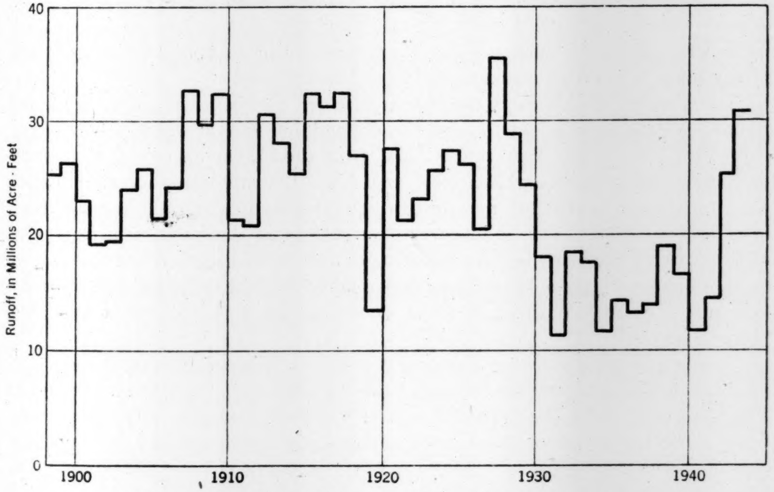

FIG. 10.—VARIATIONS IN ANNUAL RUNOFF AT YANKTON, S. DAK.

This requirement for flood control, quite obviously, is very similar to the annual flow regulation requirement for navigation during normal years. To exploit this opportunity, extensive study programs of historic and potential floods and of variations in seasonal flood control storage requirements were undertaken, and practical possibilities of prediction of seasonal and annual inflow to the reservoirs were considered. It has been found that a large proportion of the storage capacity that would be required for flood control, alone, also can be utilized to serve the requirements of navigation as effectively as if the storage capacity were reserved exclusively for navigation, without detriment to effective flood control. However, it should be noted that this particular joint-use storage space for both flood control and navigation cannot be used to carry water over from one year to another for navigation. To obtain assured flood control, all the flood control storage space must be vacated in advance of the yearly flood season.

The need for long-range carry-over storage of water supply for navigation at the main stem reservoir sites is paralleled by comparable needs for much of the new irrigation proposed for the Missouri Basin (about 2,000,000 acres of proposed new irrigation to be supplied by water directly from the main stem reservoirs), and also for flow regulation to generate firm power, as well as for other purposes. To meet all these various requirements, large blocks of multiple-purpose carry-over storage are to be provided in the main stem reservoirs, to supplement the natural river flow during low-flow years and droughts. This storage, which is not allocated to any single purpose, will be used primarily to provide a dependable minimum water supply: (1) To supply new irrigation in the basin, and to offset the increasing depletions by new irrigation in the upper basin of the limited natural water supply now available to the lower Missouri Basin during low-flow periods; (2) to furnish the remaining water with maximum efficiency for the navigation project; and (3) to generate a very large amount of marketable hydroelectric power.

The amount of this carry-over storage which can be provided at the main stem sites will be sufficient to raise the dependable annual water yield of that part of the basin above Yankton, S. Dak., from the 11,000,000 acre-ft which was available in years such as 1931, 1934, and 1941, to a minimum average annual yield of almost 17,000,000 acre-ft throughout a critical period such as that from 1930 to 1942. This joint carry-over storage in the main stem reservoirs will thus serve to reconcile the conflicting demands of irrigation and navigation in the Missouri Basin, providing dependably for the reasonable minimum demands of each. Without it there would not be sufficient water available "to go around."

The relationship between navigation and power is a special feature of the joint use of multiple-purpose storage at the Missouri River main stem reservoirs. The head created by the impoundment of water for multiple purposes such as flood control, navigation, and irrigation, coupled with releases through the dams to serve these purposes, results in a large amount of potential power. On first thought, it might appear that the relationship between navigation and power at the main stem reservoir sites would not be harmonious, since large releases for navigation are not regularly needed more than about 8 months a year, and much lower releases would be sufficient for all other purposes except power during the remainder of the year. However, with the system of Fort Peck, Garrison (N. Dak.), Oahe (S. Dak.), and Fort Randall (S. Dak.) reservoirs and the Big Bend (S. Dak) and Gavins Point (S. Dak.) projects (see Fig.

7), it has been possible to work out a plan for maintaining a very high firm power rate without loss of water down the river when it is not needed for navigation. This operation plan takes advantage of the proposed electrical interconnection of all these power plants, and provides for somewhat higher releases in the nonnavigation season at the reservoirs upstream from Fort Randall Reservoir, with recapture of the surplus water in the lowermost reservoirs for retention until the following navigation season. Thus, it has been possible to develop a practical solution which permits both navigation and power to use the same storage capacity jointly with the same full efficiency for navigation, and very nearly the same efficiency for power, as if this storage were utilized exclusively for either of these purposes.

The Missouri River main stem reservoirs form a system of multiple-purpose reservoirs in which nearly all the storage capacity can be used jointly and effectively to serve the needs of one combination or another of basically different purposes. As far as navigation is concerned, there will be:

(1) Top storage zones effective both for flood control and for annual flow regulation for navigation; and

(2) Large multiple-use storage zones which will enable both expanded irrigation and successful navigation, as well as large values for power and other purposes.

The latest studies available indicate that about 50,000,000 acre-ft of the total storage capacity of about 70,000,000 acre-ft proposed for the main stem reservoirs will fall into one or the other of these two multiple joint-use zones. Thus, great economies in cost, and consequent benefits to all the various purposes, result. In addition, the remaining 20,000,000 acre-ft provided for the various purposes not involved with navigation in multiple-purpose reservoir developments at these main stem sites enables this storage, too, to be secured more effectively and at substantially less cost to all purposes involved in reservoirs than would be possible if it were necessary to obtain this storage at other sites.

In connection with the navigation function of these multiple-purpose reservoirs on the main stem of the Missouri River, it is to be noted the reservoirs will supply all the water from storage required for the Missouri River navigation project at Sioux City, and also will furnish much of the water needed to supplement natural flows for satisfactory navigation on the lower Missouri River and on the middle Mississippi River. However, because of larger flow requirements resulting from wider channels and longer navigation seasons, these main stem reservoirs would not completely supply all the navigation water needed on the lower Missouri; and, with these reservoirs completed and in operation, there would still be winter flow deficiencies, from time to time, on the middle Mississippi River. Multiple-purpose tributary reservoirs, at economical reservoir sites on lower basin tributaries (where reservoirs are required as a part of the basin plan, for complete control of Missouri River floods) will serve to supply the additional navigation deficiencies on the Missouri River and, together with other reservoirs outside the Missouri Basin, will provide for navigation on the middle Mississippi River.

SUMMARY

Many supporting pertinent technical studies have necessarily been omitted from this paper. However, it should be emphasized that the detailed technical studies which have been made for the Missouri River projects have proved conclusively that, where there are varied requirements, multiple-purpose reservoirs incorporated in a comprehensive multiple-purpose basin plan can serve any individual need in the basin, such as navigation, fully effectively, with great economies in cost to the various other individual uses. Also, because of economies in reservoir sites and cost, multiple-purpose projects can be developed to serve the various needs, such as navigation, considerably better than would be possible by exclusively single-purpose project developments.

APPLICATION OF GENERAL POLICIES WHEN USED FOR IRRIGATION

BY WESLEY R. NELSON,[10] M. ASCE

Not one of the irrigation storage reservoirs constructed by the Bureau of Reclamation can be considered as a single-purpose reservoir, although the reservoirs have been planned, built, and operated for that one purpose, and the repayment of the cost of construction is being made only by the irrigation interests. Some measure of flood control, recreation, and fish and wildlife benefits is apparent in each storage reservoir.

An increasing recognition of multiple benefits has been evident since 1933. It has been especially evident in the work of the Bureau of Reclamation since the time plans were prepared for Hoover Dam. With this recognition has come the necessity of determining, within certain limits, the priorities of use among the various purposes. The state and federal governments have given, and are continuing to give, attention to this subject, and their ideas are reflected in the form of legislation. The operation of projects for irrigation and other purposes is dependent on federal and state policies and on contracts with the beneficiaries. Designs and operations of reservoirs have been changed to keep pace with these factors.

There is not much uniformity, and there is no standardization, apparent in the measures that have been adopted from year to year in the allocations to different purposes or in the methods of operation; but, to some extent, the examples given in this paper will explain the principles involved. The operating characteristics cannot be divorced entirely from the financial ones, because in Bureau of Reclamation projects the two are interdependent. Consequently, information on the cost allocations will be treated briefly in some cases.

The Rio Grande Project in New Mexico and Texas, which covers 155,000 acres of irrigated land, extends nearly 180 miles along the Rio Grande upstream and downstream from El Paso, Tex. The international boundary between the United States and Mexico, beginning at a point immediately above El Paso, is formed by the river. Water regulation and storage are provided by the Elephant Butte Reservoir, with a capacity of 2,219,000 acre-ft in 1940 (the date of the latest survey), 22 miles above the project lands, and the Caballo Reservoir with 346,000-acre-ft capacity, immediately above the project lands in New Mexico. A power plant of 24,300-kw installation is situated at Elephant Butte Dam.

Caballo Dam and the power plant at Elephant Butte Dam were added to the project works about 23 years after Elephant Butte Dam was completed. Caballo Dam was constructed in part to provide control of arroyo floods, but its principal function was to provide after bay storage capacity to reregulate releases from Elephant Butte Reservoir and make possible the generation of a large block of salable power at the Elephant Butte plant.

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[10] Asst. Commr., Bureau of Reclamation, U. S. Dept. of the Interior, Washington, D. C.

The initial contract with the project water users, signed in 1906, called for repayment to the federal government of all costs of Elephant Butte Dam and Reservoir. In 1907, $1,000,000 was declared nonreimbursable by appropriation from the General Treasury in accordance with a treaty with Mexico (signed May 21, 1906) whereby 60,000 acre-ft of water would be stored annually in Elephant Butte Reservoir and delivered to Mexico at a point near Ciudad Juárez. The treaty contains a schedule of water delivery for each month, but the supply is diminished in years of extraordinary drought or serious accident to the irrigation system in the United States. Although there was a storage allocation to flood control of 416,000 acre-ft, which covered the upper part of the reservoir from the inlet of the cylindrical gates (El. 4396) to the spillway crest (El. 4407), there were no nonreimbursable funds allotted for this purpose. After Caballo Dam and the power plant were completed (in 1938 and 1940, respectively), the cost allocations and the repayment contracts were revised. Approximately $1,500,000 was transferred to construction funds by the United States State Department to cover the flood control allocation for the Caballo Dam. This action was a part of the program of the International Boundary Commission for flood control and river rectification along the international boundary. An agreement was reached between the United States and the project water users whereby the remaining part of the costs for Caballo, the cost of the power plant and facilities, and the costs of construction of Elephant Butte Dam and appurtenant works were to be repaid solely from net power revenues. The reimbursable costs of the storage works were divided between irrigation and power; the allocation to irrigation to be repaid without interest, and the allocation to power with interest at 3%.

A summary of storage and cost allocations resulting from the various contracts and agreements is given in Table 2.

An agreement on the operation of Caballo Dam was reached with the State Department, whereby 100,000 acre-ft of storage would be available for flood control from June 1 to November 1 of each year. Although the major part of the costs of storage works must be repaid from revenues derived by sale of power, water may be released from Caballo Dam only when needed for irrigation or when spilled during floods after both reservoirs are encroaching on their flood control pools. It is obvious that the reservoirs must be operated very carefully to obtain the maximum power development, and at the same time to stay within the flood control limitations and avoid all discharge of unusable water from Caballo Dam. This situation is further complicated by provisions in the compact among Colorado, New Mexico, and Texas, which governs the distribution of use of Rio Grande waters above Fort Quitman, Tex. (80 miles below El Paso).

Benefits to fish and wildlife, recreation, and sediment detention have always been substantial at Elephant Butte Reservoir, but no allocations have been made to these uses. The City of El Paso obtains a part of its water supply from the project works. When the project was first planned, the city was given an opportunity to participate and to receive a water supply. It elected to continue to procure its supply from underground sources. Recently, the increased population of the city, the influx of industry, and the establishment

TABLE 2.—STORAGE AND COST ALLOCATIONS RESULTING FROM VARIOUS CONTRACTS AND AGREEMENTS INVOLVING THE RIO GRANDE PROJECT IN NEW MEXICO AND TEXAS

Item No. (1)	Use allocation (2)	Elephant Butte Dam and Reservoir (New Mexico) (3)	Project diversion, distribution, and drainage system (4)	Power system (direct cost) (5)	Caballo Dam and Reservoir (New Mexico) (6)	Totals (7)
1	Purpose	Multiple[a,b]	Irrigation	Power	Multiple[a]	....
	(a) STORAGE ALLOCATIONS, IN ACRE-FEET					
	Initial Allocations—					
2	Flood control	416,000[c]	....	....	100,000	516,000
3	Irrigation	2,222,860	....	....	245,000	2,467,860
4	Total	2,638,860	....	....	345,000	2,983,860
	Allocations as of 1940 Survey—					
5	Flood control	388,700[c]	....	....	100,000	488,700
6	Irrigation	1,830,300	....	....	245,000	2,075,300
7	Total	2,219,000	....	....	345,000	2,564,000
	(b) COST ALLOCATIONS, IN DOLLARS, RESULTING FROM CONTRACTS AND AGREEMENTS					
	Prior to 1937—					
8	Storage for Mexico	1,000,000	....	....	....	1,000,000
9	United States irrigation	4,246,408	11,269,087	....	....	15,515,495
10	Total	5,246,408	11,269,087	....	....	16,515,495
	Prior to October 31, 1946—					
11	Storage for Mexico	1,000,000	....	....	....	1,000,000
12	United States irrigation	2,123,204[d]	11,402,130	....	383,954[d]	13,909,288
13	Flood control and river rectification	....	....	....	1,510,654[e]	1,510,654[e]
14	Power	2,123,204[d]	....	2,921,290	383,954[d]	5,428,448
15	Total	5,246,408	11,402,130	2,921,290	2,278,562	21,848,390
16	Total estimated construction cost	5,246,408	11,778,244	5,549,575	2,278,562	24,852,789

[a] Flood control, irrigation, power, and fish and wildlife benefits. [b] Recreation and sediment detention. [c] Considered as available for flood control but not specifically allocated by federal regulation or legislation. [d] To be repaid from net power revenues. [e] The United States State Department allotted $1,498,240 of Caballo Dam funds for flood control and contributed $12,413.73 for preliminary surveys and investigations. These funds are not reimbursable.

of a large army post produced a much greater demand on the underground reservoir than could be met by recharge. Since all the waters stored at Elephant Butte were required for project lands, it was necessary for the city to purchase lands and thus obtain rights to water that could be used for domestic purposes.

It was necessary to operate the project storage works for flood control in the spring and summer of 1942. Unusually large inflow into the reservoir in May and June, 1941, was followed by heavy runoff from the basin above the reservoir in the following spring. The outflow from both Elephant Butte and Caballo reservoirs was held substantially below the rate of 11,000 cu ft per sec as required by the agreement with the State Department; but some damage was caused in areas near Hot Springs, N. Mex., between Elephant Butte and Caballo reservoirs, where property had encroached on the flood plain.

Fig. 11 represents a chart from the "Rio Grande Project History of 1942," showing the operations during the flood control period.

Under normal conditions, the discharge from Caballo Dam is entirely for irrigation, and Elephant Butte Dam and power plant are operated to avoid

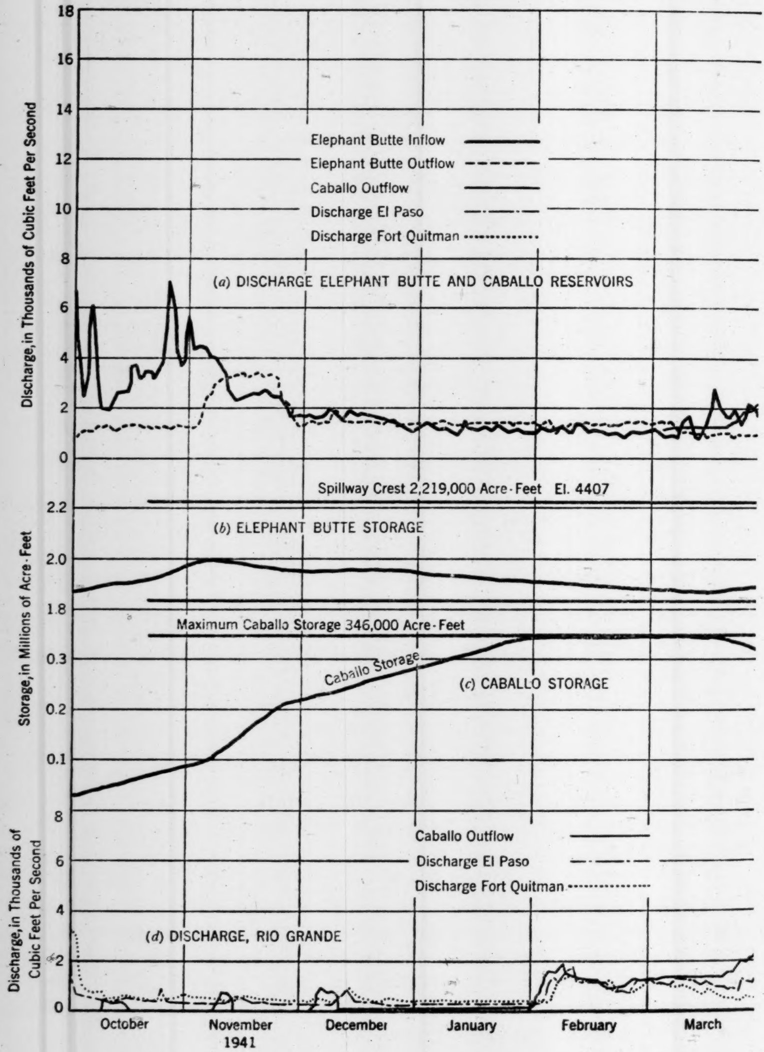
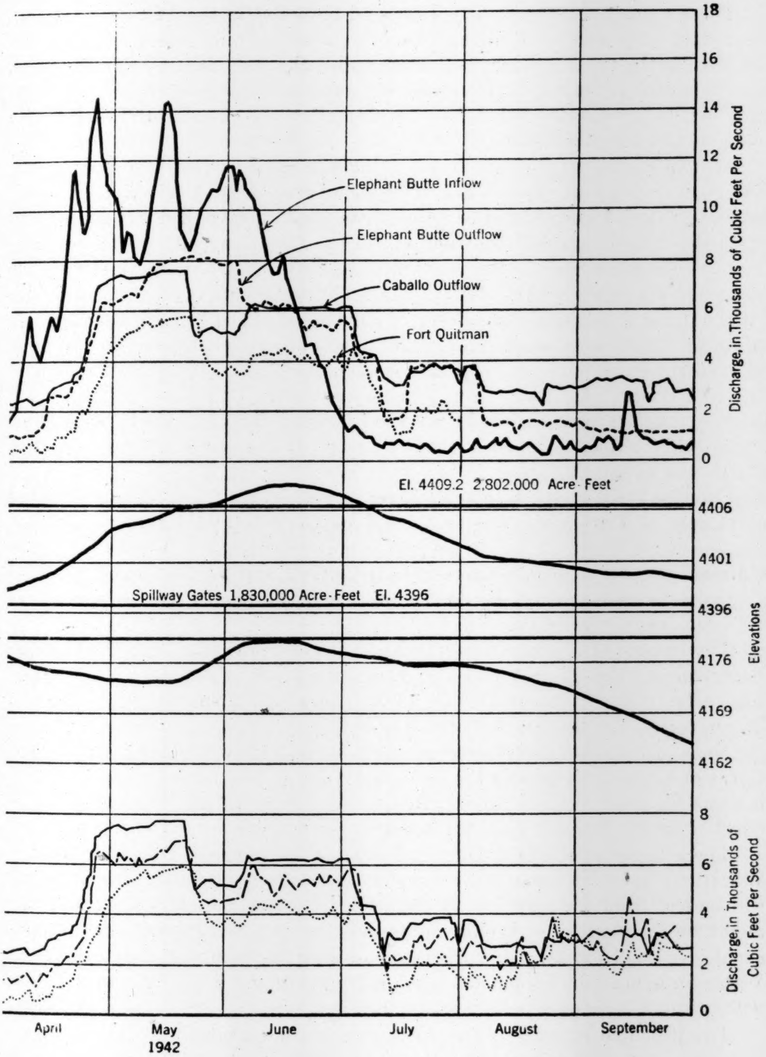


FIG. 11.—OPERATION OF ELEPHANT BUTTE AND CABALLO RESERVOIRS IN

raising the elevation of Caballo Reservoir to a level that would make spills of nonirrigation water necessary. Farmers on the project inform the ditch rider when they will need water and how much will be required. This information is passed on to the project office where data are compiled and orders given for



New Mexico for the Period from October 1, 1941, to October 1, 1942 the release of water from Caballo. The operator at Elephant Butte Reservoir is notified of the quantity of water that can be spilled from the reservoir during each month. On some occasions, particularly in the spring, when it is necessary to evacuate the flood control pool at Caballo Reservoir, operations of the power plant are restricted.

These factors are the principal ones to be noted in the Rio Grande case. Decisions, with regard to allocations, were reached only when exigency demanded, as the circumstances indicated, and without an analysis of all benefits and potential allocations. However, the project works are so designed that they can be operated satisfactorily for irrigation, flood control, power development, and other purposes.

Although considered originally as a single-purpose project, the Alamogordo Reservoir on the Pecos River, 247 miles above the Carlsbad (N. Mex.) Project, which it serves, has captured or reduced the peak of several floods since 1937, when the dam was under construction. The present capacity of the reservoir is about 132,000 acre-ft. Recognition of its flood control potentialities is expressed in a federal act (Act of August 11, 1939; 53 Statute 1414) which provides for the determination, by the Corps of Engineers, of the benefits the reservoir affords. There are some fish and wildlife and recreational benefits, and prior to 1946 the permissible drawdown was such that only 5,000 acre-ft remained in the reservoir. In 1946, the water shortage on the Carlsbad Project became so acute that the reservoir was lowered to 1,500 acre-ft, the maximum drawdown that could be accomplished without damage to the outlet valves. No loss of fish was apparent.

Conchas Dam on the Canadian River in east central New Mexico, 35 miles northwest of Tucumcari, was completed by the Corps of Engineers in 1939. Storage capacity amounting to 600,000 acre-ft was assigned: 100,000 acre-ft to sediment detention; 300,000 acre-ft to irrigation; and 200,000 acre-ft to flood control. Storage capacity for flood control is in that part of the reservoir above the permanent crest of the dam spillway.

The Bureau of Reclamation is building the 38-mile canal from the Conchas Reservoir to the project lands and the irrigation distribution and drainage systems for a project area of 45,000 acres which surrounds the City of Tucumcari.

None of the costs for the reservoir is to be repaid by the water users. The Corps of Engineers operates the dam, reservoir, and all outlet works including the one for the irrigation canal. Releases of irrigation water are made as requested by the Bureau of Reclamation.

There is considerable fishing and boating at Conchas Reservoir, but no allocations have been made for such benefits. It is probable that some fish will be lost from the reservoir through the irrigation outlets. This condition might be rectified by screening the outlets or by placing a trap in the main canal. A small 150-kw plant provides power for the dam and camp operations. Water for operation of the plant is taken from any water available in the reservoir, without reference to a specific allocation.

The principal features of the Altus Project in southwestern Oklahoma are the Altus Dam and Reservoir on the North Fork of the Red River and a canal and distribution system for the irrigation of approximately 60,000 acres south of the dam. The reservoir capacity is 188,414 acre-ft. The allocations made by the Bureau of Reclamation for regulation purposes, and the cost allocations, as established in the "Findings of Feasibility" (dated January 21, 1941) were as follows:

Allocated to:	Acre-feet	Costs
Silt	43,000	....
Municipal supply	4,800	$1,080,000
Irrigation	103,850	2,000,000
Flood control	36,764	1,130,000
Unemployment relief	....	1,390,000

The allocation of space for flood control is above the crest of the ungated spillway. The Work Projects Administration participated in the unemployment relief. The estimated cost of the project has increased since the "Findings of Feasibility" but the allocations have not been changed.

This project is one for which the Secretary of War is to prescribe regulations for use of storage allocated to flood control under the terms of Section 7 of the Flood Control Act of 1944 (58 Statute 890). The tentative agreement is that the flood control discharge facilities shall be operated under the direction of the district engineer of the Corps of Engineers whenever the pool stage reaches the crest of the ungated spillway. The district engineer then directs the operations of the discharge facilities to prevent flood damage below the reservoir and to limit the pool stage to an elevation 3 ft above the crest of the ungated spillway as far as possible.

Hoover Dam, one of the most successful multiple-purpose projects in the United States, is located on the Colorado River, 420 miles above its mouth, where the river forms the boundary between Nevada and Arizona. The dam is 8 miles east of Boulder City, Nev., and 71 miles northwest of Kingman, Ariz. The reservoir, Lake Mead, has a capacity of 31,142,000 acre-ft. The power plant, located at the toe of the dam, will have a total capacity of 1,317,500 kw, of which 1,030,000 kw is now (1948) installed.

The purposes for which the reservoir and power plant were built, as set forth in Section 1 of the Boulder Canyon Project Act (Act of December 21, 1928; 45 Statute 1057), are:

1. Control of floods;
2. Improvement of navigation and regulation of the flow of the Colorado River;
3. Provision for storage and for the delivery of the stored waters for reclamation of public lands and other beneficial uses within the United States; and
4. Generation of electrical energy to make the project a self-supporting and financially solvent undertaking.

Engineering studies indicated that the lower 300 ft to 360 ft of the reservoir should be allotted to silt and sediment detention; the next 155 ft to 215 ft to active storage or river regulation; and the upper 75 ft to flood control. The approximate reservoir capacities are 3,000,000 acre-ft for silt detention;

20,000,000 acre-ft for active storage and river regulation; and 9,000,000 acre-ft for flood control.

The initial legislation provided that the cost of all the works would be repaid in 50 years with 4% interest through the sale of stored water and electrical energy, but that no charge should be made for water stored for the Imperial and Coachella valleys in California. The sum of $25,000,000 of the construction cost was allocated to flood control and was to be repaid to the United States out of a percentage of the revenues which were in excess of the amount needed to meet periodic payments during the period of amortization. The Boulder Canyon Project Adjustment Act (Act of July 19, 1940; 54 Statute 774) revised the repayment plan. It provided that the payment of the $25,000,000 allocation to flood control was to be deferred to the period starting June 1, 1987, and to be repaid without interest. The remainder of the cost of the dam and power plant was to be repaid with 3% interest from revenues from the sale of power and water. The costs of operation and maintenance were to be met by an advanced payment each year of generating charges. As of May 31, 1945, the end of the project's fiscal year, $37,700,000 had been returned to the Treasury from the sale of water and power. Generating charges for 1945 totaled approximately $1,630,000.

Among the unique provisions of the act were those providing for payment of $300,000 per yr from 1938 to 1987 to each of the states of Arizona and Nevada and for the transfer to a special fund in the Treasury, designated as the Colorado River Development Fund, of $500,000 a year from 1938 to 1987 for studies, investigations, and construction of projects in the Colorado River Basin.

In accordance with the terms of the initial authorizing legislation, contracts for the sale of water and power were obtained before construction was initiated. After the adjustment act was passed, the contracts were amended and arrangements made for the power allottees to operate the electrical generating and transmission system. The Bureau of Reclamation, however, operates all valves and gates which control the flow of water from the reservoir.

It will be noted that the allocations of reservoir capacity do not apply specifically to the purposes for which the project was authorized. Furthermore, the policies for repayment differ from the general ones pertaining to irrigation and flood control in that repayment is required for flood control and is made for irrigation with interest. The recreational and fish and wildlife benefits are very great at Lake Mead, but allocations have not been made for these purposes.

The water stored in Lake Mead must be conserved for municipal and irrigation uses downstream; therefore, power and flood control releases must be regulated in such manner that the water may be used to full advantage on the projects below the dam.

Among the more important projects already constructed below Hoover Dam are the Parker Dam power plant and the pumping plant of the Los Angeles aqueduct immediately upstream from Parker, Ariz.; the Palo Verde Irrigation Project near Blythe, Calif.; the Headgate Rock Diversion of the Colorado River Project of the Indian Service; and the diversion dams for the Imperial Valley and Yuma projects near Yuma, Ariz. These projects are from 150 miles to 300 miles from the dam. Their needs for water fluctuate from day to day; and, on account of the long distances involved, careful scheduling must be followed to avoid wastes on the one side or water shortage on the other. In addition, water releases and river control are required for the delta area in Mexico, under the terms of the Mexican Treaty, which became effective on November 8, 1945. These requirements make it desirable to route all water requests to a river control officer of the Bureau of Reclamation, who compiles the data, coordinates it with the required operations of Lake Mead for power development and flood control, and designates the quantity and time of water releases from Hoover Dam.

The Davis Reservoir is being built (1948) in Arizona-Nevada 65 miles downstream for power development and river regulation. When completed, this structure should aid materially in increasing the efficiency of operation of Hoover Dam. As the development of the Colorado River Basin progresses and the states of the basin reach an agreement on the allocation of the river waters, the importance of Hoover Dam as the key structure in the river basin development will become increasingly apparent.

An enormous amount of study has been given to the design and the operation of the Columbia Basin Project, in Washington, to obtain the maximum control and use of the Columbia River waters and to provide the greatest benefits for multiple purposes.

The Grand Coulee Dam, the principal feature of the project, is situated on the Columbia River, 94 miles northwest of Spokane, Wash. Other features of this project are:

(*a*) The 9,500,000 acre-ft reservoir back of the dam;

(*b*) Power plants situated one at each end of the dam, which will have a final capacity of 1,944,000 kw, 648,000 kw of which is installed;

(*c*) The pumping plant, with a capacity of 16,000 cu ft per sec and 650,000 hp, which will raise water a maximum of 360 ft from the reservoir to the irrigation project works;

(*d*) A regulating reservoir of 1,202,000-acre-ft capacity; and

(*e*) Main canals (440 miles) to serve the project area of 1,022,000 acres, lying in the Grand Coulee south of the dam.

The power plant is connected to the high voltage power network of the Bonneville Power Administration, which markets the power from the Bonneville (Ore.) plant, built by the Corps of Engineers at a site 450 miles below Grand Coulee Dam.

The principal purposes served by the Grand Coulee Reservoir are navigation, flood control, irrigation, power and river regulation. Specific allocations of storage capacity have not been made to different purposes, the benefits being obtained through appropriate regulation of the reservoir.

An indication of the elements that will be considered in the operation of the project can be obtained from the final cost allocation determinations that were made in a report by the U. S. Department of the Interior, submitted to Franklin D. Roosevelt, then President of the United States, and to the Congress:[11]

Irrigation—including power for pumping	$341,929,994
Present commercial power	113,827,243
Downstream river regulation	30,272,991
Flood control and navigation	1,000,000
Total	$487,030,228

It was decided that the costs of constructing the project works should be met by (1) repayments by the water users within their limits to repay, (2) a nonreimbursable allocation to navigation and flood control benefits, and (3) net revenues from the sale of power generated at Grand Coulee Dam and generated at plants downstream, when the power output was benefited by the river regulation provided by Grand Coulee Dam. The basis for repayment of the reimbursable costs is given in the report.[12] For example, the cost requirements and the revenues needed to repay total reimbursable costs plus interest at 3% on investment allocated to power (over repayment period ending in 2017, inclusive), are:

Item	Description		Cost
	Requirements:		
1	Reimbursable construction cost		$486,030,228
2	Replacements		72,920,000
3	Other operating expenses		151,567,042
4	Interest on unamortized balances of investment allocated to power		70,786,815
5	Total		$781,304,085
	Revenues:		
	From Water Users—		
6	For pumping power	$50,500,000	
7	Construction cost repayment	87,465,000	
8	Total revenues from water users		$137,965,000
9	Remainder required from commercial power		$643,339,085
10	Revenue for river regulation at the Bonneville project		9,378,500
11	Remainder required from sale of power generated at Grand Coulee		$633,960,585

The reimbursable construction cost (item 1) is equal to the total cost of the project ($487,030,228) less $1,000,000 which is allocated to navigation and flood control. Items 3 and 8 do not include operation and maintenance expenses of the irrigation system, amounting to several million dollars annually.

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[11] *House Document No. 172*, 79th Cong., 1st Session, 1945, p. 21, Table 8.

[12] *Ibid*, p. 26, Table 11.

The sum required from the sale of power generated at Grand Coulee (item 11) is in excess of that required by law to the extent of about $70,786,815, which is equivalent to the interest on the investment allocated to power.

Items 1 to 11 show that there was no allocation of part of the costs to "fish and wildlife" or to "recreation." The benefits to these purposes are substantial, although there are some offsetting damages to fish propagation. The Grand Coulee Dam blocks the run of salmon into the spawning grounds in the upper reaches of the Columbia. Runs are being reestablished in tributaries below the dam, requiring a considerable expenditure of money for traps, hatcheries, transportation equipment, and operations.

Another feature of the Columbia River Project is the proposal to secure returns from the benefits to present and future downstream power plants; but, in the allocation of repayments, the only revenues anticipated at present (1948) are those from the Bonneville plant.

Several more of the fifty-seven operating projects of the Bureau of Reclamation could be described to point out certain distinguishing characteristics of design and operation, but the general principles would be similar. The Rio Grande Project is typical of those where there are definitely recognizable multiple purposes. On this project reservoirs can be operated for many purposes, without appreciable conflict, provided that the various potential uses are recognized in the planning stage and the designs are prepared accordingly.

The design, and particularly the operation, of reservoirs on federal irrigation projects are not dependent solely on engineering considerations, river discharge characteristics, or the regulation of stream flow for optimum benefits for all purposes. They are influenced in considerable degree by financial and and legislative matters which apply to each separate project.

It is extremely desirable to analyze all potentialities of proposed projects to avoid costly reconversion or duplication of construction features; and it is desirable, where feasible, to build and operate the works for all practicable purposes. At the same time, there should be recognition of all benefits in multiple-purpose projects and general uniformity and standardization should be provided when cost allocations are determined. These objectives will be attained only as fast as the states and the federal government—and especially federal agencies—make their decisions on water policies. There is much dissimilarity in the approaches of different agencies to problems of investigations and operations, with consequent confusion in the handling of water matters. This situation can be rectified only when an adequate national water policy has been adopted. Then, and only then, will it be possible to attain any appreciable uniformity in the determination of allocations for various purposes in multiple-purpose projects. It is hoped that the impetus afforded by the papers of this Symposium will develop the guidance that is needed so greatly.

COORDINATION WITH THE ELECTRIC UTILITY INDUSTRY

BY J. B. THOMAS[13]

FOREWORD

The general background of the extent, character, and cost of multiple-purpose dams in the United States and of the trends established and the problems posed is presented in this paper. A general résumé and discussion of a group of such dams in the southwest, with which the writer has had direct experience are also included.

MULTIPLE-PURPOSE DAMS IN GENERAL

The building of this type of dam has been almost solely developed as a governmentally financed device based on general public benefits. Taxpaying industry, which must finance itself and otherwise "stand on its own feet," has not found economic justification for such structures.

Relatively early in its history, the United States Reclamation Service found that there was some electrical energy available at its dams over and above that needed to operate the control gates and other equipment. It marketed this power successfully to farmers and to neighboring utilities. The proceeds of these sales were used to reduce the sums to be paid by the settlers for water to irrigate their lands. In all these earlier structures, although the power supply has become a comparatively large income producer, the reclamation objective has remained paramount, and power is still subordinate. The power supply has been coordinated, completely and efficiently, with the other power resources in the areas through sales to existing utilities.

Hoover Dam, which was started in 1930, and placed in operation in 1936, was the first of the great multiple-purpose dams to be built with power production as the primary purpose of the enterprise, and to present a coordination problem of large proportions.

After Hoover Dam was completed, many multiple-purpose dams were undertaken, and many more are scheduled for future construction, practically every one of them with power as its predominant purpose (see Table 3). In many cases, the actual cost of the projects that have been completed has overrun the estimate on which they were authorized. Taking into account such overruns and the rapidly mounting costs of construction, it would appear that the total capital expenditures for the 289 projects listed in line 7, Table 3, may reach a total of $15,000,000,000. The total installed capacity, present and prospective, in the 289 projects is 79% of the 40,360,000 kw of installed capacity of privately owned utilities as of December 31, 1946. The successful coordination of such a great amount of government power with the existing power resources obviously is no small undertaking, and if badly handled government power could be very harmful to the utility industry and costly to ultimate consumers and taxpayers.

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[13] Pres. and Gen. Mgr., Texas Electric Service Co., Fort Worth, Tex.

Obviously, the output of completed multiple-purpose dams must be marketed—either by displacing existing sources of power, or by using the dams to take care of their appropriate part of the increases in the demand of the existing market.

TABLE 3.—MULTIPLE-PURPOSE DAMS INVOLVING ELECTRIC POWER, AND FINANCED BY FEDERAL FUNDS

Line	Description	No. of dams	Kilowatts installed (ultimate)	Estimated cost
1	Completed and in service since 1933	30	6,286,500	$1,661,360,000
2	Steam stations and single-purpose hydro plants built in conjunction with, and because of, multiple-purpose dams	3	342,000	49,260,000
3	Total dams in service	33	6,628,500	$1,710,620,000
4	Under construction	28	3,142,550	872,320,000
5	Authorized and approved	132	11,915,700	3,100,440,000
6	Being actively proposed	96	10,179,300	2,615,500,000
7	Total built, being built and in line for early construction	289	31,866,050	$8,298,880,000

It should be noted that the generation and the sale of electricity in appreciable commercial quantities date from as recently as the 1880's. Invention and development were the results of private initiative and private enterprise. Both steam-electric and hydroelectric generating stations of various characteristics were built, and their operations coordinated. Step by step, large interconnected transmission networks were constructed and operating methods developed. These improvements led, in turn, to the formation of interconnected groups working together, with attendant economies and improved power supply. Today, the United States leads the world in the use of electricity. The outstanding accomplishment of the utility industry was in meeting all the power needs of the nation in full throughout World War II, despite great shortages of equipment, labor, and supplies, although every other commodity was in short supply, or strictly rationed. Morever, this need was met without increasing rates, despite a great increase in the price of supplies used by the utilities, and a large increase in utility taxes. Such results are indicative of the vigor and resourcefulness of the electric utility industry—all without tax remission or subsidy. It stands to reason that, if there were economic justification for a taxpaying individual or group to build such multiple-purpose projects, many would have been so built.

The electric utility industry is willing and able to meet the future power needs of the United States without benefit of subsidy or special privileges, and the industry has the technical skill and experience to make use of the power from multiple-purpose dams to the best advantage when and if it is called upon to do so. Every evidence points to a long-continued growth in the use of electricity. Therefore, existing efficient power resources need not be displaced by multiple-purpose dams. If the new plants are properly scheduled and designed, they can be utilized to carry the new load in conjunction with existing generating facilities and additional base load stations.

Ever since its early beginnings, the electric utility industry has been principally privately owned and partly government owned. For many years the relation between these two sections of the industry was relatively constant. About 95% of the public supply was generated by privately owned utilities, and about 5% by government owned utilities, mainly municipalities. About 1933, the proportion of government owned power production began to increase rapidly. In 1946 it amounted to 18.86% of the total produced for public use, or about 3¾ times the proportion prevailing prior to 1933. Multiple-purpose dams built by governmental financing, with the attendant tax advantage to the users and corresponding loss to other taxpayers, were largely accountable for the increase.

Much of the increased production in public power generating stations has been marketed through existing utility systems. In 1945, privately owned utilities accounted for about 90% of the "revenues from ultimate consumers" and public power about 10%, and the 1946 proportion probably was about the same. The outlet for marketing and coordinating the power from multiple-purpose dams is still predominantly through privately owned utilities.

However, in discussing the problem of coordinating this vast amount of very specialized hydro power with the existing power facilities, certain other problems must be taken into account. Multiple-purpose projects vary widely in respect to cost, desirability, usefulness, and amount and kind of power; but they all have some common economic characteristics which have a direct bearing on coordination and marketing, such as

1. All are more or less the recipients of financing and tax advantages at the cost of the general taxpayers;
2. The power from these dams, contrary to common belief, is not cheap. Its actual cost is often greater than for equivalent power from other available sources; and
3. The cost of many of these dams is so great that, to make them "money out," even under the government formulas used to justify their construction, it is necessary to design them to operate at very low load factors.

The need to show that large power values are inherent in these projects has led to some designs that seem to have little relation to the power requirements of the territory. For example, the engineers laying out multiple-purpose dams involving power and flood control or navigation are furnished a formula (usually in the form of a two-part rate) by the Federal Power Commission, with which to place a dollar value on the power and energy to be produced. These formulas vary widely with localities and with the price of fuel. The "Texas formula" in Table 4 is the one used for the Red River in Texas and Oklahoma. It sets the value of power capacity at $10.50 per kw-yr and the value of energy at 0.83 mill per kw-hr. The "Virginia formula" in Table 4 was used in connection with the Buggs Island development on the Roanoke River in Virginia. The rate under this formula is $17.20 per kw-yr plus 3.12 mills per kw-hr. Both formulas are used in fuel-producing areas.

The Division of Power of the United States Department of the Interior furnished a list of proposed multiple-purpose dams in the eastern part of the United States, the output of which must be absorbed by the load of the territory. An estimate of the firm power is given for fifty two of the projects on the long list. The load factors (ratio between average prime power and installed capacity) indicated are:

TABLE 4.—VALUE OF 1,000,000 KW-HR OF ENERGY, IN DOLLARS PER YEAR

Formula	LOAD FACTORS				
	100	75	50	25	10
Texas	2,029	2,428	3,227	5,624	12,816
Virginia	5,083	5,738	7,047	10,974	22,754

Load factor (%)	No. of multiple-purpose dams
72	1
50	3
37 to 31	4
29 to 26	4
25	24
24 to 21	5
19 to 15	5
14 to 9	6
25 (weighted average)	52

Local rainfall and runoff conditions greatly affect the design of multiple-purpose dams. A comparison between two diverse areas is instructive. The Columbia River in the northwest has a large dependable flow because it is fed by great snow accumulations and great glaciers, and is replenished by heavy rainfall. The Red, Arkansas, White, and Ouachita rivers in the southwest, on which many multiple-purpose dams are planned are flash rivers lacking dependable flow. The four rivers together have about the same drainage area as the Columbia; yet their total volume of water (floods and all) is only 38% of that of the Columbia.

The federal government has built two multiple-purpose dams on the Columbia River. They are practically completed, except for a few generating units being installed (1948) at Grand Coulee. The federal government has built or financed three, and is building or planning to build fifteen more multiple-purpose dams on the Red, Arkansas, White, and Ouachita rivers. The comparison in Table 5 is indicative of the wide variations between the two sections.

The comparison would be even more striking if estimates for the proposed plants in the southwest were available, as they would be much higher than those shown in Table 4; but, taking the data as they are, for each 1,000,000 kw-hr, of primary energy per year a capital expenditure of $23,818 was necessary on the Columbia, and $221,954—or nearly ten times as much—will be required for the multiple-purpose projects in the southwest. A gas-fired steam-electric station in the southwest, built for continuous operation will cost, at 1948 high prices, about $16,250 for each 1,000,000 kw-hr of primary energy, or only about one fourteenth of the cost of a multiple-purpose dam! Regardless of what may be done by allocating part of the cost of the dams to purposes other than power, and regardless of the difference in production cost between hydro and steam, it is evident that hydro plants in the southwest can be used effectively only if a peak load is carried exclusively. On the other hand, Columbia River power is suitable for base-load operations. Obviously, the coordination of power sources having such widely different characteristics cannot be realized by a single arbitrary formula. Each situation is different, and each problem must be solved by itself.

TABLE 5.—Comparison of Federal Multiple-Purpose Reservoir Projects in the Columbia River Area with Those in the Southwest

Line	Description	Columbia Basin	Current Projects[a]	
			Quantities	%
1	Drainage area, in square miles	237,000	238.100	..
2	Ultimate installed capacity, in kilowatt-hours	2.200,000	1,566.000	71
3	Annual primary energy, in kilowatt-hours	12.255.000,000	2,333.600.000	19
4	Average annual energy, in kilowatt-hours	16,728.000.000	3,765.600,000	23
5	Estimated cost of projects, in dollars	291,858,872	517,951,000	177

[a] On the Red, Arkansas, White, and Ouachita rivers; values in the last column are percentages of the Columbia Basin quantities.

The principles that should govern each of these separate studies have been time tested and proved through many years of experience. Few utilities are self-sufficient in regard to power. Most of them are interconnected with neighbors and, in one way or another, participate in the pooled operation of the power resources of the area. During World War II, these power pools were very extensive in size. The one in the southwest, for example, comprised sixteen privately owned utilities and seven public power utilities. It extended from western Texas to central Mississippi, and from the Rio Grande to northern Nebraska. A total of eighty steam-electric stations (ranging from highly efficient base-load plants to old plants of the "teapot" variety), ninety-four internal combustion stations, and thirty-four hydroelectric stations (the largest of them equipped with reservoirs) was involved.

Each station was scheduled for operation day by day, and sometimes hour by hour. The place on the load curve assigned to each plant gave full recognition to its characteristics, capability, efficiency, and availability, and to the prevailing transmission limitations. The most efficient steam plants were placed on base load. The hydro plants were located on the load curve at the point where the available water would enable the maximum capacity possible to be realized. Then, one by one, the other steam and thermal plants were added to the load in the order of their respective efficiencies, leaving the peak loads to be carried by the steam stations of poorest efficiencies, unless it proved best to pick off the peaks with the hydro plants. Steam units were scheduled for overhaul at the most suitable times; and, wherever possible, water was stored in the reservoirs of the hydro plants to be used at critical times, by substituting steam power for the hydro power and temporarily curtailing hydro operation.

To realize the best efficiency and to make the maximum effective use of each power resource, the operation of such a system cannot be allowed to become static. It must be flexible and subject to prompt adjustment to meet the constantly changing conditions.

The transmission system of such a coordinated operation is in reality an integral part of the power facilities, and should not be considered separately. Experience has shown that the most efficient and the least costly transmission system is one that fulfils several requirements. A well-designed network generally will:

1. Deliver power and energy to the various load centers;
2. Provide two-way service to all important points;
3. Interconnect and integrate the power resources;
4. Interconnect with neighboring utilities; and
5. Constitute an effective link in the regional power system.

When the Southwest Power Pool was formed during World War II, very little reinforcing was needed to enable the existing transmission systems to perform the necessary coordinating operations, and the job was done for a reasonable cost because the methods used were economically sound.

Although the power pools during the war dramatized the advantages to be gained by proper coordination between power resources, the principles had been developed and long been used by most of the power systems in the United States. These proved coordination methods, in respect to both generating plants and transmission lines, are precisely those needed to make the most effective use of the output of multiple-purpose dams. It is instructive, therefore, to review what has been done toward coordinating the output of the multiple-purpose dams already in service and what should be done about future dams. Should the coordination be by nationalization of the utility business, or by interaction with the taxpaying utilities that constitute a vital link in the American way of life?

To date, little if any real attempt has been made to plan, schedule, and operate multiple-purpose dams to fit the power requirements of the areas in which they are located. The federal government has confined its efforts to marketing such power, using the following two methods:

(*a*) Sales at powerhouse in wholesale quantities; and
(*b*) Establishment of regional authorities.

Selling power in wholesale lots at the powerhouse has been very successful for the most part. This plan was followed at the early Reclamation Service dams and is used at Hoover Dam, where the available power is allocated among the distributors, public and private, who have built transmission lines to the powerhouse to take the power away. Although there is no evidence available that the use of Hoover Dam power has resulted in any savings to the distributors as compared with other available sources of power, this method of marketing the power output has proved quite favorable for the federal government and it has resulted in conserving fuel in a territory where cheap fuel no longer is obtainable in necessary amounts. Hoover Dam is one of the governmental projects that is "moneying out" as planned.

The outputs of the Morris Sheppard (Possum Kingdom) Dam in Texas, Denison Dam in Texas and Oklahoma, and Norfork Dam in Arkansas are all sold at or near the powerhouse.

The sales of the output of multiple-purpose dams at or near the powerhouse, and marketing of the power through the facilities of the existing utilities, have been proved successful, especially in respect to the financial returns to the government, and this practice minimizes the cost of energy to consumers. The method will be even more successful in the future if greater care is taken to design the multiple-purpose dams in cooperation with the experienced power people in the area so that the plants will:

(1) Be scheduled to come in service as needed;

(2) Be of a type, and be built on a schedule, that will permit complete coordination with existing power facilities; and

(3) Supply the power at a net cost at least no greater than the net cost of supplying equivalent power by other methods.

Even if "coordination" by force through nationalizing the electric utility industry were the most efficient way of doing the job, the by-products of this course, its long-range effects on the social and political structure are so undesirable, in the writer's opinion, that this method should be abandoned before more harm is done. In addition, the facts all point to this method as producing less efficient results than can be realized through coordination with the taxpaying utilities and the municipally owned utilities.

EXPERIENCE IN THE SOUTHWEST

The writer is identified with a group of privately owned electric utilities which, collectively, compose an interconnected system that serves an area of some 114,000 sq miles in north-central Texas (the upper coastal plain) and sections of the state lying west and north of Dallas–Fort Worth, almost as far westward along the Texas Pacific Railroad as the Pecos River (see Fig. 12).

This territory shows wide variations in character as to: Elevation (Fig. 13(*a*)), topography, soils, climate, geological formation, rainfall (Fig. 13(*b*)), and natural resources. These varying characteristics naturally govern the economic and social status of the respective geographic divisions.

Referring to topographic and climatic features, the extreme western sections are in the southern end of the Great Plains, with elevations up to 3,000 ft and higher above the cap rock. On this Great Plains area, average rainfall varies from 10 in. to 20 in. The north and west escarpment of the coastal plain

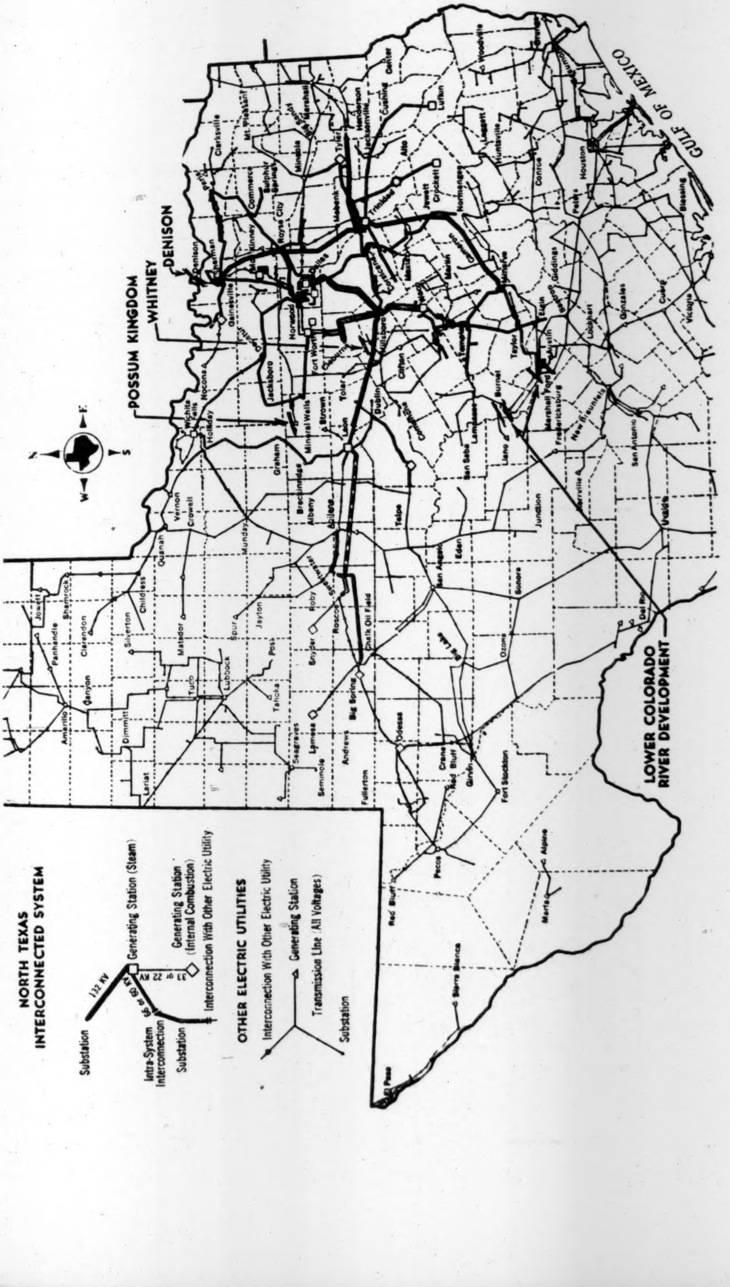
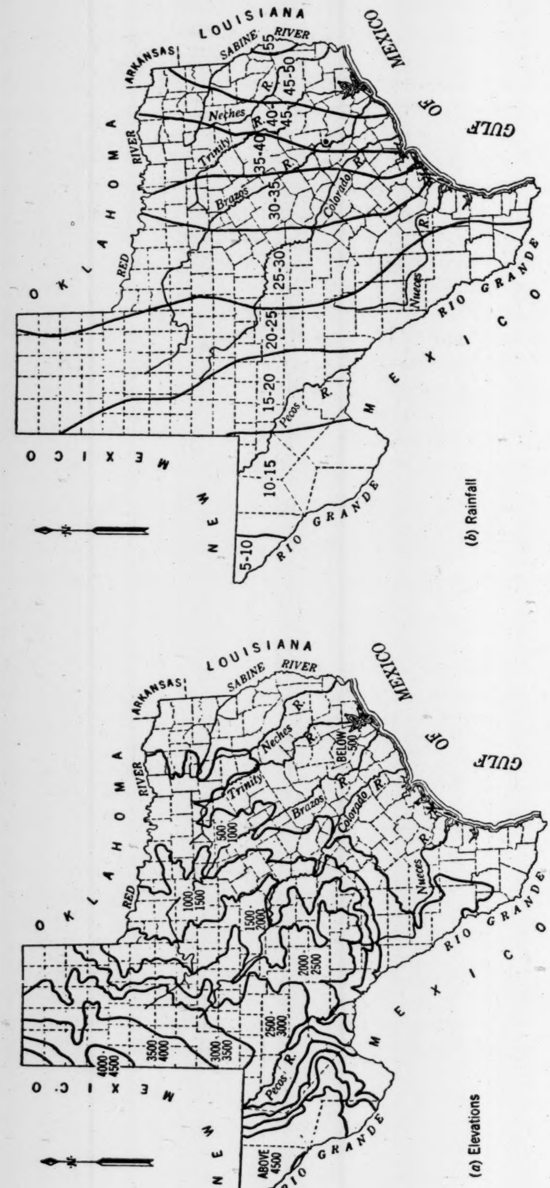

NORTH TEXAS INTERCONNECTED SYSTEM
Substation
132 KV
66 or 60 KV
33 or 22 KV
Generating Station (Steam)
Generating Station (Internal Combustion)
Intra-System Interconnection
Substation
Interconnection With Other Electric Utility
OTHER ELECTRIC UTILITIES
Interconnection With Other Electric Utility
Generating Station
Transmission Line (All Voltages)
Substation
N
E
S
W
POSSUM KINGDOM
WHITNEY
DENISON
LOWER COLORADO RIVER DEVELOPMENT
GULF OF MEXICO
Amarillo
Panhandle
Shamrock
Clarendon
Canyon
Dimmitt
Tulia
Silverton
Childress
Quanah
Vernon
Crowell
Wichita Falls
Holliday
Nocona
Gainesville
Denison
Sherman
McKinney
Paris
Clarksville
Commerce
Sulphur Springs
Mt. Pleasant
Mineola
Tyler
Marshall
Henderson
Jacksonville
Center
Lufkin
Woodville
Crockett
Palestine
Trinidad
Corsicana
Dallas
Fort Worth
Cleburne
Hillsboro
Waco
Mexia
Jackshoro
Mineral Walls
Graham
Breckenridge
Albany
Abilene
Munday
Stephenville
Dublin
Clifton
Temple
Taylor
Elgin
Austin
Lockhart
Gonzales
Cuero
Victoria
Houston
Conroe
Huntsville
Navasota
Giddings
Brenham
Burnet
Llano
Lampasas
San Saba
Brownwood
Fredericksburg
New Braunfels
San Antonio
Kerrville
Junction
Uvalde
Del Rio
Sonora
Eden
San Angelo
Ozona
Big Lake
Sweetwater
Roscoe
Snyder
Roby
Jayton
Spur
Post
Lubbock
Tahoka
Lamesa
Seagraves
Seminole
Andrews
Big Spring
Odessa
Crane
Rankin
Girvin
Fort Stockton
Fullerton
Pecos
Red Bluff
Alpine
Marfa
Sierra Blanca
El Paso
Matador
Lariat
Chalk Oil Field
Baylor

FIG. 12.—LOCATION OF NORTH TEXAS INTERCONNECTED SYSTEM AND OTHER ELECTRIC UTILITIES IN TEXAS


OKLAHOMA
ARKANSAS
LOUISIANA
SABINE RIVER
RED RIVER
NEW MEXICO
MEXICO
GULF OF MEXICO
RIO GRANDE
Pecos R.
Nueces R.
Colorado R.
Brazos R.
Trinity R.
Neches R.
ABOVE 4500
4000-4500
3500-4000
3000-3500
2500-3000
2000-2500
1500-2000
1000-1500
500-1000
BELOW 500
5-10
10-15
15-20
20-25
25-30
30-35
35-40
40-45
45-50
55

(a) Elevations

(b) Rainfall

FIG. 13.—CHARACTERISTICS OF TEXAS AREAS

generally follows a line falling west of Fort Worth, through Austin and San Antonio. Between this escarpment and the upper plains country lies what is locally termed the hill country and the intermediate plains, having elevation extremes of from, say, 500 ft to 2,000 ft and a rainfall bracket of from 35 in. at Dallas to somewhat more than 20 in. at the western boundary.

Substantially all the hydroelectric projects fall within, or adjacent to, the service area of the utility companies with which the writer is identified; all the multiple-purpose reservoirs, existing or proposed in Texas, lie within this same region on rivers that flow from the upper plains or the hill country, southward, southeastward, or eastward to the coastal plains, and thence to the Gulf of Mexico.

The rivers rise in what would be classified as an arid region and, in general, are fed by rains that travel northward and westward from the Gulf region toward the headwaters of the rivers, following much the same course as the streams, but in reverse. As in most arid countries, the rainfall is highly variable and unpredictable; but, when excessive rainfall does occur, the result can be only the occurrence of extreme floods.

Numerous instances can be cited of unprecedented stages on the larger streams which, for the most part, were caused by floods on only one or a few of the tributaries. The greatest recorded flood on the Colorado River at Austin was contributed in the main by the Llano River. The largest recorded flow on the Brazos River below Cameron was caused mainly by rainfall on 4,000 sq miles, a minor part, of the Little River watershed.

The multiple-purpose reservoirs are in the hill country, at varying distances upstream from the coastal plain escarpment.

The Projects

The existing projects are described briefly in Table 6, with supplementary comment, as follows:

1. Denison Dam is on the Red River near Denison, on the boundary between Texas and Oklahoma (see Fig. 12). This project was constructed with federal funds by the Corps of Engineers and is operated by that organization, but marketing of the power is under the jurisdiction of the Southwestern Power Authority (an agency of the United States Department of the Interior).

Since this project was placed in operation in 1944, the Texas Power and Light Company has purchased all the power output under the terms of a temporary, wartime contract. In 1947, this company negotiated a new contract, under the terms of which it will utilize most of the capacity of the present 35,000-kw to 40,000-kw unit and will have the use of one half or more of the capacity of the third unit; the second unit is reserved for sales in the State of Oklahoma. A total of three units is proposed, the second of which was scheduled for completion around the end of 1948 but may not be in service before the middle of 1949.

2. The Morris Sheppard Dam on the Brazos River, approximately 18 miles northwest of Mineral Wells, Tex., comprises the first of a projected series of developments on the upper Brazos (above Waco) by the Brazos River Conservation and Reclamation District. This is an agency of the State of Texas

but it was assisted financially by the federal government in building the first unit in the plan.

The Whitney and several other sites of the multiple-purpose type are proposed to be developed, although the DeCordova Bend site would lend itself to little other than power purposes. Included in the grand scheme are some purely flood control reservoirs on the Leon River and other tributaries. These latter assume greater importance in the flood control plan than reservoirs on the main stream because the tributaries have developed record high unit-area runoffs.

TABLE 6.—DATA ON MULTIPLE-PURPOSE DAMS IN TEXAS, FROM VARIOUS SOURCES

Dam	RESERVOIR STORAGE, IN THOUSANDS OF ACRE-FEET				CAPACITY, IN THOUSANDS OF KILOWATTS		ENERGY, IN MILLIONS OF KILOWATT-HOURS	
	Total	Power	Flood control	Other	In-stalled	Firm	Pri-mary	Aver-age
(*a*) RED RIVER								
Denison Dam near Denison, Tex.:								
Present (1948)	5,825.0[a]	1,148.0	2,745.0	1,932.0	40.0	28	140.0	218.0
Proposed (three units)	..	..	..	..	120.0	84	140.0	296.0
(*b*) BRAZOS RIVER								
Possum Kingdom Dam, northwest of Mineral Wells, Tex.	730.0[b]	524.0	..[c]	206.0	22.5	10	24.0[d]	75.0
Whitney Dam, North of Waco, Tex.:								
Present (construction started in 1947; earthwork incomplete in 1948)	2,118.4[e]	131.7	1,731.4	..	..	..	33.2[f]	85.0[f]
Future	..	386.9	1,476.2	255.3	30.0[f]	23[f]	40.8[f]	98.1[f]
(*c*) LOWER COLORADO RIVER AUTHORITY								
Austin Dam	1,950.0[a]	200.0[g]	590.0[h]	28.0	15.0[i]	..	..	..
Marshall Ford Dam	3,110.0	1,132.0	1,160.0[c]	..	75.0[i]	..	..	..
Inks Dam	1,100.0[a]	..	..	..	12.5[j]	..	..	..
Buchanan Dam	1,100.0	790.0	100.0[c]	210.0	25.0[i]	..	..	..
Total	..	..	..	..	125.0	116.0	200.0[k]	410.0[k]

[a] Below the spillway. [b] Total to top of gates. [c] Above the spillway. [d] This is the 1948 estimate, the preliminary estimate was 49,000,000 kw-hr and the estimate of the Federal Power Commission was 33,300,000 kw-hr. [e] Includes 100,900 acre-ft above the spillway. [f] Unofficial. [g] Used to store secondary energy; considered as an aid to flood control. [h] Controlled flood storage. [i] Units are kilovolt-amperes at 0.9 power factor. [j] Units are kilovolt-amperes at 0.95 power factor. [k] From a 1943 bond prospectus.

Flood control at Morris Sheppard Dam is provided by storage in the freeboard (maximum estimated height of flood crest above spillway).

The Brazos River Conservation and Reclamation District (the state agency operating the Brazos project) contracted to sell its entire output to the Brazos River Transmission Cooperative, Incorporated. The latter organization expected to market the output to the member cooperatives. Actually, the Texas Power and Light Company purchased the major part of the output from the Brazos River Transmission Cooperative and thus far has supplied the market for Morris Sheppard Dam output under terms of a wartime contract. The Texas Power and Light Company supplies the member cooperatives and makes the best possible use of the capacity and energy of the Morris Sheppard Dam.

During one year, the Texas Power and Light Company supplied more kilowatt-hours to the member cooperatives than it derived from Morris Sheppard Dam, although the aggregate peak demand of the Rural Electrification Administration (REA) probably did not exceed 2,000 kw. Future disposition of this power has not been resolved.

3. The Lower Colorado River Authority (LCRA) has developed a series of projects on the Colorado River and has two additional sites to be developed for power purposes, all upstream from the coastal plain escarpment at Austin.

The LCRA likewise is an agency of the State of Texas but was financially assisted by the federal government, having been advanced funds for the first construction project by a federal loan and grant. The loan was subsequently refinanced with bonds of the LCRA. The largest unit of the project, Marshall Ford Dam, was constructed with federal funds.

Power is marketed to many consumers directly; three of the private utilities purchase, under firm contracts, a total of 63,000 kw, the Texas Power and Light Company participating to the extent of 42,000 kw. The project cooperated fully with the wartime power program of the area.

Flood Control Characteristics

These several projects, as constructed and as proposed, are capable of considerable flood control. Perhaps it is superfluous to remark that anything short of proper operation of the facilities will deprive the public of the potential flood control benefits for which good money has been spent; conditions may even be aggravated.

The most ardent supporters of these Texas developments would not contend that the multiple-purpose projects, in themselves, are capable of controlling floods. Flood control is a compromise. Denison Dam will release all but extreme floods through turbines and sluiceways, capable of filling or probably overflowing the downstream channel; but it has been provided with a large, free discharge spillway. The Brazos River basins will have to be aided by several flood control reservoirs on the tributaries. Nevertheless, much is gained in utilizing the freeboard of uncontrolled spillways for flood control, the top storage in a reservoir frequently being the most economical unit-cost storage. The added flood control feature may bring economic justification that would be lacking on single-purpose power projects.

The writer does not propose to set himself up as an arbiter on the economic justification of the multiple-purpose dam projects that have been constructed in Texas, but he will state that—under his supervision and in some cases as long ago as 1927 when fuel prices were much higher and construction costs lower—comprehensive studies of Texas streams were made to determine feasibility for development of hydro power. Without exception, the findings were unfavorable.

As to design and operation of electrical facilities, it appears that hydro power is economical in Texas because it offsets the high operating costs of thermal plants when operating at low load factors, or when operating as spinning reserve, in which case the amounts of hydro energy are small in proportion to the installed capacities. In general, the lower the load factor carried by hydro generation (within the limits of its relation in size to the load of the system on which it operates) the greater will be the economic justification for the hydro capacity, inasmuch as the trend of thermal plant unit production costs rises sharply with a reduced load factor. It follows that the peaks should always be carried by the hydro plant if the power pool storage is sufficient to provide the necessary flexibility.

Utilities in northern Texas have loads that are capable of absorbing to full advantage the hydroelectric capacity that thus far has been installed. Analyses have shown that a system of lesser size would not be capable of fully utilizing such hydro power—thus effectively illustrating the economic desirability of marketing low load factor hydro power to established power systems of ample size and load diversities, and of scheduling the new hydroelectric installations to conform with the ability of the power systems to absorb each new increment of hydro power.

The writer's philosophy as to proper design and operation may be delineated by presenting an analysis (see Fig. 14) of hydro power utilization that was prepared not so long ago, but based on 1940 loads. In this study, it was assumed that there would become available to the utilities in north Texas, from present and proposed projects on the Colorado, Brazos, and Red rivers, aggregate power equivalent to 130,500 kw, 178,400,000 kw-hr annually, and a 15.6% load factor.

Analyses were made which, summarized, showed the following tabulated results, which are illustrated by Fig. 14:

(*a*) In 1940, the Texas Power and Light Company (Fig. 14(*c*)) could have absorbed only 77,400 kw of the capacity although there was a peak demand of 125,410 kw;

(*b*) The load of the Texas Power and Light Company would have to increase to 216% of the 1940 load or to 270,890 kw to absorb the full 130,500 kw (Fig. 14 (*d*)); and

(*c*) The north Texas group of companies had a peak demand in 1940 of 282,950 kw (Fig. 14(*a*)); and the characteristics of the aggregate load were such that the upper 130,500 kw of the demand closely matched the 15.6% hydro power output.

Viewed from another angle, a 130,500-kw peak demand system (Fig. 14(*b*)), supplying a load having a 58% annual use factor, would have annual total energy requirements of 663,000,000 kw-hr, contrasted with the hydro yield of 178,400,000 kw-hr, or 26.9% of the required energy. At a 58% load factor the hydro could carry only a 35,000-kw load. Given a system supplied from

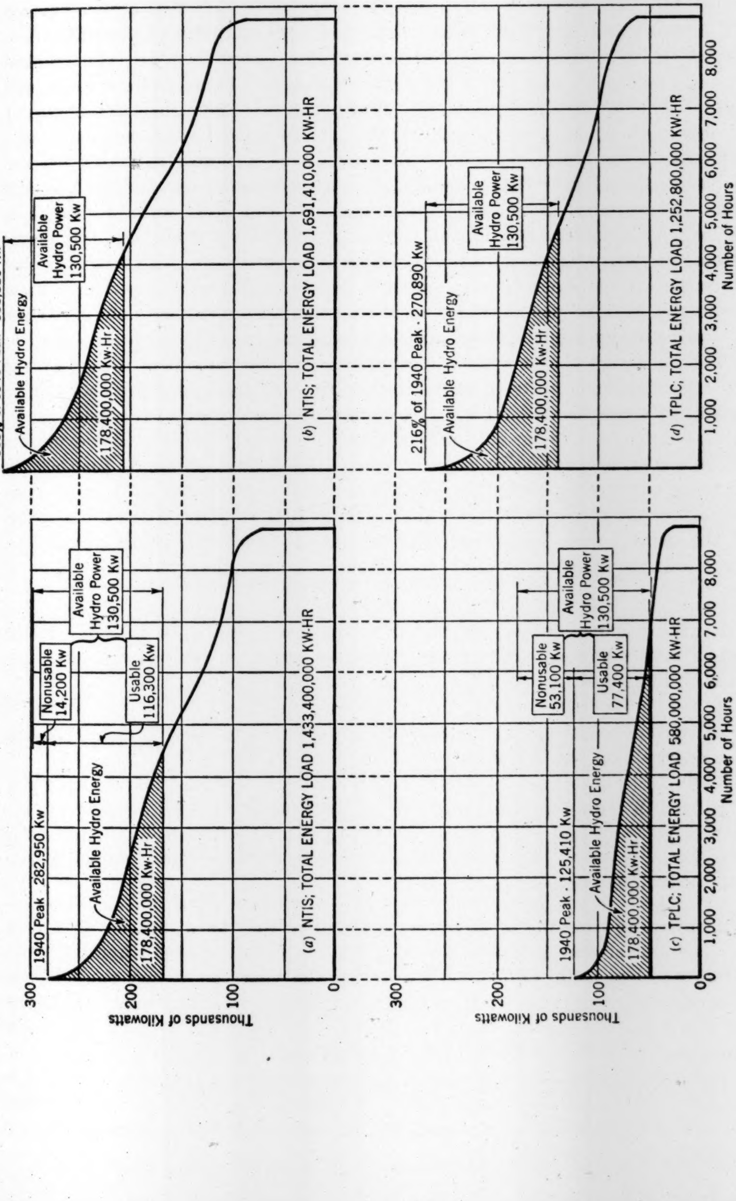


FIG. 14.—COMPARISON OF UTILIZATION OF HYDROELECTRIC POWER AND ENERGY, BY NORTH TEXAS INTERCONNECTED SYSTEM (NTIS) AND BY TEXAS POWER AND LIGHT COMPANY (TPLC)

thermal plants with a load of 333,880 kw, the total hydro capacity of 130,500 kw could be effectively used with the kilowatt-hours available.

The foregoing example and discussion cover basic facts that are well known to the power industry; they concern the "ABC's" of hydro power utilization. It is axiomatic that no commodity represents a bargain, at any price, unless there is a use for it, and hydro generation must fit the load pattern to be fully useful. The lower the load factor of generation (contrariwise, the greater the degree of capacity development), the greater must be the ratio of system peak load to the hydro capacity. This argument is increasingly true because utilization load factors have trended upward and have reached high annual totals, with almost unbelievably high seasonal and daily values. Hence, a low load factor power should not be constructed faster than the basic power system or systems can absorb it. It should be mentioned, in passing, that the determination of fitness is not so simple as in the illustrated case, utilizing a 1-yr load-duration curve. More accurate calculations require determinations by months, using monthly load-duration curves.

The mere fact that a system load can absorb the output of a hydro plant obviously does not, in itself, prove the economy of the hydro project. There are many factors that enter such an equation—for example, the existence of thermal capacity of lesser efficiency with which to carry peaks. However, it is almost a truism that hydro capacity has little, if any, economic justification unless its output can be substantially absorbed, because the unit investment in hydro capacity runs relatively high, except in a very few exceptional cases. On the other hand, hydro capacity, if wisely employed, offers advantages in addition to those commonly reflected in economic equations. Some of these are:

1. Hydro capacity may be added to the line quickly; in fact, it may be floated on the line as synchronous condenser capacity, ready for instant use, but with little cost or loss if the turbine gates are tight.
2. Under favorable conditions, energy may be stored during one season for use in another season; this storage is often justifiable in face of a risk of wasting some energy over the spillway.
3. Small hydroelectric projects, with a relatively low load factor, often may be utilized in reducing peaks imposed on the system by local or district loads (that is, geographical components of system peaks) with attendant reduction in losses, improved local voltage regulation, and perhaps yield of some worthwhile reserve capacity in the locality.

A revealing example of advantages 1 and 3 was the utilization of Morris Sheppard power during World War II. Although during 1944 when the hydro output was less than 12,000,000 kw-hr for the year the energy supplied the REA customers was greater than 12,000,000 kw-hr, the net "take" was in the red; yet Possum Kingdom Dam constituted a valuable aid in delivering power to war industries in the Dallas-Fort Worth area.

Illustrating advantage 2, during a period when additional energy was stored for the Texas Power and Light Company in the LCRA dams, the Texas Power and Light Company carried LCRA load at night by off-peak thermal generation. The stored energy was used to stretch the daily load factor of hydro generation. There are many such devices that can be resorted to, and should be used, to derive optimum results.

The North Texas Interconnected System has thus far constituted, and most likely will constitute in the future, the only economical vehicle for the marketing of the hydro power in this territory.

SUMMARY

Assuming that multiple-purpose dams are to be built for reclamation and flood control reasons (it being up to the public utility industry to make the best possible use of the power to be produced in order to be of the maximum feasible help to flood control and irrigation, and to make the burden on the taxpayers as light as possible), there is a proved course to follow which will realize the desired result in harmony with established business procedure. Such a course involves complete cooperation between the government bureaus that design and build the multiple-purpose dams, and the entire utility industry in each area in which the dams are located, so that:

a. The dams will be designed with a power output that fits in with the power resources of the area to the very best advantage;

b. The coordination is extended to include the existing transmission system, to avoid all unnecessary and wasteful duplication of facilities;

c. The rates for power and energy from multiple-purpose dams are equal to or less than the cost of producing equivalent power and energy by alternate available methods;

d. The cost to the taxpayer is kept at the lowest possible level; and

e. The experienced selling organization of the established utilities can market the additional power quickly to the greatest possible number of people.

Coordination in this manner will result in the maximum beneficial use of power and energy from multiple-purpose dams, the maximum return to the government on its investment, the lowest possible cost to the taxpayers, and the lowest rates to the users of electricity. In addition, it will benefit the causes of reclamation and flood control. Obviously, duplication and wasteful expenditures for power facilities mean less money for flood control and reclamation. Skill will be needed for the proper application of this solution to the problem, but it is abundantly available in the electric utility industry and the industry is willing and anxious to do its part in coordinating the output of the multiple-purpose dams with the other power resources of the United States.

INFLUENCE OF FEDERAL POWER COMMISSION ON DESIGN AND OPERATION

By E. Robert de Luccia,[14] M. ASCE

The influence of the Federal Power Commission on the design and operation of multiple-purpose reservoirs has probably been greater than the average engineer realizes. The Commission has broad responsibilities in connection with the improvement of those rivers under the jurisdiction of the United States, particularly as they relate to hydroelectric projects. In its work the Commission performs duties prescribed under the Federal Power Act with respect to the licensing of hydroelectric projects, and advises with other government agencies on river basin improvement programs. The commission is not engaged in actual construction work; nor does it operate any projects.

All the reservoir projects that the War Department has studied or constructed since the passage of the 1938 Flood Control Act, and about 350 projects, with a total of 5,600,000 kw of ultimate installed capacity, under license from the Commission since 1920 have been subject to the advice and consultation, or the approval, of the Commission before final designs and plans of operation have been adopted.

As an aid in understanding the influence of the Federal Power Commission on design and operation of multiple-purpose reservoirs, it will be helpful to review briefly a little of the history of the legislation relating to the conservation and development of rivers in the United States.

The authority of the United States to deal with the development of water power arises from the provisions of the Constitution with respect to interstate and foreign commerce and from its proprietary interest in public lands and reservations. The Act of June 6, 1866, confirmed rights of appropriation of waters for mining and provided for further acquisition of similar rights for "mining, agriculture, manufacturing and other purposes." Other acts modifying and amplifying the act of 1866 followed and, as a result of the initiation of hydroelectric developments in about 1890, the Act of May 4, 1896, was enacted to authorize the use of government lands for the generation of electric power. A number of subsequent acts amplified and supplemented the 1896 act, to include transmission lines among other matters.

During this period several acts were adopted for the protection of navigation by requiring certain permits to be obtained from the War Department with respect to the erection of structures in, or over, navigable waters. As a result of a multiplicity of acts, and the fact that new uses for water resources, such as the development of electric power, were becoming evident, attempts were made to give expression to the need for conserving water resources. Probably the first definite steps in the general conservation of water power resources were the reservation and the withdrawal from entry under the public land laws of government lands considered to have value for power purposes under the Act of June 25, 1910.

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[14] Chf., Bureau of Power, Federal Power Comm., Washington, D. C.

The growing consideration of the conservation of water power resources and the unsatisfactory conditions with respect to their protection and development under the then existing legislation kept the problem before Congress and resulted in the River and Harbor Appropriation Act of August 8, 1917. This act is of particular interest in that it established for the first time a commission with responsibilities for bringing into—

> "* * * cooperation and coordination the engineering, scientific and constructive services, bureaus, boards * * * that related to studies, development or control of waterways * * * with a view to uniting such services in investigating with respect to all watersheds of the United States questions relating to the development, improvement, regulation and control of navigation as a part of interstate and foreign commerce including therein the related questions of irrigation, drainage, water storage, control of floods, prevention of soil erosion, and utilization of water power."

Because of the start of World War I in 1917 that commission was never appointed.

In place of the 1917 act, the Federal Water Power Act was enacted by Congress on June 10, 1920, "to create a Federal Power Commission to be composed of the Secretaries of War, Interior and Agriculture." In 1930 the act was amended to provide for the independent five-man commission as it is generally known today. An important provision of the act of 1920 and which is related directly to the subject of this Symposium is Section 10(*a*) which provides that all licenses issued should be on the following conditions:

> "(*a*) That the project adopted, including the maps, plans, and specifications, shall be such as in the judgment of the Commission will be best adapted to a comprehensive scheme of improvement and utilization for the purposes of navigation, of water-power development, and of other beneficial public uses, and if necessary in order to secure such scheme the Commission shall have authority to require the modification of any project and of the plans and specifications of the project works before approval."

This section was amended in 1935 to include recreational purposes among "other beneficial public uses." The Flood Control Act of 1938 and the several flood control and river and harbor acts since that time require:

> "* * * that penstocks or other similar facilities adapted to possible future use in the development of hydroelectric power shall be installed in any dam herein authorized when approved by the Secretary of War upon the recommendation of the Chief of Engineers and the Federal Power Commission."

It is abundantly clear that it is the purpose of Congress that the water resources of the United States be conserved and that the development of the rivers of the nation shall include maximum provision for the use of waters for the development of hydroelectric power wherever feasible and consistent with other important uses.

The policy of the Federal Power Commission with respect to these matters is the policy of Congress as expressed in its many acts, and all measures and actions taken by the Commission with respect to its duties and responsibilities relating to the rivers of the United States are in accordance with the statutory duties and responsibilities laid down in these acts.

Probably the best way of illustrating the policy of the Commission and its influence on the design and the operation of multiple-purpose reservoirs is to give a few examples of actual practice.

As an example of the work done with the War Department under the flood control and river and harbor acts, consider the Bull Shoals multiple-purpose project on the White River Basin in Arkansas. This project was included in a survey report of the Corps of Engineers transmitted to the Commission for review and comments in accordance with routine machinery which has been set up between the Corps of Engineers and the Commission. The project is immediately downstream from the existing privately-owned Ozark Beach hydroelectric plant which forms a reservoir known as Lake Taneycomo. In the survey report preliminary studies had placed the top of the flood control pool at the elevation of the tailwater of the Ozark Beach plant, and the maximum power pool at an elevation 83 ft lower which, of course, resulted in a loss of potential power head of that amount.

With a view to capturing all the head possible for development of power on the river at this location, the Commission's engineers determined that the dam should be raised and the power pool placed at about the elevation of Ozark Beach tailwater, with maximum water surface, under conditions of the reservoir design flood, at about the elevation of the spillway crest of the Ozark Beach Dam. Storage capacity allocated to flood control would not be lessened. Of course, there would be total loss of head and power at Ozark Beach under conditions of extreme flood. This situation occurs, however, under present conditions of maximum floods as the dam is relatively low. Also, if Ozark Beach is "drowned out" by Bull Shoals, maximum heads would obtain at the Bull Shoals power plant and generation there would be more than sufficient to compensate for the power lost at Ozark Beach.

By raising the Bull Shoals Dam about 42 ft, the maximum static head would be increased about 84 ft, the usable power storage increased about 10 times, the minimum regulated flow nearly doubled, the primary or continuous power nearly trebled, and the average annual generation about doubled. Such improvements in the power at this project were well worth the effort.

These matters were discussed most thoroughly with the Corps of Engineers, joint visits were made to the site; full consideration was given to the adverse effects from inundations; and costs, power markets, and power values were studied. After thorough investigation, the Corps of Engineers adopted substantially the suggestions of the Commission's staff, and greater benefits will be obtained.

Another example of the influence of the Commission is illustrated by the studies relating to the development of the Alabama-Coosa River. This stream flows from the headwater mountain streams in northwestern Georgia south through the length of the State of Alabama to Mobile Bay after it has joined with the Tombigbee River a few miles north of Mobile.

In 1941, the Chief of Engineers requested the Commission's comments on a comprehensive report of the district and division engineers on the Alabama-

Coosa River, in which about twenty-five projects had been recommended for the control of the river for the purposes of navigation and development of hydroelectric power. The plan of development outlined in the report included several small tributary reservoirs and a complete chain of dams on the main stem of the river from Rome, Ga., to the mouth of the Alabama River. Slack-water navigation was to be provided by a system of locks at proposed and existing dams, with power to be developed at those dams which were sufficiently high to warrant power installation. The generation of power at existing power projects on the river averages about 2,000,000,000 kw-hr per yr, and the prospective development suggested in the report would result in the generation of an additional 1,000,000,000 kw-hr.

In response to the request of the Chief of Engineers, the Commission in June, 1941, forwarded the results of a preliminary study, by the Commission staff, of the proposed waterway project and suggested that the plan be modified to include fewer but higher navigation dams and rearrangements of power installations. With these modifications and other changes to effect the full coordinated operation of the entire system of projects, the Commission staff found that the additional generation on the river could be increased from 1,000,000,000 kw-hr to 2,000,000,000 kw-hr.

The Chief of Engineers stated in his reply that the Board of Engineers for Rivers and Harbors was substantially in accord with the views of the Commission and that revision of the report appeared desirable. Additional studies were then undertaken by the Corps of Engineers, in cooperation with the Commission's engineers, resulting in plans for the comprehensive development of the river. In addition to other water use purposes, this development will make possible an ultimate generation on the river greatly in excess of even that indicated in the preliminary studies by the Commission. The authorization for the Alabama-Coosa development was included in the 1945 river and harbor act which in specific language envisages this greater generation of power found feasible by the later studies.

As part of its duties the Commission is required to assure itself as to the safety and adequacy of the dams, powerhouses, and appurtenant works which it licenses. In discharging these duties, the Commission frequently requires major structural revisions in design. These revisions cover such items as increasing spillway capacity, providing greater freeboard for wave action, prescribing different or more complete foundation treatment, changes in detail design items relating to size and number of reinforcing bars, strength to concrete, suitability of aggregates, and many others. Sometimes the Commission finds that a higher dam and an increased installed capacity are required so that the project will be best adapted to a comprehensive scheme of development.

An example of this phase of the Commission's work is illustrated by the Saluda Dam of the South Carolina Electric and Gas Company, Federal Power Commission licensed project No. 516. This project is located on the Saluda River about 18 miles above Columbia, S. C., and consists of a semihydraulic earth fill structure about 212 ft high and about 7,900 ft long, a large reservoir, known as Lake Murray, 30 miles long with about 2,500,000 acre-ft of storage capacity, and a powerhouse with an installed capacity of 135,000 kw.

The Commission's staff was at one time concerned over the stability of the dam and the adequacy of the spillway. Passage of the floods of 1929 and 1936 indicated inadequate spillway capacity. After the 1936 flood the downstream part of the dam became saturated. The Commission, therefore, ordered that the operating level of the reservoir be lowered and required that settlement readings and water level records be taken within the downstream face of the dam and that corrective measures be adopted to assure the safety of the dam.

A plan of remedial action was agreed on and approved by the Commission providing for raising the dam 3 ft to restore design grade. Thus the spillway capacity was increased by about 140%, and the stability of the dam was improved by flattening the downstream slope. Approximately 1,000,000 cu yd of earth fill and rock fill was placed on the downstream face of the dam. As a result of these improvements the licensee has been permitted to increase the normal water surface elevation of the reservoir from 355 ft to 360 ft.

Another example is found in the Wilder Development, of the Bellows Falls Power Company, on the Connecticut River in New Hampshire and Vermont, which was not constructed under license, but which was later brought under license. The project consists of a dam 30 ft high, a small pond, and two powerhouses with turbine capacity of 5,400 kw operating under a head of about 36 ft. Studies made by the staff of the Commission indicated that the existing project did not make maximum use of the project site. With increased capacity greater flow would be available and a higher dam would be best adapted to a comprehensive plan for river development. Provision was made in a license issued April 22, 1944, for the redevelopment of the project in accordance with such plans as would be found by the Commission to conform best to a comprehensive plan for improving and developing the Connecticut River, for the use and the benefit of interstate commerce, for the improvement and the utilization of water power development, and for other beneficial public uses, including recreational purposes. The licensee is now (1949) redeveloping the project by constructing a new dam about three quarters of a mile downstream from existing Wilder Dam with the pond level about 16 ft higher than present level and a new powerhouse with installed capacity of 33,000 kw operating under a head of about 51 ft.

In 1923, the Commission issued a license for a project on the Ouachita River in Arkansas to the Arkansas Power and Light Company. The company's plans were approved with the requirement pursuant to Section 10(*a*) of the act with respect to proper and comprehensive development of reservoir sites for production of hydroelectric power and other beneficial public uses, that further studies be made for a larger development to permit full utilization of the site and the water resources for power, navigation, and flood control. The Flood Control Act of 1938 authorized the War Department to participate in an amount not to exceed $2,000,000 in the construction of a multiple-purpose reservoir, at the Blakely site, this participation was to have been with the power company in the interests of navigation and flood control. However, the company's plans for commencement of construction were indefinite and the Flood Control Act of 1944 authorized construction of the entire project by the Corps of Engineers.

To summarize briefly, the Federal Power Commission's influence on the design and the operation of multiple-purpose reservoirs results from its work in the licensing of hydroelectric projects and its studies of river basins since 1920 under the Federal Power Act, its duties under the flood control and river and harbor acts since 1938, and, more recently, its active participation in the cooperative work done under the Federal Inter Agency Agreement procedure.

The basic contributions of the Commission have resulted in making engineers realize that one project is only a unit of a whole series of projects in an entire river basin; and that the operation of multiple-purpose projects can be so arranged, after appropriate methods are included in the design stage, to increase greatly the power yields of a stream without in any way impairing the use of the stream for other primary purposes such as irrigation, flood control, navigation, and domestic uses.

THEIR RELATION TO FISH AND WILDLIFE

BY RUDOLPH DIEFFENBACH[15]

INTRODUCTION

Until recently, few dams could claim to have "multiple purposes" in the present accepted meaning of the term; that is, they were not designed to make complete economic use of every available unit of water, fall, or land. In the past few years, a great movement has begun to develop practically every important river basin in the land, with emphasis on flood control, irrigation, power, navigation, water supply, and silt control. Congress has recognized that the fish and wildlife resources affected by the dams are worth millions of dollars, and has strengthened the laws that give to the Fish and Wildlife Service, U. S. Department of the Interior, the responsibility for conserving these resources. This federal agency was formed in 1940 by consolidation of the former bureaus of Fisheries and of Biological Survey, which had been established in 1871 and 1885, respectively. Under the new status, the service is rapidly formulating policies and building up the machinery necessary to make them effective. It is the purpose of this paper to define these policies and to explain their underlying purposes.

BASIC PRINCIPLES

The Fish and Wildlife Service assumes that full development of a multiple-purpose project can be achieved only if all its resources are taken into account in the plans for development. Not only should plans be devised to preserve existing resources wherever possible, but every feasible procedure for increasing them should be adopted. These public values will afford great benefits to the people and proper use of water is essential to their perpetuation.

It is conservatively estimated that not less than 20,000,000 people in the United States annually indulge in the sports of hunting and fishing, thus obtaining food and healthful recreation in forests and fields, and on lakes and streams. Each year these people spend about $2,000,000,000 for equipment, guide fees, transportation, and related items. Much of this money goes into local communities and small industries. A small part supports state and federal agencies charged with the responsibility for perpetuating the supply of fish, birds, and game animals. The food value of noncommercial fish and game annually taken exceeds $500,000,000. After World War I, hunting and fishing pressure immediately increased 30%. Statistics today indicate that there will be not less than a 40% increase in hunting and fishing pressure, with the strong likelihood that the increase will be 50% or more. Viewing this situation from the economic side alone, hunting and fishing must be classified as big business.

It is unfortunate that, in the past, many reservoir projects have been planned, constructed, and operated without consideration of the interests of fish and wildlife. To permit the maximum development of river basins, this policy is being replaced by a more rational view of all natural resources. Reservoirs should be planned to serve multiple purposes, including fish and wildlife. Such plans will involve, in many cases, additional expenditures for structures and operation, but the investments will pay dividends throughout the life of the projects.

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Law Putting New Policy Into Effect

To implement this more enlightened concept Congress enacted Public Law No. 732 (House of Representatives Bill No. 6097), approved on August 14, 1946, to amend the Act of March 19, 1934, entitled, "An Act to promote the conservation of wildlife, fish and game, and for other purposes." The new law is referred to as the revised "Coordination Act." Its scope is illustrated by the following extracts:

> "Whenever the waters of any stream or other body of water are authorized to be impounded, diverted, or otherwise controlled for any purpose whatever by any department or agency of the United States, or by any public or private agency under Federal permit, such department or agency first shall consult with the Fish and Wildlife Service and the head of the agency exercising administration over the wildlife resources of the State wherein the impoundment, diversion, or other control facility is to be constructed with a view to preventing loss of and damage to wildlife resources, and the reports and recommendations of the Secretary of the Interior and of the head of the agency exercising administration over the wildlife resources of the State * * * shall be made an integral part of any report submitted by any agency of the Federal Government responsible for engineering surveys and construction of such projects.
>
> "The cost of planning for and the construction or installation and maintenance of any such means and measures shall be included in and shall constitute an integral part of the costs of such projects: * * * In the case of construction by a Federal agency, that agency is authorized to transfer, out of appropriations or other funds made available for surveying, engineering, or construction to the Fish and Wildlife Service, such funds as may be necessary to conduct the investigations required. * * *
>
> "Whenever the waters of any stream or other body of water are impounded * * * adequate provision consistent with the primary purposes of such impoundment * * * shall be made for the use thereof, together with any areas of land, or interest therein, acquired or administered in connection therewith, for the conservation, maintenance, and management of wildlife, resources thereof, and its habitat thereon. * * *
>
> "The provisions of this Act shall not apply to the Tennessee Valley Authority."

A thorough study of the "Coordination Act" by all engineers who are interested in multiple-purpose dams is suggested. The foregoing quotations merely touch the highlights sufficiently to indicate that the policies expressed by this act give a definite measure of responsibility to the Fish and Wildlife Service to take a constructive part in planning the multiple-purpose use of reservoirs. Previous laws had no such force, but the Flood Control Act of 1944 did enable the Service to set up a special staff to make studies of, and reports on, river basins.

River Basin Studies

A river basin study may involve surveys, investigations, and reports on a single reservoir, on a group of projects, or even on all the projects in coordinated plans for an entire river basin. It is to be hoped that eventually every project may be studied in the light of the effects on the fish and wildlife of an entire river basin, as well as on those of the immediate locality, since a single project near a basin headwater may conceivably affect the entire valley down to the mouth of the main river. In fulfilling the intent of the "Coordination Act," a study usually embraces the following procedure:

1. A through examination of the project as outlined by the sponsor, to gain a complete understanding of the engineering plans and proposed methods of operation;
2. A biological field survey, including mapping of vegetational types on the affected areas, to determine the present quantity of fish and wildlife and their habitats;
3. The collection and study of all available fish and wildlife statistics applicable to the vicinity (in conjunction with state agencies);
4. An investigation of the effect of the project on fish and wildlife after completion (based largely on knowledge of the history of other developments);
5. An estimation of the present and future dollar value of fish and wildlife on the area, and determination of the net value;
6. The formulation of suggestions for modifying plans and methods of construction and operation which would prevent unnecessary losses and insure maximum gains to fish and wildlife—the more common recommendations in such reports being concerned with stabilization of water levels, maintenance of a permanent pool, maintenance of minimum flow below reservoir, fish protective devices, food and cover planting, wildlife management areas, and public access; and
7. The preparation of a concise, understandable report which will embody all the foregoing steps.

The most difficult part of this program is step 5, the determination of dollar values for fish and game. The need for such a valuation becomes evident in view of the fact that the river developments planned by the Corps of Engineers and the Bureau of Reclamation are measured in terms of money. These projects stand or fall after having been measured by this yardstick. Therefore, in order that the findings of the Fish and Wildlife Service may be expressed in terms understandable to the sponsors of the projects and to the Congress of the United States, the Service must also convert fish and wildlife values into monetary terms.

Although the methods employed to determine the money values of fish and game, which seem basically sound, have been accepted widely by conservationists, the Service is constantly seeking improvement. In particular, assurance is needed that adequate consideration is given to the esthetic values associated with these resources. All esthetic values cannot be reduced to dollars and cents, but their importance should not be underestimated for "man does not live by bread alone."

PART PLAYED BY ENGINEERS

The hydraulic engineer has a definite place in the river basin studies organization. This group does not presume to pass judgment on plans prepared by the sponsor, but since the plans are couched in engineering language they often require reduction to terms and measures that the biologist cannot mistake. Therefore, an effort is made to include in every unit of the organization an engineer capable of interpreting this language to the biologist. The engineer also defines the limits of the recommendations by the Service affecting changes in engineering plans with the view to keeping them reasonable. In fact, the Service is developing a new type of technician, who might be called a "biological engineer," with a working knowledge of biology grafted onto a training in hydraulics.

ALTERATIONS IN DAM DESIGN AND OPERATION

Studies have indicated that both fish and wildlife are more likely to be affected by the methods of hydraulic operation than by the type of dam. There are occasions, however, when it appears desirable to make certain alterations or additions to the primary design. For instance, normal engineering plans may provide outlets at only one level; but, because of the tendency of water to stratify into layers of different temperatures and chemical characteristics, there may be a distinct advantage in drawing off water at other levels during certain times of the year. Obviously, such a situation calls for the addition of several outlets, at various elevations between the bottom and the top of the dam.

Where the stream under investigation is used by great numbers of migratory fish, such as the salmon, the addition of fishways over the dam may be necessary, as only thus can the fish make their periodic spawning migrations to the headwaters. On the other hand, the addition of gate screens on dam outlets may be desirable, to prevent the downstream escape of fish from the reservoir.

Hydraulic operation of a dam may affect fish and wildlife favorably or unfavorably, depending on the resultant fluctuations in the reservoir and in the rivers downstream. The following examples will illustrate the point:

The value of a reservoir for fish and wildlife depends largely on the mode of operation. It is deplorable that the needs of irrigation, power, and flood control usually demand large fluctuations in reservoir levels and discharges, as such changes are commonly in direct conflict with the needs of fish and wildlife. Extreme or rapid changes in pool stages and stream discharges tend to disturb biological conditions, and thereby impose severe limitations on the productive capacity of most aquatic animals. Uniform levels in reservoirs are not expected, but the Service frequently recommends that fluctuations be limited during biologically critical periods, such as during the spawning of fish and the rapid growth of plants.

To offset losses from a widely fluctuating reservoir, the Service may recommend subimpoundments in the shallow arms of a reservoir. These are partitions by low dams constructed to maintain controlled water levels. Such ponds create valuable fish spawning areas, feeding grounds for waterfowl, and breeding places for muskrats and other fur animals.

When water is released to the stream below in a manner calculated to eliminate or to reduce flood peaks and to provide a steady minimum flow of sufficient volume, it is reasonable to expect an increase in fish and wildlife values. The complete cutting off of flow, even for short periods, might be disastrous, since the entire fish population would probably be destroyed.

Actual examples of all the aforementioned conditions may be found in existing projects all over the Unites States. One great reservoir is filled and practically emptied yearly, with the result that a marginal mudbank is formed a mile or more in width, with practically no plant growth, and the pool is almost a biological desert. In contrast, there is another large project where operation is such as to minimize fluctuations of pool levels. There the mudbank is missing, the reservoir is teeming with fish, and waterfowl are multiplying.

Reservoirs that inundate wooded bottom lands destroy one of the most valuable of all habitats for wildlife and eliminate important areas for waterfowl and fur animals. In addition, vital winter cover, on which game of a wide area is dependent, may be destroyed. For example, proposed reservoirs on the Missouri River, if constructed, will extend almost the entire width of South Dakota. They will cause the elimination of winter cover essential for deer, and will cause substantial losses to pheasant and sharp-tailed grouse production. In an attempt to replace these losses of cover, planting recommendations are made. It is hoped that these will be at least partly adequate.

In gathering data for a river basin study involving a multiple-purpose dam, among the most important items required of the sponsor are operation graphs of proposed pool and stream fluctuations. These graphs (which should cover as many years as available stream flow records will permit) are studied to ascertain the probable future effect of operation on fish and wildlife. Where the effect is deemed injurious, suggested changes in operation are made to reduce the harm or perhaps even to bring benefits. The Service is keenly aware, however, of the fact that a single change in an operating schedule may dislocate an entire program. Therefore, suggestions for changes are made with great caution and with the understanding that they may not be adopted if inconsistent with the primary purposes of the project.

CONCLUSIONS

The Fish and Wildlife Service is convinced that the industrial development of all river basins should be realized, but not to the exclusion, or at the expense, of other aspects of regional wealth. If the plans for development too lightly consider the bountiful fish and wildlife resources found in streams, valleys, hills, and plains, resources of great economic importance and priceless esthetic value will be lost forever. The Service is optimistic in its belief that, by cooperation between governmental agencies, by intelligent planning, and by careful management, the present fish and wildlife resources can be not only preserved in large part, but in many areas actually improved.

PLANNING FOR THE RECREATIONAL USE OF RESERVOIRS

BY CONRAD L. WIRTH[16]

Since about 1946 there has been a great outpouring of literature in regard to the various proposals advanced for the control of water resources in the United States, particularly west of the Mississippi River where the problem is not only to prevent floods and to provide for navigation, but also to conserve water for agriculture, power, and other uses. Periodicals—popular, professional, and organizational—have carried articles dealing with one or more phases of the subject. Hundreds of studies have been made by interested agencies and many of the results have been published. Cartoonists and cartographers have found a fertile field for their skills. Organizations have been formed to put forward ideas in behalf of one proposal or against another. All this serves to demonstrate the interest, or concern, in the subject of water conservation.

The National Park Service, U. S. Department of the Interior, shares both interest and concern—interest, because of the inherent possibilities for the extension of recreational opportunities, and concern because of the possibilities of injurious effects on national or state park areas. To explain these attitudes, it is necessary to outline some of the background of the Service as regards protecting such areas from abuse or unwarranted intrusion, and fostering the provision of opportunities for outdoor recreation in natural surroundings. Under Congressional mandate, the National Park Service protects and administers the 168 areas of the National Park System—

> "* * * to conserve the scenery and the natural and historic objects and the wild life * * * and to provide for the enjoyment of the same in such manner and by such means as will leave them unimpaired for the enjoyment of future generations."

It is evident from this definition that the Service would be vitally concerned with any proposal which might result in major changes to, or the inundation of, the lakes or streams in a national park since the natural conditions would be seriously altered in either case. The Service cannot tolerate such alterations. Furthermore, there are many scenic and recreational values that would be adversely affected by reservoir construction. The destruction of scenic, scientific, and cultural features can be of serious import when balancing benefits gained against benefits lost. The same is true of the destruction of fishing waters, a fact which is constantly being emphasized by the Fish and Wildlife Service and the Izaak Walton League.

The preservation of historic and archeological sites and remains is an important function of the Service. To assist in discharging this responsibility, provisions of the Historic Sites Act of 1935 include authority to make necessary surveys to determine those prehistoric and historic resources which are entitled to federal, state, or local protection. Since the routes of travel and habitats of man throughout the ages have been in the river valleys, it is easy to understand the urgency of preserving or salvaging any remains or artifacts that may help to interpret earlier American culture and that are in danger of being lost forever with the rise of the reservoir waters. Therefore, historians and archeologists of the National Park Service and archeologists of the Smithsonian Institution under a cooperative agreement are diligently studying the areas to be inundated in an attempt to salvage valuable source material.

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Under authority of the Park, Parkway, and Recreational Area Study Act of 1936, the National Park Service cooperates with other federal and state agencies in solving park and recreation problems. This act made official a relationship that had developed in a limited manner since the inception of the National Park Service in 1916. Its greatest impetus was gained with the creation of the Civilian Conservation Corps and the emergency relief program, when for the first time federal funds were made available to assist in the development of state parks and related areas. The Service guided these programs, in many instances aiding in the establishment of park systems where none existed before, and, pursuant to the aforementioned act of 1936, cooperated with forty six of the states and the Territory of Hawaii in conducting state-wide recreation surveys. It quite naturally follows that the Service considers that it should be alert to assist in averting, if possible, any threat to the park areas of the states or of their political subdivisions which might arise from proposed inundations.

The interest of the National Park Service in the field of recreation, therefore, is much broader than mere concern with the areas under its administration. As reservoirs are brought into existence, some new recreational opportunities usually are created. A public, increasingly vigilant for more opportunities to use leisure time, demands that these areas be put at its disposal. Although not indispensable, bodies of water greatly enhance and stimulate outdoor recreation, particularly in the more arid sections of the United States, and this fact has been recognized in the establishment of the various state park systems. These systems, however, have seldom been developed to the point where all recreational needs of the region are met. Consequently, impounding water for other reasons usually necessitates studies of the recreational potentialities of the area involved and the requirements of the population within a reasonable travel distance.

Recognition of the recreational advantages to be gained incident to reservoir construction is almost universally granted by writers of papers concerning the accelerated water control programs, and the Flood Control Act of 1944 specifically mentions authority for the provision of recreation areas and facilities. Pursuant to this act and under agreements with the Bureau of Reclamation and the Corps of Engineers, the National Park Service, using the powers granted by the Park, Parkway, and Recreational Area Study Act of 1936, investigates the recreational resources of reservoir sites and makes recommendations for their protection, development, and use. Such activity is initiated only on the request of the particular agency concerned. After necessary field studies have been made, a report to the agency is compiled in which the Service identifies, appraises, and evaluates the scenic, scientific, historical, and archeological features and recreational potentialities, and recommends appropriate preservation and development. In this report it is important to note that existing and proposed recreational developments, plans, and programs of other agencies and estimated recreational needs are taken into consideration. Basin-wide consideration of all recreational resources is essential so that federal, state, and local agencies will work together in achieving a coordinated and integrated system of areas, facilities, and programs. Only thus can wasteful and irrational overlapping be avoided while, at the same time, recreational opportunities are provided as equitably as possible for all the people of the basin.

After it is decided that a proposed reservoir will offer needed advantages for recreation, the question immediately arises as to what agency should properly be the one to assume responsibility for operation of the area and facilities when provided. This question is settled on the basis of its comparative importance and the ability and willingness of the appropriate agency or agencies to assume the obligation. In some few cases, such as the Boulder Canyon and Grand Coulee projects, where the recreational resources are of more than regional significance, the National Park Service, by request, is undertaking administration of the recreational aspects of the areas under cooperative agreements with the agency having primary jurisdiction over the reservoir area. In other cases, where reservoir developments may be located partly or entirely on lands of other federal agencies, these agencies might properly administer the areas. In general, however, reservoir sites are largely of state or local interest, and responsibility for management of recreational developments logically falls on the appropriate state or local agency. It is evident that, where only a few picnic tables are indicated, for instance, the community which will use these facilities should be expected to take care of them. An area for which a more ambitious program of development is planned and which presumably would draw visitors from various parts of the state should be maintained and operated by funds derived from the state at large. Where the dam site is accessible by automobile, there is likely to be an influx of visitors drawn to the location by curiosity during the construction period and later by interest in operation, particularly where power is generated. Provisions for the care and convenience of such visitors should be the responsibility of the primary operating agency.

As to the procedure involved in the cooperative work on river basin recreation surveys, it is basic that agreements must be consummated and funds provided. Except for the Missouri River Basin, funds are made available by an appropriation specifically for "River Basin Studies." The Missouri River Basin work, however, is financed by an allotment from the Bureau of Reclamation from funds available for the purpose, as a part of the departmental program for the basin. The National Park Service in turn reallots funds to the Smithsonian Institution for the conduct of archeological investigations and preliminary archeological salvage under the terms of an interbureau agreement.

With the necessary agreements executed and the equally important matter of funds attended to, the Service is ready to begin the studies which lead to conclusions and recommendations as to what recreational developments, if any, are considered desirable. The field work is directed from one of the four regional offices maintained by the Service. Whenever a basin-wide survey in cooperation with the Bureau of Reclamation has been provided, a staff is set up in the regional office concerned, or elsewhere, under the immediate direction of a chief of that particular survey. Actual field studies are then initiated for designated reservoir sites according to established priorities. After considering topography, natural advantages of cover or scenery, historical or other scientific interests, population distribution, accessibility, and any existing or proposed neighboring recreation areas, the various factors are weighed and a determination is reached as to whether the site under investigation has recreational possibilities that warrant development and, if so, to what extent. A preliminary report is then submitted to the Bureau of Reclamation. Reservoir sites are similarly investigated on request of the district engineers of the Corps of Engineers; but, except for occasional consulting assistance in plan preparation, the resulting reconnaissance or preliminary report is all that is expected of the Service since planning personnel are available in the district offices of the Corps of Engineers.

Because the Bureau of Reclamation has no professional personnel to plan for recreational developments, cooperative agreements are extended to cover more comprehensive studies and the actual planning of the recreational areas and attendant facilities. If desired, the Service may also furnish technical advice and supervision during construction stages. As may readily be suspected, landscape architects, architects, engineers, historians, archeologists, foresters, biologists, and geologists must be used to explore all phases and to plan for the complex developments necessary to accommodate an area to intensive human use. Since the National Park Service has long been engaged in similar or related work, the organization of such a force requires only an expansion of personnel. The application of long-familar principles then produces the desired results. As stated in another paper of this Symposium, the Fish and Wildlife Service of the U. S. Department of the Interior investigates the fish and wildlife aspects.

Although the evaluation of recreational benefits in monetary terms is difficult and unsatisfactory because of such variables as the intangibles involved and the lack of data, the Service is often called on to attempt such evaluations. This work is undertaken reluctantly and with the full knowledge of all concerned that the evaluations are only estimates. However, until more is known about this subject, any estimates prepared will be scrupulously conservative. These evaluations are required principally by the Bureau of Reclamation. The attitude of the Corps of Engineers is less sanguine, as evidenced from this quotation from *Circular Letter No. 4231:*

> "In many cases the most important benefits from the recreational use of a reservoir area cannot be evaluated satisfactorily on a monetary basis, and it is not necessary in most cases that a monetary value be determined. These important benefits should nevertheless be fully described and discussed in the report. If desirable, an approximate or partial monetary evaluation of the recreational benefits may be included as a means to indicate their magnitude."

The Missouri River Basin program is further advanced at this date (1949) than are similar programs for other river basins, and it is interesting to note that coordination has been the keystone to progress of the work. The interrelations of the many responsibilities of diverse agencies have been recognized by the formation of federal inter agency committees, both in the field and in Washington, D. C., and by the establishment of a departmental field committee for the U. S. Department of the Interior. Thus, the duplication and the overlapping that might otherwise be encountered are avoided, and, at the same time, all phases of the vast problem of planning for the control of the water resources of a great river basin receive attention. It is expected that programs for other river basins will be coordinated in a similar manner.

It may fairly be asked whether there is any sound basis for assuming that the considerable expense involved in the creation of so many new recreational facilities is warranted. The answer may be found in several directions. Cooperative studies under the Park, Parkway and Recreational Area Study Act resulted in completed reports in thirty-seven states in which the present situation and the deficiencies were described. It became apparent immediately that there are large segments of the population which either are not served by any area within reasonable travel distance, or are inadequately served. Attendance records or trends also give indications as to the need for expanded opportunities. The total attendance for the last prewar year at state parks and related types of recreational areas approximated 90,000,000. Estimates based on the curve for preceding years indicate that provision of adequate facilities and programs will attract an attendance of at least 200,000,000 by 1960. Present accommodations would fall far short of meeting this demand. It should also be borne in mind that the ardent fisherman is certain to wet his hook in any body of water that is home to a fish. Many new reservoirs provide excellent fishing and therefore draw anglers from considerable distances as well as from the immediate vicinity. This interest dictates the provision of facilities for boating, fishing supplies, and, in many cases, overnight accommodations where fishing is good regardless of the possible proximity of other recreation areas.

A hint has been given of the economic aspects of park and outdoor recreation. The subject could be considered at much greater length if space were adequate. It could be shown that many and varied forms of occupations and business enterprises are provided by the use of recreational facilities. Data could be produced putting recreation in the big business class with total annual expenditures running into billions of dollars. It is not at all absurd to argue that outdoor recreation has an economic influence in the health-giving effect of rest and relaxation—intangible, to be sure, but none the less real and important. The National Park Service feels that it has a contribution to make to the economy of the United States by its participation in the development of new recreational areas where they are justified.

A social significance is attached to the development and the use of parks and related recreational areas. It is natural and desirable that citizens who are pent up in the cities and industrial centers should seek release and relaxation in the open air on their holidays. The need for contact with nature is strong within nearly everyone. Although perhaps subconscious, the feeling is there, and its satisfaction is important to a well-rounded, normal existence. The individual who craves hikes or reveries in solitude, the family that picnics on the seashore or in the woods, the groups that hold reunions, and the great gatherings for patriotic celebrations or various kinds of pageantry—all gravitate naturally to the open spaces called parks. The link with nature must be established. In assisting in the development of areas where activities of this kind can be indulged in, the National Park Service considers that it is exercising a function of the utmost importance.

HYDROMETEOROLOGICAL SERVICES

BY MERRILL BERNARD,[17] M. ASCE

The theme of this brief paper could well be, "How to eat your cake and have it, too." The "multiple-purpose family," made up of hydroelectric power, irrigation, municipal water supply, and navigation, was "getting along" most amicably until flood control came to live with it officially. Until then it had been the common purpose to maintain full reservoirs and to hold withdrawals from storage to business-like schedules. Furthermore, there had always been reasonable time for the adjustment of control works as the needs of changing power load, or a retarded irrigated crop, or deeper navigation waters were disclosed by these operations themselves. In other words, the conservation of stored water was to the common interest of all who were participating in a multiple-use project. However, flood control, demanding not a full, but an empty reservoir, has called for compromises which place on the operator the requirement that he "eat his cake" one minute and in the next have it, or wish he had.

Surface storage is created and sustained by the runoff from precipitation—in the west, largely from the seasonally controlled melting of snow deposited throughout the winter at the higher elevations; in the east, from storms confined to relatively short periods of practically continuous rainfall, the longest of which usually lasts 3 days or 4 days. It does not take much imagination to understand how easily optimum operating conditions could be maintained if the operator of a multiple-use reservoir had before him a record of the actual weather immediately ahead just as he has of the weather which has just passed—in other words, a perfect weather forecast. Such a forecast would have the range necessary to prepare for the beginning of runoff producing rains, the ending of the runoff period, or the beginning and ending of an extended rainless period. Unhappily, the operator can be provided with no such complete previsional operating schedule. However, the question can still be asked—have not the considerably less than perfect weather forecasts now available some operational value? The operator's answer to this question is that he has taken the fullest advantage of the national weather services but has found that the forecasts available to him are too short in range, not sufficiently detailed as to time and place, and subject to so low a verification as to be untrustworthy and even dangerously misleading. To support his case the operator points to the fact that the Corps of Engineers, the TVA, and the Bureau of Reclamation have made extensive use of the meteorologist, in both the design and the operational phases of multiple-purpose projects. Such claims are enough supported for the planning phase, as these agencies have supplemented their design techniques with hydrometeorological studies which have contributed much to structural security and program economy. However, when the comprehensive weather forecast (obviously the primary requisite of successful multiple-purpose operations) is needed, the tendency is to "let George (the

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Weather Bureau) do it." It is not the writer's objective to alibi for "George," but rather to call the engineer-operator's attention to a responsibility that he himself has not altogether met.

The prevailing criticisms of weather forecasts, as they are issued to the public, are that they are too generalized, that they lack detail and continuity, and, therefore, that they cannot be used as operational forecasts. Such faults, it must be agreed, have no place in multiple-purpose reservoir operations which are sensitive to weather changes because they attempt on one hand to control the excesses of weather and at the same time offset the ill effects of its deficiencies on the other; but why has the engineer allowed the case to rest here? Modern aviation, whose entire operation rests on weather forecasts, did not allow the technical limitations of an applied science, or the service deficiencies of an old-line federal agency, or a less than perfect use of weather forecasting to prevent an accomplishment which engineers having the responsibility for river control can scrutinize with profit.

Weather forecasts in the United States are based on some 700 synoptic observations—the term "synoptic" indicating that these observations, being taken at the same time, give a picture of weather conditions "as of the moment." Having been plotted and analyzed, the observations serve as the basis for a wide variety of forecasts, the most immediate being the operational forecasts for aviation of all types—commercial, military, and private. First, there are the short-range trip and terminal forecasts which, on a hit or miss verification, score more than 90%. Next, are the airway regional forecasts used for flight planning as far ahead as 6 hours or 8 hours. These forecasts are translated into operational terms by the air line meteorologists and dispatchers; and, from them, estimates of arrival and departure times, pay load, flight levels, icing, and other operational hazards are determined. The operational forecasts for aviation are supplemented by state and local forecasts covering a range of from 12 hours to 48 hours. These forecasts, which are designed to serve the needs of the general public, are accurate in from 65% to 85% of the cases, depending on the range. A 5-day forecast is issued as a guidance forecast and for the assistance of those interests whose operations can be benefited even though verification for the fifth day shows little skill—that is, little improvement over the probabilities that can be derived from the statistical use of climatological data.

The forecasts discussed thus far have been based on current reports of meteorological conditions. Multiple-purpose operations in the west are served directly by the water supply forecasts of the Weather Bureau which are synoptic in the sense that they are based on observed conditions, even though the time interval is 30 days long. The forecasts begin on January 1, when about half of the data used in the computations is observed and half is statistical. Forecasts are then issued as of the first of each succeeding month, through May, progressively substituting observed data for the statistical at which time practically all data are observed. Water supply forecasts are issued for about 270 points in the Columbia, Missouri, Colorado, Sacramento, Great Basin, Platte, and Rio Grande basins.

The writer has discussed the "synoptic forecasts" applicable to weather immediately ahead. Despite their limitations, these forecasts must be used as the basis for any operation demanding quick adjustment of controls designed to maintain the highest reservoir levels commensurate with the storage of flood runoff when such occurs. Another kind of forecast is the "statistical forecast." That little use has been made of the statistical forecast as a basis for planning and as a base on which to superimpose the short-range synoptic forecast is attested by the fact that the climatological data necessary for the development of such techniques now lie dormant and practically unused in the repositories of the Weather Bureau. The explanation is simple. Throughout the 50-odd-year period of record more than one billion items of climatological data have accumulated. Such a mass of material, in its present unorganized state, defies processing and analysis by manual procedures. Long since have the average or so-called "normal" values published by the Weather Bureau demonstrated their inadequacy and inapplicability to the present-day statistical problem. Only now (1948) have funds been provided for the Weather Bureau to place the currently collected data on punched cards, an operation which is being carried out at seven weather records processing centers throughout the United States. Even then, it will be a long time before the huge backlog of data has been reduced to decks of punched cards available for summarization, analysis, and application to the solution of engineering problems.

The larger engineering organizations responsible for operating control works which must serve both for regulating floods and for utilizing water have recognized the weather forecast as an essential operational need but, in the writer's opinion, are limiting themselves too restrictedly to those forecasts that can be provided by the Weather Bureau. The business of taking observations and preparing the resulting data for analysis and the issuance of weather advices to the public are basic functions of the Weather Bureau. However, as demonstrated by aviation, the beneficiary of a specialized and detailed operational forecast must be prepared to accept the cost as he would any other item of operational expense. The course to be followed is not one of duplicating the services of the Weather Bureau but of supporting operational staffs with meteorological "interpreters" and in developing techniques which will express the forecasts in operational terms. Just as cooperative hydrometeorological studies of the Corps of Engineers, Bureau of Reclamation, and Weather Bureau have provided design criteria so could the same cooperative effort provide the tools needed by the weather forecaster to make successful forecasts for reservoir operation.

STATE AND FEDERAL GOVERNMENT PARTICIPATION

BY DON McBRIDE[18]

This paper presents some rather important aspects of the relations between the federal government and the respective state governments as they pertain to the question of multiple-purpose water resource development—one of the broadest of the several fields of joint federal-state interest. In fact, it would be difficult to name another that has a more direct bearing on the economic welfare and the security of so many individuals. In recent years this relationship has been the topic of considerable controversy and discussion which probably will grow in tempo as federal water control and development projects become increasingly more important in changing and shaping the economy and the national security. This matter is of particular concern to the engineering profession.

The types of activities concerned with the physical control and utilization of water include irrigation, navigation, flood control, hydroelectric power, fish and wildlife protection, drainage, recreation, municipal water, sewage and industrial waste disposal, pollution control, and sediment control. These activities are highly important to the people of all the states, and are generally of vital economic significance throughout the seventeen western states. The federal government has moved into these fields with great vigor in recent years and has embarked on wide-scale programs of planning and construction, principally under the Corps of Engineers and the Bureau of Reclamation. Some of the state governments feel that the federal government has encroached too much on state jurisdictions, or "State rights," in carrying on these programs. They feel that the federal agencies executing the programs are attempting to dictate to the states the kind, the extent, and the programming of developments, regardless of state or local views.

Such a situation will surely affect all resource programs adversely. If there is any field of endeavor in which there should be complete understanding and agreement and effective cooperation between the state and federal governments, it is in the use, control, and development of water resources. Both state and federal governments, in the interest of the economic welfare and the security of the nation, have a vital stake in this undertaking.

The United States at present is in a period of postwar transition during which the economic and political structure is undergoing some drastic changes born of conditions that grew out of, or were accelerated by, World War II. To use a cliché that expresses the situation quite well, the nation is "at the crossroads." This is a good time to examine clearly the past, present, and future of that structure. Engineers should understand the "whys" so that they may do a better and quicker job of defining and attaining their goals in resource conservation and development. They must be realistic and must deal intelligently and opportunely with water resource problems. Throughout the United States, citizens cannot continue to suffer huge human and economic losses from floods; and the western states have reached a point at which present stability and future growth will be measured by the expansion of irrigation and hydroelectric power, regardless of what agency does the job. These facts need to be faced now, and workable solutions found. The federal-state relationship is particularly in need of clarification. In the early part of the twentieth century, there was little question about state responsibility for water resource development. The United States stayed quite closely to fields of undisputed federal jurisdiction, such as navigation and the reclamation of public lands.

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The nation was growing fast and the economy was expanding without much thought of centralized planning for irrigation, power development, flood control, and other water resources, or for the other elements of national economy. Things just seemed to grow of their own accord, and the few who raised warning voices that the nation was heading into trouble because of unplanned development and use of resources were either ignored, defeated, or shouted down as malcontents and radicals not worthy of an audience of serious men.

Projects sprang up wherever a few people got together and formed irrigation districts, or wherever a few communities brought out some concrete proposal for flood protection. Few people gave much thought to the day, now here, when "the cream would be skimmed off" potential irrigation developments and when careful engineering and economic planning of water uses would be necessary. Soil and forest depletion, the devastation of the grazing lands, and other conservation problems were matters for interesting discussions and heated arguments, but little was done about them. Urban and rural sections suffered recurring floods. People scooped the mud out of their homes and shops, cursed the river, said "somebody ought to do something about it," but never quite got around to the point of getting the job started. Farmers suffered recurring droughts and economic ruin while the precious water they needed flowed down the river systems and wasted into the seas, incidentally carrying a good measure of soil with it.

During all this time, the states had opportunity and incentive to move into the field of intelligent planning and development of water resources but very little was done. It is true that some of the states did make attempts in that direction, but their efforts did not produce the results demanded by the situation.

In 1917, Congress passed the Newlands Act, which provided for coordinated state-federal planning of water resource development, with full provision for preserving the rights and jurisdictions of state and local governments. This law was repealed in 1919 as a result of the activities of myopic individuals who failed to comprehend, or chose to ignore, the conditions looming on the horizon that eventually led to federal action.

The writer does not seek to condemn or reprimand the states, and he dislikes very much to point to this situation with a critical finger; but it is high time that a little soul searching was done in regard to this topic. Engineers must look at it as technicians, and without the emotional responses that sometimes handicap a scientific approach to political and social questions.

In retrospect, one can see that the nation has allowed a critical water resource situation to grow, which cried louder and louder for sound public leadership and planning. What was needed in the semiarid and arid west was patently a sound inventory of potential water resources, a realistic appraisal of water needs, current and long time, looking toward ultimate full development, and a programming of work that would enable development to keep step with needs. Unquestionably federal aid and participation in such an undertaking would have been necessary, but the basic "spadework" and the responsibility for taking the initiative in planning and programming, and for a high degree of control over the entire undertaking, rested with the states. It is not necessary to discuss why the states failed to meet this challenge with the speed and vigor demanded.

When the crisis precipitated by the economic depression in the 1930's brought about a public examination of the entire economic and political life of the nation, the federal government moved into the province of planning resource development and conservation where the failure of state action had created a vacuum. The National Resources Planning Board and other federal agencies attacked the problem on an engineering basis, by an inventory of what there was, a computation of what was needed, an over-all plan of action, and a program of necessary steps.

How the federal government launched an expanded program for reclamation, flood control, improvement of rivers and harbors, soil conservation, and other resource measures is a well-known story. The United States Supreme Court smoothed out the legal way with decisions, in the New River and Red River cases, that recognized the theory of navigability and federal jurisdiction over streams from source to mouth. Then came legislation in 1933 establishing the TVA. This new system of resource development plagues many today in view of its impact on what are considered the time-honored fundamentals of government.

The threat of change in procedures has made this nation great; it hangs over all heads like the "Sword of Damocles," and greatly disturbs many. The threat can be removed if methods are devised for implementing river resource development within traditional democratic principles and practices of government, using traditional government administrative agencies. Excellent progress has been made in demonstrating that the job can be done without resorting to the valley authority system, and the risk implied; but there is much room for improvement.

Today, there is considerable confusion, hesitation, and much downright squabbling over the further development of water resources. The planner is confronted with conflicting laws bearing on the matter, with differences of opinion as to the hydroelectric power aspect of the irrigation and the flood control programs, with the need of a comprehensive national water development policy that would keep planning agencies in step and help them reach their goals, and with the failure of too many states to meet their responsibilities. Out of this confusion a stronger and stronger sentiment is growing to rectify this situation. Without positive action there are better and better prospects that the public, in some time of crisis, may call for and obtain a full-fledged valley authority system of administering developments. There is manifestation of this confusion and disorganization in the Eightieth Congress in the several bills seeking to adjust the development of hydroelectric power as a working and paying side partner of irrigation development and as an aid to economic justification of flood control projects. If the matter is not straightened out fairly quickly it is a good bet that valley authorities will become accepted as normal.

Two of the most important subjects that need clarification are the federal-state relationship and state responsibility. Engineers should approach this problem with the realization that the federal government is in the water resource development business to stay. Their task, then, is to build the best possible cooperation and coordination between the federal and the state and local governments. The prospects for success in this connection, and for a beneficial effect on resource development, are bright.

The writer prefers, as a cornerstone, Section 8 of the Reclamation Act of 1902, which states, in part: "That nothing in this Act * * * [is] intended to interfere with the laws of any State relating to the control, appropriation, use or distribution of water. * * * "

Even as it moved broadly into the field of resource development and conservation coincident with the depression of the 1930's, the federal government recognized State rights and provided for state participation. The National Resources Board urged the states to establish permanent state planning boards and to develop, as promptly as possible, adequate state planning programs. Continued cooperation with, and encouragement of, state and local planning work and assistance in handling interstate and regional problems were urged on the federal government by the board.

A few of the states responded by establishing engineering planning organizations and forming programs for making effective use of federal aid, but most of them did not. As a result, the river resource programs developed were largely the result of thinking and planning by federal agencies, with a minimum of influence from state agencies. It is well to remember that the first flood control act (1936) required states to secure right of way, and to maintain and to operate flood control reservoirs. The act was amended in 1938, without opposition from the states, to make all these functions the responsibility of the federal government. The states apparently were not willing to accept responsibility and to bear the financial obligation that the 1936 act imposed upon them.

The 1944 flood control act is a federal law which puts it squarely up to the state governments to discharge their responsibility for participating in the planning of river basin developments. That act contains an outright declaration of Congressional policy—

> "* * * to recognize the interests and rights of the States in determining the development of the watershed within their borders and likewise their interests and rights in water utilization and control, * * *."

The law requires that the Corps of Engineers and the Bureau of Reclamation give the interested states full access to data gathered during the course of the resource development investigations. It requires that the states be given an opportunity for consultation regarding federal plans and proposals and, to the extent deemed practicable, an opportunity to cooperate in the investigations. It requires a similar degree of cooperation between the two federal agencies with respect to the investigations that each conducts.

The law requires that the originating federal agency submit a copy of its proposed report to each of the affected states before submitting it to Congress, as well as to the other federal agency. It provides that the states and the other federal agency shall send comments on the report to the originating agency, and that the latter shall forward the report and all comments to Congress, along with the comments of the originating agency.

The federal agencies interested in water, and related resource development, review and coordinate their programs through a federal inter agency river basin committee. To provide for even closer cooperation in the localities where problems arise, subordinate local inter agency committees have been established for the Missouri and Columbia river basins, with provision for representation by the state governments.

A recent survey conducted by the writer indicates that very few of the western states are taking any appreciable advantage of the opportunity afforded them by this law for vigorous participation in the planning and the programming of resource developments. Many of the states, for lack of funds and personnel, are not in a position to make thorough reviews of federal reports. In other cases, the states are so thorough in their investigations and reports that the result is almost the equivalent of sending two different reports to Congress on the same subject.

Some of the states have procedures for holding hearings for the purpose of ascertaining the views of the people directly affected, as an important guide to state review of federal plans; but most of them do not provide their people with this kind of service. To discharge its obligation fully, each state should take an inventory of its water resources, establish definite goals for their development, establish procedures for ascertaining public views on federal projects, set up and use effective engineering machinery to gear federal programs to state needs, and do all this economically. Cooperation of this type at the inception of planning would be highly beneficial and is earnestly desired and provided for under federal law and procedures.

J. E. Hobson, director of the Armour Research Foundation, Chicago, Ill., in an address has stated:

> "Government is a servant of the people and therefore should be told what its role is to be. Government should not tell the people. It is for the people to decide how Government can best serve them."

This is a fine expression of sound government policy; but, when the people stand mute, the government can be expected to move ahead without the benefit of their expressions. Progress cannot be held back in the development of river basin resources. In other words, as long as the states fail to undertake a program of vigorous and effective cooperation and participation in resource development, the federal government will maintain the initiative and anticipate a minimum of state influence. Without clear-cut definitions of goals and expressions of public views and needs, and without strong state leadership along sound engineering lines, there will be areas of friction in federal-state relations, rather than smooth coordination.

The location, design, operation, and purposes of reservoirs present problems that should not be too difficult to solve to the satisfaction of all requirements—local, state, and federal—if the states give appropriate engineering attention to federal proposals and are willing to approach the matter in the old-fashioned spirit of the greatest good for the greatest number of people.

With the passing of time, multiple-purpose projects will become more and more complex, and planning agencies will be faced with more complications from the standpoints of design and operation than heretofore. Therefore, such projects must be scrutinized very carefully to produce sound economic and beneficial use.

From the standpoint of the state engineer, certain principles should be basic in the design of multiple-purpose projects. If the costs of the project are to be repaid from earnings, it is particularly necessary to check proposed plans carefully in the interest of preserving the most stringent economy. There may be a tendency to overdesign structures if the project is entirely or heavily nonreimbursable, and sound public policy requires that all care be used to ascertain economies that might be effected without impairment or risk. A careful study may show that to stand the property losses from a flood stage that might be reached once or twice in a century would be much cheaper than to design a structure for such an unusual occurrence. Such matters as spillway designs, destruction of agricultural values by the inundation of lands in the reservoir area as compared with the creation of agricultural values below the reservoir, and the effect of such changes in land values on local government subdivisions are some of the problems that affect the state engineer's viewpoint.

In response to inquiries from the seventeen western state engineers as to the basic principles of design and operation of multiple-purpose projects, the following seem to predominate:

1. Multiple-purpose operation of reservoirs should not be attempted unless conditions are favorable;
2. Operation should be planned to result in the optimum benefit being derived even if operation for one use may be detrimental to another;
3. Reservoirs should not be operated to produce a benefit if that benefit can be obtained at a lesser cost by some other procedure;
4. Separate allocation of storage capacity for each use is essential; and
5. Preference of use should be the controlling feature of design and operation of any reservoir.

The need for a comprehensive national policy is crystal clear. The federal government, at the outset, was concerned solely with navigation. The close connection between that function and flood control led the federal government into the latter field. Reservoirs built for reclamation were suitable for irrigation, hydroelectric power, and other uses. As new situations were encountered, including drainage, pollution abatement, sediment and debris control, soil conservation and erosion control, water supply, recreation, and fish and wildlife conservation, the government was led inevitably into participation in the entire field of water resource development.

Governmental entry into these fields was usually marked by an act of Congress, establishing the government's policy to meet the particular need in question. In this manner, the national water policies and activities have been developed piecemeal and have lacked careful coordination and codification into an integrated instrument. As a result national policies are frequently inconsistent and conflicting, jurisdications overlap, and gaps occur in the over-all policy structure. Leaders of water resource development have long felt the need of a consistent national water policy which will result in full economic development in the interest of stable growth of the region, state, and nation.

The Subcommittee on Appropriations for the U. S. Department of the Interior, in reporting the appropriations bill, cited the need for a national public power policy, as follows:

> "At present the Appropriations Committee has to consider requests from various Government departments for funds for the production and distribution of power and each department and agency has a different plan or system from the others. The issues involve the type of transmission systems, priorities of customers, interest charges, and rate structures; therefore, in the opinion of the committee a sound uniform and coordinated power policy should be written into law to guide the Congress and its committees in making appropriations and authorizations for projects which contain hydro-electric power incidental to water resource development. If the power is to aid the projects, it must be marketed to assure economic stability of the project.
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> "The desirability of an over-all policy is clear. This is a legislative matter, however, beyond the jurisdiction of this committee, but it agrees that there should be such legislation, particularly with respect to both the application of the interest component and the public power policy, and in the opinion of the committee this should be the subject of immediate legislative action."

The controversy over the development of hydroelectric power has injected much confusion into the water resource development picture. Everyone must realize that hydroelectric power is an essential and inseparable part of any water resource development. If the nation is to go forward with large-scale irrigation and related resource developments, it must depend upon the widespread sale of hydroelectric energy made at federal dams and, where necessary, carried by federal transmission lines to points of wholesale delivery. This procedure is necessary to defray project costs. Power is needed to build the industrial economy that must go hand-in-hand with western agricultural growth. It lightens the burdens of rural life and will brighten many a western home.

The 1944 flood control act was "a step in the right direction," containing provisions dealing with hydroelectric power, irrigation, priority of water use, water supply, and recreation. Much more remains to be done, however, and should be done expeditiously to unite all interested agencies in a common, coordinated effort.

Briefly, to summarize all the foregoing:

a. Instead of holding back or obstructing federal programs, the states should strive together and separately to improve the coordination and the effectiveness of federal work.

b. The states should exercise fully their responsibility for participation in the inception of projects, and for adequate engineering reviews and public expressions in the interests of promoting economy and adaptability to state and local needs.

c. The states should cooperate in the establishment of a federal water policy that would define the rules of the game so that all agencies can work together more effectively.

d. Continuance of the present high degree of indifference, and failure to come forward and exercise the responsibility attending State rights, is to risk perpetuating and increasing the present state of confusion. This result certainly would promote the germination of the already planted seeds from which valley authorities might some day grow.

In his rather frank criticisms of the states, it is the writer's hope that he has not minimized the fact, nor left the false impression, that the federal agencies interested in the development of national resources have a lesser responsibility than do the states.

SUMMARY AND REVIEW OF PRINCIPLES

By Raymond A. Hill,[19] M. ASCE

This Symposium contains thirteen papers on the subject of multiple-purpose reservoirs. Some of them review the development of the principles governing designs; some illustrate the application of such principles in actual operating practice; and others, by authors who influence the design and the operation of such reservoirs, explain functions and viewpoints.

In this final paper it seems appropriate to call attention to a certain divergence in views expressed in the papers and to emphasize some of the basic factors that should be considered in the design and the operation of multiple-purpose reservoirs. It should be understood that any opinions expressed are those of the writer and are not necessarily endorsed by the Joint Committee on Design and Operation of Multiple-Purpose Reservoirs as such.

Definition of Multiple-Purpose Reservoirs

In the "Introduction" made on behalf of the committee, it was stated that:

> "* * * the term 'multiple-purpose reservoirs' should include all reservoirs actually designed and operated to serve more than one function and that it should exclude those whose design and operation are controlled by a single function, even though other benefits accrue as by-products."

It would appear that those authors representing the various federal agencies consider that this definition is too restrictive and that every reservoir serves, or at least should serve, more than one purpose. As a corollary to this proposition, it would follow that indirect benefits will result from the operation of any reservoir and that these benefits may be used to offset the costs of construction.

Conflicting Requirements

When the concept of designing a reservoir for more than a single purpose first became popular, the problem was only one of reconciliation between the regulation of stream flows over a short period for flood control and the regulation over extended periods for irrigation, power production, or navigation. In recent years, however, fish and wildlife protection and propagation, recreational uses of reservoirs, and other functions have come to influence, materially, the design and the operation of multiple-purpose reservoirs.

The requirements of reservoir operation to satisfy these several uses are fundamentally conflicting; thus:

(*a*) Regulation for flood control is best accomplished when reservoirs are kept empty in anticipation of floods and are evacuated as rapidly thereafter as circumstances permit.

(*b*) Conservation for irrigation or domestic use requires that floodwaters be held in storage, sometimes over a period of years in the semiarid west, that the release of water be in conformity with seasonal demands, and that at times all water be withdrawn for use.

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(*c*) Regulation of stream flows for the production of hydroelectric power requires that reservoirs be kept as nearly full as practicable, that they never be emptied, and that the release of water be made in accordance with demands for power and energy.

(*d*) Maintenance of a stable reservoir level is most favorable to fish and wildlife protection and propagation and in no event may such a reservoir be emptied.

(*e*) Recreational purposes are best served when a reservoir is kept full, simulating conditions at natural lakes.

There is less conflict between irrigation and power than between either of these and flood control, in that conservation of floods and regulation of releases over extended periods of time are essential to both. However, except in limited areas where the power load happens to conform to irrigation demands, reregulation of water released for power production is necessary, as is cited by Mr. Nelson in his description of Elephant Butte and Caballo reservoirs on the Rio Grande. Without Caballo Reservoir below it, Elephant Butte Reservoir could be operated only to satisfy irrigation needs and power would have been but a by-product; but, with power releases reregulated by storage in Caballo Reservoir, firm power can be produced. In this case it took the coordination of operation of two reservoirs to constitute a multiple-purpose project. In general, it would appear that, where the requirements of irrigation and power must both be met, a series of reservoirs is needed and that no single reservoir can serve these multiple purposes without waste of water. Such waste cannot be permitted where agriculture is dependent on irrigation.

Coordination of the requirements for fish and wildlife propagation and for recreational uses of reservoirs with the requirements for either flood control or irrigation is almost impracticable. The fluctuations in stage characteristics of reservoirs operated for such purposes make them unsuitable for recreational use and fish can be protected only if a minimum pool is maintained. Flood control reservoirs are particularly subject to rapid changes in level; irrigation reservoirs must be emptied of all water whenever it is needed for delivery to farms.

Obviously, the conflicting requirements of flood control and irrigation, or of flood control and power production, could be avoided by pyramiding capacities. For example, if 1,000,000 acre-ft of storage was deemed necessary for conservation and 500,000 acre-ft was needed for flood control, a reservoir of 1,500,000 acre-ft could be built and the component parts could be operated in effect as separate reservoirs.

Such superimposition of capacities would not be inconsistent with the philosophies of the Corps of Engineers; on the other hand, it would be contrary to the principles advanced by other federal agencies who hold that dual use of reservoir capacity is practicable and economically desirable. Various illustrations were given of the feasibility of using flood control storage for seasonal regulation of stream flows to conform to the requirements of irrigation, hydroelectric power, or navigation. In this connection it was shown by Mr. Bernard that a wealth of data has been accumulated from which statistical forecasts could be made.

ECONOMIC FACTORS

In the design of any reservoir, therefore, which is intended to serve more than one purpose, such conflicting requirements must be compromised, unless there is to be a pyramiding of capacities and costs. No such compromise of conflicting requirements can be made intelligently without consideration of the relative values involved. Hence, the design of a multiple-purpose reservoir is fundamentally an economic problem.

Although little was mentioned about costs and benefits by the authors representing the various federal agencies which construct such reservoirs, it is a matter of common knowledge that there is little agreement among them as to the basis of determining benefits. Some people in government service believe that any expenditure of taxes is justified if it does not exceed the total of all benefits deemed to flow from that expenditure. Others in private industry maintain that the economic system depends on rigid application of the rule that every investment should earn a rate of return commensurate with the risk involved. Reconciliation of these opposing philosophies is just as important as compromise of the conflicting requirements of operation of multiple-purpose reservoirs.

The rule of private industry must remain applicable to the investment of private capital because such investments are dependent on direct benefits only. On the other hand, governments can afford to make greater capital expenditures because of the indirect benefits accruing to the taxpaying public; but there is reason to question expenditures beyond the point where the incremental cost exceeds the incremental benefit.

Benefits arising from the construction of any multiple-purpose project approach an upper limit as the capacity provided in the reservoir is increased. Costs of construction, on the other hand, normally include some substantially fixed items and become tremendous whenever the size of the reservoir exceeds the limitations of the site. If typical curves are plotted showing the relations between benefits and reservoir capacity and between costs and reservoir capacity, it will be found that these curves intersect at two points; below some small capacity the costs will exceed the benefits and beyond some much greater capacity the benefits again become less than the costs. Between these limiting amounts there are two values of particular significance:

(1) The reservoir capacity at which the ratio of benefits to cost is a maximum; and

(2) The capacity at which the difference between benefits and costs is a maximum.

Referring to Fig. 15, it will be found that, for any reservoir capacity between 0.3 unit and 8.3 units, the benefits (curve 1) exceed the costs (curve 2). The maximum dollar difference between benefits and costs occurs for a reservoir 3.3 units in size, which is 40% of the upper limit where the costs become as great as the benefits; the maximum ratio of benefits to costs (curve 3, Fig. 15) occurs for a reservoir of 1.9 units of capacity, which is only 23% of the same upper limit.

The lower limit at which costs exceed benefits is of little practical significance but reservoir capacities corresponding to the other three values determined in this illustration must be given consideration in the design of any multiple-purpose reservoir. There is usually ample justification for providing storage capacity up to the quantity at which the ratio of benefits to costs is a maximum, and reasonable justification for increasing the capacity beyond this size up to the point where the difference between benefits and costs is a maximum. There would seem to be little justification, however, for providing any reservoir capacity in excess of that required to produce the maximum difference between benefits and costs (curve 4, Fig. 15), because beyond that point it takes more than a dollar of taxes to derive another dollar of benefits.

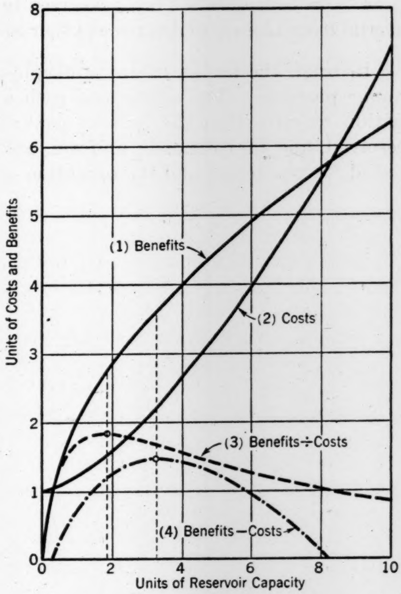

FIG. 15.—RELATION BETWEEN BENEFITS AND COSTS OF MULTIPLE-PURPOSE RESERVOIRS

In any such comparison of costs and benefits of multiple-purpose reservoirs, the benefits deemed to arise out of the reservoir must be as tangible as the costs themselves. When a dam is built, capital is then invested and it is immaterial whether such capital be obtained directly from the public by the sale of securities or whether it be obtained indirectly from the public by the levy of taxes; in either case real dollars must be used. It follows that, if such benefits are to outweigh the costs, these benefits must have substance. Furthermore, whenever those who benefit directly from a multiple-purpose reservoir do not bear its cost and the project must be subsidized by the public at large, consideration should also be given to competitive benefits—that is, to benefits that would be derived from the expenditure of equal sums for other purposes or at other locations.

SUMMARY

It would seem to be evident, if all secondary uses and by-products of a multiple-purpose reservoir are to be considered in its design, that:

1. The conflicting requirements of all possible uses of the reservoir must be compromised;

2. Compromise of such conflicting requirements will depend on evaluation of each direct and indirect benefit that will actually be realized in the operation of the reservoir;
3. The sum of all such benefits must be substantially more than the costs of construction and operation; and
4. Due consideration must be given to the competitive benefits which would accrue from like expenditures at other locations or for other purposes.

In brief, the design of any multiple-purpose reservoir is basically an economic problem. The solution of such a problem in the best interests of the nation requires that the present confusion of philosophies and practices be resolved, and that rational, uniform, and sound economic principles be established for the design and the operation of multiple-purpose reservoirs.

DISCUSSION

Herman A. Hilton,[20] M. ASCE.—In those parts of the world where data are available (referring to such items as precipitation, evaporation, size of basin, and runoff factors), the planning and design of multiple-purpose reservoirs can be based on rather exact engineering principles and methods. Where insufficient data exist (for example, where periods are too short to permit reliable conclusions), the problem becomes much more complicated. If other very erratic conditions prevail, such as frequent change of watercourses, extensive disintegration of rock and soil, and enormous sedimentation, and if general difficulties must be considered, such as the lack of skilled and ordinary labor, equipment, and funds, then the problem becomes really grave and even experiences elsewhere, and detailed reports, will be of only limited assistance. This is exactly the situation in many South American Andean countries, where the questions of flood control and protection, irrigation, power, and drinking water supply are of the utmost importance. The writer has described these conditions briefly elsewhere.[21,22]

In such cases, the well-known, and well-proved, engineering methods of planning and design are not sufficient, as the necessary basic information is not at hand, or only partly so. The problems must be solved, however, and the engineers concerned with them found ways, apparently with success as far as the results show. The writer knows of a case in which pluviometric data were compiled by a procedure based on the theory of similitude. One of the stations had precipitation records covering 50 years. The precipitations were plotted for 1-year, 5-year, and 10-year periods, the year being taken as starting with the end of the dry season in October. All these curves had practically the same parabolic form; they differed only in level, according to dry and wet years. The regularity of the curves is explained by the local meteorological conditions. This phenomenon raised a possibility that, if the precipitations of the first months were known, it would be possible to predict, at the end of December or January, the probable precipitation for the remainder of the year, including the period of highest precipitation, from January to April. When precipitation data were needed at another site some distance away (about 350 miles), the existing records of a few years were examined, and it was found that these curves also had the parabolic form; but, compared with the first place, precipitations for the same years were reduced in a certain proportion. This proportion offered a method of computing the average values of probable precipitations and then determining runoffs. This experience occurred in the spring of 1947, and the writer has no information as to the results of a check by subsequent observations.

In cases where other than meteorological data were lacking, the following procedure has been used. One unknown fact (or even two) was set aside first. The design was made in the regular form; then the influence of the omitted fact or facts was investigated, for the worst conceivable case. If the resulting necessary or desirable changes were not too considerable (above all in time and money), they were included (or at least the possibility of a later change was provided for as far as possible). If the desirable changes were too great, a compromise or a negative decision had to be made; perhaps the entire project had to be postponed.

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[20] Civ. Engr., Michigan City, Ind.

[21] *Ingenieria* (Mexico), September and October, 1945, p. 273.

[22] *Boletin de la Administracion Nacional del Agua* (Argentina), March, 1945, p. 222.

If the lack of some data threatens to make a design impossible, it may sometimes be feasible, without too excessive cost, to modify the influence of such data or to limit that influence to certain values and thus to obtain a basis for planning the project.

After such a project has been completed and is at the very beginning of its operation experience, the routine steps in the operation must be based on the conditions assumed in the design. Thereafter, by progressive modification, a definite form and a standard method of operation must be evolved by experience. Sometimes there exist possibilities of providing for later adaptations in design and construction.

Under such conditions it is not sufficient for the engineer to have a theoretical knowledge of hydraulics and experience in hydraulic and general structures. He needs more than knowledge—that is, he needs an instinctive feeling and understanding for all the problems involved. In order to give young engineers this understanding, many technical universities or colleges in South America have a special 1-year course, obligatory for all civil engineering students, entitled "Stream and Sea Hydraulics," of which three fifths or three fourths is concerned with stream hydraulics—both theory and laboratory practice. The course is given in addition to courses in fluid mechanics, sanitary engineering, and other subjects. The writer does not know if any colleges in the United States include a similar special course or if the necessity for such a course exists.

The writer read the Symposium papers with these unique and special conditions in mind, and fails to find any direct reference to this point. It seems to him that all the reservoirs and systems considered by the authors were based on more or less complete data and knowledge of the prevailing conditions. It would certainly be interesting to read something more concerning experiences with reservoirs designed and originally operated on the basis of insufficient data and under special conditions as outlined in this discussion. The writer felt obliged to raise this point for the benefit of the many engineers in countries other than the United States who read the ASCE *Proceedings*.

WESLEY E. GILBERTSON,[23] ASSOC. M. ASCE.—The importance of multiple-purpose reservoirs is readily acknowledged, and the divergent points of view from which the subject is discussed in this Symposium are quite comprehensive. A serious omission from the Symposium is a paper on the health aspects of reservoirs and water impoundments. It should scarcely be necessary to recall the importance of this consideration. The location, design, and operation of reservoirs may have several important effects, both beneficial and detrimental, on the health of human populations residing above, adjacent to, or below the actual site of the impoundment.

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[23] San. Engr., Director, Communicable Disease Center, U. S. Public Health Service, Federal Security Agency, Atlanta, Ga.

Much work has been done on the control of malaria mosquitoes relative to water impoundments. In fact, this consideration has been a major factor in the location, design, and operation of water impoundments. Some of the results have been to increase costs, others have caused environmental changes that have actually produced savings to people living in the locality. Furthermore, it is now known that water impoundments as related to irrigation have a definite effect on the incidence of eastern and western encephalitis, a mosquito-borne disease. Thus, the geographic area in which the mosquito potential of water impoundment must be considered is enlarged.

The continuous stream flow to provide dilution water for the disposal of sewage or the effluents from sewage treatment plants is an extremely important economic and public health consideration and, alone, has a serious impact on the design of water and sewage treatment plants.

Both state and federal health agencies have recognized the public health significance of reservoirs and have conducted a number of field studies. Careful analysis has been made of the relationship of various health factors to other uses of reservoirs, such as fish and wildlife, recreation, power, and irrigation. The interrelationships of these programs is of great significance; often, changes in one may seriously affect the others.

ROGER E. AMIDON,[24] ASSOC. M. ASCE.—The Symposium on Multiple-Purpose Reservoirs poses a number of important questions to the engineering profession and to all technicians involved in water management, including foresters, economists, and conservationists. An important question not adequately treated in the Symposium is the problem of sedimentation.

People occupying the United States in future centuries will hold a dim view of the reclamation and flood control projects of the twentieth century if needed water control and storage space in these great reservoirs is filled with silt and debris. The question arises, of course, of how distant a view should be taken by planners of reclamation and flood control projects. With exhaustible natural resources, such as large reservoir sites, the probable length of need by man should certainly be given consideration.

Hydro power will probably always be needed. Transportation on navigable rivers will always be economically desirable. Water for domestic and irrigation use will be necessary as long as civilization exists. Sea water may be cheaply converted for use in seaboard areas but its transportation to interior states will always involve costs exceeding the costs of distribution from near-by rivers. There seems little likelihood that precise control of precipitation will obviate the need for control of floodwaters. Fresh-water reservoirs will always be in demand for recreational use.

These considerations point to the permanent need of reservoir space for domestic, industrial, and irrigation waters; the control of floods; recreation; power development; and river navigation.

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[24] Civ. Engr., California Forest and Range Experiment Station, Berkeley, Calif.

Silt control and debris control are as much parts of reservoir programs as water control. This principle is recognized under the flood control act and therefore no further legislation is needed to deal with the problem. Without silt and debris control water control becomes a temporary expedient—in many cases an extremely short-lived expedient. In fact, silt control or sediment control in Southern California is a prerequisite to the reduction of flood damage. Although especially important in Southern California, this condition is true elsewhere; if reservoirs are to be applicable to flood control, adequate reservoir capacity not only must be obtained, but also must be maintained. The total and correct cost of water control for multiple use must include the cost of silt and sediment control.

The allotment of space for silt and sediment detention in reservoirs is a recognition that a problem exists; but it is not a solution.

No discussion or apparent thought has been given to the "life of the project" in the Symposium. The size of reservoirs needed to meet the objectives of the planner is one of the major problems of design; but the size for any given set of conditions is directly dependent on the length of time the reservoir is to be used. Encroachment of other uses on flood control space is generally recognized by the authors as reducing the effectiveness of the reservoir. Encroachment of debris and silt on the design space also must reduce the effectiveness of the reservoir for its designed function. Eventually this encroachment will make the reservoir worthless for each and all of its uses. Certainly the engineers conceiving the huge projects being planned and constructed throughout the world are not thinking only in terms of the "economic life" (from 50 to 100 years) of the structure. Studies indicate that the useful life of 64% of all reservoirs in the United States is less than 100 years and that only about 15% will serve for more than 200 years.[25]

Three problems relative to sedimentation should be considered by the planners of multiple-purpose reservoirs: (1) Rate of sedimentation; (2) sediment removal from reservoirs; and (3) reduction of the sediment rate by upstream stabilization measures.

(*1*) *Rate of Sedimentation.*—Sufficient data are usually not available to make accurate estimates of sedimentation. Since 1929, however, data have been collected throughout the United States which, when properly analyzed, will provide a basis for estimating with much greater accuracy than was formerly possible. Sedimentation research, too, will undoubtedly improve procedures and methods for making more precise estimates.

(*2*) *Sediment Removal from Reservoirs.*—Eventually sediment deposited in many reservoirs will probably be removed by sluicing and mechanical excavation. Where sluicing is possible, it will undoubtedly be most economical; but the effects of sluicing on downstream installations should be studied. Where mechanical excavation appears necessary, consideration should be given to the availability and accessibility of dumping space. Space should be procured and reserved at the time of construction to prevent intensive development of adjacent areas. The length of haul might dissipate the economy of an otherwise good dam and reservoir site. The cost of sediment removal should be included in the total cost of the project.

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[25] "Sedimentation in Reservoirs," by Berard J. Witzig, *Transactions*, ASCE, Vol. 109, 1944, p. 1047.

(*3*) *Reduction of the Sediment Rate by Upstream Stabilization Measures.*—In considering this item the sources of sediment production must be determined. A program must be laid out to reduce the production from these sources. The degree of reduction must be estimated. The cost of the program must be included in the total cost of the reservoir.

Measures involved in upstream stabilization will vary from area to area, and within watersheds. They will generally include intensive fire control in forested areas, vegetation of actively eroding areas, road slope stabilization by vegetation supported by the necessary structures, road drainage for the harmless disposal of water crossing and concentrating on roads, agricultural land treatment, and stabilization of channels. The stabilization of channels will consist mainly of engineering works. In many cases, the stabilization of upstream gullies and severely eroding channels with raveling side slopes will reduce, permanently, the volume of material available for transport to reservoirs. The construction of debris dams, upstream from reservoirs, may be a necessity to provide immediate and positive storage control and to anchor in place large volumes of channel bed material. Reduction of the channel slope by the construction of debris dams and barriers will reduce the transporting power of the flow. Upstream sediment storage may be extremely desirable and necessary but should not be considered a permanent solution.

Various combinations of many measures will ordinarily be needed to reduce reservoir sedimentation. The measures will usually be costly, but in many cases they will be less costly than the continual excavation of material from the reservoirs.

Summary.—In determining the economic justification of a multiple-purpose project, evaluation of the sediment problem over a long period is a requisite to correct analysis. The cost of erosion control measures on the watershed, and the cost of maintaining the reservoir space in whatever quantitative combination is most practical and economical, must be included in the total project cost. The benefits (whomsoever they may help and whensoever they may accrue) should equal or exceed the cost.

L. C. Bishop,[26] M. ASCE.—One feature of multiple-purpose reservoirs that has been somewhat neglected in the past is silt control. This important consideration has been omitted from the Symposium; yet, unless some plan is devised for holding back most of the silt by better farming practices and erosion control, and then providing a method for washing the remainder through the reservoirs, the day will come when all the fall of the streams is taken up by dams and the basins created behind them will be filled with silt. It will then no longer be economical to operate such reservoirs.

Mr. McBride, in the twelfth Symposium paper, is correct in blaming the states, to some extent, for lack of cooperation with federal agencies. For the most part, the Bureau of Reclamation has made investigations of irrigation projects on interstate streams without regard to the fact that the equities of the states in the right to the use of such waters have not been determined. After considerable agitation by the states in the Upper Colorado River Basin, the United States Department of the Interior did require a compact for the division of the waters, and assisted materially in negotiating it. This should be done in every instance.

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[26] State Engr. of Wyoming, Cheyenne, Wyo.

A. H. Davison,[27] M. ASCE.—Admittedly, there is confusion as to the philosophies and practices discussed in this Symposium that needs to be resolved. Establishment of rational, uniform, and sound economic principles is needed to control the expenditure of public funds. Naturally, since fully 20% of the entire national income is required to finance the federal budget, a tough public will be more than usually critical of the behavior of federal agencies and their management of public funds.

Currently reported in an engineering periodical is the developing controversy as to which government agency shall have control over streams on which public works are projected. This is reminiscent of the "squabble" between military agencies of the Department of National Defense. The public interest is not helped thereby and the responsible agencies earn a poor reputation.

As a fundamental principle, it seems appropriate to emphasize that public agencies are public servants—not public dictators to exploit pet philosophies at public expense. Public resources are not unlimited, and carelessness in their investment will be attended by public distrust, resentment, and loss of good will toward the responsible agency.

Honest engineering in the public interest suggests that departmental philosophies and rivalries should be discouraged. They are more likely to compete with the public interest than to support it.

A justifiable approach to multiple-purpose reservoirs would emphasize objectives that aid people to preserve and improve their means of livelihood. According to the testimony advanced in the Symposium, such limited objectives are most easily coordinated. During a period when the tax burden caused by past emergencies is heavy, every dollar of investment cost should earn at least a dollar's worth of tangible benefit and as much more as is consistent with legitimate objectives. Flood control, as the primary justification for investment (because public safety is the legitimate objective), should not be exploited to include speculative intangible benefits. In some parts of the United States, the provision of sanctuaries for fish, wild life, and recreation at appreciable public expense would be considered wasteful and frivolous by many people who are seriously concerned about their own livelihood. On many streams that are highly developed industrially, navigation no longer exists above tidewater, and any representation that navigation should be considered an important factor in that area is "humbug." It is extravagant to design for the development of inherent power possibilities that provide for two or three times as much installed capacity as the stream can possibly operate, at a load factor consistent with the prevailing load factor of the electric system in that area. Experience has proved that such design simply results in public veto of the entire project including its worthwhile flood control feature.

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[27] Civ. Engr., Glens Falls, N. Y.

In the northeastern part of the United States, a multiple-purpose reservoir project usually emphasizes the flood control feature as justification for investment of public funds. Provisions for power storage and power development at the site are added under favorable circumstances. The added functions are not allowed to impair the primary function of flood control operation to protect damage centers downstream. An example is a state-owned-and-operated reservoir in the Adirondack section of New York State, which (since about 1930) has been operated to control floods and to increase low stream flow to power plants and industries downstream. The capacity of the reservoir is about 690,000 acre-ft or about 650 acre-ft per sq mile of controlled drainage area.

It has been stated that the primary operating purpose of a reservoir should be clearly defined, and the problem of design is to provide facilities to realize that purpose.

To secure the expected tangible benefits, intelligent operation must be guaranteed, particularly with respect to flood control. Intelligent flood control operation of a reservoir that controls not more than one fourth or one third of the total drainage area above important damage centers downstream depends considerably on shrewd appraisal and understanding of the runoff from that part of the drainage area that is uncontrolled. Prompt closing of reservoir gates at the beginning of a storm, consistent with the maintenance of minimum flow required by industries and other facilities, followed by gradual release of stored water after the uncontrolled runoff has passed the damage centers, is generally indicated. Reservoir operation is not a matter of finicky rule curves devised by theorists.

It seems axiomatic that: (1) A multiple-purpose reservoir should never be caught full to spillway crest at the beginning of a storm; and (2) the storage space reserved for floods should be sufficiently empty at the beginning of a storm to provide adequate flood control. All business is a calculated risk. The degree of "emptiness" of flood storage space might vary with the season of the year when major floods may be expected in that locality. Certainly, the flood storage space should be practically empty at the beginning of the "design storm."

Effective control of a reservoir considers the three elements of inflow, storage, and outflow. Maximum inflow is reflected by the "design storm," the magnitude of which is determined by consideration of the records, hydrology, and meteorology applicable to the locality. The designed capacity of the reservoir represents the storage factor. The discharge capacity of spillway and gates represents maximum outflow, which for satisfactory results should not threaten the safety of damage centers downstream. Cost is budgeted to provide for effective controls and their adequate and continuous maintenance, including the salary of a full-time attendant.

It is the function of design to make those controls effective and dependable at all times. A design assumption that considers gates closed or otherwise inoperative and the reservoir full to spillway crest at the beginning of a storm may be permissible in the case of a power storage reservoir because that condition is occasionally an incidental aspect of power storage operation. Such a design assumption does not make sense in the case of a flood control reservoir, especially when applied to the flood storage space of a multiple-purpose reservoir where the chances are excellent that the power storage space is also somewhat depleted at the beginning of a storm.

Extravagant and illogical assumptions result in extravagant design and construction of spillways. An actual flood occurrence that would test the capacity of such a spillway would destroy the damage centers downstream that had trusted in "flood control." If such actual occurrence is unthinkable, why provide the expensive and excessive spillway capacity?

If spillway design applicable to power storage reservoirs is applied to flood control or multiple-purpose reservoirs, the public may suspect that the avowed flood control feature may later disappear and the entire reservoir may ultimately be acquired as power storage by a special interest at public expense. Experience proves that, if design procedure is inconsistent with avowed operation procedure, the public will become suspicious of motives and will probably ascribe the least favorable motive to the public agency involved.

It seems evident that the public interest must be treated fairly by federal agencies or the public may prefer to take its chances with natural forces that appear more trustworthy in behavior and less expensive in results. Such an outcome might be more unfortunate for federal agencies than the reorganization advocated by the report of the Hoover Commission on its recent investigation into the activities of government departments with respect to development of natural resources.

JOHN S. COTTON,[28] M. ASCE.—There is certain to be conflict between the various uses incorporated in the operation of a true multiple-purpose reservoir. The greatest conflict is usually between irrigation and power, unless reregulation is provided. There is less conflict between flood control and irrigation, or between flood control and power. The least conflict is probably between navigation releases and power.

There is no direct approach to the planning of multiple-purpose reservoirs. The best approach is to "cut and try"—that is, to assume various heights of dam and, for each height, to operate the reservoir taking into account the various requirements (to the degree possible) and the rule curves of operation. It is usual to find that it is not economical to provide for full flood protection, complete firm power delivery, or full irrigation releases in critical years. If more than 35 years of runoff records exist which show a reasonably definite and consistent flood flow pattern, it is not reasonable to provide definite flood control storage.

The next problem that arises is the yardstick to be used in arriving at the optimum size of reservoir. If the rate of annual return is used as an index, a very small reservoir will result. If the ratio of annual benefits to annual costs (annual benefits divided by annual costs) is used as an index, a larger reservoir will result. If the annual benefits minus annual costs is used as an index, a much larger reservoir will result. The writer is of the opinion that public projects should approach but not reach the limit set by annual benefits minus annual costs, because beyond this point the annual costs increase more rapidly than do the annual benefits. In the writer's opinion, this problem of basic economics, because of the very wide variations involved, should have as much or more attention than most of the other current problems concerning multiple-purpose projects.

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[28] Cons. Engr., San Anselmo, Calif.

In computing the annual benefits of a multiple-purpose project it is unnecessary to allocate the capital costs to the various uses. It is sufficient to lump all the benefits whatever they may be and to whomsoever they may accrue. It is logical to consider probable future increases in values of benefits and also to make allowance for indirect and intangible benefits. Great caution should be taken to avoid duplication of benefits, and to arrive at benefits that are truly practical and realistic. On the other hand, in all cases the annual costs, aside from operating and maintenance expenses, should include interest and depreciation (or amortization) charges.

In many instances private capital has provided some multiple-purpose benefits to the public through construction of projects developed for single purposes. It is unfortunate that some way has not been devised for the government to compensate private owners for the development of public benefits. If such compensation were possible, many private projects would have been built to optimum size instead of to some smaller size suitable for the private company's uses. This procedure would have obviated the underdevelopment of many fine sites, and would protect the future development of the relatively small remaining number of good sites.

The problem of reservoir silting is a serious one. Many ideas of silt control have been advanced, but nearly all will not meet the acid test of economic justification. It is true that proper farming practices will help the situation, but unfortunately the world's most important reservoirs are not in farm belts. The writer is aware of only a few rare cases in which excavating the deposits in the reservoir would be economically justified. Assuming that no money needs to be spent for storage of silt removed, it would cost at least $150 per acre-foot of silt excavated.

Ordinarily, the control of erosion on large steep areas by setting out plants, shrubs, and trees is not justified economically. The writer has examined vast areas so treated by the Japanese in Korea and Formosa. This work, especially in Korea, was done on an heroic scale. Many of the shrubs survived, but were dwarfed because the soil mantle had washed away. There is virtually nothing growing between the planted rows of shrubs and, as a consequence, the erosion is still severe. The writer has explored the Yellow River (China) drainage area. Nowhere in the world are erosion conditions so severe; and, at times, the Yellow River carries as much as 40% solids. The loessal area which provides the silt is 288,200 sq km in extent and the river drainage area is 771,500 sq km. It was found that this great erosion rate is caused by lack of vegetation, which in turn is very largely a result of overgrazing. A region was visited which the inhabitants had left 5 years before because of bandits. Surprisingly, a fine stand of grass and weeds was found even though the annual rainfall was only 6 in. The erosion from this area was practically nil. Some of the erosion came from very large deep gullies, but these could be stabilized by blasting rock canyon walls (where rock was present) to form rock-fill dams. In other cases, soil saving dams built of loess could be constructed to form barriers. Unfortunately the current war in China ended the program.

The writer has also observed the devastating effect of overgrazing in Mongolia, Ceylon, India, and Turkey. Unless carefully controlled, sheep do much damage, but goats do much more damage. Therefore, on large unfarmed areas the economic approach to prevention of erosion is (1) controlling grazing very carefully, (2) spreading seeds in fertilized pellets from airplanes, (3) planting grass and shrubs in selected areas, (4) preventing fires, and (5) constructing cheap dams for gully control by blasting hillsides and by constructing soil saving dams. In short, let nature do most of the work.

Even if erosion is very severe and cannot be economically controlled, it is justifiable to construct a multiple-purpose dam, provided that the project can amortize itself during its lifetime. After the reservoir is filled with debris, run-of-river power can be obtained. By providing concrete pipes up the bottom of the reservoir and controlling them preferably from the downstream end, bed loads can be passed during the silting up period. These pipes could be used after the reservoir is silted up to drain off water from the debris. The concrete should be very porous, or should have holes in it and be covered with a rock and sand filter. The water yield would depend on the porosity and the size of the debris particles. It is obvious that this device would be suitable only for short reservoirs having relatively steep bottoms.

At times there is a tendency to install too much electrical production capacity at multiple-purpose projects. There is room for only one or two low-capacity-factor power plants near the peak of a system load curve. Hence, other low-capacity-factor plants, because of insufficient or undependable energy production, cannot be fully utilized. Studies should be made to see if the firm power output, together with the firm output of existing and future plants, fits into the future integrated load curve of the system under all conditions. If, in addition, blocks of secondary energy can be produced, investigation should be made to determine if such energy is actually salable, or if it can be profitably made firm by steam support.

ABRAHAM STREIFF,[29] M. ASCE.—"Empty storage" (storage space kept empty in anticipation of floods) appears to be the most conflicting and least economical of multiple purposes for which reservoirs are used.

Until about 1910 flood control in the United States was obtained principally by the use of levees. In Ohio after the Dayton disaster in 1913 and the construction of the Miami project, flood control reservoirs in the upper reaches were introduced. Since then the flood control reservoir has descended into the plains combined with storage for other purposes. In the last Symposium paper, Mr. Hill has clearly set forth the existing conflicting requirements and has called for further discussion, which the writer undertakes, under five headings:

1. *Requirements of Flood Control.*—To prevent inundation a river channel must be of sufficient size to carry off maximum floods without overflowing its

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[29] Cons Engr., Austin, Tex.

banks. Most of the rivers in the United States are still in a wholly unimproved native state, and are generally far too small to carry occasional great floods. As has been emphasized by Gerard H. Matthes, Hon. M. ASCE, J. C. Stevens, Past-President, ASCE, and others, river channels deteriorate rapidly when floods are absent. Vegetation chokes and blocks the channel, covering the sand bars. There is unrestricted encroachment on the river channel by riparian owners. Entire city sections are within the flood areas. The flood storage reservoir aims to reduce flood heights, but thereby creates only further neglect of and encroachment on the river channel.

The improved or "normalized" river, on the other hand, is maintained in an artificially corrected and protected channel, with increasing sectional area as it approaches the sea. The Rhine River in Europe is an example of such a "normalized" river. There are winter dikes, summer dikes, surface and submerged groins, longitudinal groins, bank revetments, maintenance dredging, and policing along its entire course. The Rhine River has ceased to overflow its normalized channel, and is at all times confined within that channel. "Empty storage" for flood control is not found along its entire course from Switzerland to the sea, and could not be tolerated on account of the great value of the land for other purposes. The Lower Mississippi is another example of a "normalized" river of very great magnitude and is virtually still under construction. "Empty storage" has been, and is being, provided on the tributaries of this stream.

2. *Value of Empty Storage in a Single Dam.*—At best, what is the value of an "empty storage" reservoir? There are four possible cases of flood damage:

(*a*) There may be flood damage to the uplands due to torrential rains and gullying, such as occurred near Taylor, Tex., in 1921. (Damage to the lands in such cases is enormous and wholly beyond protection by a reservoir.)

(*b*) There may be flood damage in the valley upstream from the reservoir.

(*c*) There may be damage in the valley downstream from the reservoir due to floods originating downstream from the reservoir.

(*d*) There may be damage downstream due to floods originating upstream from the reservoir.

Only in case (*d*) is the reservoir able (by reducing the flood height over a certain distance downstream from the dam) to protect the land areas between former and lowered flood heights. The "quick release" following storing of floodwater is generally not as simple a problem as might appear. For example, to release a million acre-feet in 10 days will require a release of 50,000 cu ft per sec for 10 days, which, in most large streams is a sizable flood in itself. Proponents of the method refer to a "bankfull" discharge as allowable. This tolerance, in an unimproved river, causes riparian owners, who watch their riverbanks being caved in by a boiling river under a sunny sky, to file damage suits immediately.

3. *Value of Empty Storage in a Chain of Dams.*—When the river has been transformed into a succession of reservoirs, however, the one protective function of four possible cases of flood damage cited under case (*b*) also disappears completely.

In such a system, like the Tennessee River below Chattanooga, Tenn., hundreds of square miles of the valley area below maximum flood lines were purchased over a length of some 630 river miles (nearly the straight line distance between New York, N.Y., and Chicago, Ill.) and the entire population, including even the occupants of cemeteries, was moved out of the flood area. This system of flood "protection" thereby becomes identical to the levee system, where maximum flow lines are also the demarcation of the public domain which is reserved for the passage of the floodwaters, and it becomes the very system of the Mississippi River, of the Rhine, Seine, Po and Tisza rivers in Europe, and of many other rivers. As in the case of levees, such protection can be accomplished, of course, without building a single dam. By descending into the natural valley storage space, the "empty storage" reservoir becomes superfluous. At maximum flood heights (which are the most important) such "empty storage" reservoirs are just as uncontrollable as the original valley storage itself.

4. *The Dam Chain as a Normalized Channel.*—The present policy of federal agencies is to extend the Tennessee Valley Authority system to other watersheds. Leaving out of the discussion the storage dams themselves, the accompanying purchase and clearing of all the lands below maximum flow lines in itself, will serve an important purpose. The process of clearing these lands is equivalent to the establishment of a normalized channel, large enough to carry maximum floods, which is exactly what is needed. In other words, the levee system is extended far upstream into the main tributaries in a different form. The normalized channel now appears in the form of a series of reservoirs. Thereby the problem of maintaining a long, normalized channel is concentrated in the maintenance of a number of dams instead. This solution exactly fits the requirements of the United States, where a small population density makes maintenance of thousands of miles of normalized river channel uneconomical. Agricultural land area, on the other hand, is abundant. Where the levees begin in the lower stretches, the normalized channel must be maintained and dimensioned as before.

The maintenance of "empty storage" in dams that are completely contained within the valley storage is useless and does not merit repeating. Instead, the dams should (and could) be operated under full head for other multiple purposes. Carrying full storage will rout floods quicker; but, as long as the normalized channel is adapted to the requirements, the results will be satisfactory.

5. *Storage on the Watershed and in the Upper Reaches.*—For a dam containing storage space not only within but also above the valley storage, the conditions are as mentioned under case (*b*), for a single dam. Under such circumstances, improvement of flood storage efficiency can be achieved by a multiplicity of storage all over the watershed. For centuries this storage technique has been in existence in India. The province of Madras contains, on an area equal to that of New Mexico, 43,000 storage reservoirs with a total length of earth dike of 30,000 miles, containing some 200,000 masonry outlets. Runoff is stored during the wet monsoon season for use during the dry monsoon season. In this instance the storage is for irrigation purposes. When storage space is available in such quantity, the need for "empty storage," whether

within or above the valley storage, is automatically eliminated and the stored waters can be allocated for all other multiple purposes without considering flood control.

Dana M. Wood,[30] M. ASCE.—Considering the papers of this Symposium as a whole, the reader is immediately impressed by the number and authority of agencies having an interest in federal multiple-purpose development in particular, and a nearly equal interest in private developments. Assuming, for example, that the Corps of Engineers, United States Department of the Army, under Congressional authorization, has a paramount interest in planning and design, because of flood control or navigation functions or both, it must: (1) Obtain the approval of the Federal Power Commission on the design and the comprehensiveness of the river development scheme; (2) cooperate with the Bureau of Reclamation if irrigation is involved or arrange for the United States Department of the Interior to sell any power available; (3) perhaps coordinate power planning with private utilities if an independent market is not available; (4) secure recommendations from the Fish and Wildlife Service of the Department of the Interior; (5) determine the extent of participation of the National Park Service of the same department with respect to recreational possibilities; (6) procure the hydrometeorological services of the United States Weather Bureau and the United States Geological Survey; (7) ascertain public health requirements; and then (8) receive cooperation and assistance at lower state and municipal levels. Historic and archeological investigations must be included by the National Park Service, assisted by the Smithsonian Institution in Washington, D. C., and perhaps by local universities. The soil conservationists have an interest. All this requires special appropriations for each agency, or a shifting of funds from one to the other, necessitating government budget requests and reviews, or special agreements. How complicated the life of the government planning and design engineer is becoming and how divided the authority!

Little has been said about the necessity of obtaining local public approval, coordinating, and compromising the appeals of special pressure groups, or overcoming selfish and individual attitudes at all levels. Of course, the greatest good to the largest number of individuals is supposed to be the controlling theory, but political and financial controls can violate this. One of the greatest evils of today is an unwillingness to compromise issues on any level from the individual to even the international. The question is too apt to be raised, "How does this affect the individual, or the community, or the State?"

Yes, a national water policy is needed, but how is one to be obtained? Is Congress ready to modify its stipulations that one agency sell its power at the switchboard while another builds its transmission lines to the market, or is any particular agency willing to depart from its own pet theory of allocation of costs of multiple-purpose objectives in determining project economics? Is flood control of greater or less importance than power except for reasons of political expediency? Are local benefits sufficiently large to outweigh local taxes on government projects? Should theoretical rather than actual interest be charged against a government project which competes with private business? Will nonreimbursable appropriations be made at the expense of the United States at large so that purchasers of water and power will not be required to pay for services provided for the general public in the same restricted area? Shall large landowners receive the same benefits from irrigation as small landowners? Shall residents within a flood control district contribute to the project on some equitable basis or shall the federal government bear the entire expense? Shall governmental planning be allocated to numerous bureaus or be consolidated in a new bureau? One can go on indefinitely in raising questions requiring compromise which probably never will be actually compromised until the character of the individual and the groups is improved and selfish interest is largely eliminated.

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[30] Chf., Power Studies Branch, Div. of Water Control Planning, TVA, Knoxville, Tenn.

A national water policy must be flexible because no two cases are exactly alike. In one case, it may be necessary to sell power at the switchboard to the one available utility, thereby continuing a monopoly, because any other market could not be built up readily. In another case, the public interest might be hurt by following such a policy. The residents within a proposed flood control district might not have the financial ability to participate, no matter how desirable the flood protection might be, because of already existing financial responsibilities. Many municipalities are overburdened with heavy debt, perhaps as a result of mistaken past policies, and cannot undertake badly needed improvements. This condition, of course, raises the important question as to whether a community which has been thrifty by maintaining a pay-as-you-go basis should now be penalized to assist through taxes a less thoughtful community. Water laws and practices in one section of the United States differ from those in another. Basic fundamental questions are woven all through the problem of a national water policy.

Desirable as some national water policy may be, it seems likely that it will be a long time before one can be developed. In the meantime, the step-by-step progress made in recent years should be continued. Indeed, any fixed policy may be undesirable because of the constantly changing conditions in the United States.

CLARENCE S. JARVIS,[31] M. ASCE.—Relatively few water storage reservoir projects of large capacity and major importance have been developed to serve a single purpose. Ordinarily one principal interest was stressed, with incidental benefits to others, and possibly some disadvantages and inconvenience accrued unless prior rights were recognized and fully protected.

The legislation embodied in *House Document No. 308*,[32] which was enacted into law in Section 1 of the River and Harbor Act of January 21, 1927, initiated a new era of practical hydrology. It directed the Chief of Engineers to prepare technical reports on the principal river systems of the United States, except the Colorado River (on which the Bureau of Reclamation, United States Department of the Interior, had already made notable progress), from the fourfold standpoints of navigation, flood control, power development, and irrigation. Naturally, the interests of municipal water supply, manufacturing industries, recreation, fish and game life, and other outstanding items of national life and economic structure were duly investigated and included in the final reports. The impressive array of "House Document No. 308 Reports," published during the following decade, helped to establish the concept of "multiple-purpose reservoirs," in which both the principal and the incidental beneficiaries were recognized, along with their eligibility and probable ability to share in the capital cost of the undertakings. This new concept has made possible the launching of several important key projects in the realms of flood control, navigation, hydroelectric power development, irrigation, and municipal water supply. Without the hydroelectric energy made available by such projects as Grand Coulee Dam (on the Columbia River), Bonneville Dam (on the Columbia River), Hoover Dam (on the Colorado River), the notable features of the Tennessee Valley Authority, and other monumental achievements, just how could this nation have accomplished its tasks during the years of World War II?

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[31] Cons. Engr., Salt Lake City, Utah.

[32] *House Document No. 308*, 69th Cong., 1st Session, 1928.

If a substantial part of the total costs of such projects (with their demonstrated strategic value and importance during national crises) must be charged to the general welfare, while subscribers under irrigation, municipal water supply, and hydroelectric power projects pay all that can rightfully be assessed against them, such a basis of financing is both justifiable and commendable.

Alleged conflicts of major interests in a multiple-purpose reservoir project are too often grossly overdrawn. Conceding that the flood control ideal inclines toward an empty reservoir at certain seasons, and that hydroelectric power interests prefer a high storage head, both ideals must readily submit to limitations and to modifications in their own behalf, and should make even greater concessions in the spirit of compromise to promote conservation and utilization of basic natural resources. If properly designed and operated, a reservoir filled nearly to spillway level is proof of a mission accomplished for that particular season or period. It is still capable of storing and automatically releasing a regulated volume of floodwater over the spillway, with but minor or negligible damage to the pasture or woodland below the designed high-water contour, for which flowage rights are already secured. A remarkably large proportion of the total reservoir capacity is ordinarily located above the spillway level. With assurance that automatic release will continue at the predetermined rates, over the open spillway, the operators are relieved of needless worries and the pitfalls of immature decisions regarding emergency drawdown.

Recent advances in the field of hydrometeorological services by the United States Weather Bureau and cooperating agencies have effectively removed the probability, if not the possibility, of unwarned approach of catastrophic storms (and resulting floods) except those associated with structural failures. This situation certainly justifies the adoption of an intelligent and liberal policy in the field of reservoir drawdown. With long-range weather forecasts attaining higher percentages of accuracy and dependability, it is becoming more justifiable to maintain considerable storage volume of floodwater for orderly release to promote beneficial use. On the other hand, demand for hydroelectric energy may justify a drawdown below the ideal storage level when there is reasonable assurance of increased volume of river discharge in the near future, capable of replenishing the storage to the desired level or even above that level.

Some of the most important and valuable storage capacity of a reservoir may consist of what is known as bank storage. Lateral infiltration into coarse granular deposits or other readily permeable geological formations above Elephant Butte Dam on the Rio Grande in New Mexico accounted for a return flow, during the final drought year of a series ending in 1940, which was sufficient to offset all reservoir losses and to yield an additional dividend of approximately 47,800 acre-ft. The 46-year mean annual loss had been calculated as 198,700 acre-ft from records of the Bureau of Reclamation, as summarized, in tabular form[33] and graphical form,[34] by the writer in 1943. In some cases, geological reconnaissance may determine the advisability of opening trenches, cuts, or shafts to penetrate the potential underground storage strata, especially where the topsoil mantle would retard the inflow and outflow unduly.

The Symposium is of comprehensive scope and constitutes a valuable contribution to the subject of multiple-purpose reservoirs.

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[33] "Supplementary Gleanings from the Field of Hydrology," by Clarence S. Jarvis, Brentano's Inc., Washington, D. C., 1943, pp. 40–41.

[34] *Ibid.*, p. 37.

AMERICAN SOCIETY OF CIVIL ENGINEERS
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Paper No. 2413

AN ENGINEERING CONCEPT OF FLOW IN PIPES

By Charles W. Harris,[1] M. ASCE

With Discussion by Messrs. E. L. Harrington, Ralph W. Powell, J. C. Stevens, George A. Whetstone, G. H. Hickox, James F. Halsey and William M. Owen, Tsen-Ding Chen, Lynn Perry, Andre L. Jorissen, Weston Gavett, Arthur L. Collins, and Charles W. Harris.

Synopsis

Today enough is known about flow in pipes, both fundamentally and experimentally, to form a secure basis for consolidation. Holding the physical contacts and keeping the rational elements open to inspection are the secrets of safe advancement.

The relations of smooth and rough pipes, and of large and small pipes, are here expressed by correlated equations rather than by separate empirical formulas. A specific roughness coefficient, by which a surface may be designated and which remains fixed for extreme ranges of diameter replaces a variable coefficient. An index of pure turbulence indicates the velocity at which the friction factor becomes constant. A formula for smooth pipes is recommended also for rough pipes under certain circumstances. Correlation is shown between artificially prepared random roughness and a variety of commercial pipes. A method is furnished for distinguishing surface losses from localized losses, thus exposing the latter for corrective measures. All the equations are related through simple algebraic expressions.

Introduction

The following facts and principles are arranged to promote a correct method and, in so far as possible, a rational procedure, for computing flow in pipe lines. The inclusion of oils and other fluids need not defeat this purpose; nor does it imply that, to select a friction factor, the viscosity of water must always be known. However, a few old traditions must be firmly set aside.

A reader who is overconscious of the meagerness of his preparation in hydraulics should not despair. It is the nature, not the extent, of the groundwork that counts. He should see in the Chezy formula, $V = C\sqrt{R\,s}$, the embodiment of simple physical meaning whether the equation is solved for

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Note.—Published in May, 1949, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Prof., Hydr. Eng., Univ. of Washington, Seattle, Wash.

velocity, as commonly found for open channels, or for lost head as H. Darcy[2] and Julius Weisbach[3] preferred for pipes. He should know that a constant coefficient in either case confirms the Newton principle of similarity which demands the dissipation of energy in proportion to the kinetic energy involved. If he chances to be a mature engineer, he may find the article mainly corroborative of his early training, although at times it may be sharply contradictory to it. When contrary to his training, it may still agree with his experience.

One who is well grounded in fluid mechanics should not presume that the writer is renouncing that subject when at times his interpretations are not conventional. Absence of logarithmic formulas for resistance factors does not imply that empirical links render them too inaccurate, but that a complicated formula is unnecessary.

The agreement of friction factors for partly turbulent flow in a smooth and in a rough pipe should be emphasized, but not to the extent of obscuring the equally important fact that extreme contrast exists between the two at those higher velocities which produce complete turbulence in a rough pipe. This condition applies to all fluids. No one should feel secure in the knowledge of his limited field since this overlapping is so universal.

An attempt is made herein to express the physical meaning in a simple manner, using only the mathematics with which the average reader is familiar.

General Principles

The flow of fluids through pipes has long been computed by a closely organized combination of two distinct types of formulas—one rational and the other empirical. An efficient use of such a combined method requires keen discrimination as to the nature of these component parts. To make full use of the rational aspects, one should observe that a fluid follows one or another of three distinct modes of physical behavior depending on the circumstances. These three modes are revealed successively when the flow through a moderately rough pipe is changed from zero to some high velocity.

Roughness does not appreciably change the resistance in nonturbulent flow because in this parabolic distribution there is no velocity at the surface of contact. However, when the velocity is increased until the viscous drag along the wall becomes greater than the internal stability, the slightest cross component of velocity permits the streamlines to buckle. The resulting turbulence is projected throughout the cross section of the pipe with the exception of a thin boundary layer that still resists the penetration of cross components. This mixed type of flow constitutes partial turbulence, the boundary layer still being nonturbulent. Roughness here, also, does not materially change the resistance, since the roughness elements are shielded by the boundary layer.

A velocity is eventually reached at which the boundary layer has become thinned to a negligible film and the shape of any roughness becomes exposed to the direct action of the moving fluid—the rougher the surface the lower the necessary velocity to establish this condition. The pattern of flow, once established by the topography of the solid boundary, remains unchanged for all higher velocities and the friction factor remains essentially constant. There may be a minor secondary influence of viscosity tending to render the pipe wall smoother; but with roughness elements varying widely in size and shape this influence is negligible at any one velocity.

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[2] "Recherches expérimentales relatives au mouvement d'eau dans les tuyaux," by H. Darcy, Mallet-Bachelier, Paris, 1857.

[3] "Principles of the Mechanics of Machinery and Engineering," by Julius Weisbach, 1st American Ed. edited by Walter R. Johnson, Lea and Blanchard, Philadelphia, Pa., 1848, Vol. 1, p. 391.

Most engineers who are familiar with the use of the Reynolds number do not care to use it simply to show familiarity. They should however recognize in it the extent to which large physical dimensions dwarf the influence of viscosity, and should review the following facts: First, viscosity plays no part in the loss of head in the fully turbulent flow most frequently encountered in moderately rough pipes and channels. Second, viscosity is the direct source of loss in nonturbulent flow such as found in heavily viscous oils or even in water in small pipes and some laboratory models. Third, for small changes in viscosity in partly turbulent flow, a given percentage of change in viscosity causes approximately one fourth of that percentage of change in lost head. For Reynolds numbers greater than 100,000 the ratio is less than one fourth. An approximate Reynolds number for water at 55° F is 76,500 $V\ D$—in which V is the velocity, in feet per second; and D is the diameter, in feet. It should be remembered, however, that the viscosity of water is doubled by lowering the temperature from 85° F to 37° F, thus changing the Reynolds number from 114,000 $V\ D$ to 57,000 $V\ D$.

TABLE 1.—TEST DATA FROM 2-IN. STANDARD BLACK PIPE

Velocity, v (ft per sec) (1)	Measured head, h_f (ft) (2)	Reynolds number, R (3)	Friction factor, f (4)
27.4	18.90	306,000	0.0276
25.3	16.50	282,000	0.0281
23.5	13.90	262,000	0.0277
21.75	11.95	248,000	0.0277
19.40	9.51	208,000	0.0277
16.50	6.99	174,000	0.0280
12.05	3.70	125,000	0.0277
8.88	2.01	106,000	0.0279
7.00	1.25	93,000	0.0279
6.24	0.990	83,500	0.0278
5.46	0.750	73,000	0.0277
4.15	0.435	55,500	0.0277
3.22	0.265	43,000	0.0278
2.65	0.176	35,500	0.0274
2.24	0.125	30,000	0.0274
1.92	0.0925	25,700	0.0274
1.734	0.0770	23,200	0.0280
1.623	0.0665	20,700	0.0278
1.440	0.0530	17,500	0.0281
1.332	0.0450	16,200	0.0278
1.121	0.0333	14,300	0.0290
1.015	0.0280	12,400	0.0298
0.874	0.0217	11,100	0.0313
0.737	0.0162	9,440	0.0324
0.601	0.01145	7,700	0.0345
0.395	0.00535	5,140	0.0375
0.280	0.00291	3,580	0.0405
0.232	0.00185	3,160	0.0376
0.1915	0.001083	2,620	0.0324
0.1677	0.000713	2,300	0.0277
0.1406	0.000598	1,920	0.0331
0.0915	0.000370	1,320	0.0486

A Typical Test.—All three of the modes of flow appear in the following test of a moderately rusty 2-in. standard black pipe. The test specimen used for this experiment was 10.0 ft long with an average diameter of 2.039 in., preceded by another 10 ft of the same 2-in. pipe. Conditions during the test were kept as nearly constant as practicable, with the exception of the velocity which was varied by a gate valve placed at a remote point. Small friction heads were measured with a tilting mirror gage developed for the purpose. The results are shown in Table 1. The symbol f, as usual, represents the friction factor to be applied through the Chezy principle by the Darcy-Weisbach formula, $h_f = f \frac{L}{D} \frac{V^2}{2\,g}$—this form seeming to have returned permanently to the engineering profession with its rational embodiment of pure dynamic laws.

The expanded form of the Reynolds number is $\mathbf{R} = \frac{V\,D}{\mu}\frac{w}{g}$—in which μ is the absolute viscosity (preferably in pound-seconds per square foot); and w is the unit weight of fluid.

Three Methods of Presenting the Same Data.—The results of this test are plotted in three forms for reasons that will become obvious. The graph shown in Fig. 1 is drawn to natural scale to indicate relative values and thus give proper weight to the higher velocities where the friction factor f remains constant. This range in which the pure law is undistorted by viscous resistance is much greater than is suggested by the conventional logarithmic chart. Fig. 2 is a logarithmic plotting of the data shown in Fig. 1. The distortion is too often overlooked through a desire to magnify the smaller values. The same data are again shown in Fig. 3. This latter logarithmic plotting (loss of head with respect to velocity) is a still more dangerous expedient, particularly for extrapolation. The small break in the line of turbulent flow near the midpoint of Fig. 3 should be noted. By this method, the line is often carelessly drawn straight, with a slope depending on the particular range of velocities observed.

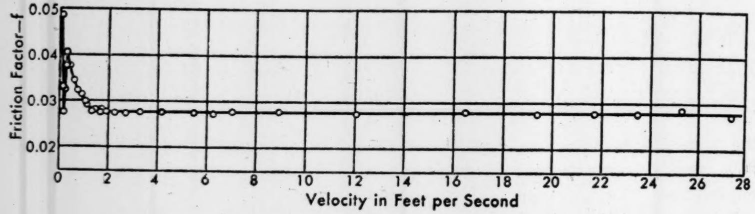

FIG. 1.—NATURAL-SCALE PLOTTING OF DATA FROM TEST ON 2-IN. STANDARD BLACK PIPE, 5 YEARS IN SERVICE

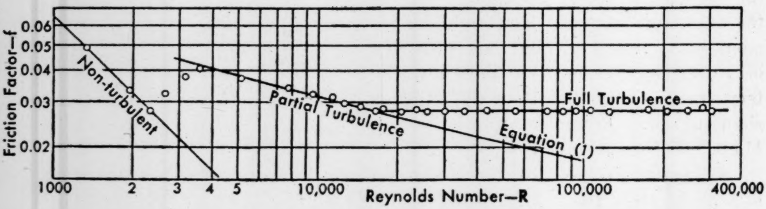

FIG. 2.—LOGARITHMIC SCALE OF DATA IN FIG. 1, REYNOLDS NUMBER REPLACING VELOCITY

In general, any plotting of data that leaves obscured within the scale such a major law as that of Chezy dwarfs the features that should be emphasized. When reviewing old experiments, it should be noted that an excessive number of low velocity readings have been recorded. These readings were readily secured, often to an abnormally low value, whereas those equally above the needs were probably not attainable. Had the data in Fig. 3 ended at 1.5 ft per sec, the error from producing the curve to 27 ft per sec would have been about 50%.

The choice of one or another of the three forms of plotting is one of emphasis, since the same facts are simply viewed in different perspective. All these forms show that in the higher velocities the loss is proportional to the square of the velocity, that is, the friction factor f is constant in response to the dominant law.

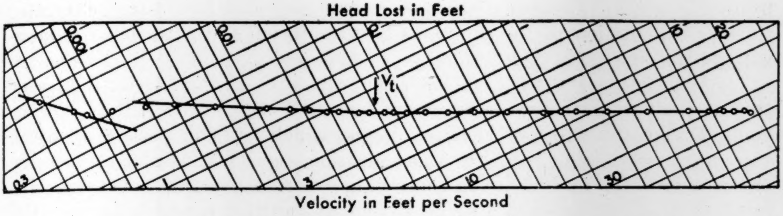

Fig. 3.—Logarithmic Scale of Velocity and Head Lost for Data of Fig. 1

At intermediate velocities, where the loss responds in part to the major dynamic law and in part to the various influences of viscosity, the lost head does not vary with any constant power of the velocity. It is customary and proper to use the Chezy principle (the Darcy-Weisbach formula) here, however, since the loss varies more nearly with the square of the vleocity than with the first power. The exponent shown in Fig. 3 is the common 1.75 in this middle range, which is equivalent to the value of f varying as the -0.25 power of the velocity. This exponent is known to apply only at the lower Reynolds numbers, and a more accurate formula will be given subsequently for all Reynolds numbers, including those less than 100,000.

At velocities low enough for the flow to remain nonturbulent, the dynamic law is replaced by an equally exact but entirely different law. Solved for drop in pressure, the fundamental Poiseuille formula is $p = \frac{32\, V\, L}{D^2}\, \mu$; and, since no f is required, it is regrettable that a fictitious f is so often used to circumvent this correct formula. The subterfuge, a method of including all velocities and their respective coefficients in a single graph, serves mainly for detecting the change from nonturbulence to partial turbulence. The implication of physical meaning is quite misleading since this highly variable f (where in reality there is no f) is simply the mathematical consequence of using a wrong formula. Nonturbulent flow being independent of the roughness of the pipe, the Poiseuille formula is a complete answer; and its range of application is reasonably defined. Although by providing local excitation one may secure partial turbulence factors at Reynolds numbers of 1,500 or less, under most circumstances the turbulence is damped out as rapidly as it is produced with Reynolds numbers even greater than 2,000. A Reynolds number of 2,000 is a reasonable lower limit of partial turbulence, however, since (for the purpose of computing losses) it is on the safe side.

In Fig. 2 there is no complex rising transition sometimes thought to be necessary for Reynolds numbers less than approximately 300,000, for this particular roughness. On the contrary a faint suggestion of a dip persistently reveals itself in this natural pipe.

The Transition from Partial Turbulence to Full Turbulence.—When there is a gradual transition between the two types of turbulent flow, it is usually explainable. Friction factors in the vicinity of the change from partial turbulence to full turbulence may be somewhat above the constant *f* if the total resistance is the result of two or more segregated degrees of roughness, or is somewhat below the constant *f* if the surface is roughened with a uniform size of elements. Further explanation of the dip caused by uniformity of artificial roughness will appear later.

An illustration of the gradual transition caused by two degrees of roughness that are well separated in the same pipe appears in the results of the following test. Unlike the effect of localized losses, which is to increase *f* an equal amount at virtually all velocities, this transition begins with the lowest velocity that supports full turbulence in the rougher section. A 20-ft length of 2-in. pipe was tested throughout a range of velocities with the results shown in the middle line of Fig. 4. A decided transition was apparent. The pipe was then cut into two 10-ft sections and each section was tested separately. By chance, one section proved much rougher than the average of the entire pipe; the rougher section showing practically no transition. Had the separate friction factors for each pipe been known, the graph of the 20-ft length could have been made in advance of the test, since it is simply the mean of the two halves.

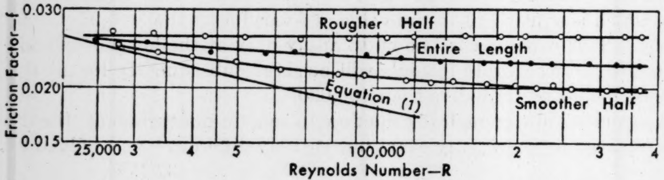

FIG. 4.—COMPONENTS OF TRANSITION FROM PARTIAL TURBULENCE TO FULL TURBULENCE IN 2-IN. NATURAL ROUGH PIPE

It is important to note that these gradual transitions can be reproduced by a properly weighted average of the constant *f* of the rough part of a pipe and the variable *f* of a perfectly smooth pipe. The two 10-ft lengths of pipe in the foregoing experiment were repeatedly tested together in all combinations of direction of flow and of connection one with the other, the graph of *f* remaining substantially as shown. This type of transition is not uncommon. New iron or steel pipe installed in the conventional manner may show its effect. The general surface is smooth enough to remain in partial turbulence at all ordinary velocities, but there are fully turbulent areas.

INADEQUACY OF SOME EARLY METHODS

The physical implications of the Chezy formula should not be treated lightly by empirical methods. Some of the older formulas ignore these rational features, with serious consequence. It is interesting to speculate as to what might have been Darcy's final conclusion had his elaborate program not been interrupted by his death in 1858, for he was in the midst of an intensive study of flow along prepared surfaces having artificial roughness. His assistant, H. Bazin, was dominated by E. Ganguillet and W. R. Kutter[4] who attempted to evaluate the partly completed work, and neither of these men showed an eagerness for physical meaning in their mathematics.

No influence during the nineteenth century reacted more strongly against clear thinking than did Kutter's failure to distinguish between the two types of turbulent flow. Only his lower velocities or smaller conduits were in partial turbulence; yet the data were averaged indiscriminately. This weakness permitted such fallacies as that of the coefficient being always a function of the slope, and that of the rate of change in the coefficient being independent of relative roughness. Robert Manning[5] removed the former fallacy but retained and emphasized, in his formula, the latter and more serious fault. Fortunately the use of these old formulas is becoming less prevalent for pipes. The growing tendency to segregate coefficients into pipe coefficients, f, and channel coefficients, C, should be encouraged. Using C only for open channels with the formula $V = C \sqrt{R\,s}$ prevents such features as shape of cross section and wave action from marring the coefficients to be used for pipes.

Bazin maintained[6] that the coefficient C did not normally change with changing slope, a fact entirely consistent with a constant f for those surfaces and velocities that were pertinent to his field of application. There the principle of dynamic similarity was undisturbed by viscosity. Despite its lack of dimensional consistency the Bazin formula does supply what Manning's formula does not—a greater percentage of change in coefficient for rough surfaces than for smooth surfaces, for any given change in size of conduit.

Weisbach,[3] being impatient with theories, was prone to discourage any rational approach unless quickly confirmed by superficial experiment. His methods did not tend to explain why his f varied. Thus it is not surprising that in the half century following, a few experiments were considered sufficient to establish a "law." The constant coefficient with fractional exponent like that which led to the formula of H. Blasius[7] originated in this atmosphere. The fractional exponent carried to this extreme obscures the rational elements, and makes the number of formulas unlimited.

At the beginning of the twentieth century the method of constant fractional exponent was at its zenith. One small volume that found its way to almost every library shelf, "Hydraulic Tables" by Gardner S. Williams and Allen Hazen,[8] Members, ASCE, was based on a fallacy that should have been apparent at the time. A constant-exponent formula was developed, selecting only the data from the smoothest pipes attainable, the exponent of V reducing to 1.85. It was then assumed that all rough pipes would follow that same exponential "law," the coefficient depending only on the degree of roughness.

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[4] "A General Formula for the Uniform Flow of Water in Rivers and Other Channels," by E. Ganguillet and W. R. Kutter, translated by Rudolph Hering and J. C. Trautwine, Jr., John Wiley & Sons, Inc., New York, N. Y., 1889.

[5] "On the Flow of Water in Open Channels and Pipes," by Robert Manning, *Transactions*, Inst. of Civ. Engrs. of Ireland, Vol. 20, 1891, p. 161, and Vol. 24, 1895, p. 179.

[6] "Étude d'une Nouvelle Formule pour calculer le Debit des Canaux Découverts," by H. Bazin, *Annales des Ponts et Chaussées*, 4th Trimestre, 1897.

[7] "Das Ähnlichkeitsgesetz bei Reigungsvorgängen," von H. Blasius, *Physicalische Zeitschrift*, Vol. 12, 1911, p. 1175.

[8] "Hydraulic Tables," by Gardner S. Williams and Allen Hazen, John Wiley & Sons, Inc., New York N. Y., 3d Ed., 1920.

Fig. 5 illustrates the consequence. The graph shows the formula applied to a 9.63-in. pipe with C_w equal to 90. The test points are from an experiment by Darcy on an old cast-iron pipe as compiled by J. T. Fanning,[9] M. ASCE, and made available in English. Although it is approximately correct for lost head at a velocity of 1.5 ft per sec, the formula with the same coefficient is 26% in error at 12.6 ft per sec. This is not an isolated case. Fanning[9] published the results of other tests that showed the same characteristic. The Chezy formula with a constant f would have served these pipes with no appreciable error at any reasonable velocity.

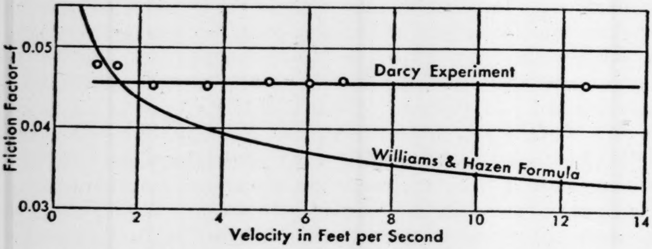

FIG. 5.—FAILURE OF WILLIAMS AND HAZEN FORMULA FOR A ROUGH PIPE TESTED BY DARCY

ARTIFICIAL ROUGHNESS VERSUS NATURAL SURFACES

Uniform Sand Grains.—Attention has frequently been called to the failure of experiments[10] by J. Nikuradse on uniformly roughened pipes to conform to experiments on ordinary pipes. The cause and the meaning of this disagreement are more apparent if one considers Nikuradse's work as an intensive study of the several components of one average degree of roughness. A composite of all his experiments conforms reasonably well to the pattern found for a single natural surface of this average roughness.

The solid line of Fig. 6 shows Nikuradse's work reduced to one graph for equal Reynolds numbers. The dotted line is his individual graph for approximately that average degree of roughness. The composite graph remains within 3% of a constant 0.035 for all Reynolds numbers above 12,000. Nikuradse's equivalent individual graph varies from 4% above to 14% below that value.

In this connection, one might examine a frequently quoted experiment by E. W. Schoder and H. A. Gehring,[11] on a 2-in. rough pipe. It shows practically a constant f of 0.034 for velocities ranging from 17 ft per sec to 4.6 ft per sec, well into the region of the Nikuradse dip; but, instead of a dip, there is simply a negligible tendency to rise.

The disagreement between random and uniform roughness cannot be avoided. A dip in the graph must occur when all elements require the same range of velocities for the minimum resistance factor that precedes their fixed pattern of flow. It is a well-known fact that even spheres, in a certain range of Reynolds numbers, are streamlined by the viscous fluid, mainly behind them; but it is not necessary to visualize the influence of individual elements. These elements are all shielded by the boundary layer until the velocity is sufficient to expose them. The guiding boundary, as modeled crudely through the nonturbulent fluid, becomes rough enough to cause pure turbulence when it is still smoother than the solid wall itself. When stronger vortices of turbulence within the shielded depressions deplete the remaining nonturbulent fluid, the flow is controlled by the solid elements. The fixed pattern is thus established and the friction factor has risen to its normal constant.

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[9] "Practical Treatise on Hydraulic and Water Supply Engineering," by J. T. Fanning, D. Van Nostrand Co., 13th Ed., 1896, p. 240, Table 59.

[10] "Laws of Fluid Flow in Rough Pipes," Pt. 2, by J. Nikuradse, English translation in *The Petroleum Engineer*, May, 1940, p. 78, Fig. 6.

[11] "The Flow of Water in Spiral Riveted and Other Pipes," by E. W. Schoder and H. A. Gehring, *Engineering Record*, Vol. 58, 1908, No. 9, p. 240.

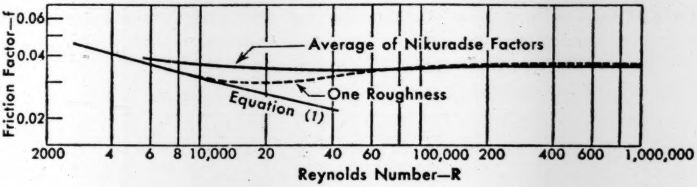

FIG. 6.—ONE UNIFORM ROUGHNESS VERSUS THE AVERAGE OF SIX DEGREES OF ROUGHNESS

Since the tendency to dip occurs at higher velocities for the smaller elements, it follows that only when many roughness elements are of one size does the dip become perceptible; and, since each size of roughness element contributes only a minor tendency, the integrated effect does not materially lower the graph as a whole. Even if the lowering were appreciable, no correction would be necessary, since the selected value must in any case come from a natural pipe possessing that quality.

The most logical procedure therefore is to assume the perfect dynamic law with its horizontal graph, and to modify f only if a reason exists—rather than to supply some variable general function presumed to be required, and on most occasions to reduce it to a constant. The constant f is a complete designation of the relative roughness of a pipe; so it would be unfortunate indeed if an artificial surface with a maximum departure from the pure law should be chosen as standard for comparing other surfaces that in general show no such departure.

Artifically Prepared Random Roughness.—To simulate the inside surface of a rough pipe and at the same time to insure that pipes of various diameters would present the same absolute roughness, the writer outlined an investigation based on artificial linings. The material selected was No. 20 sheet steel, on which was sprayed a uniform coating of stainless steel. Large sheets were prepared in this manner with a controlled adjustment, and the various linings were cut from these sheets. The lining for each of the five pipes was annealed and formed by a special die so as to fit inside its respective standard pipe. For sizes of $\frac{1}{4}$ in., $\frac{1}{2}$ in., 1 in., and 2 in. (iron pipe size before lining), brass pipe was used for the casing. The 4-in. size was made to fit 4-in. (inside diameter)

Shelby tubing. The predominating length of lining was 2 ft and the total length of each test specimen was 60 diameters for the largest size, and 100 diameters or more for the other sizes.

The characteristics of this surface were entirely satisfactory for the purpose. The random nature of the roughness elements was sufficient to prevent most of the dip at the beginning of full turbulence, the $\frac{1}{4}$-in., 1-in., and 4-in. sizes showing substantially none. The fact that the roughness was well distributed longitudinally along the pipe prevented a long transition. The results were in close conformity with those for moderately rusty iron pipes. The surface was somewhat rougher than that of the smoothest rusty pipe, but decidedly smoother than the Schoder 2-in. rough pipe previously mentioned and even smoother than the rusty pipe of Fig. 1. The results taken from the 4-in. size are shown in Fig. 7. The other four pipes showed approximately the same characteristics. The probable deviation for an individual surface is 2.1%. This type of surface also serves to relieve the impression that there is merit in knowing the exact size or shape of the roughness particles.

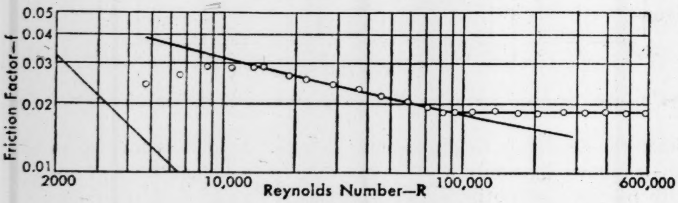

FIG. 7.—ARTIFICIAL RANDOM ROUGHNESS IN A 4-IN. PIPE

In keeping with the nature of rough and moderately rough iron pipes, the change from partial turbulence to full turbulence came at substantially the same velocity for all diameters, when corrected proportionately for viscosity. The breakaway with this particular surface came at somewhat less than 4 ft per sec. This break will be shown to correlate with the roughness coefficient.

A SMOOTH PIPE FORMULA AND ITS DOUBLE PURPOSE

The need for a simpler method of finding the resistance to flow in smooth pipes gained added emphasis through the fact that flow in a rough pipe conforms so closely to that same law within the range of partial turbulence. A single formula for f, when properly adjusted for localized losses, thus serves for all smooth pipes, and also for rough pipes if the velocities are less than the velocity which causes full turbulence.

A number of simple formulas is available within a given range of Reynolds numbers; but the attempt to extend the range indefinitely by rationalizing as in some treatments has produced unnecessary complexities. Such methods still include empirical steps which prevent a purely rational interpretation. In this respect, one might question the validity of assuming that an infinite Reynolds number would be accompanied by a zero friction factor, since this assumption reacts against simplicity. Consider the use of various fluids with progressively decreasing viscosity. The flow with each higher Reynolds number is of course more turbulent than with the preceding one, yet the layer of fluid in contact with the wall is still interlocked mechanically with the molecular structure. The turbulent flow in contact with either this boundary layer or the surface itself furnishes the condition for impact without compensating reflection. The resistance cannot therefore approach zero with the approach of zero viscosity. A similar conclusion is reached for those large Reynolds numbers that result from large physical dimensions. Since f is the loss of head (in terms of velocity head) that results from a length of pipe equal to one diameter, the length approaches infinity in the same manner that the diameter approaches infinity. Thus the abstract f here also approaches some positive finite value. This limiting value is dependent only on the fact that the fluid attaches itself to the walls of the pipe.

One should be reluctant to discard the advantages of the type of formula recommended by Charles H. Lees[12] $\left(f = A + \dfrac{B}{\mathbf{R}^n}\right)$, provided the exponent n and the other constants A and B are correct for the required range. Within the range of present knowledge (which extends up to a Reynolds number equivalent to a velocity of 5 ft per sec in an 8-ft pipe), the variable part of f may be assumed to change inversely with the cube root of the Reynolds number. Whether or not this condition extends to the theoretical limit is not pertinent to practical needs. The important fact is that this simple function is correct throughout the usable range and approaches 0.0061 as a limit.

The experimental data that led to the following formula were observed in connection with the calibration of some equipment installed in 1945 at the laboratory of the University of Washington in Seattle. The pipe was of brass, 19.5 ft long with an average inside diameter of 2.050 in. The water was weighed in a calibrated tank scale and the head was measured by open columns of water and mercury. The results are presented in Table 2. The Blasius "line" having been so repeatedly confirmed at a Reynolds number of 10,000, this one point was retained and the new data were accepted for the higher Reynolds numbers, ranging from approximately 100,000 to more than 1,000,000.

TABLE 2.—TEST DATA FROM A 2-IN. SMOOTH PIPE

Velocity (ft per sec)	Measured head (ft)	Reynolds number, **R**	Friction factor, f	Gage used
7.02	1.55	105,000	0.0177	Water
10.70	3.29	156,000	0.0162	Water
15.20	6.21	221,000	0.0152	Mercury
22.49	12.60	328,000	0.0141	Mercury
32.75	25.15	476,000	0.0132	Mercury
47.51	49.6	693,000	0.0124	Mercury
62.97	83.4	916,000	0.0118	Mercury
69.43	99.3	1,040,000	0.0116	Mercury

With $f = 0.0316$ for a Reynolds number **R** of 10,000 and the observed values of f taken from Table 2,

$$f = 0.0061 + \frac{0.55}{\sqrt[3]{\mathbf{R}}} \qquad (1)$$

The advantage of a pure cube root from the standpoint of easy application is obvious, and the rational aspect may be more than implication.

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[12] "On the Flow of Viscous Fluids Through Smooth Circular Pipes," by Charles H. Lees, *Proceedings*, Royal Soc. of London, Series A, Vol. 91, 1915, p. 46.

Even more important than its discovery through a simple experiment, however, is the agreement of Eq. 1 with other reliable data. Intersecting the Blasius line at a Reynolds number of 10,000, the graph follows closely below that line, recrossing it at 49,000. Continuing throughout the data that supported T. E. Stanton and J. R. Pannell,[13] it agrees with T. B. Drew, E. C. Koo, and W. H. McAdams[14] who included in their study the noteworthy experiments on brass pipes by John R. Freeman,[15] Past-President and Hon. M. ASCE. It also conforms to the Nikuradse experiments[16] up to 3,200,000, within a normal error of those points. These facts are easily verified with greater refinement than necessary. It is not more precise information that is needed, but a more workable method of interpreting that already at hand. (No comparison is made with the Nikuradse formula itself. To secure the small exponent of 0.237 one must ignore the disagreement at low Reynolds numbers and assume an agreement, at points beyond physical verification, with a formula that makes f approach zero with the approach of an infinite Reynolds number. When reaching beyond the range already examined, it would seem unwise to sacrifice expediency and to retain a feature even mildly against safety.)

In addition to the obvious purpose of Eq. 1 for smooth pipes and for partly turbulent flow in rough pipes, its graph also furnishes the stem from which the constant full turbulence factors spring, as shown in Fig. 2, and subsequently in Fig. 9. If there is a transition, the junction point can be located more definitely by extending the horizontal line back from the higher velocities where it is firmly established. This point on the graph of smooth pipes thus marks the division point between partial turbulence and pure turbulence of a rough pipe.

Solving for the Reynolds number in Eq. 1, and knowing that complete turbulence is indicated by the appearance of a constant factor, one finds the Reynolds number $\mathbf{R}_t$ at which pure turbulence is pending:

$$\mathbf{R}_t = \left(\frac{0.55}{f_t - 0.0061} \right)^3 \quad \text{..................... (2)}$$

in which f_t is the constant f for full turbulence.

Rough Pipes of Same Material and Various Diameters

A method of correlating pipes made of the same material, and thus of evaluating the influence of relative roughness, is furnished through the fact that the excitation for full turbulence depends on the size and the shape of the pattern of roughness rather than on the diameter of the pipe. Eq. 1 is based on the pipe diameter, whereas the break to full turbulence responds to the Reynolds number of the pattern. Hence, no numerical value is required for this Reynolds number of the pattern. The only essential fact is that it remains constant with changing diameter. Therefore, with the constant friction factor known for one diameter, the corresponding factor for any other diameter with this same pattern of roughness can be taken from Eq. 2, but with a Reynolds number of the pipe in proportion to the diameter. Accordingly, if the constant friction factor f_1 is known for a pipe whose diameter is D_1, the corresponding value of f_2 for a pipe whose diameter is D_2 is $f_2 = 0.0061 + \dfrac{0.55}{\sqrt[3]{\dfrac{D_2}{D_1}\mathbf{R}_1}}$; and, since the controlling Reynolds number $\mathbf{R}_1$ for the factor f_1 is $\mathbf{R}_1 = \left(\dfrac{0.55}{f_1 - 0.0061}\right)^3$, an expression results which is independent of the Reynolds number: $f_2 = 0.0061 + \dfrac{f_1 - 0.0061}{\sqrt[3]{\dfrac{D_2}{D_1}}}$.

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[13] "Similarity of Motion in Relation to the Surface Friction of Fluids," by T. E. Stanton and J. R. Pannell, *Philosophical Transactions*, Royal Soc. of London, Series A, Vol. 214, 1914, p. 199.

[14] "The Friction Factor for Clean Round Pipes," by T. B. Drew, E. C. Koo, and W. H. McAdams, *Transactions*, A. I. C. E., Vol. XXVIII, 1932, p. 56.

[15] "Experiments Upon the Flow of Water in Pipes and Pipe Fittings Made at Nashua, New Hampshire, June 28 to October 22, 1892," by John R. Freeman, A. S. M. E., New York, N. Y., 1941, p. 83.

[16] "Gesetzmässigkeiten der turbulenten Strömung in glatten Rohren," by J. Nikuradse, *Forschungsheft*, Verein Deutscher Ingenieure, No. 356, September–October, 1932, p. 30, Table 9.

TABLE 3.—TEST DATA FROM PIPES WITH SPRAYED STAINLESS STEEL LININGS

Velocity (ft per sec)	Degrees centigrade	FRICTION FACTORS: f	f_1
(*a*) DIAMETER D = 0.294 IN.			
30.2	18	0.0360	
37.0	17	0.0360	
43.7	17	0.0360	0.0148
(*b*) DIAMETER D = 0.542 IN.			
24.0	14	0.0281	
25.8	13.5	0.0281	
30.8	13.5	0.0281	
36.6	13	0.0287	
41.3	13	0.0281	
45.4	13	0.0282	0.0140
(*c*) DIAMETER D = 0.977 IN.			
29.7	10	0.0247	
32.8	10	0.0246	
36.9	10	0.0247	
41.5	9	0.0247	
45.5	9	0.0248	
48.5	9	0.0247	0.0142
(*d*) DIAMETER D = 1.955 IN.			
32.8	8	0.0229	
36.6	9	0.0232	
40.4	9	0.0233	
45.9	9	0.0233	
50.6	9	0.0234	
41.7	20	0.0224	
44.9	20	0.0228	
47.6	20	0.0232	0.0154
(*e*) DIAMETER D = 3.898 IN.			
....	For data see Fig. 7		0.0145
....	Average for five diameters		0.0146

Defining the friction factor f_1 as the specific roughness coefficient for a unit diameter of 1 ft, the foregoing formula becomes

$$f = 0.0061 + \frac{f_1 - 0.0061}{\sqrt[3]{D}} \dots (3)$$

in which D is the diameter, in feet; and f is the friction factor to be used with that diameter. A test of any size of pipe having a given character of surface reveals the same value of f_1.

Uses of the Specific Roughness Coefficient.—Although no experiments on rough pipes were required in the derivation of any of the foregoing formulas, quantitative values of f_1 must be derived from tests made on actual surfaces.

The average specific roughness coefficient f_1 for the five pipes having the surface metallized with a spray of stainless steel was 0.0146, as given in Table 3. The five individual points, taken from the higher velocities only, are shown in Fig. 8. The graph is determined by Eq. 3 and gives the friction factors for all pipes made with that same interior surface.

The identification of this surface by its specific roughness coefficient is further illustrated in Fig. 9 The horizontal line indicates the average of the foregoing five specific roughness coefficients. The points are the individual specific coefficients computed from fifty-three observations, ten or more on each of the four larger pipes. (The respective diameters were 0.541 in., 0.976 in., 1.955 in., and 3.899 in. Flow in the fifth pipe with a diameter of 0.294 in. was nonturbulent below a Reynolds number of 6,000.) All velocities were adjusted to a temperature of 55° F, making the velocity for full turbulence equal to 3.5 ft per sec. The inclined line is computed by Eq. 1 for smooth pipes.

The average specific roughness coefficient, as computed by Eq. 3 for Nikuradse's smallest sand grains that were used in all three of his pipes, is 0.0152. These three points, as shown in the lower line of Fig. 10, fall reasonably on the graph. Nikuradse's experiments with the largest sand grains reveal the average specific coefficient of 0.0277 shown in the upper line. The effective diameter of the smallest pipe when used with the largest sand grains is sensitive to the size of roughness particles. To assume a section of average area rather than a clear opening, in the extreme limit, may result in a negative sectional area for a positive diameter. If the radius were assumed smaller by even one fourth of the diameter of the sand grains, that point would be materially lowered.

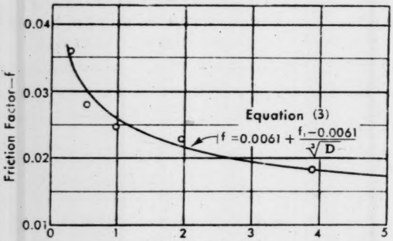

FIG 8.—FRICTION FACTORS FOR FULL TURBULENCE IN FIVE DIAMETERS WITH RANDOM ROUGHNESS

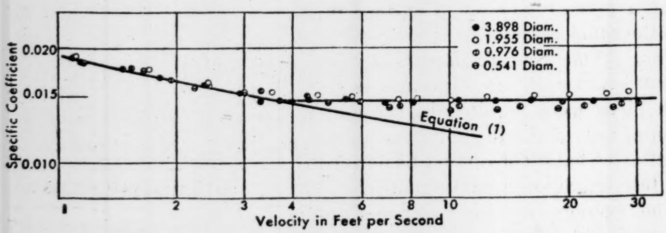

FIG. 9.—CORRELATION OF ADDITIONAL TESTS WITH THE AVERAGE FOUND FROM HIGHER VELOCITIES AS USED IN FIG. 8

The Index of Full Turbulence.—Solving Eq. 2 for the V_t that is embodied in the Reynolds number and preserving the relation of f and D as in the derivation of Eq. 3, one finds the index velocity for full turbulence:

$$V_t = \nu \left(\frac{0.55}{f_1 - 0.0061} \right)^3 \quad \text{(4)}$$

As usual ν is the kinematic viscosity obtained by dividing the absolute viscosity μ by the mass density w/g. (The kinematic viscosity of water at 55° F is 0.00001307 sq ft per sec.)

Since the roughness coefficient f_1 is the same regardless of the diameter from which it is found, the index velocity V_t that fulfils the conditions for incipient full turbulence is independent of the diameter. This point is even more stable than the critical point that indicates nonturbulence. The velocity marked by the intersection of graphs in Fig. 9 can be computed by Eq. 4 using the average specific roughness coefficient. Its correlation with the observed points should be noted. When applying Eq. 4 one should make certain, of course, that the roughness factors do not include localized losses.

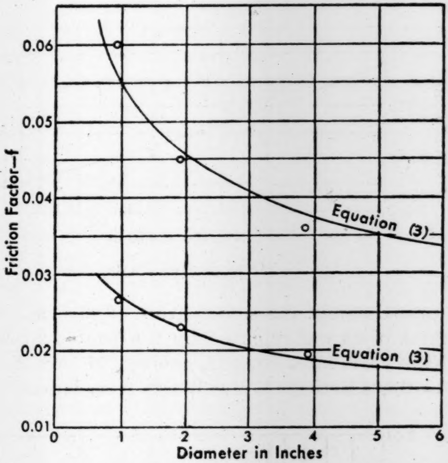

FIG. 10.—NIKURADSE'S THREE PIPES WITH TWO SIZES OF SAND GRAINS

Although Fig. 17 will be presented subsequently for rough pipes at velocities above V_t where friction factors are independent of velocity or temperature, it could be used in comparing pipes at the same velocity whether they are rough or smooth. The specific coefficients in Fig. 9 for example were all adjusted to a diameter of 1 ft by Eq. 3 whether they were the constant roughness factors of full turbulence or the variable factors in partial turbulence. Plotting these latter points according to the velocity instead of the Reynolds number keeps them on the inclined graph because of the relation of Eqs. 3 and 1. When adjusting for a significant temperature difference observe that the velocity at which these variable friction factors are correct is in proportion to the viscosity.

If V_t is greater than the highest velocity ever to be used for pipes with some particular surface, those pipes act as smooth pipes, and making them smoother will not increase the flow. The friction factor should then be taken directly from the smooth pipe formula. On the other hand, in very rough pipes, V_t is often so small that partial turbulence ceases to be a significant feature. If there is a question, try both methods and select the larger value of f.

Concrete drain tile presents a rough surface, particularly when the open joints are included. The irregularity of manufacture and of placement makes this condition a severe test of a specific roughness coefficient. A series of experiments at the University of Iowa[17] at Iowa City serves to indicate the correlation in this extreme case. Fig. 11 shows the six friction factors measured for diameters of 4 in. to 12 in., inclusive, for velocities above the index for the surface alone. The graph is computed by Eq. 3 using the average of their specific roughness coefficients. The probable deviation for a pipe selected at random, even with all its crudity, is less than 7%.

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[17] "The Flow of Water in Drain Tile," by D. L. Yarnell and Sherman M. Woodward, *Bulletin No. 854*, U.S.D.A., Washington, D. C., 1920, p. 1.

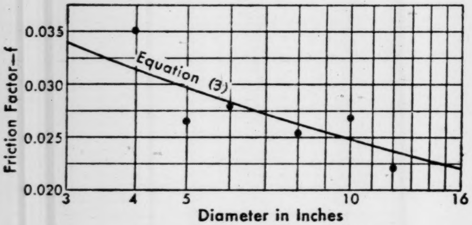

FIG. 11.—OPEN-JOINTED DRAIN TILE

Concrete Pipes.—The laws that control *f* being of universal application for models in general, the identification of a surface by its breaking away from the influence of viscosity is supplied by Eq. 1, and therefore may be widely extended by Eq. 3. Concrete pipes furnish an example of a large range of diameters, with approximately the same nature of surface. In 1920 an exhaustive study of these pipes was reported by the United States Department of Agriculture.[18] Nine years later Fred C. Scobey, M. ASCE, gave a digest of these findings for the American Water Works Association.[19] The pipes in this latter report are included in the following study except those described as probably containing deposits of sand or vegetable growth, short reaches questioned in the digest, jointed pipes less than 3 ft in diameter, and one old test by Fanning, having an insufficient description.

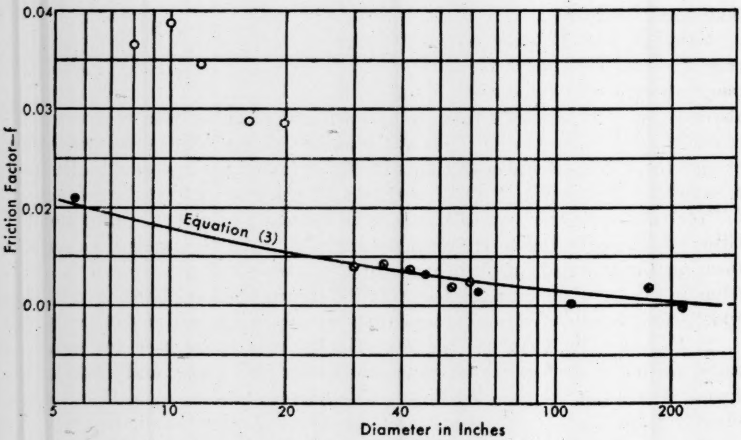

FIG. 12.—CONCRETE PIPE

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[18] "The Flow of Water in Concrete Pipe," by Fred C. Scobey, *Bulletin No. 852*, U.S.D.A., Washington, D. C., 1924, p. 95.

[19] "Coefficients of Flow in Concrete Pipe," by Fred C. Scobey, *Journal*, A.W.W.A., January, 1929, p. 1.

Mr. Scobey found the exponent of V to be substantially 2. By accepting this perfect square, one accepts a constant f for all velocities, and classifies the data as coming from pipes in pure turbulence.

The average specific roughness coefficient for these typical concrete pipes is 0.0171 ± 0.0002. Fig. 12 shows the graph for this specific roughness. The solid circles indicate cement lined pipe. The general agreement of the points with respect to the graph (the shape of which comes from Eq. 3) indicates the reliability of the data and of the method. Semilogarithmic plotting is used in Fig. 12 because the ratio of diameters is more important than their actual magnitudes.

The pipes included in this study do not necessarily represent materials and workmanship yet to be developed. They are chosen as typical examples to illustrate the use of a single specific roughness coefficient for all pipes of a given material.

It is well to note again why Kutter's n is larger for the larger pipes, inasmuch as n was conceived to be a factor remaining constant for all diameters. The formula, by making substantially the same percentage of correction for relative roughness whether the pipe be rough or smooth, overcorrects even these moderately smooth pipes. The result of this defect has usually been explained away by assuming the larger pipes to be actually rougher, an assumption generally contrary to fact. F. C. Finkle[18] has stated that the proper n for a 50-in. concrete pipe is 10% greater than that for an 18-in. pipe of the same workmanship. This 10% increase in n is no small matter. If applied to one of two pipes of the same diameter, it would mark that one for approximately 20% more loss. The variation of n being in reverse also leads to the erroneous conclusion that the larger a pipe is made the greater its roughness becomes. Kutter's formula serves no useful purpose for smooth surfaces, not even for concrete pipes. The n in this example varies as much as the C which it is supposed to correct. The Manning formula has this same characteristic.

Analysis with Eqs. 3 and 4 shows that the placing of concrete pipes in the category of full turbulence is justified. At ordinary temperatures with a specific roughness of 0.0173, the index velocity V_t from Eq. 4 is approximately 1.6 ft per sec.

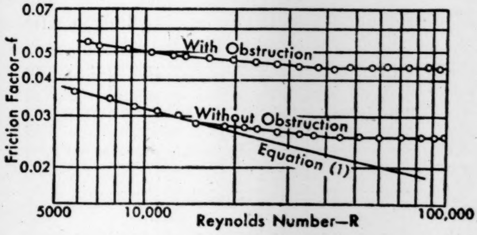

FIG. 13.—THE EFFECT OF AN OBSTRUCTION AT THE MIDPOINT OF A 20-FT LENGTH OF 2-IN. PIPE

Effect of Localized Losses.—When applying Eq. 4 it should be observed that losses resulting from obstructions or defects so widely separated as to act individually do not affect the index velocity. Friction factors as reported are not always free from such influence. Since these localized losses usually vary with the square of the velocity, they add a constant amount (natural numbers, not logarithmic) to the value of f whether the flow be partly turbulent or fully turbulent. The augmented graph should not be confused with a real transition resulting from two or more degrees of roughness acting separately in the same length of pipe.

Fig. 13 shows the effect of an obstruction at the midpoint of a 2-in. pipe 20 ft long. The lower line is the graph of the pipe before the obstruction was introduced. The upper line is the result of a partly closed standard gate valve at this midsection. The important fact is that the surface loss remained substantially unchanged.

The average specific roughness coefficient for the five bell-jointed pipes, shown by plain open circles in Fig. 12, is 0.033. This value is approximately double that for continuous pipe. The index velocity V_t, however, responds more nearly to the surface roughness. This fact is confirmed by the higher values of f found below that velocity. The data on the drain tile previously mentioned include more low velocities and therefore show more definitely that the point of change corresponds to the specific roughness coefficient of continuous pipes.

Old cast-iron pipes that have become tuberculated with large, widely spaced blisters fall in the category of pipes with localized losses and should not be analyzed by Eq. 4 until a reasonable estimate of these losses has been made and the friction factors reduced by that amount; but, since these old pipes operate so predominately in full turbulence, V_t is not so important.

New Iron Pipes.—Black iron or steel pipes are usually listed under the old name, "wrought iron." Until these pipes have rusted, the friction factor for the greater part of the surface is the same as that for any other smooth pipe; but with ordinary defects of manufacture, including roughness along a welded rib, there is generally an appreciable area rough enough to be in full turbulence. The net result, therefore, is a long transition resulting from two separate degrees of roughness.

The experiments by Mr. Freeman[15] show a wide variety of surfaces in the new iron pipes of an early period. Despite this variation, his measured results can be secured by supplying approximately three fourths of the length with the smooth pipe coefficient and the remaining length with a specific roughness coefficient.

It is not so advantageous to have a smoother surface as it is to keep a larger percentage of the surface smooth. In other words, uniformity of surface and the prevention of corrosion are more important than high polish.

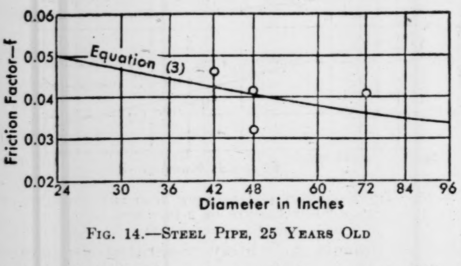

FIG. 14.—STEEL PIPE, 25 YEARS OLD

Old Steel Pipes.—The rate at which unpainted steel surfaces become rusty and tuberculated when exposed to water with various chemical impurities is beyond the scope of this paper; but an impression of what may occur in 25 years can be obtained from a description by Charles H. Capen,[20] M. ASCE, in 1941. Four pipes described by Mr. Capen, are: The 42-in. Pequannock conduit No. 2, the 48-in. Pequannock conduit No. 1, the 48-in. Pequannock conduit No. 2, and the 72-in. Rockaway conduit No. 1. Fig. 14 shows the friction factors of these pipes, interpolated for the age of 25 years. They exhibit irregularity, even for the two pipes of the same size, but clearly indicate a correlation with diameter. The specific roughness coefficient is 0.0617, indicating a loss of head approximately two and a half times that in a concrete pipe. Much of the indicated "roughness" may result from neglecting the change in diameter caused by internal surface deposits.

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[20] "Trends in Coefficients of Large Pressure Pipes," by Charles H. Capen, *Journal*, A.W.W.A., January, 1941, p. 1.

Wood Stave Pipes.—Wood staves present a surface that is reasonably the same for a wide range of diameters; but contradictions in published data cause one to hesitate in drawing general conclusions. Fortunately the presence of a localized loss does not change the general shape or the slope of the graph but simply lifts the graph to a higher level. When a small percentage of the length is in partial turbulence, the otherwise horizontal graph inclines only slightly; so an approximate analysis is possible.

An early impression that wood stave pipe remains smooth throughout a long life was furthered by placing too great emphasis on data from low velocities where considerable roughness makes no difference. There is unmistakable evidence that a specific roughness coefficient develops in wood stave pipes within a few months. With an approximate value of 0.015 in mind one should not expect any considerable departure from smooth pipe coefficients at velocities less than 3 ft per sec (3.08 ft per sec at 55° F).

Examination of the data compiled by Mr. Scobey[21] in determining his exponential formula reveals, among other things, the ill effect of vertical and horizontal curves in sequence in continuous stave pipes, as well as the ill effect of poor alinement in jointed pipes. Since these and other defects are so often present, there should be added to the coefficient for either partial or full turbulence an amount that may exceed 0.008. There are deviations from this value but it would seem unwise to cover poor workmanship or the effect of a disintegrated surface by a sweeping factor of safety.

Open Channels in Relation to Closed Pipes.—In the foregoing treatment of flow in pipes, all equations are referred to the diameter. Hydraulic radius is not used for pipes because it is an unsatisfactory correlating factor. For example, it is common knowledge that even a semicircular flume does not correspond to half of a closed pipe of that size.

Adaptation to open channels, in so far as pipes and channels agree, can be made by the well-known relationship between the Darcy-Weisbach f and the Chezy C. The identity of the equations, $h_f = f \frac{L}{D} \frac{V^2}{2g}$ and $V = C\sqrt{R\,s}$, is too often overlooked. The equation $C = \sqrt{\frac{8g}{f}}$ is simply a means of conversion.

Although this paper is not one in which to expand the subject of open channels, the writer will not refrain from suggesting the use of a specific channel

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[21] "The Flow of Water in Wood-Stave Pipe," by Fred C. Scobey, *Bulletin No. 376*, U.S.D.A., Washington, D. C., 1916.

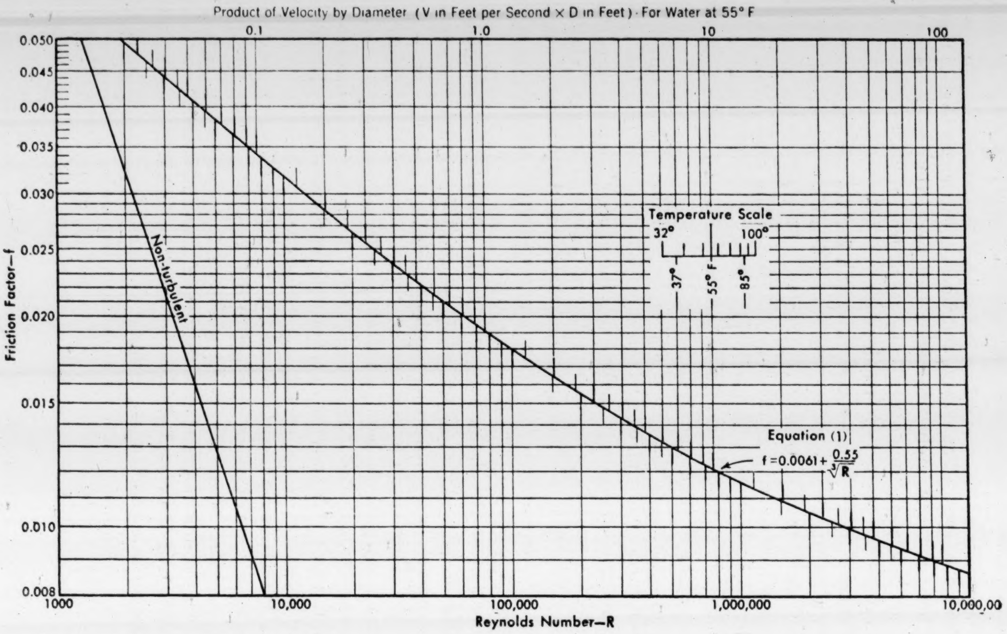

FIG. 15.—FRICTION FACTORS FOR ALL SMOOTH PIPES AND FOR OTHER PIPES IN PARTIAL TURBULENCE

coefficient based on the hydraulic radius of 1 ft. The simple mathematics shown herein, when accompanied by a "shape factor," furnishes the method of application. With each surface identified by its specific coefficient, the need for a Kutter's or a Manning's *n* or a Bazin's *m* will disappear. The specific channel coefficient C_1 for a given surface is the Chezy C corresponding to the friction factor for a 48-in. pipe having that same surface. Of course, the C_1 should be found by measurements on open channels, in which case the error of reduction to round pipes is avoided. Each surface has its own channel coefficient C_1 and, other things being equal, the flow is proportional to it.

Diagrams for Smooth and Rough Pipes

A method of applying the foregoing treatment is incorporated in the following diagrams. Pipes fall into two categories: (1) Smooth pipes, or rough pipes below the index of pure turbulence; and (2) rough pipes in pure turbulence.

Smooth Pipes.—The graph of Eq. 1 in Fig. 15 serves all smooth pipes and all fluids whose viscosities are known—oil, air, water, etc. The upper scale of $V D$ is a convenience for use with water only. It reads directly for water at 55° F, and can be adjusted for other temperatures by shifting the upper grid an amount and direction indicated by the temperature scale shown on the chart.

Rough Pipes.—With velocities less than the index velocity V_t, as taken from Fig. 16, rough pipes require the same coefficients shown for smooth pipes. They are accordingly served by Fig. 15, provided any localized loss is added. A surface with roughness reasonably distributed does not of itself produce a localized loss.

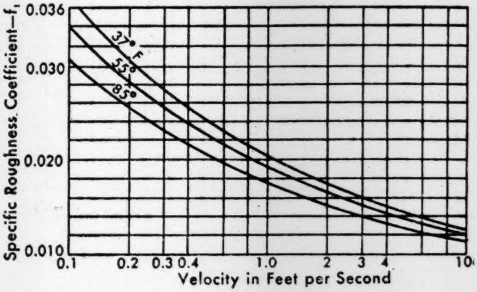

Fig. 16.—Influence of Temperature of Water on Velocity Producing Full Turbulence

Friction factors for velocities greater than V_t depend only on the diameter and the specific roughness coefficient of the surface. With these known, the desired constant friction factor is found in Fig. 17. This case covers a broad field of civil engineering in which the friction factor is not influenced by velocity or viscosity. This field is broadened because the controlling velocity for designing a pipe line is its highest velocity; and the specific roughness coefficient is generally adopted for future years.

A possible type of diagram that would include both rough pipes and smooth pipes could be prepared by superimposing the horizontal lines from Eq. 3 on the graph of Eq. 1. Making the interval between horizontal lines correspond to the change in friction factors caused by successively doubling the diameter would give a diagram similar to the one usually shown for the Nikuradse data. No experimental information would be required in making this diagram for rough pipes. Experiments would only be necessary to make it applicable to specific surfaces. The dip would naturally be lacking, as it should be for random roughness. Such a method implies a need for a linear measure of the roughness pattern, and suggests the finding of a Reynolds number whether it is pertinent or not. It is simpler to depend on a friction factor measured for one diameter, and to proceed by relative diameters, as in Fig. 17.

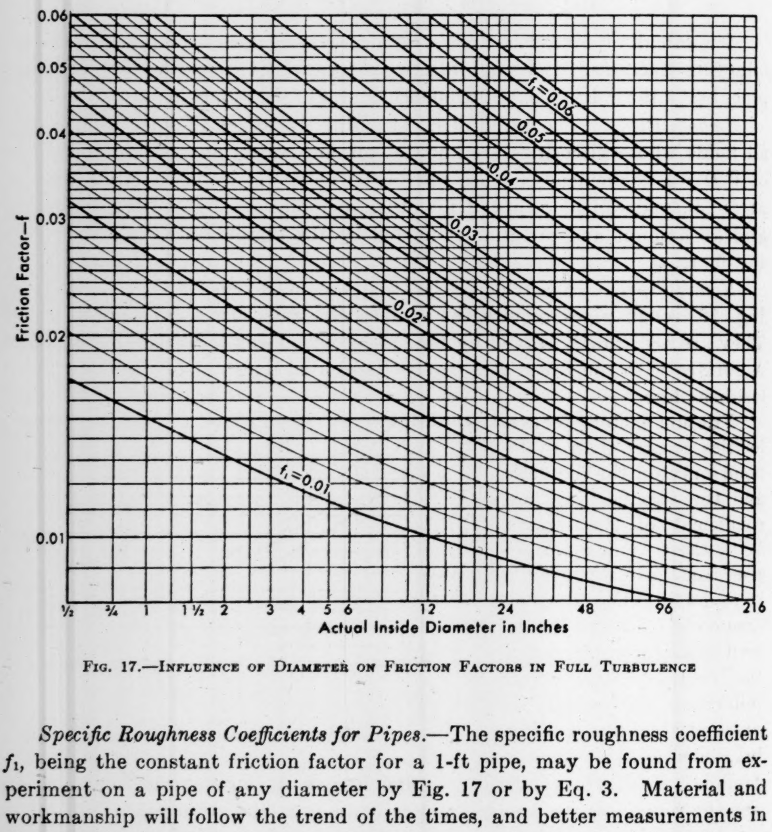

FIG. 17.—INFLUENCE OF DIAMETER ON FRICTION FACTORS IN FULL TURBULENCE

Specific Roughness Coefficients for Pipes.—The specific roughness coefficient f_1, being the constant friction factor for a 1-ft pipe, may be found from experiment on a pipe of any diameter by Fig. 17 or by Eq. 3. Material and workmanship will follow the trend of the times, and better measurements in the field will strengthen the accumulated data. In the meantime, the following typical examples will aid in securing a reasonable design:

Item	Material	f_1	V_t at 55° F
1	Metallized stainless steel in the rough	0.0146	3.5
2	Wood stave pipe, old surface	0.0150	3.0
3	Concrete pipe, straight continuous; also cement lined steel pipes	0.0173	1.6
4	Black pipe, 5 years old, surface only	0.018	1.5
5	Cast-iron pipe, bare surface	0.023	....
6	Wood stave pipe, continuous with horizontal and vertical curves, and jointed pipe with average alinement	0.023	3.0
7	Concrete drain tile	0.024	1.6
8	Cast-iron pipe, average for permanent city mains	0.028	....
9	Concrete pipe, 24 in. and under, bell jointed	0.032	....
10	Cast-iron pipe, old and foul	0.042	....
11	Riveted steel, 25 years old and tuberculated	0.062	....

(In items 6 and 7, the turbulence index V_t responds to the surface roughness, not to the localized losses.)

CONCLUDING REMARKS

The rational aspects of flow in pipes deserve more confidence than has usually been given them. The arguments presented in the preceding pages have accordingly been directed by the relation of topics and not by the chronological order in which the factual material has appeared in print. Any simplification has come from adhering closely to the basic physical laws rather than from attempting to circumvent those laws by individual experiments. Much of the complexity that so often results from meticulously examining too narrow a field can thus be avoided.

DISCUSSION

E. L. Harrington,[22] Assoc. M. ASCE.—The Darcy equation is probably the best method for computing flow of liquids in pipes, but the Hazen-Williams formula, in spite of its fallacy, as expressed in the paper, is an excellent formula for computing flow of water in pipes. The Hazen-Williams formula is usually expressed as

$$V = 1.318\, R^{0.63}\, s^{0.54}\, C \qquad (5)$$

The formula may also be expressed for flow in circular pipes in the following manner:

$$Q = 0.01037\, D^{2.63}\, H^{0.54}\, C \qquad (6)$$

in which Q is the rate of flow in cubic feet per second; D is the diameter of the pipe in feet; H is the head loss in feet per 1,000 ft of pipe; and C is a coefficient. In this form Eq. 6 may be solved directly for Q if D, H, and C are known; or, it may be solved directly for D if Q, H, and C are known, which avoids the trial and error solution necessary in the Darcy equation.

An experiment on the flow of water in a pipe made on the water supply system of the Port of Manzanillo, Colima, Republic of Mexico, from which the following information was obtained, is of interest.[23] The basic data were as follows:

Description	Quantity
Length of Pipe—	
In kilometers	16.93
In miles	10.52
Diameter of Pipe—	
In meters	0.20
In feet	0.6562
Available Head—	
In meters	138.00
In feet	452.75
Rate of Discharge—	
In cubic meters per second	0.042
In cubic feet per second	1.483

The pipe was described as being of, "cast iron, of very smooth surface, and of uniform diameter." Using the given data—$V = 4.39$ ft per sec, $f = 0.0179$ by the Darcy equation, and $C = 140$ by the Hazen-Williams equation. As early as 1905 a coefficient of 140 was predicted for "very best cast iron pipe, laid perfectly straight, and when new" by Gardner S. Williams and Allen Hazen,[24] Members, ASCE. The experiment at Manzanillo, on a pipe approximately 10 miles long, indicates the accuracy of the predicted coefficient.

Flow of water in pipes may be computed by the use of the Hazen-Williams equation and checked by the Darcy equation using appropriate friction factors. The Hazen-Williams formula apparently gives good results and is popular among practicing engineers.

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[22] Associate Prof., Dept. of Civ. Eng., Agri. and Mech. College of Texas, College Station, Tex.

[23] "Hydraulica," by Pedro J. Dozal, Talleres Gráficos De La Nación, Mexico, 1938, p. 322.

[24] "Hydraulic Tables," by Gardner S. Williams and Allen Hazen, John Wiley & Sons, Inc., New York, N. Y., 1st Ed., 1905, p. 3.

Ralph W. Powell,[25] M. ASCE.—To present the more recent knowledge regarding the loss of head in pipes, in a form that the practicing engineer can understand and use, is most commendable. The writer finds much in the paper with which he is in entire agreement including several worthwhile original ideas. The point that old experiments contained many more values at low velocities than at high velocities, and that this led to the notion that f always decreased with increasing velocity, is very good. Fig. 5 certainly shows the short-comings of the "exponential" formulas that resulted.

The explanation as to why the "dip" at the transition from smooth to rough pipe flow, which J. Nikuradse found with uniform roughness, does not occur in "natural" pipes, is praiseworthy; but, the author's idea that most pipes with which engineers deal are like the pipes in Tables 1 and 3, is questionable. Apparently these pipes reported in the paper had roughness particles of varying sizes, but with the variation very uniformly distributed along the length. As the author shows, this leads to a sudden change from smooth pipe flow to rough pipe flow at a certain critical Reynolds number, which is simultaneous throughout the pipe. Mr. Harris points out that, if certain reaches of the pipe are rougher than others, the total drop in head for the entire length will indicate a friction, f, intermediate between the f-values of the separate reaches, and therefore will show a transition such as found by C. F. Colebrook.[26] Assuming that this is the correct explanation, many pipes used in water supply and other fields must then have this unequally distributed roughness because the experiments of John R. Freeman,[27] Past-President and Hon. M. ASCE, and many others show such a transition. It seems, therefore, that engineers should be conservative and use graphs[28,29] based on Mr. Colebrook's transition—

$$\frac{1}{\sqrt{f}} = 1.74 - 2 \log_{10}\left(\frac{2\,\epsilon}{D} + \frac{18.7}{\mathbf{R}\sqrt{f}}\right) \ldots\ldots\ldots\ldots\ldots (7)$$

—for naturally roughened pipe, until this transition region has been investigated further. (In Eq. 7 ϵ represents the equivalent sand size roughness, originally called k by Mr. Nikuradse, and must be in the same units as D.)

If ϵ in Eq. 7 is made equal to zero, Eq. 7 becomes the Nikuradse formula for smooth pipes. Eq. 1 is in remarkable agreement with this formula for all values of **R** from 36,000 to 3,000,000, the maximum difference in the value of f by the two formulas being 0.0002; but they begin to deviate rapidly for higher values of **R**. The author states that the present knowledge extends only to a Reynolds number equivalent to a velocity of 5 ft per sec in an 8-ft pipe, which at 58° F gives a Reynolds number of 3,200,000; but Fred C. Scobey,[18] M. ASCE, has reported tests on the Ontario Power Tunnel at Niagara Falls in Canada at

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[25] Prof. of Mechanics, Ohio State Univ., Columbus, Ohio.

[26] "Turbulent Flow in Pipes, with Particular Reference to the Transition Region Between the Smooth and Rough Pipe Laws," by C. F. Colebrook, *Journal*, Inst. C. E., Vol. 11, 1938–1939, pp. 133–156.

[27] "Experiments Upon the Flow of Water in Pipes and Pipe Fittings Made at Nashua, New Hampshire, June 28 to October 22, 1892," by John R. Freeman, A.S.M.E., New York, N. Y., 1941.

[28] "Friction Loss in Pipes," by Ralph W. Powell, *Experiment Station News*, Ohio State Univ., Columbus, Ohio, April, 1946, p. 16.

[29] "Elementary Fluid Mechanics," by Hunter Rouse, John Wiley & Sons, Inc., New York, N. Y., 1946, p. 211.

20 ft per sec and in 18-ft-diameter pipe. Assuming a temperature of 55° F, this gives a Reynolds number of 27,500,000 and f was found to be 0.00694. The Nikuradse formula gives 0.00703 and the Harris formula, 0.00792. Therefore, Eq. 1 should not be used for Reynolds numbers greater than 3,000,000 and, since a graph will be ordinarily used rather than a formula, it would seem that the Nikuradse formula is the best. Fig. 15 gives a plotting of Mr. Nikuradse's formula in the middle part of the range. If an algebraic formula is desired, the writer suggests

$$f = 0.0049 + \frac{0.42}{\mathbf{R}^{0.3}} \quad \text{(8)}$$

The author seems troubled by the fact that the Nikuradse formula gives a zero value of f for an infinite value of **R**. There is something logically disturbing in this fact, but when one computes that it is necessary to go up to a Reynolds number of about five hundred million to bring f down to 0.005, and to nearly a million million to bring it down to half that amount, the practical engineer need not be greatly worried.

The derivation of Eq. 3 is based on the assumption that the "index" velocity, V_t, which divides smooth pipe flow from rough pipe flow, is the same for all diameters of pipe as long as the absolute roughness remains the same. This fact, according to the author, was shown "substantially" by the experiments listed in Table 3—that is, the "breakaway point" came at "somewhat less than 4 ft per sec." In Eq. 7, let $\mathbf{R} = \infty$ and divide by 2, then:

$$\frac{1}{2\sqrt{f}} = 0.87 - \log_{10} \frac{2\,\epsilon}{D} \quad \text{(9a)}$$

which is Mr. Nikuradse's equation for rough pipes. On the other hand, making $\epsilon = 0$ and simplifying,

$$\frac{1}{2\sqrt{f}} = \log_{10} \mathbf{R}\sqrt{f} - 0.40 \quad \text{(9b)}$$

which is the Nikuradse equation for smooth pipes. Solving Eqs. 9 simultaneously to determine the critical point where smooth pipe flow changes to rough pipe flow if there is no transition,

$$\mathbf{R}_t = \frac{9.31\ D}{\epsilon\sqrt{f}} \quad \text{(10a)}$$

or

$$V_t = \frac{9.31\,\nu}{\epsilon\sqrt{f}} \quad \text{(10b)}$$

Hence, with constant absolute roughness and viscosity, V_t varies inversely as the square root of f. Thus, if V_t is 3.50 ft per sec for the 1-in. pipe in Table 3, it would be $3.50\sqrt{0.0247/0.0360} = 2.90$ ft per sec for the ½-in. pipe and $3.50\sqrt{0.0247/0.0183} = 4.06$ ft per sec for the 4-in. pipe. This is too much variation to be ignored. From this point of view Eq. 4 changes to

$$V_{t2} = \nu \left(\frac{0.55}{f_1 - 0.0061} \right)^3 \sqrt{\frac{f_1}{f_2}} \qquad (11)$$

J. C. STEVENS,[30] PAST-PRESIDENT, ASCE.—The writer is glad to find that the old and tried Chezy formula still has a champion. In this phase of hydraulics much confusion has resulted from the use of many exponential formulas, some of which have become so entrenched in current thinking that it will be difficult to eradicate them, but eradicate them we must if progress is to be made. Empiricism must give way to rationality and the complex concept to the simpler concept.

TABLE 4.—LOST HEAD h_f IN ONE THOUSAND FEET OF NEW CAST-IRON PIPE

Line	Authority	Lost head, h_f, in feet,[a] equals:	INSIDE DIAMETERS, D, IN INCHES		
			12	36	36
1	....	Velocity, in feet per second	3.0	2.0	10.0
2	Williams-Hazen...	$\left(\frac{V}{1.318 \times 130 \times r^{0.63}}\right)^{1/0.54}$	2.8	0.36	7.3
3	Tutton..........	$\left(\frac{V}{140\, r^{0.66}}\right)^{1/0.51}$	3.2	0.31	7.3
4	Unwin..........	$0.0215 \frac{L}{D^{1.168}} \frac{V^{1.95}}{2\, g}$	2.8	0.36	8.1
5	Scobey[b]..........	$\frac{V^2}{0.345^2\, D^{1.25}}$	3.4	0.37	9.25
6	Barnes..........	$0.000343 \frac{L\, V^{1.953}}{D^{1.172}}$	2.9	0.37	8.5
7	Lea............	$0.00036 \frac{L\, V^{1.93}}{D^{1.25}}$	3.0	0.35	7.8
8	Manning.........	$2.87 \times 0.012^2 \times \frac{L\, V^2}{D^{4/3}}$	3.7	0.38	9.5
9	Harris...........	$f \frac{L}{D} \frac{V^2}{2\, g}$	3.2	0.37	9.25
10	Percentage range of diversion from the Harris formula		0 to +15	0 to −16	0 to −21

[a] In addition to the notation of the paper, r = radius = $D/2$. [b] For smooth pipe.

Of special interest to the writer were the head measurements in Table 1 wherein differential water levels were determined to 0.00037 ft. Such refinement is not at all impossible; remote registering instruments in which water surfaces are recorded within an error of ±0.006 in. (0.0005 ft) by electronic probes have been specified for the Waterways Experiment Station at Vicksburg, Miss. The probe system itself is sensitive to 0.002 in.

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[30] Cons. Engr. (Stevens & Koon), Portland, Ore.

It is the twilight zone between nonturbulent flow and fully turbulent flow that has given rise to these many exponential formulas. For nonturbulent flow, wherein lost head varies with the first power of the velocity and the velocity distribution is parabolic, the Poiseuille formula is mathematically correct and needs no empirical coefficients. For fully turbulent flow, the Darcy-Weisbach variation of the fundamentally correct Chezy formula has extreme simplicity in its favor. The use of a specific friction factor which depends solely on roughness and is independent of pipe diameter is a novel concept. If it was originated by the author he surely deserves signal credit for it. The next step follows logically; the friction factor for the particular pipe in question is readily found from the specific friction factor and the pipe diameter. It may be obtained directly from Fig. 17.

The imperfect or partial turbulent phase embraces most of the pipe flows in actual use. It is the transitional nature of flow in pipes that has given rise to so many empirical exponential formulas that are applied quite indiscriminately, far beyond the limits of the experimental data on which they are based. This can readily become a most inaccurate expedient, as illustrated in Fig. 5, for values beyond the velocity that indexes fully turbulent flow. This example, however, appears to be an extreme case.

The writer was mildly surprised to find that for the velocity ranges normally encountered in engineering practice the many formulas give results not too far apart for a given pipe. To satisfy his curiosity in this regard he tabulated the flow for eight formulas for new cast-iron pipe, two diameters, and three velocities. The results are give in Table 4. The percentage range of divergence from the Harris formula (line 9, Table 4) is compared in line 10, Table 4.

GEORGE A. WHETSTONE,[31] ASSOC. M. ASCE.—Under the heading, "Artificial Roughness Versus Natural Surfaces: Uniform Sand Grains," the author states:

> "Since the tendency to dip occurs at higher velocities for the smaller elements, it follows that only when many roughness elements are of one size does the dip become perceptible * * *."

With this statement and its elucidation the author has resolved the long-standing conflict between the painstaking and undeniably accurate work of Mr. Nikuradse, and its equally obvious failure to accord with experience on commercial pipe.

In addition to this fundamental contribution to the basic understanding of flow, the author has rendered a service to engineering by his consolidation of known results, by his establishment of a criterion (his value of R_t) for the transition point to purely turbulent flow, and by his discovery of the "specific roughness coefficient."

Without commenting on the relative merits of Eq. 1 and the more usual Blasius expression for this range (a close examination of the experimental data being necessary to do this justly), it would seem to the writer that the author's pattern of reorganization and simplification of pipe friction data into three lines on a (log f − log R)-graph, two of them universal and the other a horizontal line with its ordinate depending on the specific roughness coefficient, deserves to become the standard in engineering practice. It may be that the factor 0.0061 or some of the values of f_1 in the tabulation under the heading, "Diagrams for Smooth and Rough Pipes: Specific Roughness Coefficients for Pipes," may be subject to change under closer scrutiny; that is a small matter. The important thing is that designers now have a definite rational basis for the engineering treatment of pipe friction.

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[31] Associate Prof., Dept. of Civ. Eng., Texas Technological College, Lubbock, Tex.

G. H. HICKOX,[32] M. ASCE.—In promoting the use of the Weisbach formula for head loss in fluid flow, as opposed to empirical formulas, Mr. Harris has done the engineering profession a service. The empirical formulas are based on experiments covering a limited range of flow conditions; and, although they are undoubtedly correct within this range, they are not correct when extrapolated far beyond it. For this reason the use of the Weisbach formula with friction coefficients depending on the type of surface and the Reynolds number should be encouraged. However, there are a number of items in the paper with which the writer disagrees. Although some of these undoubtedly result from personal viewpoints, others seem to be of such fundamental importance as to warrant discussion.

The author's concept of viscosity does not appear to be in harmony with the kinetic theory. Since viscosity is that property of a fluid which enables it to transmit a shearing stress, the supposition of a fluid resistance existing in a fluid of zero viscosity, or of zero shear, is simply a contradiction of terms. The "mechanical interlocking" at the wall is merely momentum exchange caused by molecular activity. The concept of zero viscosity implies the complete absence of momentum interchange by molecular action; and, consequently, the complete absence of shearing resistance, both in the interior of the fluid and at the boundary. Accordingly, the concept of a zero friction factor at an infinite Reynolds number caused by zero viscosity is not at all inconsistent.

Furthermore, Eq. 1 is not supported by actual test data. Reference is made to data for a bituminous-coated steel pipe presented by the writer[33] in which f was about 0.0082 at a Reynolds number of 10,000,000. Eq. 1 yields $f = 0.0087$. Reference is also made to the data presented by J. N. Bradley, Assoc. M. ASCE, and S. P. Wing,[34] M. ASCE, which show values of f for the tunnel of the Ontario Power Company at Niagara Falls in Canada, a tunnel of a power plant at Livenza, Italy, and the conduit of Englewood Dam in Ohio. The Reynolds number for all these data ranges between 3,000,000 and 40,000,000. The values of f are lower than those given by Eq. 1 but are in substantial agreement with the von Kármán-Prandtl smooth pipe curve.

It is difficult to understand the author's conclusion that complete turbulence exists as soon as the condition for smooth pipe flow is no longer fulfilled. To arrive at this conclusion, it is necessary to ignore the work of J. Nikuradse,[35] of C. F. Colebrook[26] and C. M. White,[36] and of the writer.[33] Test data presented by each of these indicate a zone between the region of smooth pipe flow and that of complete turbulence where the friction is not constant, but varies according to some law not yet fully understood.

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[32] Associate Director, Eng. Experiment Station, Univ. of Tennessee, Knoxville, Tenn.

[33] "Friction Coefficients in a Large Tunnel," by G. H. Hickox, A. J. Peterka, and R. A. Elder, *Transactions*, Vol. 113, 1948, pp. 1027–1046.

[34] *Ibid.*, p. 1066.

The author's statement (under the heading, "General Principles") that "* * * viscosity plays no part in the loss of head in the fully turbulent flow most frequently encountered in moderately rough pipes and channels" is somewhat misleading. The flow most frequently encountered is not fully developed turbulence with a constant friction factor. Most of the test data presented by the many experimenters in the field of fluid friction show a friction factor f varying with Reynolds number. This result holds true for all the empirical formulas in which the exponent of the velocity is other than 2. Most of the empirical formulas that have been presented are correct for the data on which they were based, and to dismiss them as inadequate because they do not represent a constant factor is at least arbitrary. It is quite possible to accept these equations for what they are, explaining their divergences, and assigning them their proper place in the general field of pipe friction knowledge.

The author also states (under the heading, "Rough Pipes of Same Material and Various Diameters: Open Channels in Relation to Closed Pipes") that: "Hydraulic radius is not used for pipes because it is an unsatisfactory correlating factor." Throughout the paper he has used diameter for correlating pipe flows; and, since the hydraulic radius of a pipe is simply one fourth of its diameter, it is equally as good a correlating factor.

The author would be helpful if he explained how to differentiate between a series of localized losses and a rough pipe. If the writer's understanding of the paper is correct, a smooth pipe may become a rough pipe if there are enough localized disturbances. The dividing line between these two concepts is not clear.

JAMES F. HALSEY,[37] JUN. ASCE, AND WILLIAM M. OWEN.[38]—A simplifying unification, in Mr. Harris' opinion, of existing methods of analyzing flow in closed conduits has been presented in this paper. The author's concept of this problem, in which three distinct types of flow are characterized by straight line functions of friction factor, f, versus the Reynolds number, $\mathbf{R}$, on a logarithmic plot, as shown in Fig. 18, can be examined further in the light of additional data not included in the paper.

The three modes of flow represented in Fig. 18—those of nonturbulent or laminar flow, partly turbulent flow, and fully turbulent flow—were first noted by J. Nikuradse[39] and are now generally accepted by the profession. (The limits of the three types in Fig. 18 are up to $\mathbf{R} = 2{,}000$, from $\mathbf{R} = 2{,}000$ to

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[35] "Laws of Fluid Flow in Rough Pipes," by J. Nikuradse, English translation in *The Petroleum Engineer*, March, 1940, pp. 164–166; May, 1940, pp. 75, 78, 80, and 82; June, 1940, pp. 124, 127–128, and 130; July, 1940, pp. 38, 40, and 42; and August, 1940, pp. 83–84 and 87.

[36] "Reduction of Carrying Capacity of Pipes with Age," by C. F. Colebrook and C. M. White, *Journal*, Inst. C. E., No. 9, 1937–1938, pp. 381–399.

[37] Research Asst. Prof. of Fluid Mechanics, Univ. of Illinois, Urbana, Ill.

[38] Research Asst. Prof. of Theoretical and Applied Mechanics, Univ. of Illinois, Urbana, Ill.

[39] *Forschungsheft*, Verein Deutscher Ingenieure, No. 361, 1933.

R = 16,200, and greater than R = 16,200, respectively.) As stated by the author, nonturbulent or laminar flow is subject to rational analysis, partly turbulent flow is governed by the smooth pipe law which has been well established experimentally, and full turbulence is characterized by a constant friction factor.

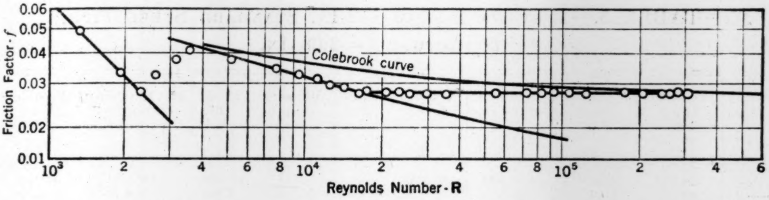

FIG. 18.—FRICTION FACTORS FOR THREE TYPES OF FLOW IN A 2-IN. STEEL PIPE

The author's experimental evidence indicates that these three types of flow were obtained in 2-in. and 4-in. pipes. However, his data show an abrupt transition from partial turbulence (that is, smooth pipe curve) to full turbulence (that is, constant *f*).

This manner of transition does not agree with the published work of C. F. Colebrook,[26] Hunter Rouse,[40] M. ASCE, and Lewis F. Moody,[41] all of whom present a gradual transition curve between partial and full turbulence extending over a considerable range of the Reynolds number. The Colebrook function[26]—

$$\frac{1}{\sqrt{f}} = -2 \log \left(\frac{k}{3.7\, D} + \frac{2.51}{\mathbf{R} \sqrt{f}} \right) \ldots\ldots\ldots\ldots\ldots (12)$$

(in which *k* is the wall particle size and the other terms are as previously defined)—was plotted in Fig. 18 to show the generally accepted gradual transition from partial to full turbulence. The value of *k*/*D* (the relative roughness factor, in Eq. 12) was determined by substitution of the averages value of *f* = 0.0278 as obtained from values greater than R = 16,200 in Table 1 in the von Kármán-Prandtl equation—

$$\frac{1}{\sqrt{f}} = 2 \log 3.7 \frac{D}{k} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

—for fully developed turbulent flow in rough pipes. It will be noted that the Colebrook function as applied to the author's data does not agree with the straight line curve used by the author until a Reynolds number of more than 300,000 is approached. This finding is consistent with those of other investigators—namely, Mr. Rouse,[40] who shows full turbulence to be developed when

$$\frac{\mathbf{R} \sqrt{f}}{D/k} = 200 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

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[40] "Evaluation of Boundary Roughness," by Hunter Rouse, *Proceedings*, 2d Hydraulics Conference, *Bulletin No. 27*, Univ. of Iowa Studies in Eng., Iowa City, Iowa, 1943.

[41] "Friction Factors for Pipe Flow," by Lewis F. Moody, *Transactions*, A.S.M.E., Vol. 66, 1944, p. 671.

When applied to the author's data for a 2-in. pipe, Eq. 14 gives **R** = 321,000 as the point at which full turbulence may be expected to be developed.

Since the author's data and curves did not agree with other investigations,[26,40,41] the writers undertook to clarify the 2-in. pipe curves by making tests on a 30-ft section of moderately rusty 2-in. steel pipe. This pipe has been in place in the Hydraulics and Fluid Mechanics Laboratory at the University of Illinois in Urbana since 1928. The test section was preceded by a straight section 28 ft long and followed by another straight section with a minimum length of 60 ft. The head loss was measured with a 5-ft differential gage using mercury, carbon tetrachloride, or a mixture of carbon tetrachloride

TABLE 5.—TEST DATA FROM A 2-IN. STANDARD STEEL PIPE; D (ACTUAL) = 2.06 IN.

Run No. (1)	Velocity (ft per sec) (2)	Measured head h_f (ft) (3)	Reynolds number (**R**) (4)	Friction factor (f) (5)
1	24.75	41.95	465,000	0.0252
2	24.10	40.40	454.000	0.0256
3	23.95	39.60	451,000	0.0254
4	23.80	38.70	447,000	0.0252
5	23.30	37.42	438,000	0.0254
6	22.90	36.00	430,000	0.0253
7	22.25	34.60	418,000	0.0257
8	21.92	33.30	412,000	0.0256
9	21.35	31.50	401.000	0.0255
10	20.75	29.60	390,000	0.0254
11	20.15	28.00	379,000	0.0255
12	19.75	27.25	366,000	0.0258
13	19.70	27.15	372,000	0.0258
14	19.62	26.55	371,000	0.0254
15	19.59	26.75	363,000	0.0258
16	19.45	26.35	368,000	0.0256
17	19.40	26.25	361,000	0.0258
18	19.15	25.65	355,000	0.0258
19	19.00	25.10	359,000	0.0256
20	18.90	24.95	357,000	0.0258
21	18.80	24.70	348,000	0.0258
22	18.05	22.80	334,500	0.0258
23	17.81	22.15	337,000	0.0257
24	16.95	20.25	320,000	0.0259
25	16.58	19.40	313,000	0.0260
26	15.40	16.81	291,000	0.0261
27	14.82	15.60	280,000	0.0261
28	14.10	14.11	266,000	0.0261
29	12.92	11.90	244,000	0.0263
30	11.63	9.76	220,000	0.0265
31	10.58	8.11	200,000	0.0268
32	10.02	7.31	190,500	0.0268
33	9.94	7.20	188,000	0.0269
34	9.10	6.04	173,000	0.0269
35	8.19	4.94	155,500	0.0272
36	7.89	4.64	149,000	0.0273
37	7.34	3.97	139,500	0.0273
38	7.16	3.79	130,200	0.0274
39	6.85	3.53	124,700	0.0278
40	6.79	3.46	129,000	0.0284
41	6.75	3.475	123,000	0.0281
42	6.40	3.110	116,500	0.0280
43	6.03	2.745	114,500	0.0278
44	5.95	2.680	108,200	0.0279
45	5.70	2.490	108,200	0.0282
46	5.69	2.515	108,000	0.0286
47	5.35	2.215	101,600	0.0285
48	5.16	2.045	97,300	0.0283
49	5.08	2.020	96,600	0.0288
50	4.79	1.800	91,000	0.0290
51	4.41	1.525	83,900	0.0288
52	4.38	1.514	83,200	0.0291
53	3.88	1.191	73,500	0.0292
54	3.88	1.207	73,500	0.0296
55	3.77	1.159	71,600	0.0300
56	3.37	0.922	64,000	0.0299
57	3.36	0.906	63,900	0.0295
58	2.995	0.735	56,600	0.0302
59	2.62	0.576	49,800	0.0308
60	2.61	0.575	49,600	0.0311
61	2.12	0.389	40,300	0.0318
62	2.115	0.384	40,100	0.0316
63	1.83	0.2915	34,800	0.0320
64	1.750	0.272	33,100	0.0327
65	1.635	0.238	31,100	0.0328
66	1.631	0.237	31,000	0.0328
67	1.63	0.2395	31,000	0.0332
68	1.478	0.200	27,900	0.0338
69	1.362	0.1724	25,900	0.0342
70	1.219	0.139	23,100	0.0346
71	1.142	0.1246	21,700	0.0351
72	1.078	0.1128	20,500	0.0358
73	0.978	0.0936	18,550	0.0360
74	0.854	0.0730	16,200	0.0368
75	0.744	0.0560	14,100	0.0373
76	0.660	0.0457	12,550	0.0386
77	0.569	0.0353	10,760	0.0402
78	0.495	0.0270	9,400	0.0466
79	0.417	0.0197	7,890	0.0418
80	0.3685	0.0163	7,000	0.0442
81	0.337	0.01392	6,360	0.0452
82	0.3065	0.01152	5,840	0.0451
83	0.2760	0.00951	5,220	0.0460
84	0.2285	0.00710	4,350	0.0500
85	0.1568	0.00368	2,960	0.0553

and gasoline (specific gravity, 1.067) as the gage fluid. Test runs were overlapped so that any error in reading small gage differences with a heavy gage fluid was compensated for by large differences with a lighter gage fluid. The discharge was measured by weight, with a filling time of 3 min or longer for nearly all runs. The results of these tests are plotted as Fig. 19.

An abrupt change from partial to full turbulence, such as the author obtained, is not present in Fig. 19. A few minor deviations appear, but it is believed that these may be attributed to experimental error. From values of **R** = 334,500 to **R** = 465,000, the friction factor, f, has a range of only 1% from the average value of $f = 0.0256$ for full turbulence (see Table 5).

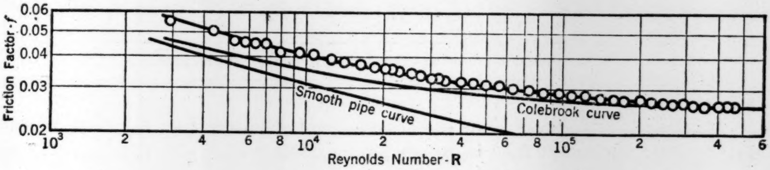

FIG. 19.—FRICTION FACTORS FOR A MODERATELY RUSTY 2-IN. STEEL PIPE

Using the value of $f = 0.0256$, Eqs. 12 and 13 were solved and the Colebrook function for the writers' data was plotted in Fig. 19. The Colebrook curve falls below the writers' curve, merging with it at a value of **R** of approximately 350,000. The application of Eq. 14 to the writers' data gave a value of **R** = 450,000 as the point at which full turbulence was developed.

The fact that the Colebrook curve falls above the curve presented by Mr. Harris and below that in Fig. 19 must not be considered a defect as there is a considerable variation in the roughness and transition curves in each class of pipe. The Colebrook function is meant to be the average of many pipes in the same class. The variation of roughness and transition curve in any one class of pipe precludes the use of the curve in the paper since it is obvious that any standard of design having wide application would necessitate the use of an average curve. The writers believe that the author's method of analyzing flow in closed conduits represents an oversimplification that can result in large errors when predicting the flow in pipes in the transition region between partial and full turbulence. This oversimplification is not desirable since most flow problems in engineering practice occur in the transition region bounded by the smooth pipe curve for partial turbulence and the curve of Eq. 14 for full turbulence.

TSEN-DING CHEN[42].—Computations of the Reynolds numbers by Eq. 1, compared with the von Kármán-Prandtl equation—

$$\frac{1}{\sqrt{f_1}} = 2 \log_{10} \mathbf{R} \sqrt{f_1} - 0.8 \ldots\ldots\ldots\ldots\ldots\ldots (15a)$$

and the Blasius equation—

$$f_1 = \frac{0.316}{\mathbf{R}^{0.25}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15b)$$

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[42] Research Asst., Saint Anthony Falls Hydr. Laboratory, Minneapolis, Minn.

—reveal that Eq. 1 yields consistent results. The values in Table 6 (determined by a 10-in. slide rule) indicate that Eq. 15*b* gives smaller values of f_1 for $\mathbf{R} > 100{,}000$.

TABLE 6.—COMPARISON OF REYNOLDS NUMBERS CORRESPONDING TO SPECIFIED ROUGHNESS COEFFICIENTS BY EQS. 1 AND 15

Authority	Item 10	Item 9	Item 8	Item 7	Items 5 and 6	Item 4	Item 3	Item 2	Item 1
Roughness coefficient,[a] f_1....	0.042	0.032	0.028	0.024	0.023	0.018	0.0173	0.015	0.0146
Harris, Eq. 1..............	3,600	9,600	16,000	29,000	34,400	98,500	119,000	236,000	271,000
von Kármán-Prandtl, Eq. 15*a*	3,380	8,760	14,600	27,200	33,900	99,200	120,000	247,000	287,000
Blasius, Eq. 15*b*...........	3,220	9,530	16,300	30,100	35,600	95,500	112,000	197,000	220,000

[a] See tabulation preceding "Concluding Remarks," in the paper.

Mr. Harris introduces V_t as the criterion of transition from partly turbulent flow to fully turbulent flow in rough pipes, a few values of V_t being given at a water temperature of 55° F. Since $f_t = f_1$ when $D = 1$ ft, and $\nu = 1.29 \times 10^{-5}$ ft² per sec (at 55° F), the value of $\mathbf{R}_t$ may be computed in two ways—by Eq. 2, or from V_t,

$$\mathbf{R}_t = V_t \times \frac{10^5}{1.29} \qquad (16)$$

A comparison of the Reynolds numbers thus computed is given in Table 7. The wide discrepancies of items 7 and 6 (see list preceding "Concluding Remarks") may lead to the necessity of further detailed study of the difference between surface roughness and local losses.

TABLE 7.—REYNOLDS NUMBERS CORRESPONDING TO FULL TURBULENCE AS GIVEN BY EQS. 2 AND 16

Item	$f_t = f_1$	V_t (ft per sec)	Eq. 2	Eq. 16
7	0.024	1.6	29,000	124,000
6	0.023	3.0	34,400	233,000
4	0.018	1.5	98,500	116,000
3	0.0173	1.6	119,000	124,000
2	0.015	3.0	236,000	233,000
1	0.0146	3.5	271,000	271,000

The writer appreciates the idea of introducing the well-established pipe flow formula to the open-channel flow field. The dimensional inadequacy of the existing open-channel formulas causes much trouble in dimensional analysis and model laws. The Manning *n*, the Chezy *C*, and the coefficients used in the Ganguillet-Kutter and Bazin formulas are not dimensionless factors. The dimensions of Manning's *n* and Chezy's *C* are, respectively,

$$[n] = [L]^{-1/3} \times [T] \qquad (17a)$$

and

$$[C] = [L]^{1/2} \times [T]^{-1} \qquad (17b)$$

The Darcy-Weisbach pipe formula—

$$h_F = f \frac{L}{D} \times \frac{V^2}{2\,g} \qquad (18)$$

—is a rational equation involving the dimensionless friction factor f. Eq. 18 applies to laminar flow as well as to turbulent flow; and it can be transformed into the famous Poiseuille equation by substituting $f = 64/\mathbf{R}$ and $\mathbf{R} = V\,D/\nu$ when flow is laminar.

The Darch-Weisbach formula is ready to apply to the open-channel flow field by using $R = D/4$ and $S = h_F/L$. The new open-channel formula is

$$V = \sqrt{\frac{8\,g}{f}}\,\sqrt{R\,S} \qquad (19)$$

Eq. 19 (the rational formula) is still similar to the Chezy formula. However, the friction factor f becomes a dimensionless coefficient by introducing the gravitational acceleration g into the expression, thus:

$$f = \frac{8\,g}{C^2} \qquad (20)$$

The introduction of gravitational acceleration g into the open-channel formula is quite reasonable because the gravitational force is usually the predominant physical force of flow in open channels.

The model law used in open-channel flow tests may be obtained directly from Eq. 19. In general, the scale ratio of velocity, V_r, can be written

$$V_r = \left(\frac{g_r\,R_r\,S_r}{f_r}\right)^{1/2} \qquad (21a)$$

If the model is "normal," $R_r = L_r$ and $S_r = 1$ since $X_r = Y_r = Z_r = L_r$; and Eq. 21a changes into

$$V_r = \left(\frac{g_r\,L_r}{f_r}\right)^{1/2} \qquad (21b)$$

If the model is built with the same material as the prototype and if the same friction factor could be used, then $f_r = 1$ and the equation of V_r practically becomes the Froude model law as follows:

$$V_r = (g_r\,L_r)^{1/2} \qquad (22)$$

In another form, the scale ratio of time, T_r, is

$$T_r = \left(\frac{L_r}{g_r}\right)^{1/2} \qquad (23)$$

in which, the value of g_r is always nearly equal to 1.0.

The Froude law is usually adopted to design open-channel models and some correction is provided for viscous force due to change of the Reynolds number between prototype and model, because it is practically impossible to satisfy both the Froude law and the Reynolds law at the same time. There is a possibility of putting the correction of the Reynolds number into the factor f_r when Eq. 21*b* is used.

In establishing a new rational open-channel formula such as Eq. 19 the determination of values of *f* depends on a detailed and complete analysis of the existing open-channel measurement data; and systematic experimental research in the laboratory is also necessary.

LYNN PERRY,[43] M. ASCE.—The modern concept of the resistance to the flow of fluids that has been developed during the past quarter of a century is revolutionary to some of the senior members of the engineering profession. Real progress has been made since the first publication of the Hazen-Williams tables.

It is refreshing indeed to read this paper in which Mr. Harris has so completely and so succinctly presented this subject. His theory is sound and his conclusions are logical. He is to be commended for stating the current position in this extremely progressive science.

It must be remembered that the data which Kutter originally had at his disposal did not warrent taking into account the slope to compute the constant *C*. In this refinement, he was influenced by the work of A. A. Humphreys, Hon. M. ASCE, and Henry L. Abbot. The elimination of the slope terms will give results that correspond more nearly to laboratory measurements. The Manning formula, for which very satisfactory tables have been published, corrects this feature.

When plotted, using discharge as the abscissa and loss of head as the ordinate, the Chezy formula with the Kutter constants produces a series of curves that require interpolation to make them continuous and to prevent intersection. The Weisbach formula, another form of the same expression, gives the same result if the factor *f* is computed from the Kutter *C* with its varying slope of the hydraulic grade line. The Hazen-Williams formula presents a continuous curve thereby correcting this error by committing another one, more serious theoretically.

During a consultation between the writer and the late Rudolph Hering, M. ASCE, these two formulas were discussed at length. Mr. Hering felt that the Weisbach formula was dimensionally correct and that until further research brought more and better values for *f* it would be well to use a series of *f*-values based on a review of published measurements. Admitting the care with which their work was done and the unquestioned integrity of the principals, Mr. Hering felt that Gardner S. Williams and Allen Hazen, Members, ASCE, must have overlooked some concept which, if taken into account, would have produced a formula much more in keeping with the dimensional theory as then known. In fact, two such concepts were later developed—fluid flow examined from a totally different viewpoint and partial turbulence. At that time, flowing water was considered either turbulent or nonturbulent. The Hazen-Williams measurements for loss of head were made on water flowing both turbulently and partly turbulently, averaged indiscriminately, and resulted in an exponent which was difficult to harmonize with theory or with former measurements. The professional standing of Messrs. Williams and Hazen, together with the usable form in which their conclusions were offered to the public, resulted in broad acceptance by the senior members of the engineering profession who were too busy with administrative details to devote much time to theory. After all, a pumping unit is manufactured and tested to meet the specifications rather than the requirements. If the specifications are in error by 10% or 20% or if the requirement changes during the life of the equipment (the usual case), the unit will still function even though it is not entirely satisfactory. In many instances, even the use of better judgment in computing the loss of head might not have obviated such unsatisfactory performance entirely.

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[43] Cons. Engr., Coral Gables, Fla.

The first two questions that will occur to the senior designing engineer are: (1) How much does the loss of head computed as suggested in this paper differ from that computed by other formulas; and (2) in the event that this rational procedure should be adopted, what simple and accurate method is suggested for its use?

An effort to answer question (1) led to Fig. 20. Three curves represent the values for f suggested by Mr. Harris and the dashed lines represent values from the Hazen-Williams tables for $C = 90$, 100, 130, and 140 (all for a 12-in. cast-iron pipe under turbulent flow with water at 55° F). The Hazen-Williams curve for $C = 90$ intersects that for $f = 0.042$ at about $1\frac{1}{2}$ mgd and gives slightly higher values for H_f for the more probable design of 700,000 gal per day or 800,000 gal per day. The curve for $f = 0.028$ for average cast-iron pipes in city mains falls between the Hazen-Williams curves for $C = 100$ and $C = 130$. The curve representing the smallest values for loss of head for turbulent flow in rough pipes ($f = 0.023$) gives values at ordinary velocities approximating the Hazen-Williams curve for $C = 140$ rather than 130 as recommended; the difference is about 25% at the rate of discharge of 1 mgd. With water at 78° F or 80° F, the viscosity being less, the initial loss of head should decrease, all other conditions being the same. This increase in temperature would add another 10% to the difference.

The answer to question (2) involves the flow of water through 36-in. and 42-in. cast-iron pipes about $4\frac{1}{2}$ miles long. The 36-in. line had been in service about 10 years; the entire inside surface was coated with a buff incrustant, mostly calcium carbonate deposited after the water had been filtered. In addition to being rough, this coating was thicker in some sections than in others, producing a washboard effect, and about $\frac{1}{8}$ in. to $\frac{1}{4}$ in. thick. Measurement of the loss of head showed 1.08 ft per 1,000 ft when discharging at 11.2 mgd. Thus, Kutter's $C = 88$, the Hazen-Williams $C = 90$, and $f = 0.036$. Using Fig. 17 and Eq. 3, f_1 must have been 0.048. A new 42-in. cast-iron pipe had been placed in service just before these measurements were made on the 36-in. line. Similar measurements on the 42-in. line indicated an over-all Hazen-Williams $C = 124$, or $C = 130$ after deducting the local losses. At a discharge of 14.3 mgd, the loss of head was 0.49 ft per 1,000 ft, or $f = 0.0216$. From Fig. 17 and Eq. 3, f_1 must be about 0.030; and in the pipe, although it was not so

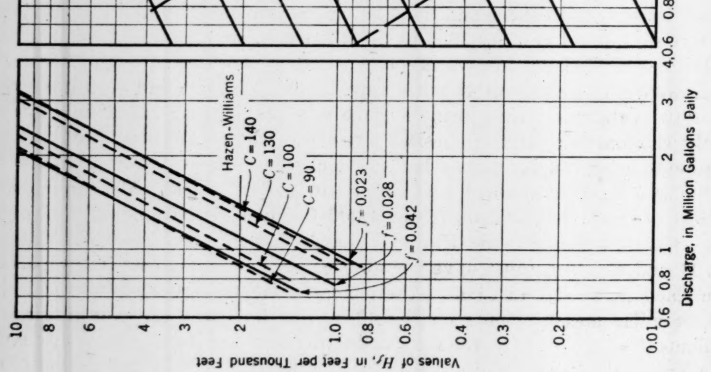
Hazen-Williams
C = 140
C = 130
C = 100
C = 90
f = 0.023
f = 0.028
f = 0.042
Values of H_f, in Feet per Thousand Feet
10
8
6
4
3
2
1.0
0.8
0.6
0.4
0.3
0.2
0.01
Discharge, in Million Gallons Daily
0.6
0.8
1
2
3
4

FIG. 20

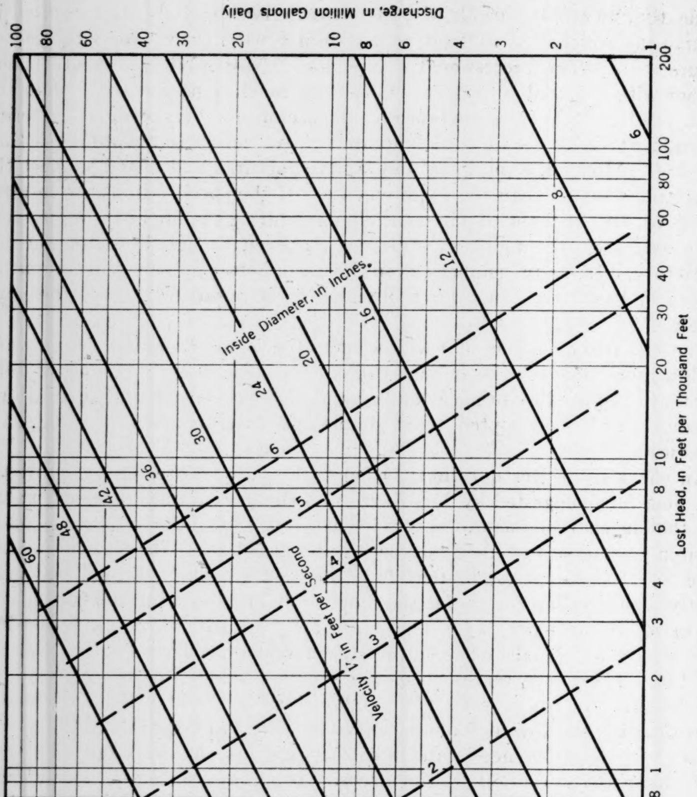
Discharge, in Million Gallons Daily
100
80
60
40
30
20
10
8
6
4
3
2
1
Inside Diameter, in Inches
60
48
42
36
30
24
20
16
12
8
6
Velocity V, in Feet per Second
6
5
4
3
2
Lost Head, in Feet per Thousand Feet
200
100
80
60
40
30
20
10
8
6
4
3
2
1
0.8
0.6

FIG. 21

rough or so completely coated as the 36-in. line, deposition was progressing rapidly with a corresponding rise in the specific roughness coefficient. For smooth pipes, Fig. 15 gives $f = 0.012$ with a water temperature of 78° F. Based on these measurements supported by a number of checks, it was considered that, although the specific roughness coefficient, f_1, for the 42-in. line was low, it would rapidly increase to the value for the 36-in. line and would remain at about that value indefinitely, any further reduction in the carrying capacity of the pipe being due to a reduction of the cross section of the flow. Fig. 21 was prepared for loss of head in pipes so incrusted and for water 78° in temperature. Using a specific roughness coefficient, $f_1 = 0.048$, in Fig. 17 and Eq. 3, values in Fig. 21 have been checked and found to comply with the theory presented in this paper.

Naturally, the writer feels that the suggested $f_1 = 0.042$ in item 10 (under the heading, "Diagrams for Smooth and Rough Pipes: Specific Roughness Coefficients for Pipes") is nearly, but not quite, large enough.

Andre L. Jorissen.[44]—A worthwhile endeavor to approach the problem of flow in pipes of natural roughness synthetically is made by Mr. Harris. The main obstacle to a comprehensive solution lies in the complexity of the roughness pattern and in the resulting difficulty of defining, and hence of measuring, roughness.

An attempt is made to characterize roughness by a specific roughness coefficient. It should be understood that this coefficient is the constant friction factor of fully turbulent flow in a pipe of unit diameter, for a given roughness pattern. No specification of the unit length used is necessary if it is stated that the value of D in Eq. 3 must be expressed in the same units.

The analysis of transition from laminar to fully turbulent flow is based on Eq. 1, derived from the author's experimental study of a smooth brass pipe. This formula (Eq. 1) is similar to a relation which was propounded, as early as 1922, by Charles Hanocq[45] of Liege, Belgium:

$$f = 0.00668 + \frac{0.532}{\sqrt[3]{R}} \qquad (24)$$

The Hanocq formula has been widely and satisfactorily used in western Europe since the early 1920's. It is based on numerous experiments, both with air and water and carried up to the Reynolds numbers of 500,000. Also, it has the advantage of a pure cube root in the computation of the second term.

A Reynolds number of 3,240,000 having been reached in the Nikuradse experiments, it is desirable to investigate to what extent classical formulas agree with experimental results. Typical formulas for smooth pipes (in addition to the Hanocq formula already mentioned) are as follows:

The Blasius[46] formula (1913)—

$$f = \frac{0.3164}{R^{0.25}} \qquad (25)$$

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[44] Associate Prof. of Civ. Eng., Pennsylvania State College, State College, Pa.

[45] "A propos du calcul des pertes de charge dans les conduites forcées," by Charles Hanocq, *Revue Universelle des Mines*, Liège, Belgium, February, 1922, Series 6, pp. 217–234.

[46] "Das Achnlichkeitsgesatz bei Reibungsvorgängen in Flüssigkeiten," by H. Blasius, *Forschung auf dem Gebiete der Ingenieurwesens*, No. 131, 1913.

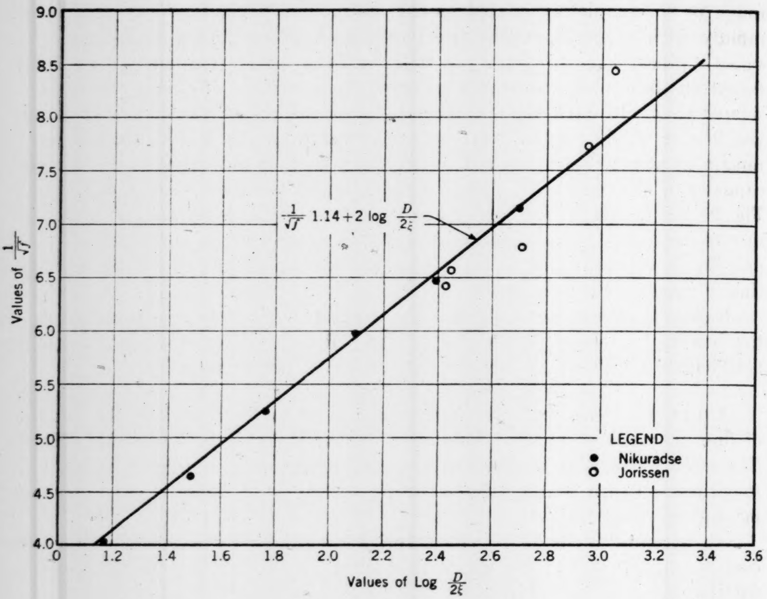

FIG. 22

for $\mathbf{R} \leqq 100{,}000$; the Lees formula[12] (1915), based on experiments by T. E. Stanton and J. R. Pannell[13]—

$$f = 0.0072 + \frac{0.6104}{\mathbf{R}^{0.35}} \quad \ldots\ldots (26)$$

for $\mathbf{R} \leqq 460{,}000$; and the Schiller-Hermann[47] formula (1930)—

$$f = 0.0054 + \frac{0.396}{\mathbf{R}^{0.3}} \quad \ldots\ldots (27)$$

—which gives results in good agreement with the Nikuradse experiments up to a Reynolds number of 2,500,000. A discrepancy is however apparent for $\mathbf{R} < 50{,}000$. The Nikuradse[48] formula (1932)—

$$f = 0.0032 + \frac{0.221}{\mathbf{R}^{0.237}} \quad \ldots\ldots (28)$$

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[47] "Widerstand von Platte und Rohr bei hohen Reynoldsschen Zahlen," by L. Schiller and R. Hermann, ***Ingenieur Archiv,*** Vol. I, 1930, p. 392.

[48] "Gesetzmässigkeiten der turbulenten Strömung in glatten Rohren," by J. Nikuradse, ***Forschungsheft,*** Verein Deutscher Ingenieure, No. 356, September–October, 1932.

is valid for a Reynolds number greater than 100,000, which is the limit of validity of the Blasius formula.

More recently, the writer has derived a comprehensive formula based on the Nikuradse measurements of velocity distribution and pressure loss. This formula[49] is expressed as

$$f = 0.0055 + \frac{0.4886}{\mathbf{R}^{0.3183}} \qquad \text{(29)}$$

TABLE 8.—PERCENTAGE COMPARISON OF FORMULAS FOR FRICTION LOSS

Reynolds' No. (R) (thousands)	f by Eq. 29 (writer)	BLASIUS		HANOCQ		NIKURADSE		SCHILLER-HERMANN		HARRIS	
		f_B, Eq. 25	$\frac{f-f_B}{f}$	f_q, Eq. 24	$\frac{f-f_q}{f}$	f_N, Eq. 28	$\frac{f-f_N}{f}$	f_S, Eq. 27	$\frac{f-f_S}{f}$	f_H, Eq. 1	$\frac{f-f_H}{f}$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
2	0.0490	0.0473	+5.51	0.0491	−0.20	0.0397	....	0.0459	+6.34	0.04970	−1.43
3	0.0437	0.0428	+2.06	0.0437	0	0.0363	....	0.0413	+5.50	0.04420	−1.14
5	0.0380	0.0376	+1.05	0.0379	+0.26	0.0326	....	0.0362	+4.75	0.03830	−0.79
10	0.0315	0.0316	−0.32	0.0315	0	0.0281	....	0.0304	+3.50	0.03165	−0.48
15	0.0284	0.0286	−0.71	0.0283	+0.35	0.0258	....	0.0275	+3.17	0.02840	0
20	0.0264	0.0266	−0.76	0.0263	+0.38	0.0243	....	0.0257	+2.65	0.02635	+0.19
25	0.0250	0.0252	−0.80	0.0249	+0.40	0.0233	....	0.0244	+2.40	0.02490	+0.40
50	0.0211	0.0212	−0.47	0.0212	−0.47	0.0202	....	0.0208	+1.42	0.02101	+0.43
100	0.0180	0.0178	+1.11	0.0182	−1.11	0.0176	+2.22	0.0179	+0.56	0.01795	+0.28
150	0.0165	0.0161	+2.42	0.0167	−1.21	0.0163	+1.21	0.0165	0	0.01643	+0.42
200	0.01553	0.0150	+3.42	0.01582	−1.86	0.01545	+0.51	0.01557	−0.26	0.01553	0
250	0.01485	0.0142	+4.38	0.01513	−1.89	0.01482	+0.20	0.01491	−0.40	0.01484	+0.07
300	0.01432	....	....	0.01466	−2.37	0.01432	0	0.01441	−0.63	0.01433	−0.07
350	0.01390	....	....	0.01426	−2.59	0.01393	−0.22	0.01400	−0.72	0.01392	−0.14
400	0.01355	....	....	0.01393	−2.81	0.01355	−0.30	0.01366	−0.81	0.01356	−0.07
500	0.01300	....	....	0.01341	−3.16	0.01306	−0.46	0.01313	−1.00	0.01304	0
1,000	0.01151	....	....	....	....	0.01156	−0.43	0.01168	−1.48	0.01160	−0.78
1,500	0.01079	....	....	....	....	0.01080	−0.09	0.01096	−1.58	0.01090	−1.02
2,000	0.01032	....	....	....	....	0.01030	+0.19	0.01050	−1.74	0.01046	−1.36
2,500	0.00999	....	....	....	....	0.00993	+0.60	0.01017	−1.80	0.01015	−1.61
3,000	0.00974	....	....	....	....	0.00965	+0.92	0.00991	−1.75	0.00991	−1.75
3,500	0.00954	....	....	....	....	0.00941	+1.37	0.00971	−1.78	0.00972	−1.89
4,000	0.00937	....	....	....	....	0.00922	+1.60	0.00954	−1.82	0.00957	−2.14
5,000	0.00910	....	....	....	....	0.00891	+2.08	0.00927	−1.87	0.00932	−2.42

and checks remarkably well with the experiments of Nikuradse and the writer for all values of the Reynolds number.

A comparison of the various formulas (see Table 8) shows fairly good agreement in the range of validity of each formula, but the agreement is not extended over this range. Eqs. 1 and 29 give close values of f up to the high Reynolds numbers, a slight discrepancy being noticeable in the extreme values of the latter parameter. However, for all practical purposes, the two formulas may be said to agree.

The proposed method of characterizing roughness is based on the consideration of the specific roughness coefficient. It is doubtful, however, if such a coefficient will prove satisfactory in the study of commercial rough pipes. In the experience of the writer, the mean height of roughness protuberances may vary considerably from one pipe to another, even for new pipes of approximately equal diameters manufactured in the same factory. This opinion

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[49] "Contribution à l'étude des pertes de charge continues dans les conduites circulaires," by Andre L. Jorissen, *Bulletin du Centre d'Etudes, de Recherches et d'Essais Scientifiques des Constructions du Génie Civil et d'Hydraulique Fluviale*, Liège, Belgium, 1949.

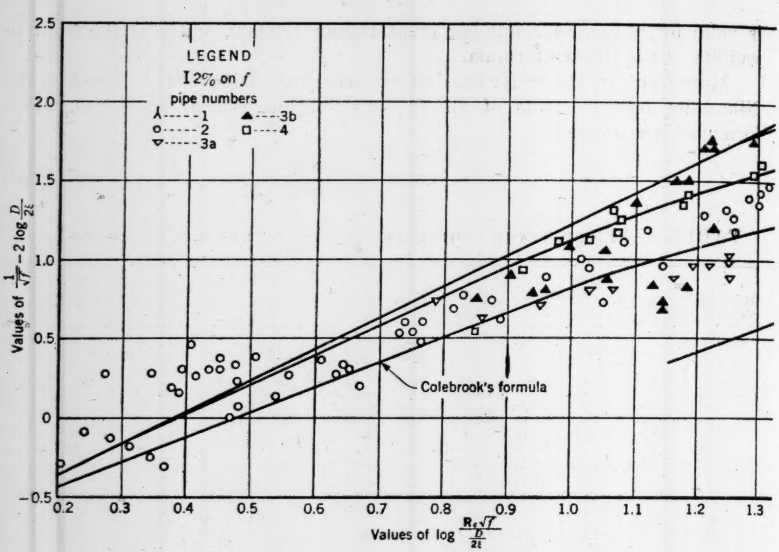


FIG.

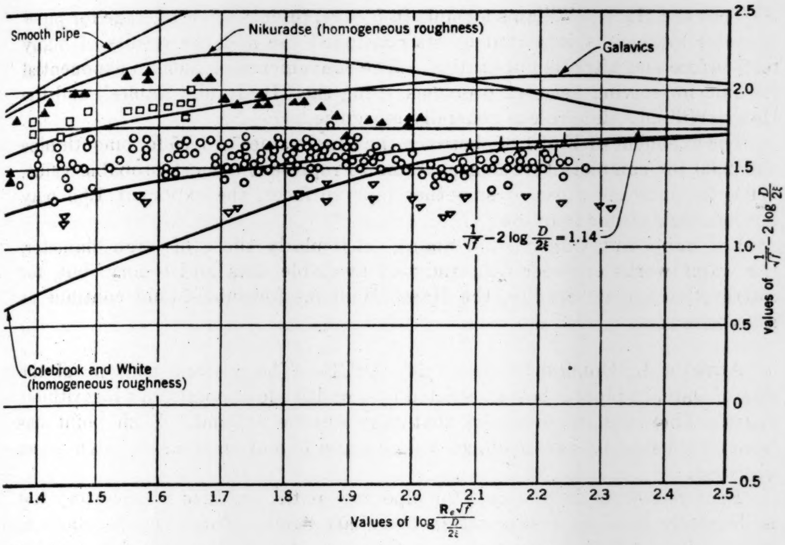
was obtained by measuring the mean roughness height ϵ by a pneumatic feeler.[49] It was shown that, in the new industrial steel pipes studied, ϵ is a characteristic value for a given pipe. The value of ϵ may vary from one cross section to another and from point to point in one section but, by averaging a large number of measurements, a characteristic mean value is obtained. As stated before, ϵ differs considerably from one pipe to another. This stresses the difficulty of using any of the classical formulas for rough pipes and seems to restrict the use of the specific roughness coefficient. Pneumatic feeler measurements, on the contrary, are easier to make than pressure loss determinations and may further be repeated from time to time to follow the evolution of the pipe roughness.

Another conclusion of the aforementioned study was that, in the zone of fully developed turbulence, the value of f was given with a very satisfactory approximation by the Nikuradse formula:

$$\frac{1}{\sqrt{f}} = 1.14 + 2 \log \frac{D}{\epsilon} \dots\dots\dots\dots\dots\dots (30)$$

in which ϵ was the mean roughness height, as measured by the pneumatic feeler (Fig. 22).

A smooth transition in the law of variation of f as a function of the Reynolds number was recorded. This is also in agreement with the Colebrook and White experiments. In this zone, the law of variation for industrial pipes is situated between the Nikuradse law for homogeneous roughness and the Cole-



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brook and White law for heterogeneous roughness. The writer's experiments show a classification following the roughness homogeneity (Fig. 23).

Weston Gavett,[60] M. ASCE.—The book on "Hydraulic Tables"[8] by Gardner S. Williams and Allen Hazen, Members, ASCE, has been described by the author (under the heading, "Inadequacy of Some Early Methods") as "based on a fallacy that should have been apparent at the time." Mr. Harris further remarks that:

> "A constant-exponent formula was developed, selecting only the data from the smoothest pipes attainable, the exponent of *V* reducing to 1.85. It was then assumed that all rough pipes would follow that same exponential 'law' * * *."

In the Introduction[61] to "Hydraulic Tables" Messrs. Williams and Hazen pointed out that—

> "* * * the values of the exponents vary with different surfaces and also their values may not be exactly the same for large diameters and for small ones, nor for steep slopes and for flat ones. * * *
>
> "The exponents in the formula used were selected as representing as nearly as possible average conditions, as deduced from the best available records of experiments upon the flow of water in such pipes and channels as most frequently occur in water-works practice."

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[60] Associate, Clyde Potts, New York, N. Y.

[61] "Hydraulic Tables," by Gardner S. Williams and Allen Hazen, John Wiley & Sons, Inc., New York, N. Y., 3d Ed., 1920, p. 1.

That the Hazen-Williams formula in fact represents a fair average for pipes in water systems is indicated by its continued use and the results of many tests before and after its publication. The convenience of such an exponential formula for solving network problems, using the "Hydraulic Tables" and the Hazen-Williams slide rule is generally recognized.

The exponent of V of 1.85, between the usual values of 1.75 for smooth pipe and 2.00 for complete turbulence, seems to represent a most probable value. With the increasing use of smoother pipe surfaces, the exponent 1.85 may become high rather than low.

For important transmission mains, particularly those of large diameter, the water works engineer will study all available data and theory; but, for distribution system studies, the Hazen-Williams formula should continue as a favorite tool.

Arthur L. Collins,[52] Assoc. M. ASCE.—The writer's concept of the flow of water in pipes is best illustrated by a multitude of points on a logarithmic chart. They are so numerous that they outline a band. Each point has been established by two readings, velocity and loss of head, as reported by an observer.

Each new formula proposed for pipe flow as the result of a laboratory test is definitely fixed by a line within the chart band. There can be about as many different formula exponents as there are stars in the Milky Way. Regardless of the formula used the data are someplace on the chart and a choice can be made after reading the testimonials.

Other factors enter into flow measurements and characteristics in the field which are not present in the laboratory where the setup includes a small pipe with perfect bore. Undulations and ovalness (out of round) are conditions that can have a marked effect upon energy losses.

The study of the velocity distribution or velocity factor should be included in pipe experiments as a method of classifying pipe conditions and making the data more comparable. It is reasonable to assume that at some time the pipe line will be designed to provide a certain velocity factor which will indirectly insure maximum transportation efficiency.

Charles W. Harris,[53] M. ASCE.—There is reason to feel that the response during the several months of discussion has been mainly favorable to the major features, or that the questions have been largely answered in the discussions themselves. For example, some resistance has been offered to the abandonment of the exponential type of formula, but both Mr. Stevens and Mr. Collins have given good and sufficient reason why that method should be discarded. To the remark by Mr. Collins, "* * * as many different formula exponents as there are stars in the Milky Way," might be added—"each exponent requiring its separate coefficient." If this fact were given due weight, the Hazen-Williams formula would fall without mention of its chief fault—namely, the use of an extravagant coefficient to supply the deficiency in full turbulence where the rational Chezy exponent takes over.

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[52] Cons. Engr., Berkeley, Calif.

[53] Prof., Hydr. Eng., Univ. of Washington, Seattle, Wash.

It is unfortunate that, with all its shortcomings, an empirical formula should still remain "popular" simply because of the impetus of long usage. Although Mr. Gavett knows that the exponents in the Hazen-Williams formula (even though constant for all ranges of size, velocity, and degree of roughness) should not be constant, he prefers not to accept a remedy; nor is he apprehensive of the prevailing temptation to average data that are not coordinate. If exponents are to be treated so lightly, there was little excuse for abandoning small whole numbers in the first place. The widest use of any formula is not to preserve experimental data for repeated use under identical circumstances but to spread known facts over a wider range. All exponential formulas have one feature in common—their logarithmic graphs are straight lines. Any two straight lines that are not parallel must agree at one point, and only one. This fact applies also to the discussion by Mr. Harrington; neither the data he quotes nor his use of $C = 140$ for the purpose is being questioned, but an understandable formula would give the same result. If pipes would remain "very smooth" as quoted by Mr. Harrington, no specific roughness coefficient would be required. The value of f from Eq. 1 or Fig. 15 is approximately 0.0150 which, subtracted from the actual test value, leaves 0.0029 to provide for joints and all other sources of localized loss. This is not an unreasonable augmenting factor for cast iron pipes. Consider another common situation: Even with moderate velocities (say, 3 ft per sec in a 12-in. pipe) the Hazen-Williams C of 100, so frequently used for distribution systems, corresponds to an f_1 of approximately 0.033, which is certainly in the fully turbulent range with 2 for an exponent. Why not then use the simple fundamental formula for distribution systems?

It would be inappropriate in a paper of this nature to write a long and detailed reply to each of the pertinent questions asked by Mr. Hickox. Briefly, it should be stated, however, that some of the so-called fundamentals would stand revision. Resistance in turbulent flow, for example, is not a shear from plane to plane as some conventional formulas would require, but rather a resistance to the acceleration of particles displaced from the boundary region by those entering. The writer's mention of viscosity in relation to the higher Reynolds numbers was to remind the reader that hydraulic losses are not entirely due to viscosity, and that some of the complicated formulas are not necessarily less approximate than the simpler ones. The concept of loss without viscosity is in no manner out of "harmony with the kinetic theory." For example, no one would call the familiar expansion loss a viscous phenomenon. As for some values of f (determined by field experiments) being smaller than those found by Eq. 1, it is thought that, in the absence of more and better field data, the simpler formula should not be excluded, particularly if it is on the safe side. The writer is acquainted with the difficulties encountered when attempting to free an important pipe line from its regular service long enough to make the repeated measurements that are found so necessary even under laboratory conditions.

Referring to Mr. Hickox's comment it should be emphasized that Eq. 1 is supported by actual test data throughout wide ranges. He is of course well acquainted with the work of H. Blasius, T. E. Stanton, J. R. Pannell, W. H. McAdams, and John R. Freeman, Past-President and Hon. M. ASCE, and indorses the von Kármán-Prandtl formula which lies within practical agreement with Eq. 1 up to the Reynolds numbers above 3,000,000. The deductions in this paper are numerically dependent on the position of that line but are equally applicable to any mathematical form of the primary equation.

The writer did not imply that transitions never occur but that when found they are usually explainable. To be able to predict a coefficient, however, one must understand its response to known circumstances. For example, scrutiny of either graph in Fig. 13 reveals a transition known by the writer to have been caused by the smoother length being in partial turbulence at the same time that the rougher length was in full turbulence. The graph became horizontal when both lengths were in full turbulence. The writer was not aware of any controversy in recent times about the friction factor in full turbulence being independent of viscosity (or of the entire Reynolds number for that matter) whether it is in the entire length of pipe or only in a part of it. He does not care to revise the statement about the flow most frequently encountered in rough pipes and channels. With increasing velocity the rougher sections go into full turbulence at the beginning of the transition, not at the end.

In referring to "hydraulic radius" the writer sought to confine its use to conduits other than round ones, and to open channels. This discrimination is made to discourage the habit of applying to round pipes the coefficients obtained from other sources.

Mr. Collins will probably agree that it is not ovalness itself (within reason) that causes a "marked effect" on friction loss, but rather the resulting abrupt offset of surface at frequent intervals, like that occurring in some vitrified pipes and in drain tile.

The writer accepts so much of Mr. Powell's discussion that he does not wish to imply any wide criticism; but it should be observed that Eq. 3 was not derived on the mere "assumption" that the velocity for full turbulence is independent of the diameter. Velocity is an essential element of the Reynolds number of the pattern of roughness, and is thus incorporated in the equations. In the various figures of the paper the graph of Eq. 3 appears six times, each time supported by separate experimental points, whereas no such points were used in its derivation. In addition to these computed curves is the graph in Fig. 9, in which the observed points were adjusted for diameter by Eq. 3. This broken curve is supported by fifty-three points taken from four relative diameters ranging from 1 to 7. The inclusion of the $\frac{1}{4}$-in. pipe with $f_1 = 0.0148$ would not broaden the experimental band and would increase the range to from 1 to 13. If V_t were to any significant extent a function of f in addition to the f_1 of Eq. 4, those experiments would have indicated it. It is no surprise that Mr. Powell could not eliminate f from Eq. 10*b*. That feature was incorporated in the von Kármán-Prandtl formula (called the Nikuradse formula by Mr. Powell) to furnish one of the necessary empirical steps in connection with the introduction of a stability factor. The resulting graph of velocity distribution was adopted by them as a reasonable approximation and, when supplied with coefficients furnished by J. Nikuradse for the purpose, the formula naturally gave friction factors agreeing with his experiments. Eq. 10*b* is not in accord with the Prandtl contention[54] that the shape of the pattern of velocity distribution near the boundary surface is independent of the diameter. For these various reasons the writer must reject Eq. 11, and can accept Eqs. 10*a* and 10*b* as applicable only to particular idealized surfaces with diameters determined volumetrically. The largest sand grains used in the smallest pipe in the Nikuradse experiments would be larger than 3-in. spherical paving stones evenly spaced in an 8-ft sewer. One might reflect in this connection that intermittent overbreak in a tunnel section increases its volume but reduces its effective sectional area.

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[54] "Applied Hydro- and Aeromechanics," by L. Prandtl and O. J. Tietjens, McGraw-Hill Book Co., Inc., New York, N. Y., 1934, p. 70.

To review the range of experimental verification, turn to Fig. 8. All points of that curve have the same velocity for full turbulence by virtue of the relation of Eqs. 3 and 4. The relative diameters for the five observed points however range from 1 to 13. Such a range, with f_1 established for a 12-in. pipe, would include a 1-in. pipe on the one hand and a 156-in. pipe on the other.

It would be of service here if one could know in advance how many readers would consider the foregoing lengthy discussion to be unacademic should they be informed that the divergence of these two methods would result in no difference at all in the quantity delivered by a 12-in. pipe for a given value of f_1, and less than 3% for a 4-in. pipe or a 72-in. pipe. It would seem inexcusable to thwart a logical analysis by injecting a questionable and irrational factor.

Had Mr. Powell's exponent 0.3 in his Eq. 8 been made $\frac{1}{3}$, and the constants adjusted accordingly, the graph would have followed almost the same curve. Then if one should choose to adjust the constants, in case it should be thought desirable for the Reynolds numbers "greater than 3,000,000," these new constants could be traced in all the related equations by inspection. It is understood, of course, that Mr. Powell did not intend to state that the graph in Fig. 15 is a plotting of the Nikuradse formula, but rather that it agrees with it in the middle range. Some confusion has grown out of referring to the von Kármán-Prandtl formula as that of Nikuradse after the introduction of his coefficients. The only Nikuradse formula mentioned in the paper was the Nikuradse exponential formula with the small exponent.

The writer is indebted to Messrs. Halsey and Owen for their fine data on a 2-in. pipe. They may be pardoned for failing to find an "abrupt change" to full turbulence, for two reasons: First, the change came at a higher Reynolds number than superficial examination might indicate, and, second, a slight transition similar to that in Fig. 4 relieves the abruptness.

The pattern of their data in Fig. 19 is not unusual. Most of the surface has remained reasonably smooth. The nature of the graph is similar to that of Fig. 13 in which the friction factors were augmented artificially by a partly closed valve. In fact, this present example illustrates perfectly the reason for inserting Fig. 13 in the paper with the comment, "* * * the surface loss remained substantially unchanged." Large imperfections such as pits, nodules, wavy surfaces, screw couplings, unreamed pipe ends, and other widely spaced irregularities have the same influence, since they cause eddy currents that do not, in their formative period, involve any considerable length of pipe. As long as these local disturbances result in losses that vary with the square of the velocity, the friction factors are increased an equal amount at all velocities. The same type of increase of f is indicated when the incrustation of scale has made the diameter smaller than that assumed. The exaggeration of any one of these localized defects, or a number of them combined, produces a parallel and elevated graph (parallel in natural numbers, not logarithms).

The data from Fig. 19 as given in Table 5 are transferred to Fig. 24 in which the normal graph is drawn and then elevated by the average amount that the observed points are found to be above the smooth pipe curve in the range where these distances are reasonably constant. The line is accordingly elevated 0.0097. The horizontal part of the elevated graph lies at $f = 0.0256$ as used by Messrs. Halsey and Owen. The only transition in this example lies in the vicinity of their intersection. It shows a maximum variation from the predicted f of less than 5%, which in turn is explainable as previously indicated.

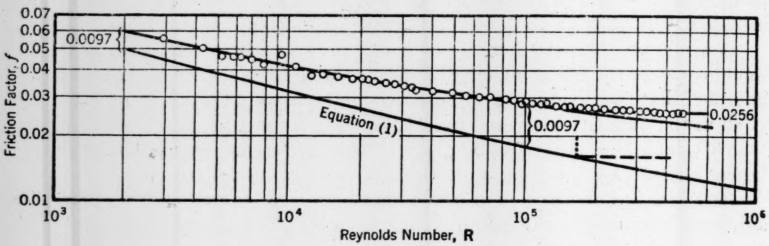

FIG. 24.—REPRODUCTION OF DATA FROM FIG. 19

Note that the graph of Fig. 24 would have been identically placed had there been but three points observed—two sufficiently spread in the partly turbulent range to show that the line was parallel to the smooth pipe curve, and one near the highest velocity which, because it is an additional elevation above the curve, indicates that full turbulence had been reached.

The existence of some localized influence is also indicated in the foregoing experiment because the graph remains parallel to the smooth pipe curve at such a low Reynolds number as 3,000. Without some induced turbulence, in addition to that normally caused by a uniform surface, the break to nonturbulence usually begins at a higher Reynolds number. The "oversimplification" in this case was in the practical interpretation of data, not in the rational procedure. A glance at Figs. 18 and 19 will serve to remind the reader that the Colebrook function (which is a mathematical method of constructing a curve tangent to a straight line and a curve, when fundamentally it should be tangent to neither) only confuses both diagrams. Some have the mistaken idea that "most flow problems in engineering practice occur in the transition region." This impression is due to an exaggerated concept of the length of the transition. One wrong impression grew out of the long Nikuradse "dip," the secondary deductions from which linger on although the results as a whole have long been abandoned as a pattern for commercial piping. In this problem it would require laboratory technique to disclose a transition.

There is no better indication of poor material or workmanship than that of a long transition or an elevated graph. The former indicates a wide variety of internal roughness either longitudinally or circumferentially; and the latter, some localized imperfection of measurable consequence. Extreme cases are chiefly valuable as indicating conditions to be avoided. The writer recalls one order of poorly galvanized small pipe containing two lengths through which no water could be forced. This incident was not the occasion for condemning a formula; instead the pipe itself was condemned.

The famous Freeman[15] experiments on iron pipes were seriously impaired by the fact that the surfaces were of widely varying roughness. For example, instead of finding a smaller friction factor for his 1½-in. pipe than for his 1¼-in. pipe, Mr. Freeman found it to be much larger. His experiments naturally revealed some long transitions. It is permissible to use a factor of safety if necessary to protect one against such extremes, but not a formula that would make those cases appear to be normal. Attempting to write each influence into a mathematical expression is accountable for a vast number of empirical formulas. Fundamental equations are the underlying strata on which these additional physical ideas rest. They should be preserved intact. The characteristics of materials deserve more attention in the light of a better understanding of their influences; and engineers are entitled to more refinement than the resort to an abnormal factor of safety applied to all kinds of pipes in order to cover the most extreme.

The request for additional data is a reasonable one. The demand for new experiments should continue. Some additional data, along with many of those from which the abstracts for the paper were taken, were presented during 1949.[55] These experiments confirmed but did not furnish or modify the coefficients in the various equations. They include another verification of that familiar transition caused by two degrees of roughness in a single length of pipe. The curve is (as it should be) virtually a straight line, logarithmically—not an asymptotic curve.

It is well to note that the form of Eq. 1 is not at all unusual. Eq. 24, presented with attending references by Mr. Jorissen, is of that form even to the extent of including the cube root of R. It is also interesting to note that subsequent to the presentation of the writer's paper at the Duluth (Minn.) convention of the ASCE, in July, 1947, an article[56] by Lewis F. Moody appeared in which not only the form was suggested but also the recommended constants were practically identical with Eq. 1.

Attempts have so often been made to solve all the problems of fluid motion from a single point of view. This kind of solution cannot be fully accomplished if any one of the physical assumptions is approximate, a condition likely to prevail for all predictable time. For example, the aeronautical engineer is concerned with velocity distribution as a common method of analyzing resistance coefficients and accepts approximations that are permissible for his purpose, where as the present, more direct, operation is accomplished without reference to any particular shape of the distribution and thus avoids the approximation.

Tests of full-scale commercial pipes, although resulting in a broad band of data indicative of experimental errors, and often of failure to respond to some presumed exponential pattern, furnished a reliable although not precise guide throughout a formative period. The laboratory with its better control, but limited scale, gave the necessary refinement of principle; but neither of these alone can fill the needs of the present economy. The theorist and the practical man should not plod smugly on their separate ways. Indeed, it would be better if the two indispensable qualities—theory and practice—were combined in all men who dare express a strong conviction.

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[55] *Bulletin No. 115*, Univ. of Washington Eng. Experiment Station, Seattle, Wash., 1949.

[56] "An Approximate Formula for Pipe Friction Factors," by Lewis F. Moody, *Mechanical Engineering*, December, 1947, p. 1005.

Procedure With or Without Localized Losses.—The simplicity of selecting a friction factor f for a given purpose may have become obscured by these rather long and technical discussions. To meet the common situation in which the flow may or may not prove to be fully turbulent, it must be understood that in this method, as in all other methods yet proposed, a test must be made, or must have been made, at a velocity high enough to determine the constant f for that surface for one or more diameters. For convenience and for comparison with other pipes, such values of f can be, or may have already been, reduced to a specific roughness coefficient f_1 (that is, made applicable to a 1-ft pipe) by Eq. 3, or its equivalent Fig. 17. By the same equation or diagram the correct f is found for the diameter under consideration. There is no additional constant for this point, since even if some small localized loss were present in the original experiment the effect would be extended to the case at hand with reasonable approximation. If by inspection, or by other methods, the flow is known to be fully turbulent, the study is now complete.

If the flow might conceivably be partly turbulent, a second friction factor is read directly from Fig. 15 for the proper velocity and diameter, or from the Reynolds number. To this value of f should be added whatever constant for localized losses the particular kind of pipe or piping has been found to require. Uniform surfaces need no such augmenting constant even if rough. If this second value of f is larger than the first, it is the correct one to use for the flow is only partly turbulent.

The influence of a possible transition normally becomes absorbed in a reasonable "factor of caution." Why should an engineer who is reconciled habitually to using a factor of safety of 2 (which means adding a factor of caution of 100%) object to a factor of 5%? In other words, if the diagram of Fig. 15 reads $f = 0.020$, one might use 0.021. This corresponds to an increased flow of $2\frac{1}{2}$%.

With a reasonable concept of the nature of flow, the engineer not only can solve his immediate problems with more reliability but also can view the deviations of old experimental data as indicating something to be avoided rather than followed. Specifications should include a strong clause relating to surface uniformity; and the search must be intensified for better and more permanent materials.

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Paper No. 2414

DEFLECTION OF PLYWOOD BEAMS DUE TO MOISTURE CONTENT CHANGE

By W. E. Wilson,[1] Assoc. M. ASCE, and Laurence G. Olson[2]

With Discussion by Messrs. M. W. Jackson, and W. E. Wilson and Laurence G. Olson.

Synopsis

This study was undertaken to determine reasons for the deformation of long floor panels of plywood box girder construction. Preliminary consideration revealed that the cause must be either temperature or moisture content changes. A theory was developed to show the relationship between deflection and change in temperature or moisture content for **I**-beams and **T**-beams. This development required knowledge of the relationship between length and moisture content of plywood panels. Experiments were conducted to establish this relationship for plywood of various thicknesses; and, to check the validity of the theory, experiments were also conducted to determine the deflection of beams as a function of moisture content. The results of these experiments indicate close agreement between theory and experiment limited, as may be expected, by the variation from piece to piece and from time to time of the physical characteristics of wood.

As a result of the analysis and experimental work it is concluded that, because plywood shows marked dimensional change with change in moisture content, its use in the construction of long box girders or **T**-beams requires the exercise of precautions to secure constant moisture content. It is further concluded that if proper moisture control is not feasible the length-depth ratios of the beams and girders must be made small enough to insure that, should moisture content change occur, the resulting change in the shape of the beam or girder would not be objectionable.

The analysis and experimental work indicate that it is feasible to calculate in advance the magnitude of the deflection to be expected in a girder or beam for a given change in moisture content. It is recommended that further experi-

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Note.—Published in April, 1949, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

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mental work be conducted to make possible the complete determination of the physical characteristics of the plywood in so far as the relationship between moisture content and dimensions is concerned.

Introduction

The construction experience which led to this study of the effect of moisture content changes in plywood beams on the deflection of that type of member is based on fabrication of plywood box girders, by two different methods, for prefabricated houses. Panels of one type, which are used in walls, are 8 ft long, 3 in. deep, and approximately 3 ft wide. These are glued by high frequency electric heating and contain nails for tacking purposes only. The second type of girder, which is 6 in. deep, 3 ft wide, and 23 ft long, is used for both floor and roof construction and is manufactured by gluing and nailing the plywood to the webs. Experience with the second type of girder led directly to the analysis and experimental work described herein.

These box girders, consisting of ½-in. top skin, four ¾-in. by 5½-in. web members, and ¼-in. bottom skin, had been constructed and were found to undergo a warping action sufficient to produce as much as a 1-in. to 1½-in. deflection from their original straight form. Deflections of this magnitude were found in storage, and after the panels were installed in houses there occurred deflection changes that produced highly undesirable results including breaking of the asphalt tile which covered them.

This investigation had as its primary purpose the determination of the cause of the distortion of the plywood box girders previously described. To attain this objective it was necessary to carry out several studies having the following individual objectives:

(1) To determine the relationship between the change in the length of a piece of plywood and the change in its moisture content;

(2) To determine the theoretical relationship between the deflection of a beam or girder and the moisture content of its plywood flanges;

(3) To determine experimentally the relationship between deflection of a beam or girder and the moisture content of its plywood flanges;

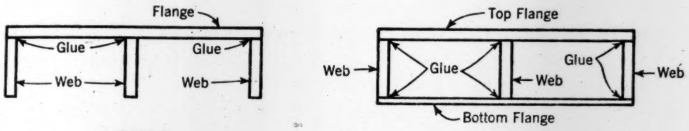
(4) To compare theoretical results from the second study with experimental results from the third study, noting any discrepancies;

(5) To determine limitations on the size of plywood box girders imposed by the effect of moisture content on the beams or girders; and

(6) To determine if the deformations due to moisture content change might be controlled or if the changes in moisture content might be controlled.

Theory

To maintain as great simplicity as possible in developing the theory of the T-beam and the I-beam, which consist of solid timber webs and plywood flanges, certain simplifying assumptions are made. Because of algebraic complications, the I-beam theory yields much more complex equations than does the T-beam theory. It was deemed advisable, therefore, to simplify the I-beam theory further. As a result of this situation, the T-beam theory is developed much more fully than is the I-beam theory, and the experimental check on the T-beam theory is more complete.

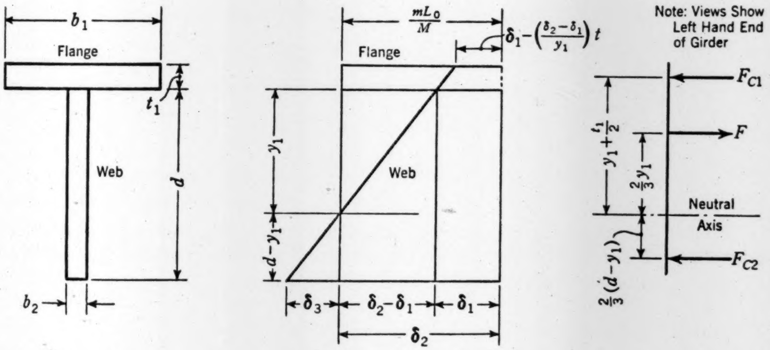

Flange
Glue
Glue
Web
Web
Top Flange
Web
Glue
Glue
Web
Web
Bottom Flange

(a) T-BEAM (b) I-BEAM

FIG. 1.—TYPES OF PLYWOOD BEAMS INVESTIGATED

For purposes of this paper a T-beam is considered to be a structural member consisting of solid timber webs glued to a plywood flange as indicated in Fig. 1(*a*). An I-beam differs from the T-beam in that it has both a top and a bottom flange (Fig. 1(*b*)).

T-*Beams*.—In developing the theory of the deformation of a T-beam due to change in moisture content the notation of Fig. 2 is employed. The following assumptions are made:


b_1
Flange
t_1
Web
d
b_2
$\frac{mL_0}{M}$
$\delta_1-\left(\frac{\delta_2-\delta_1}{y_1}\right)t$
Flange
y_1
Web
$d-y_1$
δ_3
$\delta_2-\delta_1$
δ_1
δ_2
Note: Views Show Left Hand End of Girder
F_{C1}
$y_1+\frac{t_1}{2}$
F
$\frac{2}{3}y_1$
Neutral Axis
$\frac{2}{3}(d-y_1)$
F_{C2}

(a) CROSS SECTION (b) LONGITUDINAL DEFORMATION (c) LONGITUDINAL FORCES

FIG. 2.—T-BEAM DIAGRAMS

1. The longitudinal deformation of a piece of plywood due to moisture content change is given by the expression:

$$L = L_0\left(1 - \frac{m}{M}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

in which L is the length of the plywood at moisture content corresponding to m; L_0 is the length of the plywood at zero moisture content; m is the moisture content expressed as the ratio of the weight of moisture to the oven dry weight of the plywood; and M is a dimensionless coefficient experimentally determined as explained subsequently;

2. In bending, plane sections remain plane;
3. The deflection of the beam is small compared to its length; and
4. The plywood and the lumber are elastic, obeying Hooke's law.

Considering a beam which has been deformed, as shown in Fig. 2, because of an increase in moisture content of the flange, the first assumption and the geometry of the bent beam yield the relationship:

$$\frac{m\,L_0}{M} = \delta_2 \quad \text{(2)}$$

in which δ_2 represents the total elongation of the unrestrained flange due to moisture content. The second assumption yields the equation:

$$\frac{\delta_2 - \delta_1}{y_1} = \frac{\delta_3}{d - y_1} \quad \text{(3)}$$

in which δ_1 is the shortening of the flange due to the compression induced by the resisting action of the adjacent dry lumber; δ_3 is the elongation of the extreme fibers of the web; and y_1 is defined in Fig. 2(*b*). The fourth assumption yields the expressions:

$$F_{c1} = \left[\delta_1 - \frac{(\delta_2 - \delta_1)\,t_1}{2\,y_1}\right]\frac{E\,b\,t_1}{L_0} \quad \text{(4)}$$

$$F_{c2} = \frac{\delta_3}{L_0}\,\frac{E\,b_2}{2}(d - y_1) \quad \text{(5)}$$

and

$$F_T = \frac{\delta_2 - \delta_1}{L_0}\,\frac{E\,b_2}{2}\,y_1 \quad \text{(6)}$$

in which F_{c1} is the compressive force in the flange; F_{c2} is the compressive force in the web; F_T is the tensile force in the web; and E is the modulus of elasticity.

The conditions of static equilibrium require that

$$F_{c1} + F_{c2} - F_T = 0 \quad \text{(7)}$$

and

$$F_{c1}\left(y_1 + \frac{t_1}{2}\right) - \frac{2\,F_T\,y_1}{3} - \frac{2\,F_{c2}}{3}(d - y_1) = 0 \quad \text{(8)}$$

Eqs. 2 to 8, inclusive, are sufficient for the determination of the relationship between the seven dependent variables δ_1, δ_2, δ_3, y_1, F_{c1}, F_{c2}, and F_T in terms of the eight independent variables E, t_1, b_1, b_2, d, m, M, and L_0.

The geometry of the deflected beam (Fig. 3) yields the following expression for the maximum deflection D in terms of the elongation of the fibers and the length of beam and the location of the neutral axis:

$$D = \frac{d_2\,L_0}{8\,y_1} \quad \text{(9)}$$

Solution of Eqs. 2 to 9, inclusive, yields

$$D = \frac{m\,L^2_0}{8\,M\,d}\left[\frac{3\,\frac{t_1}{d}\left(1 + 2\,\frac{t_1}{d}\right)}{6\left(\frac{t_1}{d}\right)^3 + 9\left(\frac{t_1}{d}\right)^2 + 4\,\frac{t_1}{d} + \frac{b_2}{b_1}}\right] \qquad (10a)$$

or

$$\frac{8\,M\,d}{L^2_0}\,\frac{D}{m} = \frac{3\,\frac{t_1}{d}\left(1 + 2\,\frac{t_1}{d}\right)}{6\left(\frac{t_1}{d}\right)^3 + 9\left(\frac{t_1}{d}\right)^2 + 4\,\frac{t_1}{d} + \frac{b_2}{b_1}} \qquad (10b)$$

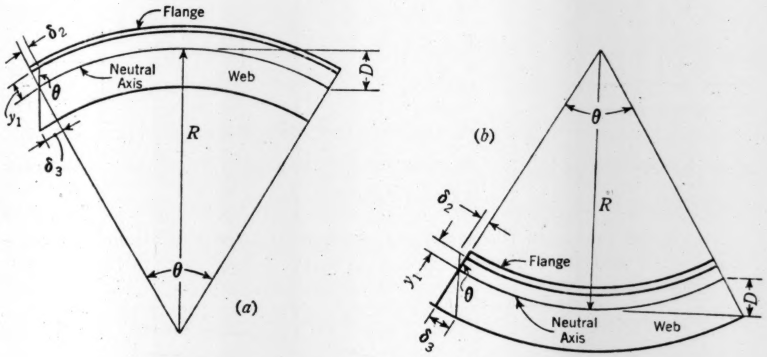


FIG. 3.—DEFLECTION OF T-BEAM

—which is represented graphically in Fig. 4, where the left-hand side appears as the ordinate; and the ratio t_1/d, as the abscissa.

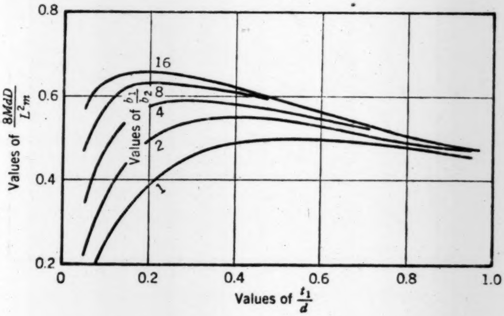
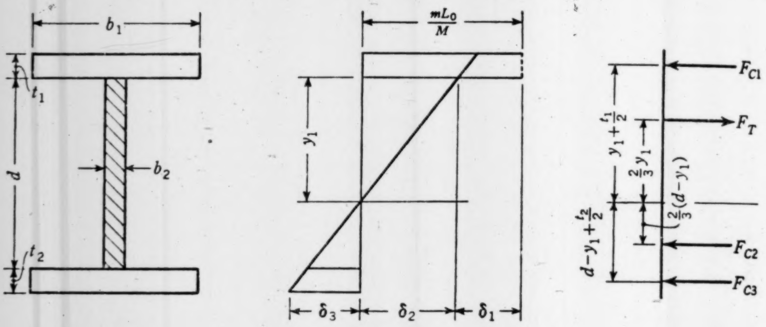


FIG. 4.—THEORETICAL DEFLECTION OF T-BEAMS

I-*Beams*.—The analysis of the mechanics of the deflection of I-beams is similar to that for T-beams and starts from the diagrammatic representation of a beam of this type (Fig. 5). The assumptions made in the analysis for an **I**-beam differ from those for a **T**-beam only in one simplifying point related to the second assumption. It is assumed that plane sections between flanges remain plane, but that the deformation of the flanges is constant throughout the flange depth, which is small compared to the beam depth.


b_1
t_1
d
b_2
t_2
$\frac{mL_0}{M}$
y_1
δ_3
δ_2
δ_1
F_{C1}
F_T
F_{C2}
F_{C3}
$y_1+\frac{t_1}{2}$
$\frac{2}{3}y_1$
$d-y_1+\frac{t_2}{2}$
$\frac{2}{3}(d-y_1)$

(*a*) CROSS SECTION (*b*) LONGITUDINAL DEFORMATION (*c*) LONGITUDINAL FORCES

FIG. 5.—I-BEAM DIAGRAMS

As a consequence of the assumptions made and the conditions of equilibrium, Eqs. 2 and 3 are directly applicable to the **I**-beam, as are Eqs. 5 and 6.

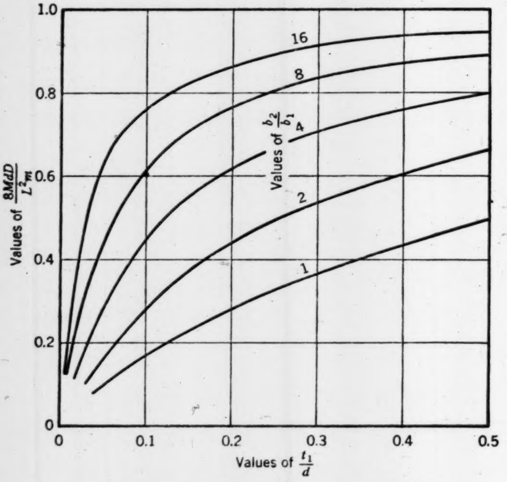


FIG. 6.—THEORETICAL DEFLECTION OF I-BEAMS

In addition, Eq. 4 is replaced by

$$F_{c1} = \frac{\delta_1 E b_1 t_1}{L_0} \quad \text{(11)}$$

and an equation is required to describe the compressive force in the lower flange:

$$F_{c3} = \frac{\delta_3}{L_0} E\, b_1\, t_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

Also the conditions of equilibrium require that

$$F_{c1} + F_{c2} + F_{c3} - F_t = 0 \ldots\ldots\ldots\ldots\ldots\ldots (13)$$

and

$$F_{c1}\, y_1 - 2\, F_t \frac{y_1}{3} - \tfrac{2}{3} F_{c2} (d - y_1) - F_{c3} (d - y_1) = 0 \ldots\ldots (14)$$

Solution of Eqs. 2, 3, 5, 6, 11, 12, 13, and 14, in conjunction with Eq. 9, yields

$$\frac{m\, L^2_0}{8\, M\, d} \frac{7}{6\left(1 + \frac{5}{12} \frac{b_2\, d}{b_1\, t_1}\right)} > D > \frac{m\, L^2_0}{8\, M\, d} \frac{1}{1 + \frac{b_2\, d}{2\, b_1\, t_1}} \ldots\ldots\ldots (15)$$

Because of the complexities of the algebraic relationships it is not feasible to obtain an exact expression for D which is workable. In Fig. 6, Eq. 15 is illustrated graphically to show the minimum value of D.

EXPERIMENTATION

The experimental work consisted of three parts—namely:

1. Determination of the relationship between moisture content and the length of strips of plywood which were approximately 3 in. wide, 8 ft long, and of various thicknesses;
2. Determination of the deflection of I-beams as the moisture content of one or both flanges changed; and
3. Determination of the deflection of T-beams as the moisture content of the flange changed.

The techniques employed in these programs and the data obtained are presented herewith.

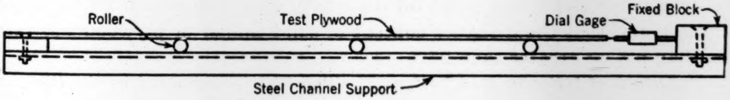


FIG. 7.—PLYWOOD TESTING APPARATUS

Shrinkage with Change in Moisture Content.—The change in the length of a piece of plywood caused by changes in moisture content was determined with the equipment illustrated in Fig. 7. A strip of plywood 3 in. wide and approximately 96 in. long was bolted to a steel channel as shown. Rollers were placed under the plywood to insure freedom of movement. A block was bolted to the channel to serve as a measuring point, and a dial gage was mounted to contact this block and a steel pin which had been driven into the plywood specimen.

The plywood, which had been prepared by soaking in water for a period of several days, was mounted in the jig and a determination of the moisture content was made with an electrical resistance type moisture meter. At regular periods, measurements of moisture content at from ten to twelve points in the plywood specimen and of the shortening or the lengthening of the piece were made.

Three thicknesses of plywood were used in the study—¼ in., ⅜ in., and ½ in. The ¼-in. and ⅜-in. thicknesses were of exterior grade; the ½-in. thickness was moisture resistant. No data were taken on other sizes since the sizes mentioned were used in the construction under study at the time of the experimentation.

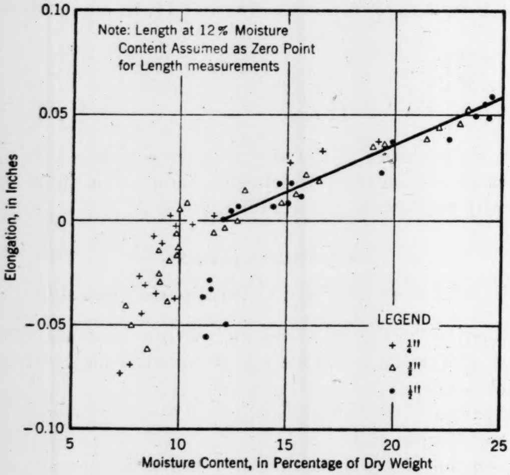


FIG. 8.—EXPERIMENTAL PLYWOOD ELONGATION DATA FOR VARIOUS THICKNESSES

The results of these measurements are shown graphically in Fig. 8, in which the length of the plywood specimen at 12% moisture content has been arbitrarily selected as the zero point for the length measurements of the specimens. It was found that the moisture content of solid timber or plywood was rarely less than 6% or greater than 25%.

Deflection of I-*Beams and* T-*Beams.*—The test beams used in the beam deflection experiments were constructed as follows: The I-beams had dry (6% to 8% moisture content) 1-in. by 6-in. solid timber webs, with flanges of plywood, ¼ in., ⅜ in., and ½ in. thick, in 12-in. widths. The plywood flanges varied in moisture content, which covered two general ranges—the dry range (6% to 10% moisture content), and the wet range (20% to 25% moisture content). Six combinations of the dry webs and the wet and the dry flanges of different thicknesses were used. These beams were cold glued and nailed at 6-in. centers throughout the entire 8-ft length of the beams. The T-beams were constructed in a manner similar to that used for the I-beams, except that the depth of the web varied from 1 in. to 6 in. and the width of the flanges varied from 3 in. to 12 in. Nine combinations of the webs and the flanges were used in the **T**-beam construction.

The deflection of the **I**-beams from their original shape was determined with the apparatus illustrated in Fig. 9, the distance from the surface of the steel channel to the bottom of the beam being measured at 12-in. intervals along the length of the beam. The shape of the steel channel surface was determined by the dial gage and the surface of a body of water at rest. The curve of the bottom of the **I**-beam or the **T**-beam was determined by calculating the distance from the bottom of the beam to a plane surface by reference to the measured distances to the steel channel surface.

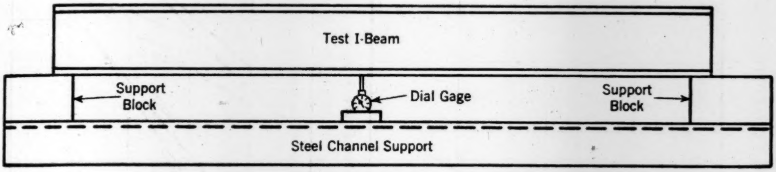


Fig. 9.—Beam Testing Apparatus

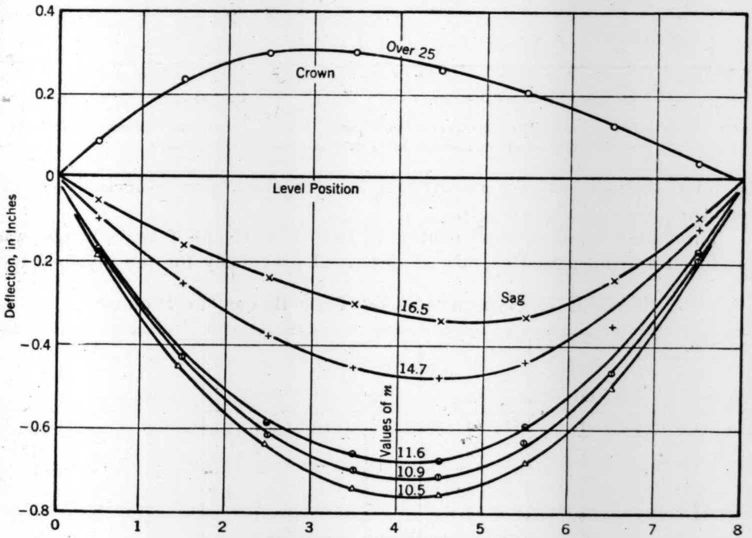


Fig. 10.—Deflection Curves for Test Beam 4 Under Various Moisture Content Conditions

A typical series of deflection curves for a beam is shown in Fig. 10. The maximum deflection *D*, scaled from curves such as these, is shown graphically for **I**-beams and **T**-beams in Fig. 11 as a function of the moisture content of the test flanges, the physical dimensions of which are given in Table 1. It will be noted that all beams were 8 ft long, a convenient length, since plywood is readily available in this length.

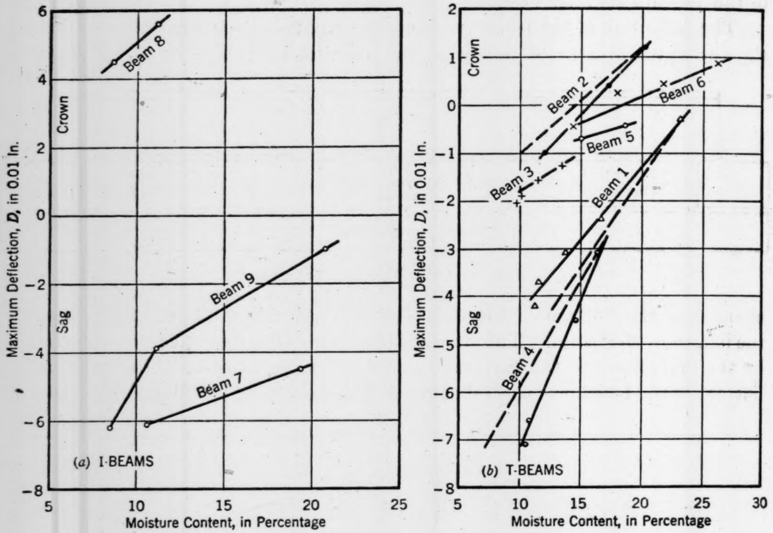


FIG. 11.—MAXIMUM DEFLECTION OF TEST BEAMS UNDER VARIOUS MOISTURE CONTENT CONDITIONS (Solid Lines, "Experimental"; Broken Lines, "Theoretical")

The variation of moisture content of both **I**-beam and **T**-beam flanges was produced and measured exactly as described previously for the plywood test specimens using one flange of an **I**-beam and the only flange of a **T**-beam as the test flange where the moisture content varies. The moisture content of the beams decreased at such a rate that successful measurements on the specimens

TABLE 1.—DIMENSIONS OF TEST BEAMS, IN INCHES[a]

Specimen No. (1)	Type (2)	b_1 (3)	b_2 (4)	t_1 (5)	t_2 (6)	d (7)
1	T	12	$\frac{3}{4}$	$\frac{1}{2}$	..	$\frac{3}{4}$
2	T	12	$\frac{3}{4}$	$\frac{1}{2}$	..	$1\frac{1}{2}$
3	T	12	$\frac{3}{4}$	$\frac{1}{2}$	..	$3\frac{1}{2}$
4	T	12	$\frac{3}{4}$	$\frac{1}{4}$	..	$\frac{3}{4}$
5	T	3	$\frac{3}{4}$	$\frac{1}{2}$	..	$3\frac{1}{2}$
6	T	6	$\frac{3}{4}$	$\frac{1}{2}$	..	$3\frac{1}{2}$
7	I	12	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$5\frac{1}{4}$
8	I	12	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$5\frac{1}{4}$
9	I	12	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$5\frac{1}{4}$

[a] All beams were 96 in. long.

easily could be obtained by making observations at intervals of from 12 hours to 24 hours.

ANALYSIS OF DATA

To compare experimental results with theoretical predictions, it was necessay to determine the slope of the curves shown in Figs. 10 and 11(*a*) and the value of the coefficient M. Reference to Eqs. 11 and 15 indicates that the slope of a curve showing deflection as a function of flange moisture content should be a constant for any given beam, being a function of t_1/d, b_2/b_1, M, and d only.

Besides its dimensions, the value of the coefficient M is the only physical characteristic of the plywood that enters into consideration. The value of M is the slope of the curve shown in Fig. 8 multiplied by the length of the specimen, or the ratio of the elongation per unit change in moisture content to specimen length. Since the curve is not a single straight line, M is not a constant. However, in the range above a moisture content of 12% a straight line fits the experimental points reasonably well. The value of M corresponding to the solid straight line shown in this region is 225.

Using the value of M and the physical dimensions of the beams, the theoretical deflection D has been computed from Eq. 10. The data shown in Fig. 11(*a*) represent I-beams whose deflection was due to moisture changes in one flange only. Although it was true that the moisture content of the other flanges varied a small amount, the variation was so small that it might be neglected. The measured and calculated values are compared in Fig. 11(*b*). The broken lines were calculated using Eq. 10.

It will be noted that the experimental points are located close to the theoretical lines, whereas strict agreement between theory and experiment would require that the two lines coincide. It is possible that the explanation lies in a value of M different from that determined from the experimental results shown in Fig. 8.

In most cases the greater value of the measured deflection might well indicate a smaller value of M for the plywood used in the beams as compared with that used in the shrinkage experiments.

CONCLUSIONS

On the basis of the experimental and analytical work previously described, the following conclusions appear to be justified:

1. Although plywood in general shows dimensional stability, the elongation and the shrinkage which accompany moisture content change are sufficient in magnitude to cause significant deflections in both I-beams and T-beams.
2. The effect of moisture change can be minimized by designing short, deep beams rather than long, shallow ones.
3. Control to prevent changes in the moisture content of the flanges of a beam does not appear to be feasible.
4. Design of girders and beams of plywood box girder type must take into consideration the maximum probable change in moisture content of the plywood and the permissible deflection of the beams.

DISCUSSION

M. W. Jackson,[3] Jun. ASCE.—An excellent job has been done by the authors in studying the effect of moisture change on the deflection of plywood beams. The experimental data presented in Fig. 8 may be used in calculations for the dimensional change in plywood due to moisture.

Since specific data about the plywood used were not given, it will be assumed to be the commonly used five-ply Douglas fir. For Douglas fir wood, longitudinal shrinkage is about 0.1%, and values for radial and tangential shrinkage are 5.0 and 7.8, respectively, from green (36% moisture) to oven dry condition.[4] The average shrinkage of cross plies in which the grain is variable could be assumed as about 6.4%. The wood does not begin to shrink from the green condition but remains unchanged in dimension until the fiber saturation point has been reached in drying—at about 25% moisture.[5] Consequently, for a 1% change in moisture, the shrinkage across the plies would be 6.4/25 = 0.255%, or about 0.0025 in. per in. per 1% moisture change.

Internal stresses are set up in the plywood by moisture changes because of the tendency of the cross plies to shrink, while the length of the parallel plies tends to remain practically unchanged. Unrestrained, the cross plies would shrink about 2½ in. in an 8-ft panel for a moisture decrease of 10%. An examination of these internal stresses across any given section in a panel indicates that the induced load on the cross plies, P_c, must be equal to the load on the parallel plies, P_p; that is, $P_c = P_p$. Consequently,

$$P_c = A_c\, s_c \quad \text{(16a)}$$

and

$$P_p = A_p\, s_p \quad \text{(16b)}$$

in which A and s are the areas and stresses in the respective plies. Since $A_c = 0.240$ sq in. and $A_p = 0.360$ sq in. per in. of width in a normal panel, then $s_p = \frac{2}{3}\, s_c$. Expressing the stress on the section in terms of the unit deformation, ϵ, and the modulus of elasticity, E,

$$s_c = E_c\, \epsilon_c \quad \text{(17a)}$$

$$s_p = E_p\, \epsilon_c \quad \text{(17b)}$$

and

$$E_p\, \epsilon_p = \frac{2}{3}\, E_c\, \epsilon_c \quad \text{(17c)}$$

The modulus of elasticity for parallel plies is 1,600,000 lb per sq in.; but, across the grain, it is only 80,000 lb per sq in. or less, depending on the direction of grain. If ϵ is the unit deformation of the parallel plies produced by shrinkage of the cross plies, the unit deformation of the cross plies is $0.00255 - \epsilon$. Evaluating, 1,600,000 $\epsilon = \frac{2}{3}$ (0.00255 − ϵ) 80,000, and ϵ = 0.000082 in. per in. per 1% moisture change. The total deformation for an 8-ft panel for 10% moisture change is 0.079 in. This result almost doubles the values from the experiments given in Fig. 8. It should be noted that, although the value for E across the grain is usually given as 80,000 lb per sq in., it may vary from 12,000 lb per sq in. to 76,000 lb per sq in., depending on the direction of grain across the cross section.[6] If an average value of 44,000 lb per sq in. is used for E, then the total deformation becomes 0.044 in., in accordance with the actual test data shown in Fig. 8.

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[3] Asst. Prof., Civ. Eng. School, Georgia Inst. of Technology, Atlanta, Ga.

[4] "Strength and Related Properties of Woods Grown in the United States," by L. J. Markwardt and T. R. C. Wilson, *Technical Bulletin No. 479*, U.S.D.A., Washington, D. C., 1935.

[5] "A Primer of Chemical Seasoning," by D. F. Hill and A. L. Mottet, West Coast Lumbermen's Assn., Seattle, Wash., 1945.

This theory does not explain the erratic results for the lower moisture contents. Possibly they may be caused by the facts that the effect of moisture content on the modulus of elasticity is little understood and that there is no experimental assurance that E is constant for all moisture contents. At the lower moisture contents possibly the moisture was not distributed uniformly through the specimen, and also possibly variations in humidity may have introduced discrepancies.

The internal stresses in the plywood on the basis of the foregoing analysis are of interest. In the parallel plies, for a change of moisture content of 10%, $s_p = E_p \epsilon_p 10 = 1{,}600{,}000 \times 0.000046 \times 10 = 740$ lb per sq in.; and, for the cross plies, s_c = 1,110 lb per sq in. These internal stresses are of considerable magnitude. The stress in the cross plies is apparently roughly equal to the ultimate strength for standard tests of small clear specimens in tension and compression across the grain, but the effect of size in transferring stresses from such standard tests to the condition in the plies is unknown. Since the modulus of elasticity in direct tension and the modulus of elasticity in direct compression for wood are not equal and since some permanent deformation probably results from each cycle of moisture change, plywood will eventually deteriorate under repeated changes in moisture content.

It would be interesting to know the effect of permanent loads on specimens like those tested, since the effective modulus of elasticity decreases with time. In Fig. 11(*b*) it is difficult to discover which set of points is based on the full thickness and which set is based on the effective thickness mentioned under the heading, "Analysis of Data."

The symbol t_2, for thickness of the lower flange introduced in Eq. 12, does not appear in Eq. 15. Since the investigation was prompted by the warping of box girders with unequal flanges, it would be of interest to compare the effects when using unequal and equal flanges.

W. E. WILSON,[7] ASSOC. M. ASCE, AND LAURENCE G. OLSON.[8]—Mr. Jackson presents an interesting discussion of the internal stresses in a piece of plywood subject to varying moisture content. He also raises a very pertinent question concerning the interpretation of Fig. 11(*b*). In Fig. 12 the data in Fig. 11(*b*) are analyzed by plotting the measured slope of the curve of Fig. 11(*b*) against the calculated slope of the theoretical curve (also shown in Fig. 11(*b*)) for beams

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[6] "Wood Handbook," Forest Products Laboratory, U.S.D.A., Washington, D. C., 1940.

[7] Pres., State School of Mines and Technology, Rapid City, S. Dak.

[8] Asst. Research Engr., Sundstrand Machine Tool Co., Rockford, Ill.

1, 2, and 4. The broken line in Fig. 12 fits the experimental points fairly well. If theory and experiment agreed precisely, the solid line in Fig. 12 would pass through the experimentally determined points.

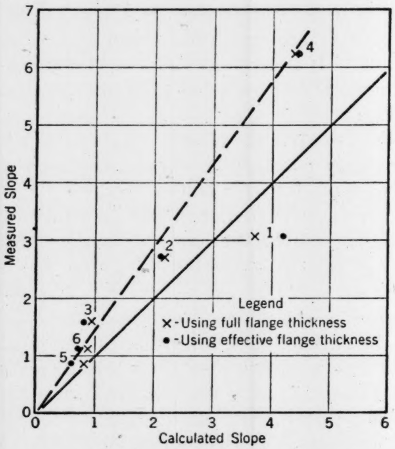

FIG. 12.—COMPARISON OF MEASURED AND COMPUTED SLOPES $\frac{\Delta D}{\Delta m}$, OF THE MAXIMUM DEFLECTION CURVE

The value of M given (under the heading, "Analysis of Data") is the slope $\frac{\Delta m}{\Delta L}$ of the solid line in Fig. 8, multiplied by the length of the specimen.

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CONTROL OF THE HYDRAULIC JUMP BY SILLS

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With Discussion by Messrs. En-Yun Hsu, Steponas Kolupaila, Charles Jaeger, Louis M. Laushey, Michele Viparelli, Ralph M. Weaver, Sherman M. Woodward and Floyce M. Yates, and John W. Forster and Raymond A. Skrinde.

Synopsis

The hydraulic jump is a phenomenon well known to hydraulic engineers as a useful method of dissipating excess energy and thereby preventing erosion below overflow spillways, chutes, and sluices. Violent turbulence inherent in the jump has also resulted in its use in mixing and aeration processes. The value of relevant data in the design of such processes has initiated many experimental investigations, so that the dimensions and characteristics of a hydraulic jump in a rectangular channel may be predicted within practical limits. Accordingly, the main difficulty in designing any structure whose functioning is dependent on the formation of a hydraulic jump lies in determining how to insure the formation of the jump and control its position, rather than in predicting its dimensions.

When the normal tailwater depth is such that supercritical flow tends to continue downstream, formation of a jump may be effected by sills placed on the channel bottom. Unless such a sill is properly designed, however, a true hydraulic jump will not form, and the desired results will not be realized. For example, a drowned jump may occur, within which the high-velocity flow does not diverge satisfactorily; the flow may simply pass over the sill as a standing wave or swell; or it may impinge upon the sill and be deflected forward and upward in a sheet or spray. In these cases the aeration or the energy dissipation, or both, may be negligible compared to that occurring in a true hydraulic jump.

The object of the study described herein was to investigate the performance of such sills in level rectangular channels, and to attempt to formulate a general method of design. The experimental results are shown to compare favorably

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Note.—Published in April, 1949, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

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with a theoretical analysis, and data of general application are presented dimensionlessly.

Two types of sills were investigated: First, a vertical nonaerated weir with a sharp crest, and, second, an abrupt rise in the channel bottom. Charts have been prepared which may be used to determine sill dimensions for preliminary design purposes and as a guide in analyzing or predicting the effect of such sills under given conditions.

THEORETICAL DEVELOPMENT

The theory of the hydraulic jump has been adequately discussed elsewhere,[3] so that a full review here is considered unnecessary. However, a brief review of the main relationships is given as a basis for the development which follows.

In the case of water forming a hydraulic jump in a horizontal flume, the ratio of unit inertial reaction to unit gravity force is a criterion for dynamic similarity. Such a ratio is comprised in the Froude number:

$$\mathbf{F} = \frac{V}{\sqrt{g\,d}} \qquad \text{(1)}$$

—in which V and d are, respectively, the mean velocity and the depth at any section, and g is the acceleration due to gravity—so that geometrically similar cases of such flow in open channels will be dynamically similar (that is, velocities and pressures will be similarly distributed) when the Froude numbers for the two cases are the same.

Other forms of this dimensionless parameter for dynamic similarity have been used; for example,[4] the "kinetic flow factor" $\frac{V^2}{g\,d}$, or the ratio of the velocity head to the depth of flow $\frac{V^2}{2\,g\,d}$. The particular form used for open-channel flow is a matter of individual preference. Owing to its general significance, the Froude number has been used in this investigation. For the condition of critical flow, $\mathbf{F} = 1$; for tranquil or subcritical flow, $\mathbf{F} < 1$; and for shooting or supercritical flow, $\mathbf{F} > 1$.

The conditions of the jump phenomenon are such that the momentum principle may be conveniently used in analyzing the problem. As a first approximation, the assumption is made of uniform velocity distribution before and after the jump. Considering the free body bounded by the end sections, the level floor, and the water surface (Fig. 1(*a*)), the only external force effective in changing the momentum is the pressure acting at each of the end sections, where the pressure distribution is hydrostatic. Considering unit width in a rectangular horizontal channel, the net force acting in the direction of flow is equated to the increase in momentum flux between the sections; thus:

$$\frac{\gamma\, d^2_1}{2} - \frac{\gamma\, d^2_2}{2} = \rho\, q\, (V_2 - V_1) \qquad \text{(2)}$$

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[3] "The Hydraulic Jump in Terms of Dynamic Similarity," by Boris A. Bakhmeteff and Arthur E. Matzke, *Transactions*, ASCE, Vol. 101, 1936, pp. 630–647.

[4] "The Hydraulic Jump in Sloping Channels," by Carl E. Kindsvater, *ibid.*, Vol. 109, 1944, p. 1109

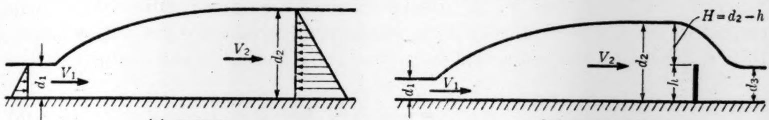
in which γ is the specific weight of the liquid; ρ is the density of the liquid; and q is the rate of flow per unit width. The equation of continuity may be written

$$q = V_1 d_1 = V_2 d_2 \ldots\ldots\ldots\ldots (3)$$

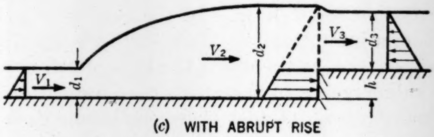
Combining Eqs. 2 and 3 to eliminate q and V_2, and introducing the value of **F** for the approaching flow, leads to the dimensionless relationship:

$$\frac{d_2}{d_1} = \tfrac{1}{2}\left(\sqrt{1 + 8\,\mathbf{F}^2} - 1\right) \ldots\ldots\ldots\ldots (4)$$

For supercritical flow on a level floor, frictional resistance dissipates the energy of the fluid, effecting a decrease in velocity and an increase in depth in the direction of flow. The water surface profile is an H3-curve.[5] At any point along the profile the flow conditions are characterized by a particular value of **F**. A hydraulic jump will form where there occurs a downstream depth d_2 that satisfies Eq. 4. If the downstream depth is then decreased, the jump will recede downstream to a point where Eq. 4 is again satisfied. Similarly, if the downstream depth is increased, the jump will be forced upstream and may be drowned at the source (sluice gate, overflow spillway, or similar device).



(a) SIMPLE (b) WITH BAFFLE



(c) WITH ABRUPT RISE

FIG. 1.—DEPTH, VELOCITY, AND PRESSURE RELATIONS IN THE HYDRAULIC JUMP

If the downstream depth is lower than that necessary for the formation of a jump, a sharp-crested baffle or an abrupt rise in the channel bottom may be used to increase it. In either case, an analysis of the flow conditions may be made similar to that for the hydraulic jump, in that only the equations expressing the principles of continuity and momentum are necessary for a solution.

Dimensional Considerations.—In the hydraulic jump phenomenon just considered, the sequent depth d_2 is primarily dependent on the depth d_1 and the velocity V_1 of the supercritical flow, and the value of g which determines the ratio of γ to ρ. Thus:

$$d_2 = f_1(V_1, d_1, g) \ldots\ldots\ldots\ldots (5)$$

Dimensional analysis then leads to the conclusion that d_2/d_1 is expressible as a function of $\frac{V_1}{\sqrt{g\,d_1}}$, or **F**, so that

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[5] "The Hydraulics of Steady Flow in Open Channels," by Sherman M. Woodward and Chesley J. Posey, John Wiley & Sons, Inc., New York and London, 1941, p. 64.

$$\frac{d_2}{d_1} = \phi_1(\mathbf{F}) \qquad (6)$$

—which functional relationship was previously developed (see Eq. 4).

Similarly, if h is the height of the sill introduced into the channel to effect the formation of a jump (see Figs. 1(*b*) and 1(*c*)) then, in general, h will depend as before on V_1, d_1, and g, and also on the normal downstream or tailwater depth d_3 and the distance X from the toe of the jump to the sill. Thus:

$$h = f_2(V_1, d_1, g, d_3, X) \qquad (7)$$

Dimensional analysis then leads to the conclusion that h/d_1 is expressible as a function of **F**, d_3/d_1, and X/d_1. However, since the exact position of the jump cannot be treated theoretically X/d_1 will be considered constant, with a magnitude sufficient to insure a complete jump. This simplifies the analysis so that

$$\frac{h}{d_1} = \phi_2\left(\mathbf{F}, \frac{d_3}{d_1}\right) \qquad (8)$$

Analysis for a Sharp-Crested Weir.—In the case of a weir being used to effect the formation of a jump (see Fig. 1(*b*)), further simplification is possible, due to the fact that, as long as the downstream depth d_3 is less than that required to produce submergence of the crest, the flow pattern upstream from the weir is independent of d_3. Under these conditions d_3 is eliminated as a variable, so that for any given position of the jump relative to the weir, h/d_1 should be a function of **F** alone, or

$$\frac{h}{d_1} = \phi_3(\mathbf{F}) \qquad (9)$$

To develop this relationship, first the momentum equation (Eq. 2) for the hydraulic jump is written as before. Then to the continuity equation for the previous case (Eq. 3) must be added a term for the discharge over the weir. For a sharp-crested, aerated weir with negligible velocity of approach, the theoretical discharge equation is

$$q = \tfrac{2}{3}\sqrt{2\,g}\,H^{\frac{3}{2}} \qquad (10)$$

However, this relationship is not expected to apply in the case of a low nonaerated weir in a rectangular channel. For want of something better, empirical coefficients may be applied to the theoretical expression in order to obtain a first approximation. The continuity equation then becomes

$$q = V_1\,d_1 = V_2\,d_2 = C_1\,C_2\,C_3\left(\tfrac{2}{3}\sqrt{2\,g}\,H^{\frac{3}{2}}\right) \qquad (11)$$

in which C_1, C_2, and C_3 are coefficients defined as follows:

$C_1 = 0.61$, which is the limiting value of the Bazin suppressed weir coefficient for no velocity of approach as H approaches infinity. It applies provided the normal downstream depth will not affect the discharge over the weir[6]—that is, $d_3 < (d_2 - \frac{3}{4}h)$.

$$C_2 = 1 + 0.55 \left(\frac{H}{d_2}\right)^2 \dots\dots\dots\dots\dots\dots\dots (12a)$$

which is Bazin's weir coefficient to provide for the effect of approach velocity.[7]

$$C_3 = 0.878 + 0.128 \frac{h}{H} \dots\dots\dots\dots\dots\dots\dots (12b)$$

which is Forchheimer's coefficient[6] to correct for nonaeration of the nappe and is valid for $H > 0.4\,h$. (This condition was found to obtain for all experimental runs during the investigation.)

Introducing these values and expressions for the coefficients (Eqs. 12) in Eq. 11 and eliminating certain variables,

$$\begin{aligned} N^2 + N = 0.664\,(N - r)^3 \biggl[\, & 0.770 + 0.847 \left(\frac{N-r}{N}\right)^2 \\ & + 0.234 \left(\frac{N-r}{N}\right)^4 + 0.225 \left(\frac{r}{N-r}\right) + 0.248 \frac{r}{N}\left(\frac{N-r}{N}\right) \\ & + 0.068 \left(\frac{N-r}{N}\right)^3 \frac{r}{N} + 0.016 \left(\frac{r}{N-r}\right)^2 + 0.018 \left(\frac{r}{N}\right)^2 \\ & + 0.005 \left(\frac{N-r}{N}\right)^2 \left(\frac{r}{N}\right)^2 \biggr] \dots (13a) \end{aligned}$$

in which, for brevity,

$$N = \frac{d_2}{d_1} = \tfrac{1}{2}\,(\sqrt{1 + 8\,\mathbf{F}^2} - 1) \dots\dots\dots\dots\dots (13b)$$

and

$$r = \frac{h}{d_1} \dots\dots\dots\dots\dots\dots\dots\dots\dots (13c)$$

This is an implicit equation for h/d_1 in terms of **F**, and may be solved by trial. The resulting plot is shown as curve (*a*) in Fig. 2.

Interpretation of Fig. 2.—Curve (*a*), Fig. 2, indicates the theoretical relationship, completed by empirical coefficients, between the Froude number **F** of the approaching supercritical flow and the $\frac{h}{d_1}$-ratio necessary for the formation of a hydraulic jump that is complete upstream from the weir (providing it is not submerged). Coordinates lying above and to the left of the curve represent conditions in which the weir is too high, so that the jump is forced upstream and may be drowned at the source. Coordinates lying to the right of the curve represent conditions in which the weir is too low, so that the jump is forced downstream and may be washed out.

The point $\mathbf{F} = 1$, $h/d = 0$ corresponds to the case of critical flow discharging over an abrupt fall in the channel bottom. If the analysis were exact, the curve would be expected to pass through this point, since for the unique conditions of critical flow $d_1 = d_2$. However, for $\mathbf{F} = 1$, the computed value of $h/d_1 = -0.15$ indicates an inexactness in this extreme limit due to the empirical coefficients used in the analysis.

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[6] "Hydraulik," by Ph Forchheimer, B. G. Teubner, Leipzig und Berlin, 1930, p. 382.

[7] "Hydraulics," by George E. Russell, Henry Holt and Co., New York, N. Y., 5th Ed., 1941, p. 162.

It is to be noted that there is nothing inherent in the solution which determines the distance X from the toe of the jump to the weir. The Froude number of the flow is that at the toe of the jump, and the only assumption relative to the distance X is that it be of sufficient magnitude to permit the establishment of uniform velocity distribution at the depth d_2 upstream from the weir. Variation of X/d_1 with $\mathbf{F}$ and h/d_1 must be investigated experimentally rather than theoretically.

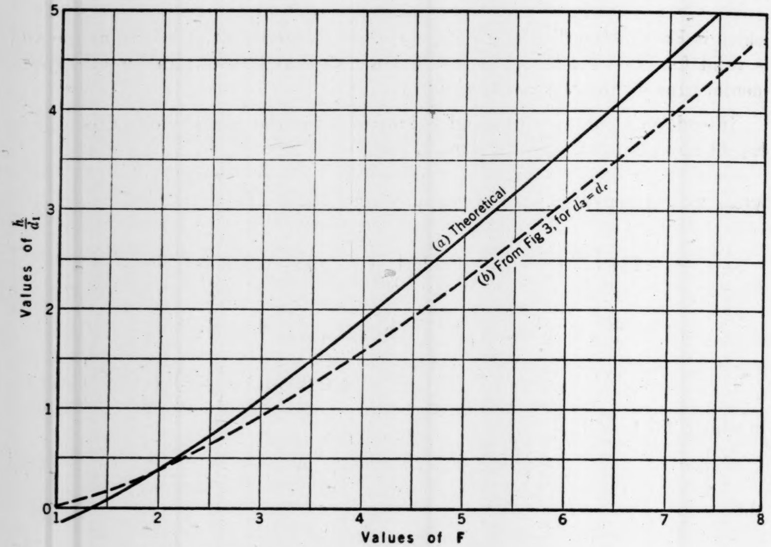

FIG. 2.—VARIATION OF RELATIVE BAFFLE HEIGHT WITH F FOR THE FORMATION OF THE HYDRAULIC JUMP

Analysis for an Abrupt Rise.—Lowering the approach section below the level of the normal channel will result in an abrupt rise in the channel bottom which may also be used to control a hydraulic jump. This case varies from that of a weir, however, in that the flow pattern is affected by the tailwater depth d_3 for all possible conditions of flow in a level channel. Consequently (see Fig. 1(*c*)),

$$\frac{h}{d_1} = \phi_4\left(\mathbf{F}, \frac{d_3}{d_1}\right) \qquad (14)$$

and this functional relationship also may be developed by use of the momentum and continuity equations.

First, the momentum equation between sections 1 and 2 is identical with Eq. 2. By assuming hydrostatic pressure distribution at the section of the abrupt rise, the force exerted by the face of the rise may be evaluated as $\frac{1}{2}\gamma h (2 d_2 - h)$ and the momentum equation between sections 2 and 3 becomes

$$\frac{\gamma d^2_2}{2} - \frac{\gamma d^2_3}{2} - \frac{\gamma h}{2}(2 d_2 - h) = \rho q (V_3 - V_2) \qquad (15)$$

Combining with Eqs. 2 and 15, the continuity equation,

$$V_1 d_1 = V_2 d_2 = V_3 d_3 \qquad (16)$$

and eliminating d_2, V_2, V_3 gives

$$\left(\frac{d_3}{d_1}\right)^2 = 1 + 2\,\mathbf{F}^2\left(1 - \frac{1}{d_3/d_1}\right) + \frac{h}{d_1}\left(\frac{h}{d_1} - \sqrt{1 + 8\,\mathbf{F}^2} + 1\right) \qquad (17)$$

For a fixed value of h/d_1 Eq. 17 is implicit for d_3/d_1 in terms of **F** and may be solved by trial. The resulting plot of d_3/d_1 versus **F** for certain constant values of h/d_1 is shown in Fig. 3.

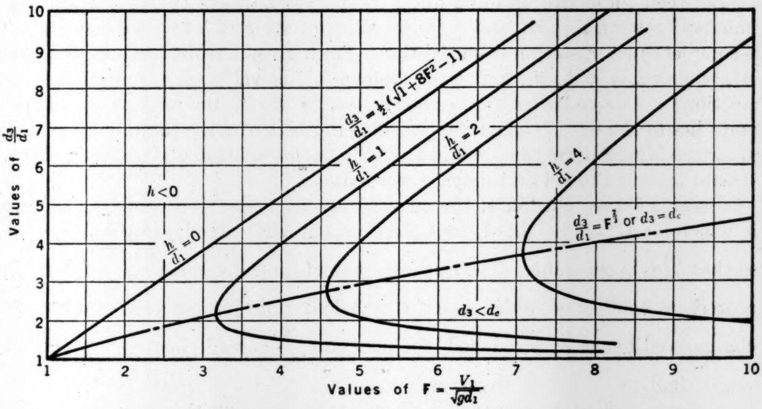

FIG. 3.—THEORETICAL VARIATION OF RELATIVE TAILWATER DEPTH WITH F AND RELATIVE HEIGHT OF ABRUPT RISE IN CHANNEL BOTTOM

Interpretation of Fig. 3.—Dividing the chart (Fig. 3) into three main regions are two lines, which pass through the intersection of the coordinate axes. By considering the conditions represented by these dividing lines, the significance of each region becomes apparent. The curve for $h/d_1 = 0$ is defined by the equation for the hydraulic jump in a level rectangular channel (Eq. 4), and thus represents equality between the normal tailwater depth and the sequent depth of the supercritical flow. Above this line, the region within which d_3 is greater than d_2 corresponds to the case in which a drop in the channel bottom (rather than an abrupt rise) is necessary to maintain a jump.

The curves of constant h/d_1 pass through a minimum value of **F**. By setting $\frac{d\mathbf{F}}{d(d_3/d_1)} = 0$, the equation of the line of minimum **F**-values is determined as

$$\frac{d_3}{d_1} = \mathbf{F}^{\frac{2}{3}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18)$$

It may be shown that this relationship corresponds to the condition that the tailwater depth is critical; that is,

$$d_3 = d_c \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19a)$$

or

$$\frac{V_3}{\sqrt{g\,d_3}} = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19b)$$

However, as flow is not normally changed from subcritical to supercritical by a reduction in the cross section of a level channel,[8] it is obvious that if a jump forms, d_3 cannot be less than d_c. Hence, the region below the line represented by Eq. 18, within which $d_3 < d_c$, cannot represent jump conditions as assumed in the analysis, but corresponds to the case in which no jump forms, the supercritical flow shooting over the sill and continuing downstream.

Between these two dividing lines, the curves indicate the theoretical relationship between **F**, h/d_1, and d_3/d_1 for the formation of a hydraulic jump that is complete just upstream from the rise. For a given abrupt rise and discharge in a channel, V_1, d_1, d_3, and h are known. The value of h/d_1 indicates the position of a curve, whereas the coordinates (**F**, d_3/d_1) define a point. If this point lies to the left of the curve, the sill is too high and the jump will be forced upstream where it may be drowned; if it is to the right, it will be forced downstream toward the sill and may be washed out.

Further interpretation of the curves may be made by considering the case in which for a fixed height of rise h the depth d_1 is also fixed, as by a sluice gate, so that h/d_1 is constant. The upper branch of the corresponding $\frac{h}{d_1}$-curve then represents conditions under which a true hydraulic jump forms. As the discharge, and hence **F**, is reduced, the constant $\frac{h}{d_1}$-curve indicates how the tailwater depth d_3 must be decreased to maintain a stable jump, so that d_3 approaches d_c. Further decrease in **F** will then result in the jump moving upstream and drowning the initial conditions. (At this point, when discharge takes place over the crest of the abrupt rise at the critical depth, the effect of the abrupt rise becomes essentially that of a weir. To illustrate this, the four values (**F**, h/d_1) defined by the line representing $d_3 = d_c$ in Fig. 3 have been replotted in Fig. 2. The resulting curve (curve (*b*), Fig. 2) agrees closely with that for the weir, and the discrepancy may be attributed partly to the error attached to the empirical constants C_1, C_2, C_3, and partly to the fact that the assumption of hydrostatic pressure distribution becomes less justified as d_3 approaches d_c.)

The lower branch of the $\frac{h}{d_1}$-curve in Fig. 3 (below the line $d_3/d_1 = \mathbf{F}^{\frac{2}{3}}$) corresponds qualitatively to conditions in which no jump forms, the supercritical flow shooting over the sill and continuing downstream. Quantitatively this section of the curve will be inexact, because the assumption made regarding the pressure distribution becomes quite unwarranted under these conditions.

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[8] "Elementary Mechanics of Fluids," by Hunter Rouse, John Wiley & Sons, Inc., New York, N. Y., 1946, p. 137.

LABORATORY APPARATUS

Experiments were carried out in a rectangular flume 10 ft long, 1 ft wide, and 28 in. deep, having glass walls and a brass floor. At one end was an entrance tank into which the water flowed through a regulating valve from a constant-level tank overhead. An adjustable sharp-edged vertical sluice gate was used to regulate the depth of flow into the flume from the entrance tank. Downstream water depths were regulated by a steel tailgate, hinged at the bottom, at the outlet end of the flume. Discharge was measured by an elbow meter, which had been previously calibrated by a sharp-crested weir, and the Rehbock discharge formula. Water depths in the flume were measured by a three-pronged point gage mounted on a carriage sliding on level rails. The prongs of the point gage were so spaced that readings were taken at the half and quarter points across the flume. The gage could be read by vernier to 0.001 ft.

Brass plates from 0.048 ft to 0.348 ft in height were used as weirs. For the abrupt rise, a false bottom extended from the top of a weir to the tailgate, a distance of approximately 2.5 ft. This false bottom could be affixed flush with the top of any of the weirs, and leveled by supports and shims. Brass plates used with this arrangement varied from 0.078 ft to 0.261 ft in height.

The maximum height of sill used was limited by the available head and the discharge measuring facilities. Minimum heights were limited to magnitudes below which probable percentage errors involved in the measurements were considered too high for the data to be of value.

EXPERIMENTAL PROCEDURE

For the Weir.—In the case of the sharp-crested, nonaerated weir it was decided to maintain an unsubmerged crest, as assumed in the theoretical development, in order to eliminate the tailwater effect, and hence the variable d_3/d_1. The investigation was then conducted to determine the variation of the form of the water-surface profile (and hence X/d_2) with **F** and h/d_1.

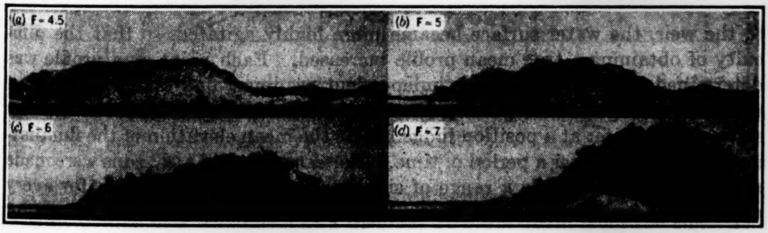

FIG. 4.—EFFECT OF BAFFLE SHOWING VARIATION OF WATER-SURFACE PROFILE WITH **F** FOR $h/d_1 = 2$

Four values of the ratio h/d_1—$\frac{1}{2}$, 1, 2, and 4—were investigated, and for each value of h/d_1 runs were made to determine the water-surface profile for certain fixed values of **F** within the practical range of discharge. With h and d_1 both fixed, a true jump formed well upstream from the baffle for a low discharge. With the increase in discharge, and hence in **F**, the toe of the jump was forced downstream toward the weir (Fig. 4) until the flow either shot over the weir as a sheet, or passed over the weir as a standing wave or surge, depending on the value of h/d_1. As the toe of the jump moved downstream and closer to the weir, the water surface became more highly agitated, so that the difficulty of obtaining a true mean profile increased. Each point on a profile was determined by averaging three independent readings, one taken for each of the three prongs on the point gage. Each individual reading was taken by setting the pointer at a position judged to be the mean elevation of the fluctuating water surface over a period of time. For each value of h/d_1, runs were made using weirs over as wide a range of size as was practicable. Thus, for every one of the four values of h/d_1 several independent profiles were obtained for each of several fixed values of **F**.

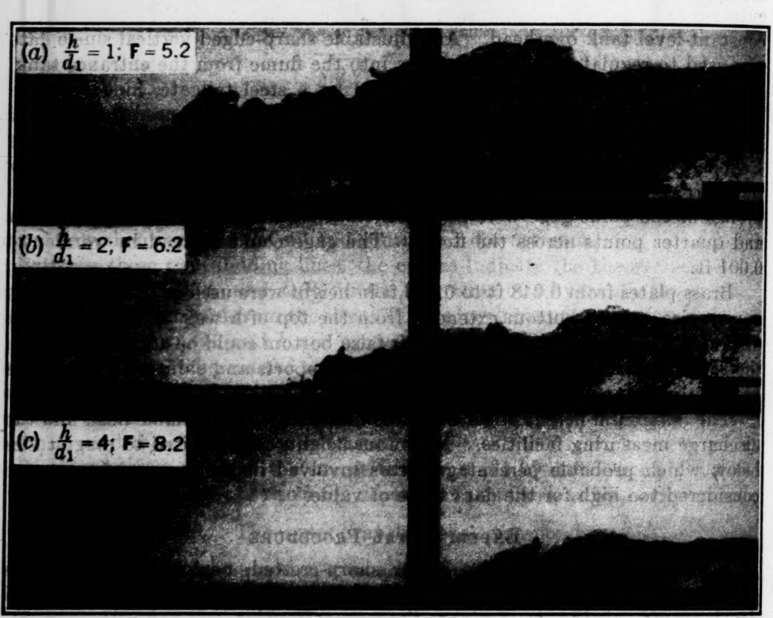
(a) $\frac{h}{d_1} = 1$; F = 5.2
(b) $\frac{h}{d_1} = 2$; F = 6.2
(c) $\frac{h}{d_1} = 4$; F = 8.2

FIG. 5.—EFFECT OF ABRUPT RISE, SHOWING VARIATION OF h/d_1 WITH **F** FOR $d_3/d_1 = 5.5$

For the Abrupt Rise.—As previously explained, tailwater effects (and hence the variable d_3/d_1) were eliminated in the case of the weir by maintaining an unsubmerged crest, under which conditions the variation of the relative position of the jump with **F** and h/d_1 was determined. In order to investigate the field as fully as possible in the limited time available, it was decided, in the case of the abrupt rise, to maintain the relative position of the jump constant (thus eliminating the variable X/d_2), and to investigate rather the effects of the tailwater. Accordingly, for the abrupt rise, the investigation was conducted to determine the variation of d_3/d_1 with **F** and h/d_1 for the condition that $X = 5\ (h + d_3)$. This value of X was selected because it seemed to provide sufficient distance for the jump to be completed before the rise, and because the tranquil flow conditions downstream from the rise permitted quite accurate measurements of $h + d_3$. Practically, this resulted in a jump, the toe of which was slightly greater than 4 d_2 upstream from the rise.

For any given abrupt rise, h/d_1 was fixed by regulating the sluice gate opening. The discharge, and hence **F**, was fixed by regulating the inlet valve. Then, by adjusting the tailgate, the toe of the jump was made to occur at a distance 5 $(h + d_3)$ upstream from the rise. For this condition, values of d_1, d_3, and the discharge were recorded, and values of d_3/d_1 and **F**, later computed. The minimum discharge under which such a jump formed occurred when the tailgate effect was removed, and the discharge took place as free fall at the outlet end of the flume. (Fig. 5 illustrates conditions in which $d_3/d_1 = 5.5$ for values of h/d_1 of 1, 2, and 4.)

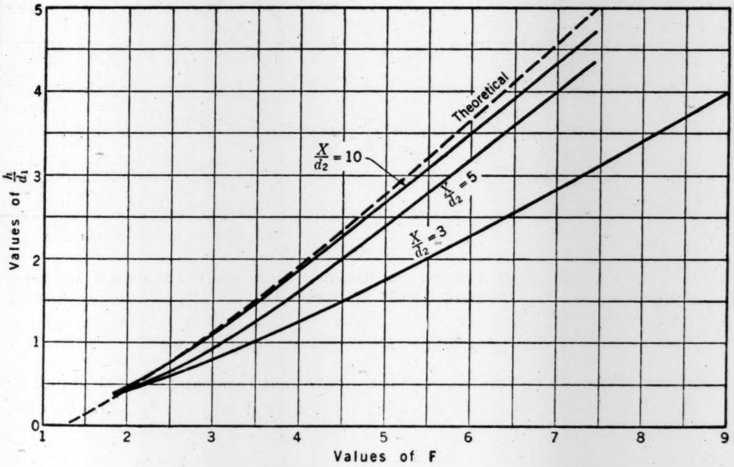

FIG. 6.—EXPERIMENTAL VARIATION OF RELATIVE BAFFLE HEIGHT WITH F AND THE RELATIVE POSITION OF JUMP

Throughout the experiments the depth measurement used to determine the recorded value of **F** was made at a distance 20 h upstream from the sill. The point so defined was convenient in that it was fixed for any given baffle, and lay upstream from the toe of all but the longest jumps.

EXPERIMENTAL RESULTS

For the Weir.—For fixed values of h/d_1 and **F**, the corresponding water surface profile was obtained by using d_1 as the unit of length and so that all recorded points could be plotted dimensionlessly for several independent runs.

Consequently, for each of the four values of h/d_1—$\frac{1}{2}$, 1, 2, and 4—profiles for four or five values of **F** were available, varying from the case in which the jump formed well upstream from the weir to that in which the high-velocity flow shot over the baffle either as a spray or as a standing wave. Fig. 4 illustrates the sequence of profiles for $h/d_1 = 2$.

For the sake of brevity these plotted profiles have not been included, only the summarized results being presented. Thus, for each of the four values of h/d_1 investigated the value of **F** corresponding to three values of X/d_2 (specifically, 3, 5, and 10) was determined from the plotted profiles by interpolation. The resulting curves are shown in Fig. 6, where the theoretical relation from Fig. 2 is also plotted to provide a comparison with experimental results.

For the Abrupt Rise.—All experimental results are for the condition that $X = 5\,(h + d_3)$. Fig. 7 shows the variation of d_3/d_1 with h/d_1 and **F** for values of h/d_1 of $\frac{1}{2}$, 1, 2, and 4.

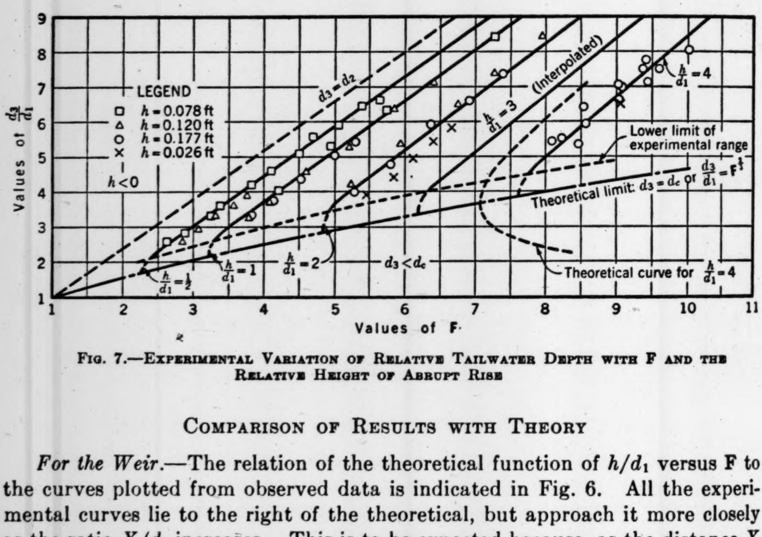

FIG. 7.—EXPERIMENTAL VARIATION OF RELATIVE TAILWATER DEPTH WITH F AND THE RELATIVE HEIGHT OF ABRUPT RISE

COMPARISON OF RESULTS WITH THEORY

For the Weir.—The relation of the theoretical function of h/d_1 versus **F** to the curves plotted from observed data is indicated in Fig. 6. All the experimental curves lie to the right of the theoretical, but approach it more closely as the ratio X/d_2 increases. This is to be expected because, as the distance X increases, the velocity distribution of the flow just upstream from the baffle approaches more closely the uniformity assumed in the theoretical development.

Also, it is obvious that the increment between the curves for $X/d_2 = 3$ and $X/d_2 = 5$ is much greater than that between the curves for $X/d_2 = 5$ and $X/d_2 = 10$. This occurs because the velocity distribution closely approaches uniformity for $X/d_2 = 5$, any further increase in X/d_2 having little additional effect. Thus, the curve for $X/d_2 = 10$ marks the closest approach that may be expected of an experimental curve to the theoretical. The remaining discrepancy must be attributed to inexactness of the theoretical development due to the empirical coefficients used, and to lack of agreement between the experimental conditions and the idealized ones of the theory.

For the Abrupt Rise.—On comparing the experimental $\frac{h}{d_1}$-curves (Fig. 7) with those of the theoretical development (Fig. 3), it is apparent that within the range of experimental data all the curves closely approximate straight lines, the slopes of which are similar in both cases. This is as would be expected, as the experimental results serve simply to fix the position of the curves for a certain relative position of the jump—namely $\frac{X}{h + d_3} = 5$.

However, the experimental curve for any value of h/d_1 is seen to be shifted somewhat to the right of the corresponding theoretical curve. This shift is almost negligible for $h/d_1 = 1$, but preceding toward the right of the chart, as the values of h/d_1 increase, the amount of shift between the corresponding experimental and theoretical curves increases. The extent of the maximum discrepancy within the experimental range, for the curve $h/d_1 = 4$, is indicated in Fig. 7.

It may be shown analytically that, as the ratio $\frac{X}{h + d_3}$ increases (corresponding to a velocity distribution approaching uniformity at the section of the abrupt rise), the experimental curves shift to the left toward the theoretical curves. Had the experimental runs been conducted with the ratio $\frac{X}{h + d_3}$ equal to 10, then, as in the case of the weir, the experimental curves would approach the theoretical more closely.

In Fig. 7 it is also apparent that the dashed line defining the lower limit of the experimental range lies above the line defining the theoretical limit. This is because the laboratory apparatus did not permit reducing the tailwater depth to the critical depth; otherwise, the lower limit of the experimental range would be shifted downward.

PRACTICAL APPLICATION OF RESULTS

In practice, d_1 and V_1 of the supercritical flow, d_2, and d_3, may be computed. Then, provided $d_3 < d_2$, a sill may be used to prevent the hydraulic jump from being forced downstream. With these factors known, Figs. 6 and 7 are useful for the purpose of either predicting the effect of a given sill on the flow pattern or of determining the height of sill required to control a jump.

For the Weir.—Fig. 6 permits an analysis of the effect of a given weir for known supercritical and tailwater flow conditions, provided that the tailwater depth does not affect discharge over the crest—that is $d_3 < (d_2 - \frac{3}{4}\,h)$. In any given case, the coordinates ($\mathbf{F}$, h/d_1) indicate a unique point on the diagram. If this point lies between the experimental curves, hydraulic jump action will result, the corresponding interpolated value of X/d_2 indicating the relative position of the jump. If the point lies above the curves, the jump will be forced upstream; when drowning results, the weir will serve to deflect any high-velocity bottom currents. If the point lies below the curve $X/d_2 = 5$, a shortened, incomplete jump or spray action will result; only a qualitative prediction of the effect is possible, depending on the distance of the point from the given curves.

For design purposes, it is proposed that the curve for $X/d_2 = 5$ in Fig. 6 be used, under conditions of maximum discharge. Then, for maximum discharge the weir will insure the formation of a jump, whereas for any lesser discharge the weir will serve to create a stilling basin in which a longer or a drowned jump occurs.

The case of the submerged weir has not been investigated quantitatively. Ordinarily, unsubmerged conditions prevail over a wide range of discharge; if submergence occurs, the jump is forced upstream with possible drowning as a result.

For the Abrupt Rise.—Fig. 7 provides a method of predicting the performance of a given abrupt rise when V_1, d_1, d_2, d_3, and h are known. Three situations are possible, as follows:

First, if the point ($\mathbf{F}$, d_3/d_1) lies above the line $h/d_1 = 0$, the conclusion is that $d_3 > d_2$, so that an abrupt rise would serve only to increase the drowning effect. Consequently, a drop in the channel bottom rather than an abrupt rise is necessary to maintain a jump. This case has not been investigated in this study.

Second, if the point ($\mathbf{F}$, d_3/d_1) lies within the experimental range (that is, between the lines for $d_3 = d_2$ and $d_3 = d_c$), then the position of the point relative to the corresponding $\frac{h}{d_1}$-curve indicates the effect of the abrupt rise on the flow pattern. Thus, as was pointed out earlier, if the point lies on the corresponding $\frac{h}{d_1}$-curve, a jump will form with $\frac{X}{h + d_3} = 5$; if it lies to the right of the curve, the sill is too low, so that the jump will be forced downstream toward the abrupt rise and may be washed out; if it is to the left of the curve, the sill is too high and the jump will be forced upstream and may be drowned.

Third, if the point ($\mathbf{F}$, d_3/d_1) lies in the region below the line $d_3 = d_c$, then the normal downstream flow is supercritical—that is, the slope is steep for the given discharge so that the control section is at the abrupt rise. Formation of a jump then results in discharge taking place over the crest of the abrupt rise at the critical depth, so that the case is analogous to that of the weir. Fig. 6 may be used, therefore, as outlined in the preceding section.

Fig. 7 may be used to determine the necessary length and depth of stilling basin when V_1, d_1, and d_3 are known. For most tailwater rating curves, the critical condition—that is, that requiring the maximum value of h—occurs for conditions of maximum discharge. It is proposed, then, that the point ($\mathbf{F}$, d_3/d_1) be first defined for conditions of maximum discharge, and the corresponding value of h/d_1 determined by interpolation. The effect of the sill for lesser flows should then be determined from the figure as previously outlined, and if necessary the sill height increased. For the expected range of discharge, this procedure will indicate the minimum height of sill necessary to prevent the jump from being washed out. For the critical condition a jump will form, and be complete upstream from the sill, whereas for other discharges a proportionally longer or drowned jump will occur; in the latter case the sill will serve to deflect any high-velocity bottom currents.

When the downstream slope is steep, and hence the tailwater depth is less than critical, either a weir or an abrupt rise of the same height may be used. Consequently, Fig. 6 may be used for design purposes.

CONCLUSIONS

For any supercritical discharge in a rectangular channel, when the corresponding d_2 is greater than d_3, a hydraulic jump may be created and prevented from receding downstream by either a weir extending across the channel or an abrupt rise in the channel bottom. When the normal downstream depth is less than critical, then an abrupt rise and a weir of the same height are interchangeable.

For conditions in which the tailwater does not affect discharge over the crest, a theoretical analysis, based on the momentum principle and completed by empirical coefficients, indicates a definite relationship between the depth and velocity of the shooting flow and the height of a sharp-crested, nonaerated weir necessary to produce a jump. A similar theoretical analysis indicates a definite relationship between V_1, d_1, d_3, and the height of an abrupt rise necessary to produce a jump for conditions in which $d_3 > d_c$.

Data from experimental investigations conducted in a level rectangular flume verify these theoretical relationships within close limits, and any discrepancies may be accounted for logically. The experimental data are presented in dimensionless charts—one for an abrupt rise, corresponding to the outlet of a depressed stilling basin, and one for a low, sharp-crested, nonaerated weir—that provide a method of predicting qualitatively the effects of such sills on supercritical flow. These charts serve also to indicate the type of hydraulic jump control applicable under given conditions, and may be used in the design of small structures, or in the preliminary design of large structures by indicating the range of sill sizes to be investigated in detail in specific model studies.

ACKNOWLEDGMENTS

This paper has been prepared from a thesis submitted by the writers in partial fulfilment of the requirements for the degree of Master of Science, in the Department of Mechanics and Hydraulics, in the Graduate College of the State University of Iowa at Iowa City. The experimental tests were made in the Hydraulics Laboratory of the Iowa Institute of Hydraulic Research at Iowa City. Grateful acknowledgment is given to Hunter Rouse, M. ASCE, for suggestions and guidance during the investigation and for constructive criticism during the writing of the paper.

DISCUSSION

EN-YUN HSU.[9]—As has long been known, the hydraulic jump provides a useful method of dissipating the excess energy of supercritical flow and consequently of reducing bed erosion downstream from transition structures. Primary difficulties in designing such a structure are to insure the formation of a stable jump and to control its position for all probable operating conditions. Since various conditions may exist at different structures, different types of control are required. The authors have provided useful information for the control of the jump by using sills or abrupt rises in the channel floor for the condition that the downstream depth is smaller than the sequent depth for a normal jump. On the other hand, if the downstream depth is larger than the sequent depth for a normal jump, a drop in the channel floor must be used—as in the case for $h < 0$ mentioned by the authors. The problem of jump stabilization, by a drop, is encountered primarily at the end of a high-velocity expansion, and was studied by the writer at the Iowa Institute of Hydraulic Research, State University of Iowa, Iowa City, under Project 67m of the Committee on Hydraulic Research of the ASCE Hydraulics Division. Since the results cover the incompleted half of Fig. 7, they are presented herewith.

Through dimensional analysis the general form of the relationship between V_1, d_1, g, d_3, and h may be indicated as follows:

$$\frac{h}{d_1} = \phi\left(\mathbf{F}, \frac{d_3}{d_1}\right) \qquad \text{(20)}$$

in which V_1 (see Fig. 8) is the velocity of approaching flow; d_1 and d_3 are the depths before and after the jump, respectively; h is the magnitude of the rise (negative for a drop) in the channel bottom; and $\mathbf{F} = V_1 \sqrt{g\,d_1}$ is the Froude number of the approaching flow. As shown by the authors, the form of the function may be determined with good approximation for positive values of h through the use of the momentum and continuity principles. This procedure also holds true for negative values, except that several conditions of flow are possible.

For a given Froude number of the approaching flow, the downstream depth may fall in any of five regions. The lower limit of region 1 (Fig. 8(*a*)) is the depth at which the jump will begin to travel upstream and the upper limit of region 5 is the depth at which the jump will begin to travel downstream. Evidently the drop does not control the jump in these two regions. In fact, the jump, as such, is stable in position and the drop is effective for its desired purpose only in regions 2 and 4. The intermediate region 3 represents an undular state of flow without a breaking front.

To apply the momentum equation to either of the two stable conditions, an assumption must be made as to the longitudinal force acting on that part of the fluid in contact with the face of the drop. To simplify the analysis, the pressure on the drop was assumed to be hydrostatically distributed, its magnitude depending on whether the jump is upstream from the drop in region 2 or downstream from the drop in region 4. In other words, it is assumed that the intensity of pressure at the drop is proportional to the downstream depth in region 2, and to the upstream depth in region 4.

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[9] Research Associate, Iowa Inst. of Hydr. Research, State Univ. of Iowa, Iowa City, Iowa.

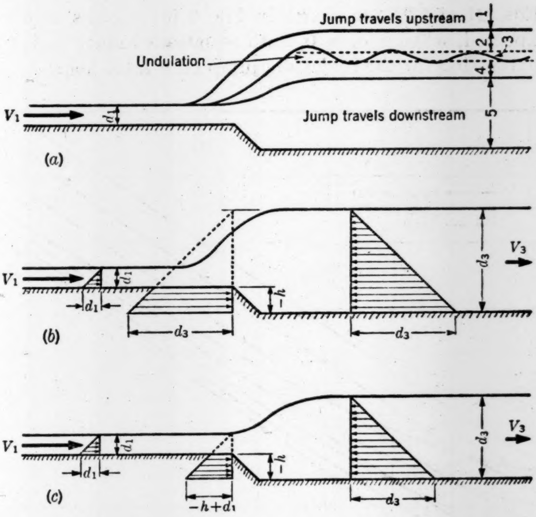

FIG. 8.—DIFFERENT FORMS OF THE HYDRAULIC JUMP AT AN ABRUPT DROP

Based on these simplified assumptions, the momentum equation for sections (1) and (2) of region 2 (see Fig. 8(*b*)) is

$$\frac{\gamma\, d_1^2}{2} + \frac{\gamma\,[d_3 - (-h) + d_3]\,(-h)}{2} - \frac{\gamma\, d_3^2}{2} = \frac{q\,\gamma}{g}\,(V_3 - V_1) \ldots\ldots (21)$$

in which γ is the specific weight of the fluid and q is the discharge per unit width of the channel. From the continuity equation,

$$q = V_1\, d_1 = V_3\, d_3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22)$$

If Eqs. 21 and 22 are combined and reorganized, the following expression is obtained for region 2:

$$\mathbf{F}^2 = \frac{1}{2}\,\frac{d_3/d_1}{1 - d_3/d_1}\left\{1 - \left[\frac{d_3}{d_1} - \left(\frac{-h}{d_1}\right)\right]^2\right\} \ldots\ldots\ldots\ldots (23)$$

In region 4 (Fig. 8(*c*)), the momentum equation has the form:

$$\frac{\gamma\,[d_1 + (-h)]^2}{2} - \frac{\gamma\, d_3^2}{2} = \frac{q\,\gamma}{g}\,(V_3 - V_1) \ldots\ldots\ldots\ldots (24)$$

If Eqs. 22 and 24 are combined, the counterpart of Eq. 23 is obtained for region 4:

$$\mathbf{F}^2 = \frac{1}{2}\,\frac{d_3/d_1}{1 - d_3/d_1}\left\{\left[\left(-\frac{h}{d_1}\right) + 1\right]^2 - \left(\frac{d_3}{d_1}\right)^2\right\} \qquad (25)$$

Plots of Eqs. 21 and 25 are shown in Fig. 9 for various values of $-h/d_1$. As is to be expected, when $h/d_1 = 0$, both equations reduce to Eq. 4, which is the familiar expression for the hydraulic jump on a level floor.

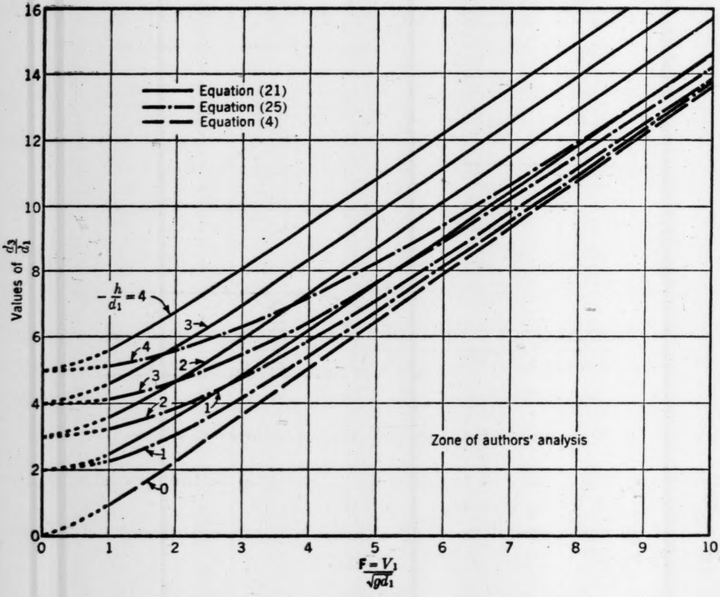

FIG. 9.—ANALYTICAL CHARACTERISTICS OF REGIONS 2 AND 4

Experiments to verify these relationships were conducted at the Iowa Institute of Hydraulic Research in the same 1-ft rectangular glass-walled flume used by the authors. The upstream end of the channel bottom was elevated by a 2-in. concrete slab, and an adjustable sharp-edged vertical sluice gate was used to regulate the upstream depth. The discharge was measured by a calibrated elbow meter. The highest Froude number obtained was approximately 10, and the h/d_1-ratio was varied systematically from 1 to 4.

Fig. 10 is the experimental counterpart of Fig. 9, only those parts of the analytical curves being retained which are followed by the measured data. The latter agree closely with the simplified analysis for both regions; and they also delineate a systematic trend of the undular transition between the two. The relative height of drop required to stabilize a jump for any particular combination of discharge, upstream depth, and downstream depth may evidently be determined from this diagram. Since the curves of Fig. 10 represent the missing part of Fig. 4, superposition of the two families on the same diagram should provide complete information for the control of the jump by either a drop or a rise in the channel floor.

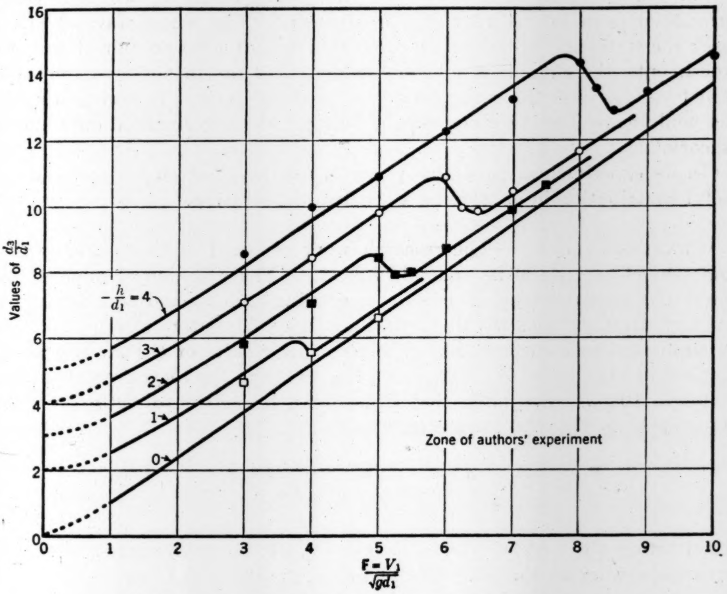

FIG. 10.—EXPERIMENTAL VERIFICATION OF THE ANALYSIS

STEPONAS KOLUPAILA,[10]—To estimate the sill head in a rectangular channel the authors use the formulas for the weir developed by H. Bazin in his investigations made from 1886 to 1895.[11] The propriety of applying these formulas for the jump conditions may be rather doubtful, because they are based on the assumption that the flow in the channel might be characterized by a comparatively regular distribution of velocities. The authors themselves ascertain this fact in their "comparison of results with theory." The application of several Bazin formulas thus has certain objections.

For example, the "theoretical" discharge formula, Eq. 10 (known as the Poleni formula of 1717) was never used without discharge coefficients, such as C_1 in the basic Bazin formula (for feet units)—

$$C_1 = 0.607 + \frac{0.0148}{H} \qquad (26)$$

In this case it is not necessary that H approach infinity.

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[11] "Recent Experiments on the Flow of Water Over Weirs," by H. Bazin, *Proceedings*, Engineers' Club of Philadelphia, Vol. 7, 1890, pp. 259–310; Vol. 9, 1892, pp. 231–244 and pp. 287–319; and Vol. 10, 1893, pp. 121–164.

This formula, like Eq. 12*a*, is also not limited by the condition $d_3 < (d_2 - \frac{3}{4} h)$. The condition (written by Bazin as $(d_2 - d_3) < 0.75\ h$) belongs to the third formula (Eq. 12*b*) only.

The third formula for C_3 (Eq. 12*b*) is not the expression for the coefficient of Ph. Forchheimer; it is another of the Bazin formulas, which is clearly formulated even in the work of Forchheimer.[6] The writer was not able to check the statement of the authors that this condition was obtained during all experiments; therefore, it is not quite evident that the authors had applied the right formulas from the entire series proposed by Bazin. It may be that this fact could cause the big disagreement between the experimental data and the "theory."

In its experimental part, this paper is worthy of attention and can be a useful contribution in developing a practical application.

Charles Jaeger.[12]—The research work reported in this paper is to be commended because of its general interest.[13] The equation of momentum is one of the most powerful devices for computing uniform and nonuniform flow, and any effort to check the theoretical results by laboratory experiment should be supported and encouraged. The principal results of the study are summarized in Figs. 3 and 7. Fig. 3 and Eq. 15 are in the form presented by the writer in 1936.[13] Assuming that Eq. 15 can be written directly for profiles (1) and (3), in Fig. 1(*c*) (assuming $\gamma = 1$),

$$\frac{d^2_1}{2} - \left(\frac{d_3 + h}{2}\right)^2 - \xi \frac{V^2_1}{2\ g} h = \alpha'_3 \frac{V^2_3\ d_3}{g} - \alpha'_1 \frac{V^2_1\ d_1}{g} \ldots\ldots\ldots (27)$$

Substituting $\alpha'_3 = \alpha'_1 = 1$ and $\xi = 0$ in Eq. 27 the result gives a momentum equation similar to Eq. 15. The factor $\alpha'_1 \cong (\alpha'_1 > 1)$ takes account of the nonuniform velocity distribution and $\xi \frac{V^2_1}{2\ g} h$ accounts for a pressure distribution that may be different from the hydrostatic pressure on the sill or weir—ξ being a coefficient. Since $\eta_2 = \frac{d_3}{d_1}$; $\phi_1 = \frac{V^2_1}{2\ g\ d_1} = \frac{\mathbf{F}^2}{2}$; and $\frac{h}{d_1} = \delta$ Eq. 27 becomes

$$\phi_1 = \frac{\mathbf{F}^2}{2} = \frac{\eta_2 \left[(\eta_2 + \delta)^2 - 1\right]}{4\ (\eta_2\ \alpha'_1 - \alpha'_3) - 2\ \xi\ \eta_2\ \delta} \ldots\ldots\ldots\ldots (28a)$$

If $\alpha'_1 = \alpha'_3 = 1$; and $\xi = 0$;

$$\phi_1 = \frac{\mathbf{F}^2}{2} = \frac{\eta_2 \left[(\eta_2 + \delta)^2 - 1\right]}{4\ (\eta_2 - 1)} \ldots\ldots\ldots\ldots\ldots (28b)$$

The curves $\phi_1 = \frac{\mathbf{F}^2}{2}$ in function η_2 are represented in Fig. 11 which is comparable to Fig. 3. A straight line in Fig. 3 becomes a parabola in Fig. 11; the curve $\frac{d_3}{d_1} = \mathbf{F}^{2/3}$ in Fig. 3 is curve $\phi_1 = \frac{\eta^3_2}{2}$ in Fig. 11; and curve $\frac{h}{d_1} = 0$ (hydraulic jump) in Fig. 3 is the curve $\delta = 0$ in Fig. 11. Fig. 12, for the case of a negative sill, should be interpreted in a similar manner, as can also the problem of river bed enlargement.[14] Figs. 11 and 12 represent all possible solutions of Eq. 27, for positive as well as for negative sills. Fig. 3 represents only part of this range of solutions. It is a satisfaction to the writer thus to find his own studies confirmed by more recent laboratory experiments; and he is in agreement with the physical interpretations presented by the authors. The writer proposes to comment on three phases of the paper:

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[13] "Der Mischungsvorgang bei ploetzlichem Querschnittsuebergang," by Charles Jaeger, *Wasserkraft und Wasserwirtschaft*, Vol. 31, No. 24, 1936.

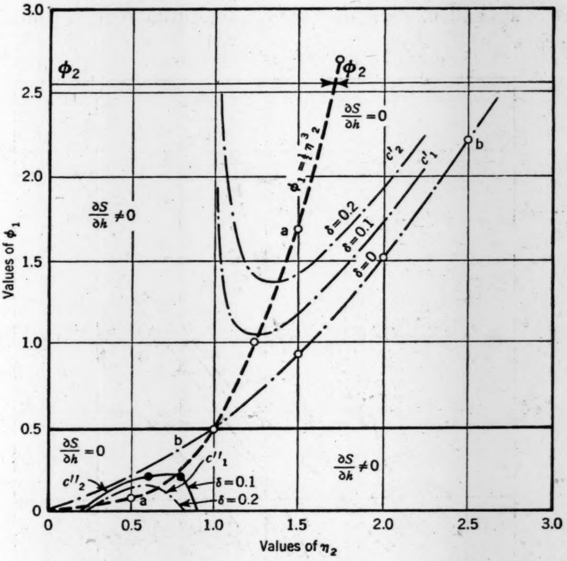

FIG. 11

(a) In Fig. 7 the experimental curves are offset somewhat to the right of the theoretical curves. The authors attribute this offset effect to the nonuniform distribution of velocities. On the other hand, if $\alpha'_1 \cong \alpha'_3 > 1$ the quantities ϕ_1 and $\mathbf{F}$ actually decrease and the theoretical curve will then, in fact, move to the left. The writer has found that α'_1 and α'_3 are always very nearly equal to 1. (One does not need to know the value of α'_2 in profile (2), Fig. 1(c).) If $\xi > 0$ (as it probably is), the theoretical curve is moved to the right to meet the experimental curve. To the writer, this translation of the curve means that the pressure distribution on the vertical sill is not hydrostatic—it cannot be hydrostatic because of the necessity of curving the stream tubes and because of the turbulence near the sill. In some experiments by students at the Imperial College, London, England, values of $\xi > 0$ were found in cases where the occurrence of bridge piers narrowed the section.

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[14] "Technische Hydraulik," by Charles Jaeger, Birkhäuser, Basel, 1949, pp. 150, 152, 158, and 159, Figs. 88, 89, 94, and 95, respectively.

(*b*) There are other parts of Figs. 11 and 12 that are not represented in Fig. 3. These characteristics have been investigated, thus indicating that real hydraulic solutions are available for these problems as well. A. Koch and M. Carstanjen[15] have done some research on the problem of streaming flow over a positive sill and L. Escande[16] spent a long time investigating conditions at negative sills.

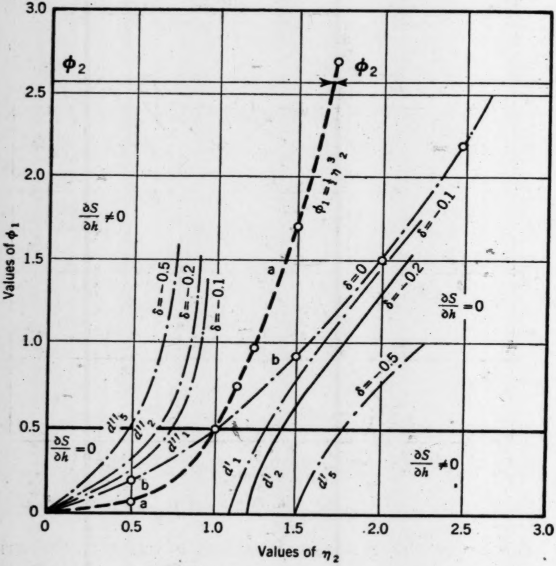

FIG. 12

When analyzing the energy of the flow that is represented by some of the other solutions of these diagrams, the investigator finds that some parts of the curves are not adaptable to real hydraulic solutions.

(*c*) There is need for further research work to explore the possibility of further solutions for the different curves in Figs. 11 and 12. To date, only parts of the entire areas have been investigated. For example, more accurate measurements of α'_1, α'_3, and ξ will probably show an even closer agreement with theory. On the whole, the results found by the momentum equation have agreed remarkably well with the results of tests, but an extension of these tests is still indicated.

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[15] "Von der Bewegung des Wassers und den dabei auftretenden Kräften," by A. Koch and M. Carstanjen, Springer, Berlin, 1926.

[16] "Etude théorique et expérimentale de l'écoulement par vanne de fond," by L. Escande, *Revue Générale de l'Hydraulique*, Nos. 19 to 21, 1938, and Nos. 25 and 26, 1939.

The momentum equation was introduced into hydraulics by J. B. Bélanger[17] in 1849 and J. A. Bresse[18] in 1860. They established a comprehensive theory, introducing the function—

$$S = \frac{h^2}{2} + \frac{Q\,v}{g} \qquad (29)$$

—or "total momentum." Experimental research followed by H. Bazin, A. H. Gibson, Mr. Koch, J. Einwachter, J. Smetana, Mr. Escande, G. Nebbia, Boris A. Bakhmeteff, Hon. M. ASCE, and others.[19,20] K. Woycicki investigated the case of the drowned flow below a gate and H. Favre studied the lateral spillway and the lateral flow of water into a primary stream. The equation of momentum has also been used for the computation of nonuniform flow (translation waves). Model tests have demonstrated how the height of the wave crest depends not only on the variation of the discharge (as in the case of the wave body) but also on the variation of impulse. No systematic research work has been done on this interesting problem.

One important case that was solved on similar lines by Eq. 8 is the computation of partly working surge tanks, as proposed in 1930 for the power station in Wettingen, Switzerland.[21] In that particular case, the computations for translation waves had to be combined with surge tank computations. These are some of the problems whose solution by means of the momentum equation represents a considerable amount of successful research work.

Figs. 11 and 12 also indicate the particular domain where the equation of momentum offers no solution. Physically, this means that the hydraulic jump becomes unstable and moves upstream or downstream. Similarly, there are certain areas in which the Bernoulli equation does not apply. Are these two domains, with their imaginary solutions, the same? What is the connection between "total momentum" and energy? These theoretical problems have led to some further research as reported by the writer in 1931,[22] 1943,[23] 1947,[24] 1948,[25] and 1949.[26]

When the investigator is dealing experimentally with particular problems as outlined in this discussion under topic (*c*), the more general aspect of the equation of momentum described in the paragraphs following topic (*c*) should be kept in mind because it is there that the explanation of many of the difficulties encountered is to be found.

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[17] "Notes sur le cours d'hydraulique," by J. B. Bélanger, *Mémoires*, École Nationale des Ponts et Chaussées, Paris, 1849–1850.

[18] "Cours de Mécanique Appliquée," Pt. 2, "Hydraulique," by J. A. Bresse, Mallet-Bachelier, Paris, 1860.

[19] "Technische Hydraulik," by Charles Jaeger, Birkhäuser, Basel, 1949, pp. 87 to 92.

[20] *Ibid.*, pp. 137 to 160.

[21] "Über die Eigenschaften von Schwällen und die Berechnung von Unterwasserstollen," by E. Meyer-Peter and H. Favre, *Schweizerische Bauzeitung*, Nos. 4 and 5, 1932.

[22] "Analogien zwischen Stützkraftminimum und Energieminimum in der Hydraulik," by Charles Jaeger, *ibid.*, April 18, 1931.

[23] "Contribution à l'étude des courants liquides à surface libre," by Charles Jaeger, *Revue Générale de l'Hydraulique*, Nos. 33 and 34, 1943.

[24] "De l'impulsion totale et de ses rapports avec l'énergie totale d'un courant liquide à surface libre," by Charles Jaeger, *ibid.*, Nos. 37–41, 1947.

[25] "Steady Flow in Open Channels: The Problem of Boussinesq," by Charles Jaeger, *Journal*, Inst. C. E., February, 1948, p. 338.

[26] "Technische Hydraulik," by Charles Jaeger, Birkhäuser, Basel, 1949.

Louis M. Laushey,[27] Jun. ASCE.—The results of this paper present unusually good data on the hydraulic jump, and the authors are to be commended for the manner in which they conducted the tests. The curves presented lead the engineer closer to the rational design of stilling basins. The discrepancy between the theoretical and the experimental curves leads to a more complete analysis than that given in the paper.

Abrupt Rise.—The authors state that the experimental curves for the abrupt rise (Fig. 7) are parallel, straight lines. An equation to fit the experimental data was easily found by plotting d_3/d_1 against h/d_1, with the parameter being straight-line curves of values of **F** all having approximately the same slope. The resulting equation fits the experimental curves within ± 5%, and had no greater error than the spread of the plotted points: $d_3 = d_2 - 1.40\,h$; or $d_3/d_1 = d_2/d_1 - 1.40\,h/d_1 = \phi\,(\mathbf{F}) - \psi\,(h/d_1)$. To complete Eq. 14—

$$\frac{h}{d_1} = \phi_4\left(\mathbf{F}, \frac{d_3}{d_1}\right) = 0.71\left(\frac{d_2}{d_1} - \frac{d_3}{d_1}\right) \ldots\ldots\ldots\ldots\ldots (30)$$

—which includes the necessary variables **F**, d_1, and h. Fig. 13 shows the agreement between the authors' experimental curves and the empirical equation.

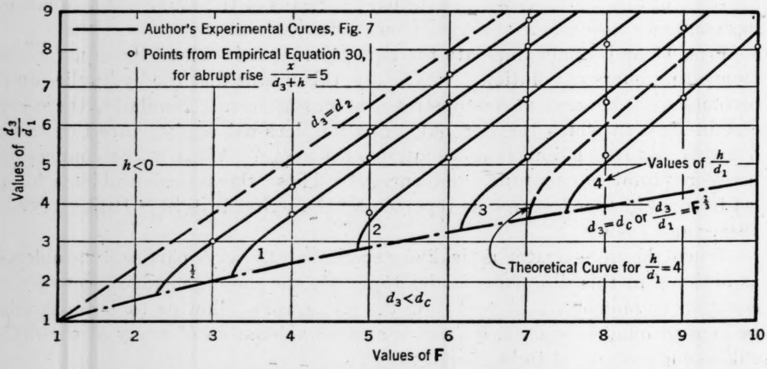

Fig. 13.—Verification of Empirical Equation with Experimental Data of Fig. 7

An adjusted equation then follows:

$$d_2 - (d_3 + h) = 0.40\,h = \Delta d \ldots\ldots\ldots\ldots\ldots\ldots (31)$$

showing the resulting drop in the water surface below that required for a jump without an abrupt rise. Within the limits of the authors' studies, it follows that the ratio $h/\Delta d = 2.5$. It may prove to be a useful rule of thumb that the drop in the water surface below that level required theoretically for d_2 would be about 40% of the height of the abrupt rise.

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[27] Asst. Prof. of Civ. Eng., Carnegie Inst. of Technology, Pittsburgh, Pa.

Fig. 14 is another plotting of the authors' data (based on the curves in Fig. 7) to show this drop. The care with which the experimental work must have been done is evident. Denoting K as the ratio $(d_3 + h)/d_2$, in which d_2 is the depth computed from Eq. 4, these observations follow:

(1) The ratio K is decreased by raising h/d_1 ($\mathbf{F}$ constant) or by decreasing $\mathbf{F}$ (h/d_1 constant);

(2) The jump and the drop in the water surface are little affected by the rise when $\mathbf{F}$ is high;

(3) The ratio K approaches 1.00 for all values of $\mathbf{F}$ and h/d_1 when $\mathbf{F}$ becomes very large;

(4) If h were increased ($\mathbf{F}$ and d_1 constant), d_3 must be lowered if the jump is not to move in the basin; and

(5) An empirical equation for K follows:

$$K = \frac{d_3 + h}{d_2} = \frac{d_3}{d_2} + \frac{h}{d_2} = 1 - \frac{1.40\, h}{d_2} + \frac{h}{d_2} \ldots\ldots\ldots\ldots (32a)$$

and

$$K = 1 \frac{-\,0.40\, h}{d_2} = \phi\,(h, d_1, \mathbf{F}) \ldots\ldots\ldots\ldots\ldots\ldots (32b)$$

A designer might consider the ratio d_3/d_2 of importance when the tailwater is low and a depressed basin is considered. The ratio of the depth of tailwater which may be used compared to the depth d_2 is computed for the plain jump. It is not to be considered a new variable, but may be useful in design as shown in the form of Fig. 15. Calling on the empirical data,

$$d_3 = d_2 - 1.40\, h \text{ or } \frac{d_3}{d_2} = 1 - 1.40\, \frac{h}{d_2} = K' \ldots\ldots\ldots\ldots (33a)$$

Again,

$$K' = \frac{d_3}{d_2} = \phi'\,(\mathbf{F}, h, d_1) \ldots\ldots\ldots\ldots\ldots\ldots (33b)$$

Fig. 15 shows the agreement between the experimental data and the curves computed from Eq. 33*a*. This was facilitated by choosing $d_1 = 1$, values being taken directly from Fig. 7 or being computed as a function of $\mathbf{F}$.

The difference between the theoretical and experimental curves was explained by the authors to be due to a nonuniform velocity distribution which existed at the ratio $X/(h + d_3) = 5$. They are correct in stating that, when $X/(h + d_3)$ approaches 10, the experimental data will check closer with the theoretical curves. However, nonuniform velocity distribution will not explain all the discrepancy.

The omission of a dynamic force on the face of the abrupt rise is not permissible. Although the force is small in some cases, it becomes exceedingly important when the jump is close to the rise.

As long as X is greater than the length of the jump, the assumption can be made that the jump is not changed by the rise that follows. Thus, the authors assumed that there is a drop in the water surface from a depth d_2 to a lower level on the rise. This problem can also be analyzed easily by the energy equation. The following basic equation applies as long as $X > L_J$:

$$d_1 + \frac{V^2_1}{2\,g} - \frac{(d_2 - d_1)^3}{4\,d_2\,d_1} = d_2 + \frac{V^2_2}{2\,g} = h + d_3 + \frac{V^2_3}{2\,g} + \frac{K\,V^2_3}{2\,g} \quad \ldots\ldots (34)$$

in which $\frac{(d_2 - d_1)^3}{4\,d_2\,d_1}$ is the head loss in jump; α is the correction coefficient for kinetic energy; and K is the coefficient for contraction loss at the rise. In fact, the exact theoretical curves can be duplicated by Eq. 34 if $\alpha = 1$ and $K = 0$. The writer computed a half dozen points which checked exactly.

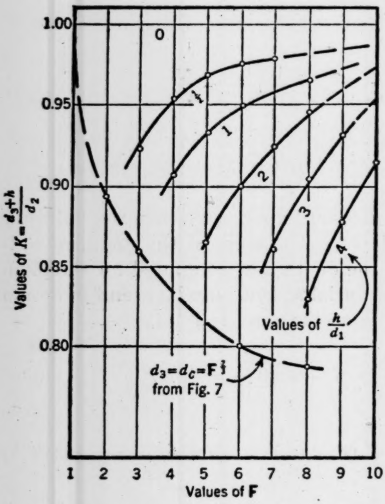

FIG. 14.—PERCENTAGE DROP IN THE WATER SURFACE OVER THE ABRUPT RISE AS A FUNCTION OF **F** AND H/d_1

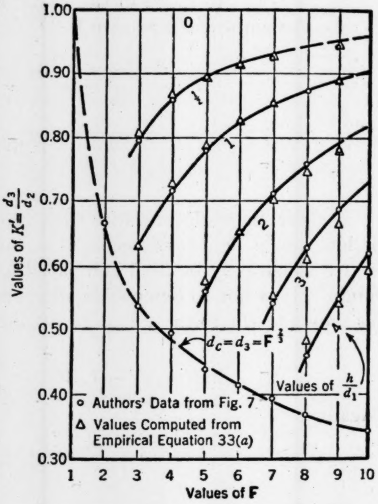

FIG. 15.—VERIFICATION OF EXPERIMENTAL DATA AND EMPIRICAL EQUATION FOR DEPTHS ON AND BEFORE THE ABRUPT RISE

It is a basic fact that the hydraulic jump can be analyzed by the energy or the momentum principles provided the head loss is known when the energy equation is applied. Thinking may be less confusing if both are applied to provide a better understanding of the problem. However, in both the energy equation and the momentum principles, it is imperative to know the velocity distribution to compute, correctly, the true momentum and energy at point 3, thus deriving an equation that checks experimental data.

The abrupt rise can easily be analyzed by the following equation as long as $X > L_J$:

$$d_2 + \frac{V^2_2}{2\,g} = h + d_3 + \frac{(\alpha + K)\,V^2_3}{2\,g} = h + d_3 + \frac{(\alpha + K)\,q^2}{2\,g\,d^2_3} \quad \ldots\ldots (35)$$

The writer computed fifteen values of $\alpha + K$ from the experimental data. This coefficient averaged 5/3 and varied very little from this value, which is a reasonable one. In Fig. 16 points are shown computed from Eq. 35, which show the close agreement, when $\alpha + K$ is taken equal to 5/3.

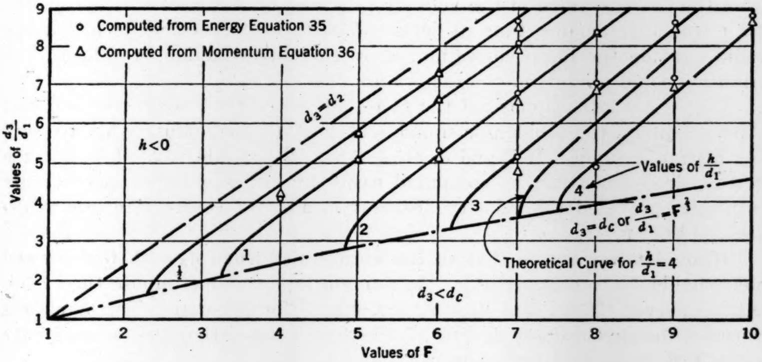

FIG. 16.—AGREEMENT BETWEEN EXPERIMENTAL CURVES, FIG. 7, AND MODIFIED MOMENTUM AND ENERGY EQUATIONS

Eq. 15, based on the momentum principle, must also be modified to include the correct coefficient for momentum at point 3 and, therefore, Eq. 15 must be modified as follows:

$$\frac{\gamma\, d^2_2}{2} - \frac{\gamma\, d^2_3}{2} - \left[\frac{\gamma\, h\,(2\, d_2 - h)}{2}\right] - \frac{2\, h\, \gamma\, q^2}{2\, g\, d^2_2} = \left[\frac{\gamma\, q^2}{g}\left(\frac{\alpha'}{d_3} - \frac{1}{d_2}\right)\right] \ldots (36)$$

in which α' is the correction coefficient for momentum; C_D is assumed to be 2.0, the value for a rectangular flat plate; and the dynamic force on the rise is

$$\mathbf{F}_d = \frac{C_D\, \gamma\, h\, V^2}{2\, g} = \frac{C_D\, \gamma\, h\, q^2}{2\, g\, d^2_2} \ldots\ldots\ldots\ldots\ldots (37)$$

Actually C_D varies, but the force is small within the experimental range, and the variation would produce a negligible difference. An average value of 1.25 was computed from the data for α'. However, it varied considerably more than $\alpha + K$, and it seemed to be a function of the Froude number. The variation of average values of α' with **F** is as follows:

α'	F
1.06	4
1.21	6
1.27	8
1.32	9
1.38	10

The use of the proper value of α' and of the dynamic force on the rise corrects Eq. 15. By using values of $\alpha' = 1.25$ and of the drag force with $C_D = 2.0$, points were computed and they are shown for comparison in Fig. 16. Of course, the correct coefficients would give exact agreement.

The fact that the momentum and energy at section 2 could be computed from the average value of the velocity also may be incorrect. However, the error would be smaller than at section 3—either the coefficients introduced could provide the lump correction, or further investigation of velocity distribution could provide more exact values.

The writer has emphasized the fact that, when the abrupt rise is placed after the jump, the problem is simple and need not be confused directly with the hydraulic jump. If **F** and d_1 are known, then d_2 and V_2 follow, and the correct drop down can be computed easily by either momentum or energy principles provided the velocity distribution and the contraction loss at the rise are known.

Weir.—It was stated that in the abrupt rise when $d_3 = d_c$ the rise and weir should be interchangeable. It is important to superimpose the experimental curves of Fig. 7 at $d_c = d_3$ onto Fig. 6 for the weir. The agreement between the two is shown in Fig. 17 where it is noted that the values for the ratio $X/d_2 = 4+$ fall perfectly into line.

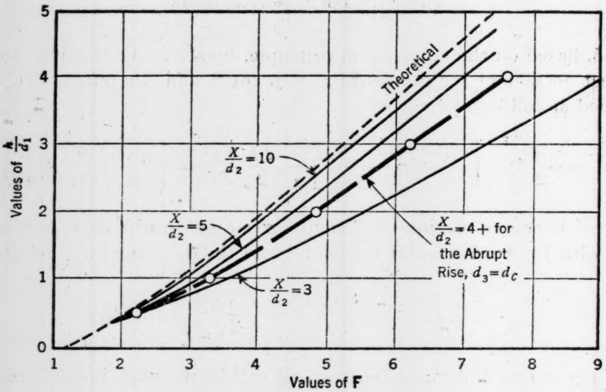

FIG. 17.—SIMILARITY OF THE ABRUPT RISE AT CRITICAL DEPTH WITH THE WEIR, FIG. 6

The two theoretical curves (at X/d_2 equals about 10), Fig. 2, also agree reasonably well. Why should not curves for the weir and abrupt rise at critical depth be the same for other ratios, as long as $X > L_J$? The same barrier height should cause approximately the same velocity distribution and force on the rise, irrespective of what happens immediately after the barrier. These observations on the close correlation between the limiting case of the abrupt rise, $d_3 = d_c$, and the weir suggest that one analysis (the abrupt rise) should suffice for both. The main criterion is whether the flow is above, or at, or below, critical depth at the rise. The trend in the analysis should be the same, although slight variations naturally may occur due to the difference between the length of the rise in the two cases. It would be most acceptable if the limiting case of the abrupt rise would eliminate the terrifying Eq. 13*a*, based on questionable coefficients, for the weir. Nothing should be that complicated.

The weir analysis is unnecessary, therefore, except to show the variation in h/d_1 in terms of x/d_2 for values of **F**. The authors state that, when the downstream slope is steep and the depth is less than critical, a weir or abrupt rise of the same height are interchangeable. Fig. 17 supports this statement.

An attempt to predict the change in the experimental curves of the abrupt rise (Fig. 7), when $x/(d_3 + h)$ is not equal to 5, is possible by combining the data for the weir with the abrupt rise. A first approximation for similar curves of other ratios of $x/(d_3 + h)$ in Fig. 7 can be obtained as follows: Project points from Fig. 6 to the line $d_3 = d_c$ of Fig. 7. Then rough curves can be sketched in parallel to the existing ones of Fig. 7.

Additional Variables.—The fact that an H3-backwater curve exists upstream from the jump may be the cause of some error if the depth d_1 is measured at a constant distance upstream from the rise, which may be upstream from the toe of the jump. As a matter of interest, if there were no backwater curves upstream and downstream from the jump, it could probably take place anywhere along the channel and would likely be unstable as to position. To move the jump upstream, the downstream depth must be increased. Some of this added depth is required simply to satisfy the new, higher Froude number (the discharge being constant). If d_1 is measured too far upstream, the depth at the toe of the jump will be higher than expected and the Froude number should really be lower. This condition, combined with a slightly lower h/d_1 than computed, would move the experimental points toward closer agreement with the theoretical curves. Hence, for precise work the depth should be measured at the toe of the jump, especially when the ratio X/d_2 varies considerably. At any rate, the backwater curve is a separate problem whose solution can be handled adequately for each situation that occurs in practice. It is not fundamentally important in the analysis of the hydraulic jump, and there is no need to duplicate model studies of the backwater curve in an already complicated problem.

The ratio d_2/d_1 must frequently be computed as a function of **F**. Ordinarily, $\mathbf{F}^2$ or $V^2_1/(g\,d_1)$ will be known by the engineer. A rapid method for approximating $d_2/d_1 = f(\mathbf{F}^2)$ mentally is as follows: Set down values of d_2/d_1 from 0 to n. Values of $\mathbf{F}^2$ are then tabulated quickly as n plus the previous $\mathbf{F}^2$-value—

$$\begin{tikzcd}
\frac{d_2}{d_1} = 0 & 1 \arrow[d] & 2 \arrow[d] & 3 \arrow[d] & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\mathbf{F}^2 = 0 \arrow[ur] & 1 \arrow[ur] & 3 \arrow[ur] & 6 \arrow[ur] & 10 & 15 & 21 & 28 & 36 & 45 & 55
\end{tikzcd}$$

The value of d_2/d_1 will be the difference between the preceding and the existing values of $\mathbf{F}^2$, or $\Delta(\mathbf{F}^2)$.

It would be of interest if the authors would state whether they measured d_2, which was assumed to exist before the rise, to see if it agreed with the computed value. In other words, is the jump preceding a barrier (as long as $X > L_J$) any different in dimensions or energy loss from a similar jump in a level channel?

The methods presented for the abrupt rise and weir cannot be used when the X-distance is less than the length of the jump. The theoretical depth d_2 would certainly not exist upstream from the rise, and possibly not at all. It would be a difficult problem. The momentum equation would be most applicable, but a study of the velocity distribution, dynamic force on the face of the rise, and assumption of hydrostatic pressure would be absolutely necessary. Research of this kind may have value in the design of even shorter satisfactory stilling basins in certain instances.

A complete study of velocity distribution is necessary in the problem presented by the authors. As long as $X > L_J$, it should be possible to express the coefficients α, α', K, and C_D as functions of X/d_1 which would complete the problem by also considering the placement of the jump with respect to the rise.

MICHELE VIPARELLI.[28]—In this remarkable study, the authors have devised a set of simple rules for determining the height, h, of a sill that will create a true hydraulic jump at a selected point in a rectangular channel, the flow being supercritical. Observations on two types of sill are reported—a sharp-crested weir and a vertical jog in the channel bottom. Of the two, the sharp-crested weir seems least practical because, more commonly, such a barrier would be in the form of a broad-crested weir. Convenient graphs are presented for the solution of equations developed from laws that govern the creation of the jump, and from laws that apply to fluid flow over weirs and in open channels. The use of nondimensional coordinates makes the application of the charts exceptionally simple.

The same problem of proportioning the sill can be solved very simply, without recourse to design graphs, with the following formula proposed by Guido Nebbia:[29,30,31,32]

$$\frac{H_2}{H_c} = 0.99 + 0.5\sqrt{\frac{\Delta H}{H_c}} - 0.025\,\frac{\Delta H}{H_c} \quad \ldots\ldots\ldots\ldots\ldots (38)$$

in which $H_1 = d_1 + \frac{V^2_1}{2\,g}$; $H_2 = d_2 + \frac{V^2_2}{2\,g}$; $H_3 = d_3 + \frac{V^2_3}{2\,g}$; $\Delta H = H_1 - H_2$; and $H_c = \frac{3}{2}\,d_c = \frac{3}{2}\sqrt[3]{\frac{q^2}{g}}$. Eq. 38 can be applied directly for the case of the abrupt rise, in which the jog in the channel is generally created by excavation of the original bottom of a depth h between sections 0–0 and 3–3 (see Fig. 18). Then the designer generally knows the value of the unit flow q, the depth d_3, and the depth d_o which is dependent on the conditions upstream from section 0–0. Fig. 18 shows that:

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[28] Asst. of the Inst. of Hydraulics, Univ. of Naples, Naples, Italy.

[29] "Sul proporzionamento di traverse a soglia prolungata per dissipatori a risalto," by Guido Nebbia, *Acqua e Gas*, Roma, 1938.

[30] "Sui dissipatori a salto di Bidone, Basi teoriche," by Guido Nebbia, *L'Energia Elettrica*, Milano, 1940.

[31] "Sui dissipatori a salto di Bidone, Ricerca sperimentale," by Guido Nebbia, *ibid*, Milano, 1940.

[32] "Sui dissipatori a salto di Bidone, Norme di proporzionamento," by Guido Nebbia, *ibid*, Milano 1940.

$$H_1 = h + d_o + \frac{V^2_o}{2\,g} = h + H_o \qquad (39a)$$

and

$$H_2 = h + d_3 + \frac{V^2_3}{2\,g} \qquad (39b)$$

As $\Delta H = H_1 - H_2$ and H_e are given implicitly, it is possible to obtain H_2 from Eq. 38 and, then, h from Eq. 39b.

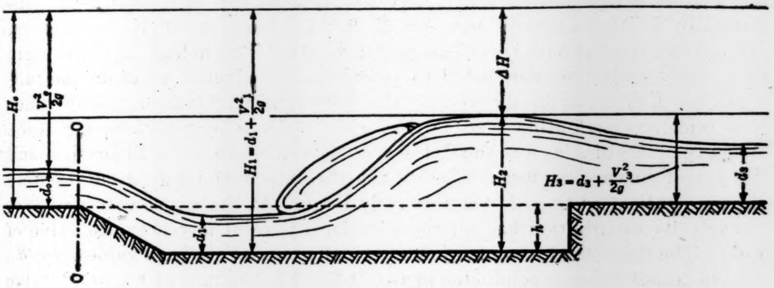
H_o
$\frac{V_o^2}{2g}$
0
d_o
0
d_1
$H_1 = d_1 + \frac{V_1^2}{2g}$
ΔH
H_2
h
$H_3 = d_3 + \frac{V_3^2}{2g}$
d_3

FIG. 18

In the case of a weir (sharp crested or broad crested) the solution for ΔH is by trial and error, solving for successively assumed values of h. Since the value of ΔH changes relatively little with corresponding values of h, the final solution generally requires no more than two trial computations.

Eq. 38 has an advantage over Fig. 6, which depends on sharp-crested weir coefficients, and is therefore convenient when the weir has a broader or other structurally resistant crest.

In their interpretation of the experimental results, the authors properly state that, had the length X been sufficient to develop the jump completely, the experiments would have checked the theory more exactly. Mr. Nebbia's experiments[29,31] confirm this observation. Under the heading, "Practical Application of Results," the authors propose "* * * that the curve for $X/d_2 = 5$ in Fig. 6 be used, under conditions of maximum discharge." Mr. Nebbia has presented evidence that there are conditions in which the highest value of h can be found at discharges less than the maximum, especially when the total head, H_1, of the flow in a lower stage is constant under varying discharges.[29,30,31,32]

RALPH M. WEAVER.[33]—In many instances it has been impossible to use a hydraulic jump because either the tailwater may have been too low, in which case the jump would not occur in the desired reach, or it may have been too high, causing the jump to be submerged. The authors have made an extensive study of two solutions for the first condition. In each, the water depth was raised, by weir action, to the depth required to form a jump.

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[33] Hydr. Engr. I, Hydr. Laboratory, Norris, Tenn.

A complete theoretical analysis, based on the assumption that submergence of the weir would not occur, was made for a weir and an abrupt rise in the channel floor. The formula for an unsuppressed weir with no velocity of approach was used with corrections applied to account for suppression of the weir, velocity approach, and nonaeration of the nappe. The theoretical analysis is interesting but, inasmuch as the values of the correction coefficients used in the derivation are approximate and the inflexibility of the formula limits its application, the practicality of the equation is questionable.

The results of the test agreed reasonably well with the theoretical analysis, but a discrepancy existed which may be explained by tests that were conducted by L. M. Laushey,[34] Jun. ASCE, R. M. Anderson,[35] J. K. Hallenburg, Jun. ASCE, and the writer[36] at Carnegie Institute of Technology, in Pittsburgh, Pa. These tests were conducted to determine the effect of a sill on the tailwater depth of the hydraulic jump. The curves in Fig. 6 indicate the presence of a velocity distribution such as that which existed in the foregoing tests. When the ratio of x/d_2 was small, the discrepancy between the theoretical and the actual curves was large, whereas the theoretical and actual curves were almost identical when x/d_2 was relatively large. At the larger values of x/d_2 the velocity distribution became more uniform than at any previous value of x/d_2. The theoretical formula applied actually only at the high values of x/d_2.

The tests[34,35,36] were conducted in a 40-ft flume 1 ft square with an adjustable inclined weir to permit fine adjustments of the tailwater depth. Sills $\frac{1}{8}$ in. wide, of varying heights extending the full width of the flume, were mounted in such a manner that the force of the water impinging on the sill could be measured by standard electrical strain gage methods. The initial depth was controlled by allowing the water to flow under an inlet plate that was fixed 0.2 ft above the floor of the flume. The general testing procedure consisted of measuring the force on the sill at various locations throughout the jump and the tailwater depths at the end of the jump. The results from the tests in which the end of the jump was over the sill were plotted as the dimensionless ratios of the initial depth, d_1, to the critical depth, d_c, and of the measured tailwater depth, d'_2, to the calculated theoretical tailwater depth, d_2, required to form a jump. It was found that the ratios of d'_2/d_2 ranged between 85% and 100% and that the values of d'_2/d_2 were a function of the ratio d_1/d_c and the sill height. The value d'_2/d_2 decreased as the sill height increased. As the sill height increased, a height was reached at which a jump would not form upstream from the sill.

It was also found that the force on the sill decreased rapidly to a minimum as the end of the jump was moved upstream to a position approximately over the sill. The force then slowly increased to a constant value as the jump was moved further upstream. It was suspected that the change in force on the sill was due to a change in the velocity distribution from one end of the jump to the other as illustrated in Fig. 19. Observations of the action in a glass sided flume partly confirmed this theory.

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[34] "Effects of Rectangular End Sills on the Hydraulic Jump, in a Level Rectangular Channel," by L. M. Laushey, Carnegie Inst. of Technology, Pittsburgh, Pa., 1947.

[35] "Forces on Rectangular End Sills in the Hydraulic Jump in a Level, Rectangular Channel," by R. M. Anderson, Carnegie Inst. of Technology, Pittsburgh, Pa., 1948.

[36] "A Study of Forces on Rectangular End Sills in a Hydraulic Jump in a Level, Rectangular Channel," by J. K. Hallenburg and R. M. Weaver, Carnegie Inst. of Technology, Pittsburgh, Pa., 1949.

The nonuniform velocity distribution increased the momentum above that existing for a uniform velocity distribution at the same tailwater depth. With the increase in momentum the pressure requirement was reduced, thereby decreasing the depth. This condition may be verified by substitution in the pressure-plus-momentum equation which must be satisfied.

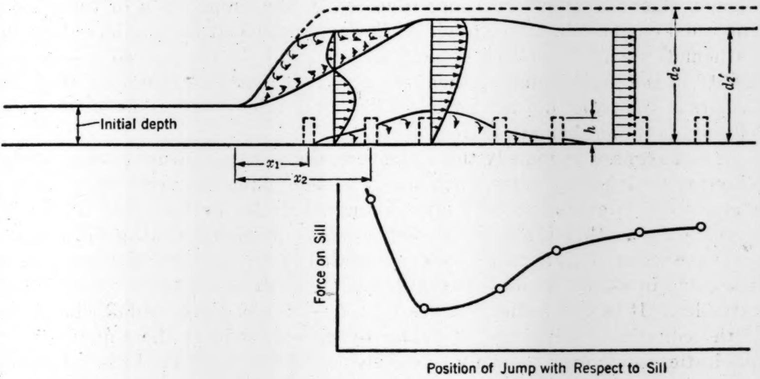

Fig. 19.—Possible Velocity Distribution Within a Hydraulic Jump with a Sill and the Force on the Sill at Various Positions of the Sill

An analysis of the results showed that: (*a*) These tests proved that placing a sill at the end of a hydraulic jump reduced the tailwater depth below the theoretical depth required to form a jump; and (*b*) the reduction in tailwater depth was caused by a change in the velocity distribution due to the introduction of the sill into the jump. The results further disclosed that the assumption of uniform velocity such as was made by Messrs. Forster and Skrinde was not entirely justified.

The tests conducted at Carnegie Institute of Technology[34,35,36] were performed under conditions similar to the tests made at the State University of Iowa in Iowa City, as reported in this paper, with one exception. In the former an attempt was made to maintain the downstream depth and the tailwater of the jump at the same level instead of using the sill to raise the tailwater. This attempt was not always successful, but it was never so unsuccessful that the sill became the control. The control was located well downstream from the sill at all times.

In view of the fact that a nonuniform velocity distribution existed in the tests conducted at the State University of Iowa, a reduction in the jump tailwater should occur. The theoretical curve should have been determined by the expected tailwater and not by the tailwater computed by the hydraulic jump formula. The tailwater that should occur would be less than that computed. Substitution of the expected tailwater into the theoretical formula will reduce the value of h/d_1, thereby lowering the curve nearer to the actual results.

Sherman M. Woodward,[37] **Hon. M. ASCE, and Floyce M. Yates.**[38]—In checking through this paper the writers discover at once that it presents an interesting mathematical exercise. They are of the opinion that every studious, serious-minded engineer, before applying the results contained in the paper to actual practice, would feel it necessary to verify the rather unusual and difficult algebraic transformations involved in the preparation of the paper. The usual engineer in active practice, however proficient he may have been in mathematics while in college, rapidly forgets many details, and soon loses the facility of the professional mathematician. It is regretful, therefore, that the authors of the paper did not include a few more brief descriptions of the steps followed in its mathematical treatment.

As an attempt to remedy this deficiency, the writers venture to suggest the following explanatory notes, particularly to the young inexperienced engineer who contemplates contributing further original studies in this field. It should be remembered that there are, of course, many possible variations in details.

If the value of V_2 from Eq. 3 is substituted in Eq. 2, the resulting equation is a cubic in d_2, which means that d_2 has three values in terms of the other variables. It is also rather obvious that $d_1 - d_2$ is a factor of both members of the equation. Dividing both sides by $d_1 - d_2$ reduces the equation to a quadratic, which may be easily solved by any of the usual methods. It must be remembered that in eliminating the factor $d_1 - d_2$ one of the roots of the equation—namely, $d_2 = d_1$, is discarded.

Substituting $g = \frac{\gamma}{\rho}$ the quadratic becomes

$$d^2_2 + d_2 \, d_1 = \frac{2 \, q \, V_1}{g} \quad \text{.....................} (40)$$

By the use of Eq. 3 the second member of this equation may also be written as $\frac{2 \, d_1 \, V^2_1}{g}$ or $\frac{2 \, q^2}{g \, d_1}$. Of these three possible forms the best one to use in the particular circumstances should be determined by the existing situation. The authors apparently chose the third or last of the three forms for their subsequent derivation of Eq. 13*a*. This is a natural procedure because Eq. 40 expresses the mathematical relation necessary for the formation of the jump, and in it the value of q given by Eq. 11 may be substituted. This substitution introduces the height necessary for the assumed controlling sharp-crested weir and so relates the effect of the weir to the formation of the jump.

In following this procedure, however, formidable and somewhat bewildering complications arise. In squaring both sides of Eq. 11, since C_2 and

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C_3 as expressed by Eqs. 12*a* and 12*b* are both binomials, their squares are each trinomials and their product contains nine terms corresponding to the nine terms contained within the large bracket in Eq. 13*a*. These terms contain powers and products of the two ratios in Eqs. 12*a* and 12*b*. It is essential sooner or later to eliminate all redundant forms of the variables. Having decided to retain as variables N and r as defined in Eqs. 13*b* and 13*c*, Fig. 1(*b*) shows that $H = d_2 - h$. Therefore, $\frac{H}{d_2} = \frac{d_2 - h}{d_2} = \frac{N - r}{N}$ and $\frac{h}{H} = \frac{r}{N - r}$.

These substitutions result in Eq. 13*a* as printed with slight variation in the third significant places. To reduce Eq. 13*a* to a standard form it is necessary to use the binomial theorem to expand the indicated powers, to perform the indicated multiplications, and to remove the denominators. The expansion produces nearly fifty terms, but by combining similar terms the number is reduced to eight. The equation then becomes

$$N^6 + N^5 = -0.113\,r^7 + 0.830\,r^6 N - 3.019\,r^5 N^2 + 6.741\,r^4 N^3 - 9.804\,r^3 N^4 + 9.206\,r^2 N^5 - 5.069\,r N^6 + 1.228\,N^7 \quad (41)$$

In plotting curve (*a*) in Fig. 2 to represent Eq. 41, it is necessary to compute a few points by the method of successive approximations. Eq. 41 is of the seventh degree in both N and r, which means that if a convenient integral value for either variable is substituted in Eq. 41 there may be seven corresponding values of the other variable which will satisfy the equation. Some of these values, perhaps most of them, may be imaginary. Fortunately the curve appears to be rather flat so that a small number of points will suffice to determine it.

It is convenient to begin with points near the origin where the values of the coordinates will be relatively small. Thus, assuming that r is to be the independent variable and N the dependent variable, let $r = 0$. Then Eq. 41 reduces to $N^6 + N^5 = 1.228\,N^7$ or $1.228\,N^2 - N = 1$.

Solving the quadratic, $N = 1.396$ or -0.583. Only positive values are probably of practical interest.

To continue, let $r = 1$ in Eq. 41. Applying Descartes' rule of signs to the resulting seventh-degree equation indicates that there may be seven positive roots or values of N which satisfy the equation. By successive trial it is found that $N = 3.56$ is one such value. Continuing similarly, sufficient points may be found to complete the drawing of curve (*a*) in Fig. 2.

It is the writers' aim to emphasize the importance of making clear the complex mathematical processes used, so that the practicing engineer may utilize with confidence the curves shown in Figs. 6 and 7.

JOHN W. FORSTER[39] AND RAYMOND A. SKRINDE,[40] JUNIORS, ASCE.—Through the discussions, for which the writers wish to express their appreciation, there has been presented some interesting material. This has contributed to the paper in three ways: Various aspects of the theoretical analysis have been presented; new interpretations of, and additions to, the experimental data have been brought to light; and, in addition, the paper has been amplified by the presentation of data for the negative sill and consideration of conditions under which the hydraulic jump does not form. These three general phases of the subject will be treated in order.

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Theoretical Analysis.—For the theoretical analysis of a sill used to stabilize a hydraulic jump (Fig. 1) there are available equations expressing the principles of continuity, momentum, and energy. Only the continuity equation, written in terms of the mean velocity and depth at any section, is independent of the velocity distribution, so that, barring air entrainment, Eq. 16 should apply in all cases. To simplify the situation it may be assumed that the jump is complete at section 2 upstream from the sill, and that the velocity distribution is uniform at sections 1, 2, and 3. Assuming negligible boundary shear, the momentum equation written between sections 1 and 2 then yields the relation between d_2/d_1 and $\mathbf{F}$ given in Eq. 4, which has been verified in previous laboratory investigations. It is still necessary to relate the sill height h to the other variables. Unfortunately, however, in the formulation of such a relation through any one of the aforementioned principles, certain difficulties arise. Thus, extension of the continuity equation entails the use of a discharge coefficient; the momentum equation between sections 2 and 3 involves the force exerted on the sill, whereas the corresponding energy equation contains a term for head loss. Such difficult terms can generally be expressed only by resorting to assumptions or empirical coefficients which are applicable over a limited range only. In this connection it is interesting to compare the ideas presented in the various discussions.

Abrupt Rise.—Considering first the energy equation between sections 2 and 3, it becomes necessary to evaluate the head loss at the abrupt rise. The assumption of zero loss would not appear unreasonable if the rise in the bottom were accomplished by a streamlined transition, but appears untenable in the case of the abrupt rise under consideration.

Nevertheless, Mr. Viparelli has suggested a solution on this basis (Eq. 39*b*). In order to determine the specific head H_2, Eq. 38 is presented, by which, it is stated: "The same problem of proportioning the sill can be solved very simply, without recourse to design graphs * * *." This equation may also be written directly in terms of the parameters used in the analysis presented in the paper:

$$\frac{\dfrac{d_2}{d_1} + \dfrac{\mathbf{F}^2}{2\,(d_2/d_1)^2}}{1.5\,\mathbf{F}^{\frac{2}{3}}} = 0.99 + 0.5\sqrt{\frac{1 + \dfrac{\mathbf{F}^2}{2} - \dfrac{d_2}{d_1} - \dfrac{\mathbf{F}^2}{2\,(d_2/d_1)^2}}{1.5\,\mathbf{F}^{\frac{2}{3}}}} - 0.025\,\frac{1 + \dfrac{\mathbf{F}^2}{2} - \dfrac{d_2}{d_1} - \dfrac{\mathbf{F}^2}{2\,(d_2/d_1)^2}}{1.5\,\mathbf{F}^{\frac{2}{3}}} \quad \text{. . (42)}$$

For several values of $\mathbf{F}$, the corresponding values of d_2/d_1 have been computed by successive approximation and plotted in Fig. 20 along with the curve corresponding to Eq. 4. The extremely close agreement reveals Eq. 38 to be simply another relation for the normal hydraulic jump, which has been complicated by the inclusion of superfluous dependent variables. Whereas Eq. 4 is explicit, Eq. 38 has the disadvantage of being implicit.

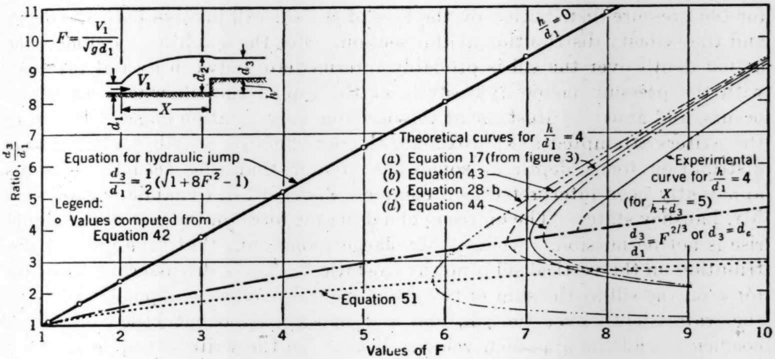


FIG. 20.—COMPARISON OF PROPOSED CURVES FOR ABRUPT RISE

By combining Eq. 39*b* (assuming no energy loss between sections 2 and 3) and the momentum equation (Eq. 2) for the hydraulic jump, the solution corresponding to Mr. Viparelli's proposal may be written as follows:

$$\frac{d_3}{d_1} + \frac{\mathbf{F}^2}{2\,(d_3/d_1)^2} = \mathbf{F}^2 \left[\frac{2}{d_2/d_1} + \frac{1}{2\,(d_2/d_1)^2} \right] - \frac{h}{d} - 1 \; \ldots\ldots\ldots (43)$$

In Fig. 20, the corresponding curve (*b*) for $h/d_1 = 4$ has been plotted for comparison with curve (*a*) presented in Fig. 3. The essential difference between writing a constant energy equation (Eq. 39*b*) and a momentum equation (Eq. 15) with hydrostatic pressure distribution for depth d_2 over the sill is embodied in the displacement of curve (*b*) to the left of curve (*a*) (Fig. 20), and away from the experimental curve.

Mr. Laushey also illustrates how the problem can be analyzed by the energy principle in Eq. 34, and states that: "* * * the exact theoretical curves can be duplicated by Eq. 34 if $\alpha = 1$ and $K = 0$. The writer computed a half dozen points which checked exactly." In considering this premise, it should be realized that the head loss in the jump $\frac{(d_2 - d_1)^3}{4\,d_2\,d_1}$ can be evaluated through the momentum equation (Eq. 2) only. Consequently, an assumption of exactly the same conditions is inherent in Eq. 2 and in the first equality expressed in Eq. 34. However, the conditions implied in the energy relation expressed on the right-hand side of Eq. 34 cannot be expected to be identical with those of the momentum equation (Eq. 15); hence, it would be a surprising coincidence if the end results of the two equations "checked exactly." Actually, Eq. 34 with $\alpha = 1$ and $K = 0$ yields the same relation (Eq. 43) as was developed from Mr. Viparelli's equations, and plotted as curve (*b*) for $h/d_1 = 4$ in Fig. 20.

Since the difference between curve (*b*) and the theoretical curve (*a*), presented in the paper, is not great (especially for lower values of h/d_1), Mr. Laushey's conclusion that they are identical is understandable.

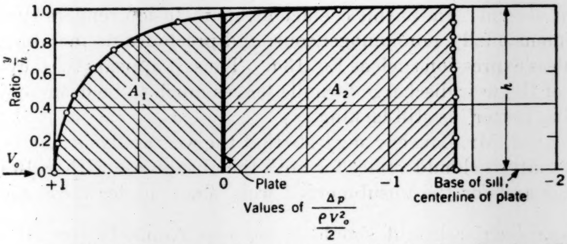
As an alternative method of relating h to the other variables, the momentum equation between sections 2 and 3 should be considered. Any rigorous analysis for the pressure distribution on the face of the sill will involve both the depth and the velocity distribution at this section. For the conditions assumed, the actual depth over the sill is probably intermediate between d_2 and $(d_3 + h)$, with the pressure below hydrostatic at the top of the sill where separation occurs, and above hydrostatic at the base due to stagnation effect. In Eq. 15 the writers attempted to approximate the net effect by a hydrostatic pressure distribution for a depth d_2 somewhat greater than the actual. This has apparently been interpreted as a complete disregard of dynamic effects, since Mr. Laushey states: "The omission of a dynamic force on the face of the abrupt rise is not permissible," whereas Mr. Jaeger points out that "the pressure distribution on the vertical sill is not hydrostatic." These discussers evaluate the force on the sill as the sum of two components—a static component based on the water depth over the sill, and a dynamic component based on a drag coefficient and the approach velocity head. To the writers it appears inconsistent to draw particular emphasis to the relatively small (for $X > L\,j$) dynamic effect while ignoring the appropriateness of the depth used in determining the static component. Thus, Mr. Laushey would superimpose a dynamic force on a hydrostatic force corresponding to the upstream depth d_2, whereas Mr. Jaeger proposes a similar procedure using instead the downstream depth. It seems reasonable to expect that the former would overestimate and the latter would underestimate the total pressure. To illustrate this point, the writers have plotted two more curves in Fig. 20 for $h/d_1 = 4$. Curve (*c*) is Mr. Jaeger's equation (Eq. 28*b*) corresponding to hydrostatic pressure due to depth $(d_3 + h)$ at the sill. Displacement of this curve to the left of curve (*a*) illustrates the considerable effect of assuming $(d_3 + h)$ rather than d_2 as the effective depth at the sill. On the other hand, to illustrate the effect of adding a dynamic force the term $0.76\, h\, \rho\, V^2{}_2/2$ has been added to the left side of Eq. 15, and the corresponding modified form of Eq. 17 is determined as follows:

$$\left(\frac{d_3}{d_1}\right)^2 + \frac{2\,\mathbf{F}^2}{(d_3/d_1)^2} = 1 + 2\,\mathbf{F}^2 + \frac{h}{d_1}\left[\frac{h}{d_1} - \frac{2\,d_2}{d_1} - 0.76\,\frac{\mathbf{F}^2}{(d_2/d_1)^2}\right] \ldots (44)$$

(The drag coefficient 0.76 corresponds to the pressure distribution on the upstream face of a submerged flat plate—see Fig. 21.) Eq. 44 is plotted for $h/d_1 = 4$ as curve (*d*) in Fig. 20. Displacement of this curve to the right of curve (*a*) embodies the effect of the added dynamic force; and it indicates the extreme right limit of the range within which experimental data would be expected to lie, when the velocity at section 2 (and this is to be emphasized) is uniform, as assumed in the analysis.

Fig. 20 reveals that differences between the various proposed solutions decrease as the relative tailwater depth increases; as tailwater depths decrease, however, the rôle of the sill becomes relatively more important, so that different assumptions lead to widely differing results.

Weir.—The energy equation appears less adaptable for the case of a sharp-crested weir than for the abrupt rise. Thus, whereas the covergence of streamlines in flow over the abrupt rise is relatively favorable to small energy losses, flow over a weir, even when submerged, is characterized by a turbulent wake; as the downstream depth is decreased, moreover, free overfall conditions are approached in which the energy losses are even more indeterminate. Thus, relating *h* to the other variables directly by an energy equation would be unfeasible.



For flat plate $C_D = A_1 + A_2 = 2.0$ For abrupt rise $C_D = A_1 = 0.76$

FIG. 21.—DRAG COEFFICIENTS CORRESPONDING TO THE PRESSURE DISTRIBUTION FOR TWO-DIMENSIONAL FLOW PAST A PLATE

It appears, however, that Mr. Viparelli had such a solution in mind for the weir in presenting Eqs. 38 and 39*b* with the assertion: "Eq. 38 has an advantage over Fig. 6, which depends on sharp-crested weir coefficients * * *." As has already been shown in Fig. 20, Eq. 38 is concerned only with the solution of the sequent depth of a normal hydraulic jump (as is the more direct Eq. 4) and, consequently, is not comparable with any figure or equation that defines a sill height. Since the auxiliary Eq. 39*b* appears meaningless when considered for the case of a weir, Mr. Viparelli's solution cannot be considered all inclusive.

The momentum equation, although applicable between sections 2 and 3 for the abrupt rise through employment of reasonable assumptions, is less flexible in the case of the weir because of the wide variation in the pressure distribution on the downstream face. If d_3 is only slightly less than d_2, application of the drag coefficient for a flat plate may be expected to approximate, closely, the total force on the sill. However, with decreasing tailwater depth, and the development of overfall conditions, the space under the nonventilated nappe will be filled with a pocket of water or water and air under reduced pressure. The obvious difficulty of formulating an expression for the force on the weir, which is applicable over any great range, renders even the momentum equation of little avail in this case.

In the face of such shortcomings of the energy and momentum equations, the writers resorted to a weir discharge equation, with the unpretentious introduction: "For want of something better, empirical coefficients may be applied * * * in order to obtain a first approximation." Admittedly, application of these coefficients has certain objections, as pointed out by Mr. Kolupaila; he also corrects certain details of the writers' statements concerning these coefficients, without, however, modifying the conditions of applicability of the complete discharge equation. Mr. Kolupaila also presents the full expression for $C_1 = 0.607 + 0.0148/H$, which the writers shortened to the approximate form $C = 0.61$. It is essential to realize that, because of the dimensional nature of the second term in this coefficient, its inclusion renders the discharge equation dimensionally nonhomogeneous, and precludes its incorporation into a dimensionless expression such as Eq. 13*a*. Thus, it appears that, even though application of the weir discharge formula leaves much to be desired, a proposal of "something better" is still lacking.

In this regard, Mr. Laushey suggests that the weir analysis may be unnecessary. One analysis should suffice for both the abrupt rise with tailwater depth below critical and for the unsubmerged weir, since, under these conditions—

> "The same barrier height should cause approximately the same velocity distribution and force on the rise, irrespective of what happens immediately after the barrier."

The writers agree with this latter statement, but would point out that the conclusion has been reached through comparison of the results of the two independent studies, and probably would not have been realized in the absence of an analysis for the weir.

Any simplification of Eq. 13*a* is to be welcomed; Mr. Woodward and Miss Yates are to be commended for developing the compact from of Eq. 41. They state that for a given value of r, solution of the corresponding seventh power equation for N is by successive approximation. Fortunately for the writers, who used 10-in. slide rules, once the trend of the rather flat curve has been established, no time need be consumed on the six superfluous roots. The need has been expressed for some description of the mathematical treatment in order to engender more confidence in the corresponding curves. In the preparation of the paper the writers assumed that the degree of confidence placed in the curves by the practicing engineer would be governed primarily, not by his understanding of the mathematical processes employed, but rather by his judgment as to the validity of the basic assumptions. It was for this reason that descriptions of the mathematical treatment were reduced to a minimum.

In view of the objections to the sharp-crested weir coefficients in an analytical treatment, and to the relative structural instability of such a sill in practice, it would appear desirable to investigate a broad-crested weir. For such a sill it has been established[41]—

> "* * * that the tail-water can be backed up to an elevation only slightly lower than the water surface at the down-stream end of the weir without changing appreciably the relation between head-water elevation and discharge."

This upper tailwater limit is defined elsewhere[42] as $d_3 = 2/3\ (d_2 + h/2)$, which condition will obtain in practically all cases in which d_3 is sufficiently low to require the use of a sill. Fortunately, a comparatively simple relation approximating the discharge over a broad-crested weir has been developed.[43] With the nomenclature of Fig. 1(*b*), this may be written

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[41] "Tests of Broad-Crested Weirs," by James G. Woodburn, *Transactions*, ASCE, Vol. 96, 1932, p. 387.

[42] "Handbook of Hydraulics," by H. W. King, McGraw-Hill Book Co., Inc., New York, N. Y., 3d Ed., 1939, p. 163.

$$q = 0.433\,\sqrt{2\,g}\left(\frac{d_2}{d_2 + h}\right)^{\frac{1}{2}} H^{\frac{3}{2}} \ldots\ldots\ldots\ldots\ldots\ldots (45)$$

Incorporating Eq. 45 into the continuity equation, and combining it with the momentum equation (Eq. 2) yields the dimensionless relationship:

$$2.667\,\mathbf{F}^2 = \frac{d_2/d_1}{d_2/d_1 + h/d_1}\left(\frac{d_2}{d_1} - \frac{h}{d_1}\right)^3 \ldots\ldots\ldots\ldots\ldots (46)$$

As d_2/d_1 is related to **F** through Eq. 4, Eq. 46 is essentially an $\left(\frac{h}{d_1}\text{-versus-}\mathbf{F}\right)$-relationship which is plotted as curve (*a*) in Fig. 22. Evidence by which the reliability of this curve for the broad-crested weir may be judged should now be brought forward. In this regard it may be observed that with $d_3 < d_c$ discharge over the abrupt rise is similarly independent of tailwater, so that the same performance from either type of sill might be expected. Substantiating this hypothesis is the close agreement with the theoretical curve (*b*)—

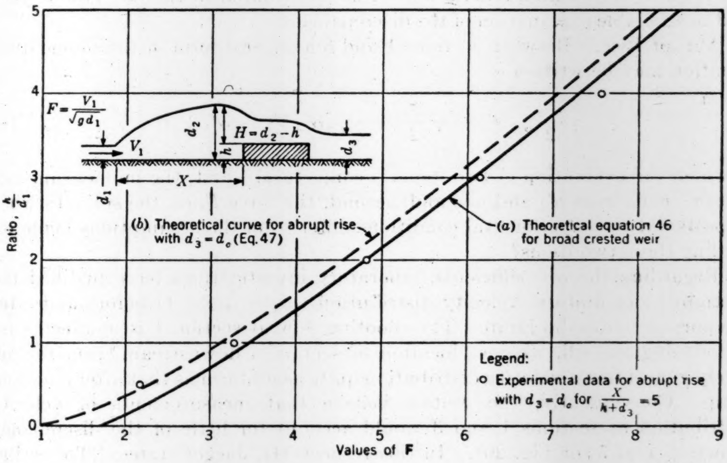

FIG. 22.—VARIATION OF RELATIVE HEIGHT OF A BROAD-CRESTED WEIR WITH F FOR THE FORMATION OF THE HYDRAULIC JUMP

$$\left(\frac{h}{d_1}\right)^2 - \left(\sqrt{1 + 8\,\mathbf{F}^2} - 1\right)\frac{h}{d_1} + \left(3\,\mathbf{F}^{4/3} - 2\,\mathbf{F}^2 - 1\right) = 0 \ldots\ldots (47)$$

—for the abrupt rise for $d_3 = d_c$ (from Fig. 2) as well as with the plotted points for the corresponding experimental data for $\frac{X}{h + d_3} = 5$ (shown thus by Mr.

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[43] "Pressure-Momentum Theory Applied to the Broad-Crested Weir," by H. A. Doeringsfeld and C. L. Barker, *Transactions*, ASCE, Vol. 106, 1941, p. 934.

Laushey in Fig. 17). Despite the lack of further experimental data, it may be surmised that curve (a), Fig. 22, may be used as a guide in proportioning such sills whenever the tailwater depth proves to be less than $2/3\ (d_2 + h/2)$. Advantages of a broad-crested sill would be its greater structural stability as compared to the sharp-crested weir, and its probable lower cost compared to the excavation necessary for the abrupt rise.

Experimental Results.—Comparison of the theoretical and experimental curves indicates to what extent the results of the combination of conditions assumed agree with those realized in the laboratory runs; it does not, however, assess the validity of each of the individual assumptions. Thus, the discrepancy between the experimental and theoretical curves in Figs. 6 and 7 may be attributed to one or more factors, and explained in different ways by assigning more or less reasonable values to those factors judged relevant. In this respect, the combined data and ideas contributed in the discussions may lead to a reliable explanation of the discrepancies.

Abrupt Rise.—Between sections 1 and 3 a general form of the momentum equation may be written

$$\gamma \frac{d^2_1}{2} - F_s - \gamma \frac{d^2_3}{2} = \rho\, q\, (\alpha'_3 V_3 - \alpha'_1 V_1) \qquad (48)$$

in which the evaluation of two items is conjectural—first, the momentum correction coefficients α'_1 and α'_3, and, second, the force F_s on the sill. In what respects did the experimental conditions differ from the assumptions made regarding these two items?

Regarding the α'-coefficients, laboratory investigations have justified the assumption of uniform velocity distribution ($\alpha'_1 = \alpha'_2 = 1$) before and after the normal hydraulic jump. The shooting flow at section 1 is unaffected by introducing the sill, whereas location of section 3 downstream from the sill surely guarantees a velocity distribution quite as uniform as that after a normal jump. Consequently, the writers believe that measurements of velocity distribution at sections 1 and 3 would account for little of the discrepancy shown in Fig. 7 (or Fig. 20). In this respect Mr. Jaeger states: "The writer has found that α'_1 and α'_3 are always very nearly equal to 1." The basis of Mr. Laushey's conclusion that α'_3-values greater than 1.33 exist (corresponding to the velocity distribution for laminar flow) must be questioned.

Regarding the second item, a uniform distribution of velocity at section 2, on which basis F_s was evaluated, can scarcely be expected for the particular jump position ($X/(h + d_3) = 5$ or $X/d_2 = 4 +$) maintained in the experimental runs. On the contrary, data for the weir indicate that the velocity distribution continues to grow more nearly uniform for $\frac{X}{d_2}$-values increasing from 5 to 10. Thus, experimental values of α'_2 considerably exceeding unity are probable; and, as Mr. Weaver has pointed out, a corresponding reduction in d_2 below that computed from Eq. 4 is required to satisfy the momentum relation. Mr. Laushey has inquired as to whether measurements were made to determine if the actual depth before the rise agreed with the computed d_2-values. Unfortunately, such data were recorded only for the weir, but regarding this question certain conclusions may be drawn from Fig. 5. Mr. Laushey's careful analysis of the experimental data has illustrated that, if the computed value of d_2 is realized, a drop in water surface $\Delta d = d_2 - (d_3 + h) = 0.40\,h$ occurs at the sill (Eq. 31 and Fig. 14). However, in the photographs no such drop in water surface is evident; on the contrary, in case (*c*), Fig. 5, the water surface appears to be rising at the sill. By re-arranging Mr. Laushey's Eq. 35 to express this drop in terms of velocity heads and head loss at the sill—

$$\Delta d = d_2 - (d_3 + h) = \alpha_3 \frac{V^2_3}{2} - \alpha_2 \frac{V^2_2}{2} - H_L \qquad (49)$$

—it may be seen that a logical explanation for this reduction in Δd lies in increased values of α_2 and V_2, and in the corresponding reduction in d_2. Accordingly it may be surmised from the photographs that in the experimental runs the velocity distribution had not yet attained uniformity before the section of the sill.

As the toe of the jump moves toward the sill, this nonuniformity of velocity distribution becomes more pronounced, while the force F_s increases correspondingly as a result of the increasing dynamic effect of the only partly diffused jet. It was this effect the writers had in mind in stating—

> "It may be shown analytically that, as the ratio $\frac{X}{h + d_3}$ increases (corresponding to a velocity distribution approaching uniformity at the section of the abrupt rise), the experimental curves shift to the left toward the theoretical curves."

The foregoing elaboration was considered desirable in view of the misinterpretation (perhaps quite justified) of this statement which was evident from Mr. Laushey's assertion that "* * * nonuniform velocity distribution will not explain all the discrepancy." In this connection also, Mr. Jaeger states: "* * * One does not need to know the value of α'_2 in profile (2), Fig. 1(*c*). * * *" Although α'_2 need not appear in the momentum equation (such as Eq. 48), actually it is this very coefficient on which the values of F_s and the correspond- position of the curves (for $X < L_j$) depend.

Mr. Laushey's attempt to affix approximate numerical values to the coefficients for velocity distribution, drag, and head loss is interesting. Thus, by writing Eq. 36 between sections 2 and 3, and making certain assumptions, he computes values of α'_3 from the experimental data. The average computed value of α'_3 is then introduced into Eq. 36, which is shown to agree with the experimental curves in Fig. 16.

The writers appreciate the enterprise of this approach, but cannot state that any evidence regarding the correctness of the α'_3-value so computed has been presented. Showing the agreement between the computed and the experimental data in Fig. 16 is analogous to testing the reliability of a general equation by applying it only to the data from which it was derived. Values of α'_3 computed from Eq. 36 are no more reliable than the assumptions on which the computations are based. These assumptions may be questioned in at least two respects: First, in Eq. 36, the value of α'_2 is assumed to be unity, with d_2 as indicated by Eq. 4 for the completed jump. As just pointed out, however, there is strong indication that α'_2 exceeds α'_3, and that the corresponding depth at section 2 is less than the assumed normal value. Secondly, selection of the value 2.0 for C_D on the basis of the drag coefficient for a flat plate does not seem justified except for the case of the baffle. As shown in Fig. 21 (which is based on a curve suggested by Hunter Rouse,[44] M. ASCE), the dynamic force on such a plate is approximately $2.0\ \rho\ V^2_o/2$; but elimination of the low pressure area on the downstream face reduces this to about $0.76\ \rho\ V^2_o/2$ for the face of the abrupt rise. Depending on the unknown magnitude of α'_2, the dynamic force on the rise may actually exceed $2.0\ h\ \rho\ V^2_2/2$, however, with V_2 defined through Eq. 4. The writers' only intention is to question the computed values of α'_3 (or $\alpha + K$ in Eq. 35) when the basis of the computations is so undependable.

It appears that all discussion has only confirmed the original premise—"* * * as the ratio $\frac{X}{h + d_3}$ increases * * * the experimental curves shift to the left toward the theoretical curves"—and has offered conjectural evaluations of the factors responsible for the observed discrepancies. In the absence of experimental data for higher relative values of X, Mr. Laushey's suggestion that an estimate be based on the data for the weir in Fig. 6 (or better, Fig. 17) should be considered. Thus, for the assumed equality of performance of the unsubmerged weir and the abrupt rise with $d_3 < d_c$, Fig. 17 indicates that increasing X/d_2 from 4+ to 5 would shift the lower end of the $\left(\frac{h}{d_1} = 4\right)$-curve in Fig. 16 to the left from $\mathbf{F} = 7.6$ to $\mathbf{F} = 7.0$, and into almost exact agreement with the theoretical curve. However, only experimental data covering a range of $\frac{X}{h + d_3}$-values can determine this effect reliably.

In Eqs. 30 and 33*a*, Mr. Laushey has presented simple relations which fit the observed data closely, within the experimental range. Use of such relations as compared to the graphs themselves would appear to be a matter of personal preference.

Weir.—Results of the tests at the Carnegie Institute of Technology, in Pittsburgh, Pa., reported by Mr. Weaver, are germane to this study. The force on the sill (and presumably the velocity of the bottom currents) appeared to reach a minimum at the end of the roller, rather than decreasing continuously from the toe of the jump to the constant value downstream as might be expected. The point where this force reaches its constant value is what the writers considered as the end of the jump in their text; apparently in this study the end of the roller has been termed the end of the jump.

Jump profiles measured in the Iowa tests, with low tailwater, seem hardly comparable with those of the Carnegie tests with high tailwater. Thus the writers found that with the toe of the jump well forward, the maximum sequent water depth occurred upstream from the weir with values ranging from 95% to 99% of the computed d_2-values. For smaller relative x-values, however, the maximum water-surface elevation occurred at, or downstream from, the weir (see Fig. 4); under these conditions the entrained air and the upward deflection of the only partly diffused jet caused greater depths than for the complete jump. Obviously, the high tailwater in the Carnegie tests would modify this flow pattern. Mr. Weaver points out that a reduced value of d_2 in the theoretical equation lowers the $\left(\frac{h}{d_1}\text{-versus-}\mathbf{F}\right)$-curve toward the experimental curves in Fig. 6.

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[44] "Elementary Mechanics of Fluids," by Hunter Rouse, John Wiley & Sons, Inc., New York, N. Y., 1946, p. 238, Fig. 121.

Mr. Weaver states "* * * an attempt was made to maintain the downstream depth and the tailwater of the jump at the same level * * *." To the writers, such an attempt would appear futile, since any net force exerted on the weir would indicate a decrease in pressure plus momentum, normally associated with a decrease in depth across the weir. If, as indicated in Fig. 19, the depth d_2 were measured downstream from the weir, this necessary drop in water surface would produce the results observed—that is, a decrease in d_2 with an increase in sill height—regardless of the presence of a jump upstream.

Mr. Viparelli correctly points out that the highest required weir does not necessarily occur under conditions of maximum discharge. The writers should have imposed the same conditions for the weir as for the abrupt rise—that the largest required value of h within the expected range of discharge be used.

Other Flow Conditions.—

Negative Sills.—Mr. Hsu's contribution was most welcome since his investigation of the negative sill has nearly doubled the scope of the paper.

The abrupt rise is in a zone of flow deflection, the force F_s on the sill varying with the relative position of the jump, which is reflected by a shift in the position of the constant $\frac{h}{d_1}$-curves. The negative sill, on the other hand, is in a zone of separation; but it would appear that, as long as the sill lies between the toe and the end of the jump, a similar variation of F_s with jump position would occur; a corresponding shift in the upper branches of the constant $\frac{h}{d_1}$-curves in Fig. 10 would be expected. Once the toe of the jump passes the sill however, F_s is independent of the jump position, so that, excepting effects of boundary resistance, no corresponding shift in the lower branches of these curves should occur.

Flow Without Jump Formation.—Before considering other flow conditions it is well to reiterate that in Fig. 3 the region lying above the line $d_3/d_1 = \mathbf{F}^{\frac{2}{3}}$ ($d_3 = d_c$) encompasses all conditions for jump formation at a positive or a negative sill. As previously explained, when the downstream channel is steep $d_3 < d_c$; then, if jump formation is desired, the sill height may be determined by considering $d_3 = d_c$. Regions lying below or to the left of this area in Fig. 3 properly represent conditions under which no jump forms and, in a strict sense, do not lie within the scope of the original title.

However, it appears desirable to make a brief review of the cases introduced by Mr. Jaeger who proposed[14] to cover all situations by two equations: (1)

For the positive sill (assuming the sill pressure governed by downstream depth), Eq. 28*b*, which forms the basis of the curves in Fig. 11; and (2) for the negative sill (assuming sill pressure determined by the upstream depth), Eq. 25, to which the curves in Fig. 12 correspond.

The writers propose that all possible cases for both positive and negative sills be represented on one diagram, the significance of each region being defined in terms of the critical depth as indicated in Fig. 23. Of the seven regions indicated, three are for positive and three for negative sills, whereas, in the seventh, the theoretical curves overlap but are without practical significance. Each of these regions will be considered briefly, with some comment as to the appropriateness of the corresponding curves in Figs. 11 and 12.

Region I represents tranquil flow over a positive sill such as occurs between sections 2 and 3 in the writers' investigation. Corresponding to the left boundary line ($\mathbf{F} = 0$) are static conditions with a level water surface over the sill. The upper limit is defined by the straight line $d_3 = d_1$ corresponding to $h/d_1 = 0$. Marking the right boundary is the line for $d_3 = d_c$, which indicates the minimum value of d_3 for equilibrium conditions.

Curves shown in Fig. 23 have an equation of the same form as Eq. 25 for the negative sill with sill pressure governed by upstream depth:

$$\mathbf{F}^2 = \frac{1}{2}\frac{d_3/d_1}{1 - d_3/d_1}\left[\left(1 - \frac{h}{d_1}\right)^2 - \left(\frac{d_3}{d_1}\right)^2\right] \qquad (50)$$

(For this case, Eq. 28*b*, assuming sill pressure governed by downstream depth, has been shown in Fig. 20 to be inapplicable; consequently, the corresponding c''-curves in the lower left of Fig. 11 appear unreliable. Two other aspects of these c''-curves must be questioned. First, their maximum $\mathbf{F}$ (or φ_1)-value does not correspond to $d_3 = d_c \left(\varphi_1 = \frac{1}{2}\eta_2^3\right)$. Second, they are extended (c''_2 branch) into the region $d_3 < d_c$ (region VII in Fig. 23) which condition demands an increase in energy at the sill, and so is an imaginary case.)

Region II represents tranquil flow over a negative sill. Here the left boundary line ($\mathbf{F} = 0$) represents static conditions with a level water surface. Defining the lower limit is the straight line $d_3 = d_1$ corresponding to $h/d_1 = 0$. The right boundary, $\mathbf{F} = 1$ ($d_1 = d_c$) marks only the transition from tranquil to shooting flow with incipient jump formation.

Judging by Mr. Hsu's investigation, Eq. 23 is applicable in this case, and the curves in Fig. 23 are plotted accordingly. The corresponding d'-curves of Fig. 12 are equivalent to Eq. 25 and the lower set of dotted lines in Fig. 9 which were discarded after the experimental investigation.

Region III, with jump formation at a negative sill, is the case investigated by Mr. Hsu. In accordance with his analysis, it may be divided into three areas, each representing a particular flow pattern. In area b, Eq. 23 still applies, the curves being continuous with those of region II, whereas in the succeeding range c, conditions are described by Eq. 25. Thus the d'-curves in Fig. 12 apply only to the narrow range c, and are equivalent to the dashed lines in Fig. 9.

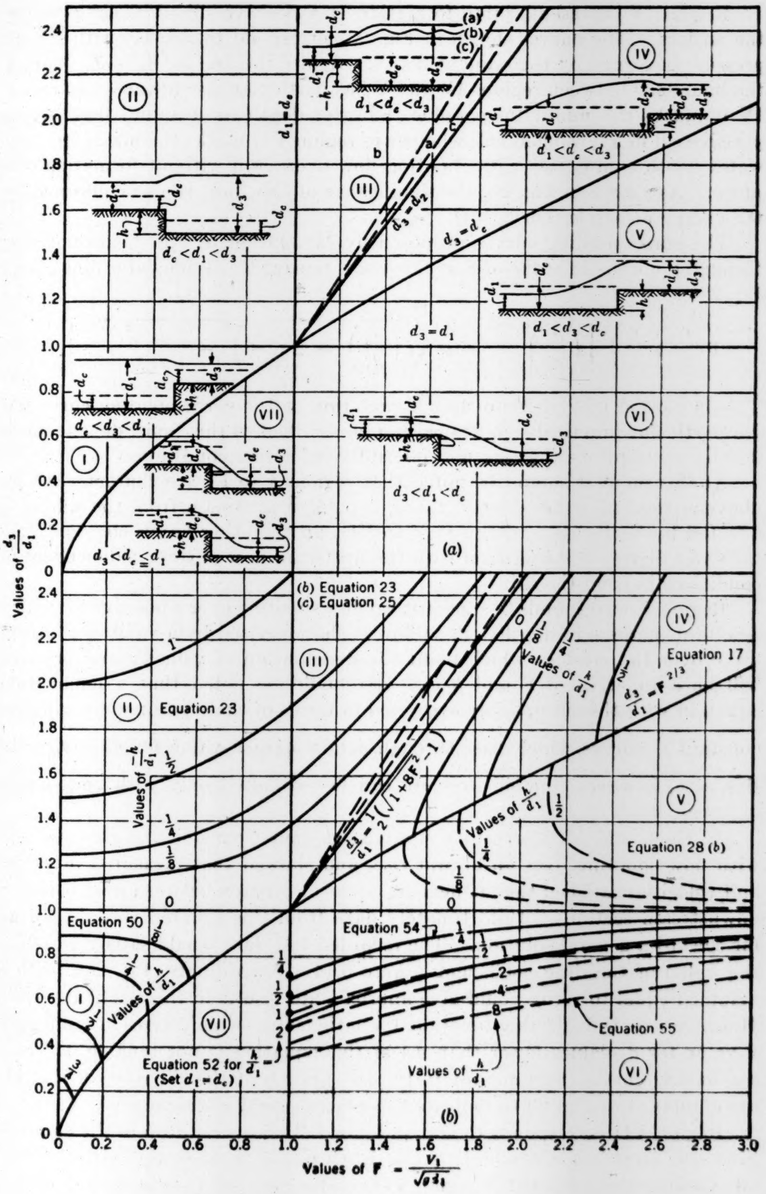

FIG. 23.—POSSIBLE CASES OF FLOW OVER SILLS

Region IV represents jump formation at a positive sill as investigated by the writers. The curves shown in Fig. 23 correspond to Eq. 17, with the sill pressure governed by the sequent depth. Again, the line for $d_3 = d_c$, defining the boundary between regions IV and V has particular significance. In region IV it marks the minimum possible tailwater depth for tranquil flow over a positive sill in a level channel; whereas in region V it marks the maximum tailwater depth compatible with shooting flow over a sill without formation of a jump. As shown in Fig. 20, the significance of this limit is also confirmed by the energy principle (curve (b)).

The corresponding curves proposed by Mr. Jaeger in Fig. 11 do not pass through a minimum value of **F** at this limit, but rather at one determined from Eq. 28*b* as

$$\mathbf{F}^2 = \left(\frac{d_3}{d_1}\right)^2 \left[\left(\frac{d_3}{d_1}\right)^2 - \frac{d_3}{d_1} + \sqrt{\left(\frac{d_3}{d_1}\right)^4 - 2\left(\frac{d_3}{d_1}\right)^3 - \left(\frac{d_3}{d_1}\right)^2 + 1}\right] \ldots (51)$$

This is plotted in Fig. 20 to show its relation to the corresponding curve (*c*). No particular practical significance can be ascribed to this limit, dependent as it is on assumptions that are not descriptive of actual conditions.

At this point it should be noted that regions I to IV are characterized by above-critical tailwater depths, the flow pattern upstream from the sill being affected by backwater. Regions V to VII, on the other hand, correspond to tailwater depths below critical with the upstream flow pattern independent of conditions beyond the sill.

Region V corresponds to supercritical flow shooting over a positive sill with a resultant increase in depth. Such flow is characterized by a swell or standing wave over the crest at which point the distribution of velocity and pressure will be a function of **F** and h/d_1. (It should be noted that a momentum equation with the sill pressure expressed in terms of these parameters will give constant $\frac{h}{d_1}$-curves which pass through a minimum **F**-value for $d_3 = d_c$.) In this case, however, the sill pressure can be estimated only with very rough approximation.

A. Koch and M. Carstanjen[15] published data from one experimental run with flow shooting over an abrupt rise, and showed that measured pressures and velocities satisfied the momentum equation written between upstream and downstream sections. Unfortunately data from this isolated case do not go far toward a general solution. The recorded test data would satisfy Eq. 27 if the value of the drag coefficient ξ were 0.14. Accordingly, Eq. 28*b* (with ξ assumed equal to zero) would give approximate results in this particular case. Hence, even though the functional relationship (which gives a minimum **F**-value for d_3-values lower than d_c) of this equation is not tenable, the corresponding curves, tantamount to the left branch of the c'-curves in Fig. 11, are employed in Fig. 23 to indicate the general trend in this region.

Region VII corresponds to tranquil flow falling over a drop in the channel. Since the crest is a control where the depth passes through the critical value, all conditions represented in region VII resolve themselves into the single case

$d_1 = d_c$ or $\mathbf{F} = 1$. Consequently, no practical significance can be attached to constant $\frac{h}{d_1}$-curves in this region.

Such flow conditions for a ventilated crest have been investigated from the point of view of energy loss at the base of a free overfall[45] by Walter L. Moore Assoc. M. ASCE. Data[46] contributed to this paper by Boris A. Bakhmeteff, Hon. M. ASCE, and N. V. Feodoroff, Assoc. M. ASCE, in connection with the nonventilated nappe reveal two regimes: (1) For relatively low falls (h/d_c less than approximately 2.5), the space under the nappe is completely filled with water; and (2) for relatively greater falls, the nappe is depressed, and the space between the nappe and the standing water is filled with air (released from entrainment) at subatmospheric pressure.

For the underfilled nappe ($h/d_c < 2.5$) Eq. 25 (to which the d''-curves in Fig. 12 correspond), assuming sill pressure governed by upstream depth, would appear suitable. However, a check with the experimental data has shown it to be even less applicable than the assumption of zero energy loss. (As explained previously, the lower branches of these d''-curves in Fig. 12 have no practical significance. Superfluous also is the extension of the curve for the normal hydraulic jump into this region, since except for regions III and IV, the case $h/d_1 = 0$ is defined by the line $d_3/d_1 = 1$.) Hence, for this case it appears feasible only to express the empirical relation determined from the experimental data:

$$\frac{1}{2}\left(\frac{1}{d_3/d_c}\right)^2 + \frac{d_3}{d_c} = \left(1.5 + \frac{h}{d_c}\right)\left(1 - 0.12\,\frac{h}{d_c}\right) \ldots\ldots\ldots\ldots (52)$$

to which the circled points in Fig. 23 for $\mathbf{F} = 1$ correspond.

For the depressed nappe ($h/d_c > 2.5$) the unknown reduced pressure and depth of standing water under the nappe make an analytical approach even more unfeasible. Results of the investigation[46] with $\frac{h}{d_c}$-values as great as about 8, lead to the empirical relation:

$$\frac{d_3}{d_c} = 0.46 + 0.014\left(\frac{h}{d_c} - 2.5\right) \ldots\ldots\ldots\ldots (53)$$

It may be noted that d_3/d_c appears to reach a minimum value of 0.46 for $h/d_c = 2.5$.

Region VI representing supercritical flow shooting over the drop has not, in so far as the writers know, been investigated systematically. Eq. 25 (to which the d''-curves in Fig. 12 correspond) appears even less applicable in this case than for tranquil flow upstream, whereas the equation based on no energy loss—

$$\mathbf{F}^2 = 2\,\frac{(d_3/d_1)^2}{(d_3/d_1)^2 - 1}\left(\frac{d_3}{d_1} - 1 - \frac{h}{d_1}\right) \ldots\ldots\ldots\ldots (54)$$

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[45] "Energy Loss at the Base of a Free Overfall," by Walter L. Moore, *Transactions*, ASCE, Vol. 108, 1943, p. 1343.

[46] *Ibid.*, p. 1364.

—would be warranted as an approximation only for relatively low drops (say, $h/d_1 < 1$). An analytical approach[47] by Merit P. White, M. ASCE, based on simplifying assumptions shows surprisingly good agreement with experimental data on ventilated nappes for the case of tranquil flow upstream $\left(\frac{h}{d_c}\text{-values as great as } 15\right)$. Extension of this analysis for ventilated nappes to the case of shooting flow yields the relation:

$$\frac{1}{(d_3/d_1)} = \frac{1}{\sqrt{2}\,\mathbf{F}} \sqrt{\frac{h}{d_1} + 1 + \frac{\mathbf{F}^2}{2} + \frac{1}{2} + \frac{1}{4\,\mathbf{F}^2}} \qquad (55)$$

In the absence of a more reliable analysis, and in order to indicate the trend of the curves in Fig. 23, Eq. 54 is plotted for $h/d_1 < 1$; and Eq. 55, for $h/d_1 > 1$.

For conditions represented in regions V to VII, there would appear advantages to plotting results in terms of a different variable—say, H_3/d_1 in which $H_3 = d_3 + V^2_3/2\,g$—inasmuch as d_3/d_1 is a relatively insensitive parameter when $d_3 < d_c$.

Summary.—Two general related ideas emerge from the foregoing review: First, it appears impossible to encompass in any specific equation the several flow regimes as defined in terms of the critical depth in Fig. 23. For instance, Eq. 27 for the positive sill (which expresses sill pressure in terms of tailwater depth and the velocity head of the approaching supercritical flow) appears generally descriptive of conditions in region V; however, it fails to represent conditions in region IV where the flow pattern is completely altered by the formation of the jump. In any rigorous analysis, ξ must be considered a drag coefficient applied to the approach velocity head—not a lump correction factor to absorb the shortcomings of the equation. It is the form of the equation, not the nature of the coefficient, that must be modified to conform to changing flow conditions.

Secondly, an analytical equation checked by experiment within a certain range of variables cannot be reliably extended even within the same region (as defined in terms of the critical depth in Fig. 23). Illustrating this fact are the three ranges revealed by Mr. Hsu's investigation in region III, as well as the different regimes observed in region VII. Similar variations in flow pattern in all probability exist in regions V and VI.

The writers hope that this review will direct attention not only to fruitful fields for further research, but also to the dangers of assuming a unique solution to be more comprehensive in scope than experimental investigation may reveal.

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[47] *Transactions*, ASCE, Vol. 108, 1943, p. 1361.

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FREQUENCY ANALYSIS OF BEAM AND GIRDER FLOORS

By Hans H. Bleich,[1] M. ASCE

With Discussion by Messrs. Robert V. Hauer, and Hans H. Bleich.

Synopsis

A procedure for finding the frequencies and modes of such three-dimensional structures as the floors in boilerhouses by utilizing the solutions of the two-dimensional vibration problems of all the beams of which the system is built up is developed in this paper. The method is explained for the case of floor systems; but the formulas apply also to any other type of structure as, for example, to multiple-story building frames or to airplanes. The method can be used with advantage for the frequency analysis of any complicated structure, and consists in resolving the system into two simpler systems, the frequencies and modes of which are easier to determine. If necessary, this simplifying process can be used repeatedly. The principle is theoretically exact but leads to the use of infinite series; for practical purposes excellent approximations are obtained by using only the first few terms of these series. A reader who wishes to apply the method should have some knowledge of the theory of oscillations and, in particular, of the fundamental notion of "normal modes" and of their properties.

The paper is presented in four sections including a numerical example (Section 4). Section 1 deals with the theory and its application to statically determinate systems. Section 2 contains an extension of the theory to include statically indeterminate systems. Section 3 is a collection of methods and formulas for reference in determining frequencies and modes of simple systems. Although some of these formulas are known, many of them are new. The purpose of Section 4 is to serve as a guide for the arrangement of practical computations. For this reason the simplest system on which all the steps of the method can be demonstrated was chosen, thus enabling the writer to discuss every detail fully. The method was devised (and has been applied successfully)

Note.—Published in October, 1949, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

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to find the frequencies of much more complicated systems than the one shown in Section 4.

Notation.—The letter symbols, adopted for use in this paper and in its discussions, conform essentially with the American Standard Letter Symbols for Structural Analysis, prepared by a Committee of the American Standards Association with ASCE participation. An important exception is that the symbols m and M in this paper are used for masses and not for bending moments. The latter are designated by $\bar{m}$ and $\bar{M}$. Symbols are defined where they first appear, in the text or by illustration. The most significant symbols are:

c = an arbitrary factor; a coefficient; c_i = a specific value corresponding with ω_i;
i = a number;
j = a number;
k = a number; a panel, or section point number on a beam;
$\bar{M}$ = moment corresponding to an externally applied load;
$\bar{m}$ = value of $\bar{M}$ for a unit load;
M = concentrated masses; dynamic reactions;
m = distributed masses;
u = deflection:
    u_d = deflection in a determinate system;
    u_X = value of u due to redundant moments X;
w = unit weight, in pounds per foot; also elastic weights;
X = redundant moments;
y = amplitude of a mode:
    $\bar{y}$ = an arbitrary function;
    $y^{(1)}, y^{(2)}, \ldots$ = successive approximations of y;
$y(t)$ = displacement, a function of t;
β = a substitution constant in Eqs. 26 and 41;
Δ = a determinant;
ϕ = amplitude of the normal mode, ϕ_i corresponding to ω_i; ϕ_A and ϕ_B = amplitudes for systems A and B, respectively;
ψ = deflection due to redundant moment $X = 1$; and
ω = circular frequency:
    ω_A and ω_B = frequencies for systems A and B, respectively;
    ω_i and ω_k = specific values of ω.

1. THEORY APPLIED TO STATICALLY DETERMINATE STRUCTURES

Fundamental Facts Concerning Oscillations.—If any mechanical system is in a state of free oscillation the only forces acting on the system are the internal elastic forces and the inertia forces. If such a system consists of the masses m and the displacements are $y(t)$, the inertia forces are

$$-m\frac{d^2y(t)}{dt^2} = m\,\omega^2\,y(t) \qquad (1)$$

in which ω is the circular frequency and t is the time, in seconds. The frequency, in cycles per minute, is

$$F = \frac{30}{\pi}\,\omega \qquad \text{(2)}$$

The displacements $y(t)$ are functions of the time and can be expressed as

$$y(t) = y \sin \omega\, t \qquad \text{(3)}$$

in which the amplitudes y are independent of the time.

A mechanical system can oscillate freely only with certain distinct frequencies ω_i. The amplitudes y_i corresponding to each of these frequencies are called the principal oscillations. It can be shown that the amplitudes y_i and y_k corresponding to two different frequencies ω_i and ω_k are "orthogonal"; that is,

$$\int y_i\, y_k\, dm = 0 \qquad \text{(4a)}$$

Eq. 4 is integrated over all the masses of the system. The amplitudes y_i can be multiplied by an arbitrary factor c, because the product $c\, y_i$ still represents amplitudes of a possible state of principal oscillation. If this factor is so chosen that

$$\int (c\, y_i)^2\, dm = \int \phi^2{}_i\, dm = 1 \qquad \text{(4b)}$$

the symbol $\phi_i = c\, y_i$ is called the "normal mode" corresponding to ω_i and Eq. 4b is an expression for the "normalizing condition." Different normal modes[2] ϕ_i and ϕ_k are orthogonal; that is,

$$\int \phi_i\, \phi_k\, dm = 0 \qquad \text{(4c)}$$

Systems with an axis of symmetry have two types of normal modes—(1) symmetrical modes for which ϕ_i has the same value in symmetrically situated points and (2) asymmetrical modes for which the value of ϕ_i in symmetrical points is equal but of opposite sign. It is usually found advantageous to consider and determine symmetrical and asymmetrical modes separately.

Two important properties of the normal modes ϕ_i are:

(a) If any mechanical system is subjected to the inertia forces $\omega^2{}_i\, m\, \phi_i$ as a static load, the resulting deflections are just the normal mode ϕ_i; and

(b) The deflections u of any mechanical system loaded with several arbitrary external forces P can be expressed as an expansion by all the normal modes of the system—

$$u = \sum_i c_i\, \phi_i \qquad \text{(5)}$$

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[2] "Mechanical Vibrations," by J. P. Den Hartog, McGraw-Hill Book Co., Inc., New York, N. Y., 2d Ed., 1940, pp. 192–196.

—provided that the forces P act only at points where masses m of the mechanical system occur. The values of the coefficients c_i are

$$c_i = \frac{1}{\omega^2_i} \int \phi_i \, dP \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

For approximate computations it is possible to consider only a few terms of Eq. 5 corresponding to the lowest frequencies of the system.

All the integrals $\int \phi_i \, \phi_k \, dm$, etc., have been written in this form to include distributed masses m and concentrated masses M. For numerical computations such integrals must be converted into an integral over the distributed masses plus a sum over all the concentrated masses; that is,

$$\int \phi_i \, \phi_k \, dm = \int m \, \phi_i \, \phi_k \, dx + \sum M \, \phi_i \, \phi_k \ldots\ldots\ldots\ldots (7)$$

Complementary Systems.—The following theoretical treatment involves two new concepts: The first is the concept of "complementary systems," which is explained in this subsection; and the other is the concept of "dynamic reactions" which will be discussed in the next subsection.

Floor systems usually consist of a number of secondary beams supported by main floor girders which rest either on external supports like columns and walls or, in complicated cases, on other beams. Restricting the subject to statically determinate arrangements, it is required to show how the complicated problem of finding the oscillations of a floor system can be solved if the frequencies and the normal modes of a simpler system, having one main girder less, are known. If the simpler system is still too complex for a direct solution, it can be reduced further until, in the end, a system is derived which consists only of ordinary beams, the frequencies and normal modes of which can be found by a number of well known methods.

One main girder is selected which, while carrying other girders of the system, rests in turn on external supports. In Fig. 1, for example, girder A should be selected. The girders normal to it would not be suitable because one end of each of these girders rests on girder A.

A simpler system is obtained by assuming the rectangular moment of inertia of girder A to be $I_A = \infty$. Thus, this girder does not deflect and all girders supported by it can be considered resting on fixed supports. This new system (simplified system B), therefore, has one girder less than the original system. It has normal modes ϕ_{iB} and frequencies ω_{iB} which, temporarily, are assumed to be known.

Apart from system B, a system A is formed which is identical with the original system as far as dimensions and mass distribution are concerned. The moment of inertia of girder A, Fig. 1, is I_A as in the original system. In all other girders, however, the moment of inertia is $I = \infty$. System A has certain natural frequencies ω_{iA} and normal modes ϕ_{iA}.

An important property of the two artificial systems A and B can now be proved:

> Theorem I.—The deflection u of any point in the original system, resulting from an arbitrary external load, is equal to the sum $u_A + u_B$ of the deflections of the two systems caused by the same load.

If the moments corresponding to the external load are called $\bar{M}$, and the moments due to a unit load, acting at the point where the deflection is contemplated, are called $\bar{m}$, then the deflection of the original system consisting of the girders a, b, c, · · ·, can be expressed as

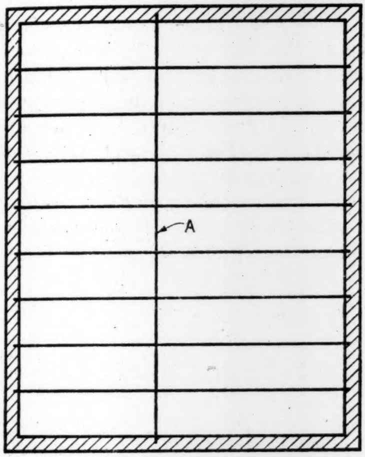

FIG. 1

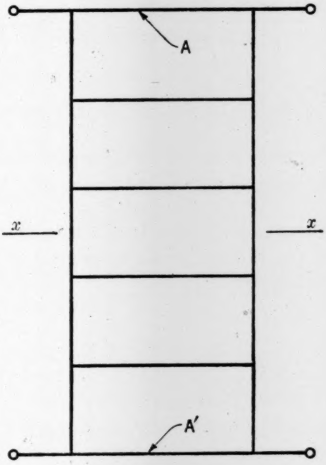

FIG. 2

$$u = \int_a \frac{\bar{M}\,\bar{m}}{E\,I_a}\,dx + \int_b \frac{\bar{M}\,\bar{m}}{E\,I_b}\,dx + \int_c \frac{\bar{M}\,\bar{m}}{E\,I_c}\,dx + \cdots \ldots\ldots\ldots (8)$$

in which the three integrals are taken over girders a, b, and c, respectively. Similar equations can be written down for systems A and B. For system A, $I_b = I_c = \cdots = \infty$, whereas, for system B, $I_a = \infty$. Consequently,

$$u_A = \int_a \frac{\bar{M}\,\bar{m}}{E\,I_a}\,dx \qquad \text{and} \qquad u_B = \int_b \frac{\bar{M}\,\bar{m}}{E\,I_b}\,dx + \int_c \frac{\bar{M}\,\bar{m}}{E\,I_c}\,dx + \cdots \ldots (9)$$

and, by comparison of Eqs. 8 and 9,

$$u = u_A + u_B \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

Because of this relation systems A and B are named "complementary systems."

In the case of symmetrical systems it is necessary to modify the procedure slightly; otherwise the advantage of the symmetry would be lost. In a case such as that in Fig. 2, system B should be obtained by making $I_A = I'_A = \infty$; system A will have moments of inertia I_A and I'_A at their original values, and in all the other beams the value of I will be ∞. The important relation expressed in Eq. 10 remains true. If the complementary systems are chosen in this way, the two systems A and B remain symmetrical and their normal modes can be separated into symmetrical and asymmetrical ones, which is of considerable advantage.

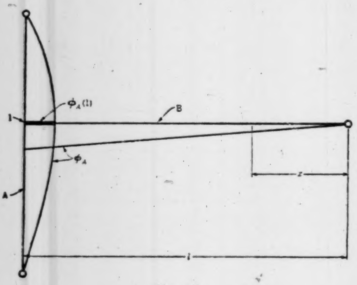

Fig. 3

Dynamic Reactions.—Except for symmetrical structures, system A consists of only one flexible girder A which supports the masses of the original system by further girders that are assumed rigid. Fig. 3 shows the simplest case possible, in which the flexible girder A supports one rigid girder B. The amplitudes ϕ_A of the normal mode are indicated in this diagram. If the amplitude of the normal mode at point 1 is $\phi_A(1)$, all the amplitudes along girder B can be expressed in the form:

$$\phi_A = \phi_A(1)\,\frac{x}{l} \qquad (11)$$

Assuming the system to be loaded with its inertia forces $\omega^2\,\phi_A\,dm$, the reaction of girder B at point 1 is

$$R_1 = \int \frac{x}{l}\,\omega^2\,\phi_A\,dm = \frac{\omega^2}{l^2}\,\phi_A(1) \int x^2\,dm \qquad (12a)$$

If system A contained a concentrated mass M_1 at point 1, the inertia force exerted by this mass would be

$$R_1 = \omega^2\,\phi_A(1)\,M_1 \qquad (12b)$$

From a comparison of the two values of R_1 in Eqs. 12, girder B is found to cause the same inertia force as a concentrated mass:

$$M_1 = \frac{1}{l^2}\int x^2\,dm \qquad (13)$$

Under the heading, "Fundamental Facts Concerning Oscillations" (Section 1), it has been stated that, when any mechanical system is loaded with the inertia forces $\omega^2_i\,m\,\phi_i$ corresponding to a normal mode ϕ_i, the deflection of the system is equal to the normal mode ϕ_i. Therefore, if the inertia force exerted by the fictional mass M_1 is equal to the inertia force exerted by the masses on girder B, it is obvious that the mechanical system A must have the same normal modes and frequencies as a system consisting only of girder A if this girder carries the additional mass M_1 assumed concentrated at point 1. For symbolic purposes the mass M_1 is termed the "dynamic reaction" of girder B.

The value of the dynamic reaction is different from the value obtained by the law of static reactions; the latter, expressed in a form similar to Eq. 13, would have been

$$R = \frac{1}{l} \int x \, dm \qquad (14)$$

For a uniformly distributed mass m the dynamic reaction becomes $m \frac{l}{3}$.

In other more complex cases the procedure is similar. The system is loaded with its inertia forces and the reactions of the rigid girders on girder A are replaced by the inertia forces of fictional masses or "dynamic reactions." The use of this concept enables the designer to reduce system A to one single girder, the natural frequencies and normal modes of which can be found by the usual methods.

In certain complicated cases the dynamic reactions become functions of the amplitudes at other points of the girder. An example of this condition is shown in Fig. 4. In system A for this structure the members will have moments of inertia $I_B = I_C = I_D = \infty$. Girder D carries a concentrated mass M_D in the center, girders B and C carry distributed masses m_B and m_C, respectively.

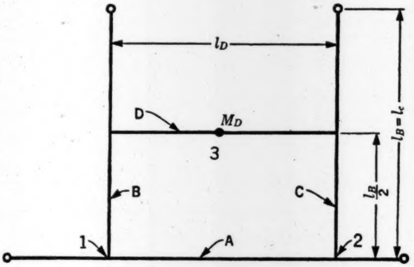

FIG. 4

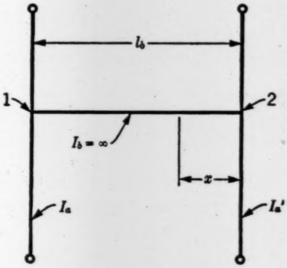

FIG. 5

The amplitudes of girder A at points 1 and 2 are $\phi(1)$ and $\phi(2)$; the dynamic reactions due to the masses on girders B and C are found, as before, to be $\frac{m_B l_B}{3}$ and $\frac{m_C l_C}{3}$, respectively. The amplitude at point 3 due to the deflection of girder A is $\frac{\phi(1) + \phi(2)}{4}$, which results in an inertia force $\omega^2 M_D \frac{\phi(1) + \phi(2)}{4}$. This force has reactions $\omega^2 M_D \frac{\phi(1) + \phi(2)}{16}$ at both point 1 and point 2. To obtain the dynamic reactions, these values must be divided by $\omega^2 \phi(1)$ and $\omega^2 \phi(2)$, respectively. The result gives total dynamic reactions at points 1 and 2 equal to

$$M_1 = \frac{m_B l_B}{3} + M_D \frac{\phi(1) + \phi(2)}{16\,\phi(1)} = \frac{m_B l_B}{3} + \frac{M_D}{16} + \frac{\phi(2)}{\phi(1)} \times \frac{M_D}{16} \qquad (15a)$$

and

$$M_2 = \frac{m_C\, l_C}{3} + M_D \frac{\phi(1) + \phi(2)}{16\, \phi(2)} = \frac{m_C\, l_C}{3} + \frac{M_D}{16} + \frac{\phi(1)}{\phi(2)} \times \frac{M_D}{16} \quad \text{..(15}b\text{)}$$

This result is not as simple as before, because M_1 and M_2 now depend on the ratio of the amplitudes at points 1 and 2. These "variable" dynamic reactions can be used without difficulty, however, to find the normal modes, applying any of the methods described subsequently in Section 3.

For symmetrical structures system A may consist of two flexible girders, but dynamic reactions can be found in a similar manner if symmetrical and asymmetrical modes are treated separately. For example, to obtain the normal modes of the system shown in Fig. 5 consider first the symmetrical modes only. The amplitudes of both girders a and a′, Fig. 5, must be alike—that is $\phi(1) = \phi(2)$. The amplitudes for all points of girder b, therefore, are $\phi(x) = \phi(1)$. The inertia loading of girder b is $\omega^2\, m\, \phi(x) = \omega^2\, m\, \phi(1)$, and the reactions become

$$R_1 = R_2 = \frac{\omega^2}{l_b} \phi(1) \int x\, dm \quad \text{.................(16}a\text{)}$$

This reaction is identical with the inertia force of a mass M_1 assumed acting at point 1 and that of a mass M_2 acting at point 2:

$$M_1 = M_2 = \frac{1}{l_b} \int x\, dm \quad \text{..................(16}b\text{)}$$

Eqs. 16 are the expressions for the dynamic reactions for symmetrical modes which, in this case, could be derived from the law of static reactions. For asymmetrical modes, $\phi(1) = -\phi(2)$ and the amplitudes of girder b are $\phi(x) = \phi(1) \dfrac{2\,x - l_b}{l_b}$. The reactions of the inertia forces are

$$R_1 = -R_2 = \frac{\omega^2}{l^2_b} \phi(1) \int x\,(2\,x - l_b)\, dm \quad \text{...........(17}a\text{)}$$

and the dynamic reactions for asymmetrical modes become

$$M_1 = M_2 = \frac{1}{l^2_b} \int x\,(2\,x - l_b)\, dm \quad \text{..............(17}b\text{)}$$

Eq. 17*b* is different from the results for symmetrical modes (Eq. 16*b*). If m is equally distributed, Eq. 16*b* yields $M_1 = \dfrac{m\, l_b}{2}$, whereas Eq. 17*b* yields $M_1 = \dfrac{m\, l_b}{6}$.

By using the concept of dynamic reactions any system A can be reduced to a one-beam system, the normal modes of which can be found.

The Normal Modes of the Original System.—Thus far, the paper has shown how a system of girders can be separated into two idealized systems A and B called "complementary systems." It remains to demonstrate that the frequencies and normal modes of the original system can be built up from those of systems A and B. The deflections of the original system due to any external load are given by Eq. 10. Furthermore, the deflections of any mechanical system can be expressed in terms of its normal modes by Eq. 5. Applying this equation to each of the two systems A and B,

$$u_A = \sum_i c_{iA}\,\phi_{iA} \quad \text{and} \quad u_B = \sum_i c_{iB}\,\phi_{iB} \ldots\ldots\ldots\ldots (18)$$

Eq. 18, substituted in Eq. 10, yields

$$u = \sum_i c_{iA}\,\phi_{iA} + \sum_i c_{iB}\,\phi_{iB} \ldots\ldots\ldots\ldots\ldots\ldots (19)$$

Eq. 19 defines the deflections of the original system due to an arbitrary external load. The coefficients c can be determined from Eq. 6; thus,

$$c_{iA} = \frac{1}{\omega^2{}_{iA}} \int \phi_{iA}\, dP \quad \text{and} \quad c_{iB} = \frac{1}{\omega^2{}_{iB}} \int \phi_{iB}\, dP \ldots\ldots (20)$$

Any normal mode ϕ of the original system can be considered as the deflection of the system due to the inertia forces $\omega^2\, m\, \phi$. Applying Eqs. 19 and 20 to this loading, Eq. 19 yields

$$\phi = \sum_i c_{iA}\,\phi_{iA} + \sum_i c_{iB}\,\phi_{iB} \ldots\ldots\ldots\ldots\ldots\ldots (21)$$

Substitution of $dP = \omega^2\, \phi\, dm$ in Eq. 20 gives

$$c_{iA} = \frac{\omega^2}{\omega^2{}_{iA}} \int \phi_{iA}\, \phi\, dm \quad \text{and} \quad c_{iB} = \frac{\omega^2}{\omega^2{}_{iB}} \int \phi_{iB}\, \phi\, dm \ldots\ldots (22)$$

The amplitude ϕ is now introduced from Eq. 21 except that, to avoid confusion, a subscript k is substituted for i in the summations:

$$c_{iA} = \frac{\omega^2}{\omega^2{}_{iA}} \left(\int \phi_{iA} \sum_k c_{kA}\, \phi_{kA}\, dm + \int \phi_{iA} \sum_k c_{kB}\, \phi_{kB}\, dm \right) \ldots (23a)$$

and

$$c_{iB} = \frac{\omega^2}{\omega^2{}_{iB}} \left(\int \phi_{iB} \sum_k c_{kA}\, \phi_{kA}\, dm + \int \phi_{iB} \sum_k c_{kB}\, \phi_{kB}\, dm \right) \ldots\ldots (23b)$$

Eqs. 23 can be written in the form:

$$\frac{\omega^2{}_{iA}}{\omega^2}\, c_{iA} = \sum_k c_{kA} \int \phi_{iA}\, \phi_{kA}\, dm + \sum_k c_{kB} \int \phi_{iA}\, \phi_{kB}\, dm \ldots\ldots (24a)$$

and

$$\frac{\omega^2{}_{iB}}{\omega^2}\, c_{iB} = \sum_k c_{kA} \int \phi_{iB}\, \phi_{kA}\, dm + \sum_k c_{kB} \int \phi_{iB}\, \phi_{kB}\, dm \ldots\ldots (24b)$$

The orthogonality relations (Eqs. 4*b* and 4*c*) supply the following equations:

$$\int \phi_{iA}\,\phi_{kA}\,dm = \begin{Bmatrix} 0 & \text{if} & i \neq k \\ 1 & \text{if} & i = k \end{Bmatrix} \ldots\ldots\ldots\ldots\ldots (25a)$$

and

$$\int \phi_{iB}\,\phi_{kB}\,dm = \begin{Bmatrix} 0 & \text{if} & i \neq k \\ 1 & \text{if} & i = k \end{Bmatrix} \ldots\ldots\ldots\ldots\ldots (25b)$$

For simplicity, substitute

$$\beta_{ik} = \int \phi_{iA}\,\phi_{kB}\,dm \ldots\ldots\ldots\ldots\ldots\ldots\ldots (26)$$

and the equations for the coefficients *c* (Eqs. 24) become simply

$$\left(1 - \frac{\omega^2{}_{iA}}{\omega^2}\right) c_{iA} + \sum_k \beta_{ik}\,c_{kB} = 0 \ldots\ldots\ldots\ldots\ldots (27a)$$

and

$$\sum_k \beta_{ki}\,c_{kA} + \left(1 - \frac{\omega^2{}_{iB}}{\omega^2}\right) c_{iB} = 0 \ldots\ldots\ldots\ldots\ldots (27b)$$

After changing the "dummy" indexes in Eq. 27*b* from *k* to *j* and from *i* to *k*, the coefficients c_{kB} can be expressed:

$$c_{kB} = \frac{1}{\dfrac{\omega^2{}_{kB}}{\omega^2} - 1} \sum_j \beta_{jk}\,c_{jA} \ldots\ldots\ldots\ldots\ldots\ldots (28a)$$

and, by substitution in Eq. 27*b*,

$$\left(1 - \frac{\omega^2{}_{iA}}{\omega^2}\right) c_{iA} + \sum_{j,k} c_{jA} \frac{\beta_{ik}\,\beta_{jk}}{\dfrac{\omega^2{}_{kB}}{\omega^2} - 1} = 0 \ldots\ldots\ldots\ldots (28b)$$

There are as many such equations as there are coefficients c_{iA}. Written fully, these equations become for $i = 1, 2, 3, \cdots, j = 1, 2, 3, \cdots,$

$$\left.\begin{array}{l} c_{1A}\left(1 - \dfrac{\omega^2{}_{1A}}{\omega^2} + \displaystyle\sum_k \frac{\beta^2{}_{1k}}{\dfrac{\omega^2{}_{kB}}{\omega^2} - 1}\right) + c_{2A} \displaystyle\sum_k \frac{\beta_{1k}\,\beta_{2k}}{\dfrac{\omega^2{}_{kB}}{\omega^2} - 1} + \cdots = 0 \\ c_{1A} \displaystyle\sum_k \frac{\beta_{1k}\,\beta_{2k}}{\dfrac{\omega^2{}_{kB}}{\omega^2} - 1} + c_{2A}\left(1 - \dfrac{\omega^2{}_{2A}}{\omega^2} + \displaystyle\sum_k \frac{\beta^2{}_{2k}}{\dfrac{\omega^2{}_{kB}}{\omega^2} - 1}\right) + \cdots = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}\right\} \ldots (29)$$

Eqs. 29 comprise a system of linear and homogeneous equations for the coefficients c_{iA}. Such equations are capable of solutions other than $c_{iA} = 0$ only if the determinant Δ of their coefficients vanishes; the condition $\Delta = 0$ is therefore the frequency equation.

To obtain Eqs. 28, it was necessary to divide Eq. 27*b* by $1 - \omega^2_{kB}/\omega^2$. This procedure is only permissible if that factor does not vanish; in other words, if ω is not equal to ω_{kB}. Therefore, the determinant of Eq. 28*b* or Eqs. 29 is not reliable as far as possible roots $\omega = \omega_{kB}$ are concerned. An investigation shows that $\omega = \omega_{kB}$ will be a frequency if all the values β_{ik} vanish, but this case is trivial and of no importance. The only other possibility that ω_{kB} may become a frequency of the original system is when ω_{kB} is a multiple frequency of system B. If $\omega_{kB} = \omega_{(k+1),B}$, the possibility of this frequency being a root ω must be investigated.

If $\omega = \omega_{kB} = \omega_{(k+1),B}$, Eqs. 27 will be satisfied if all the coefficients c_A and c_B, except c_{kB} and $c_{(k+1),B}$ vanish and provided that

$$\beta_{i,k}\, c_{kB} + \beta_{i,(k+1)}\, c_{(k+1),B} = 0 \qquad (30)$$

The number of equations similar to Eq. 30 is equal to the number of functions ϕ_{iA} used. If there is more than one mode ϕ_{iA}, Eq. 30 has solutions $c_{kB} \neq 0$ and $c_{(k+1),B} \neq 0$ only if

$$\frac{\beta_{1,(k+1)}}{\beta_{1,k}} = \frac{\beta_{2,k}}{\beta_{2,(k+1)}} = \frac{\beta_{3,(k+1)}}{\beta_{3,k}} = \cdots \qquad (31)$$

which is the necessary condition for $\omega = \omega_{kB}$ being a frequency of the system. If only one mode ϕ_A is used, Eq. 31 is automatically satisfied, and then every double root ω_{kB} creates a root of the original system. This root is not given by Eq. 28*b* and can only be deduced by referring to Eqs. 27.

The question of multiple frequencies requires attention in practical cases because similar girders, or similar groups of girders, are sometimes used repeatedly in a structure; and this fact leads to multiple frequencies ω_{kB} of system B. A case of multiple frequencies ω_{kB} is treated subsequently in Section 4.

The Frequency Equation for Special Cases.—In general, there will be as many Eqs. 29 for the coefficients c_{iA} as there are normal modes ϕ_A. As explained in the next two paragraphs, the number of these modes to be considered is only one or two.

System A is equivalent to a girder carrying a certain distribution of mass. If there are n concentrated masses such a system has n different normal modes. In addition, if part of the mass is distributed, the number of normal modes becomes infinite. All these normal modes can be arranged in order of their frequencies $\omega_{1A} < \omega_{2A} < \omega_{3A} < \cdots \omega_{nA}$. For a system consisting of one girder it will be found that ω_{2A} is considerably higher than ω_{1A}; and ω_{3A}, much higher than ω_{2A}; etc. For an equally distributed load alone the sequence of frequencies is $\omega_{1A}:\omega_{2A}:\omega_{3A} = 1:4:9:\cdots$.

If there were many coefficients, c_{iA}, the computation would become cumbersome and impracticable. Luckily, however, the expansions Eq. 21, show marked convergence and the first few terms give excellent approximations. The values of the coefficients decrease quickly due to the factors $\frac{1}{\omega^2_{iA}}$ in Eq. 22. For an equally distributed mass these factors decrease in the ratio $\frac{1}{1}:\frac{1}{16}:\frac{1}{81}:\cdots$, and the third one is practically negligible.

A rule presented subsequently under the heading, "Practical Computations, Number of Modes ϕ_A and ϕ_B" (Section 1) can be used to decide how many modes ϕ_{iA} and ϕ_{kB} should be considered to obtain reliable results. In normal cases this rule requires the use of only one or two modes ϕ_{iA}. If, in addition to a number of modes ϕ_{kB}, only one mode ϕ_{1A} is considered, the frequency equation, $\Delta = 0$, becomes simply

$$\sum_k \frac{\beta^2{}_{1k}}{\dfrac{\omega^2{}_{kB}}{\omega^2} - 1} = \frac{\omega^2{}_{1A}}{\omega^2} - 1 \ldots\ldots\ldots\ldots\ldots\ldots (32)$$

Eq. 28*a* is used to compute the amplitudes of the normal mode corresponding to any one of the roots ω of this equation; thus,

$$c_{kB} = \frac{\beta_{1k}\, c_{1A}}{\dfrac{\omega^2{}_{kB}}{\omega^2} - 1} \ldots\ldots\ldots\ldots\ldots\ldots (33)$$

By using Eqs. 25 and 26, the normalizing condition can be transformed; thus,

$$\int \phi^2\, dm = \int (c_{1A}\, \phi_{1A} + \sum_i c_{iB}\, \phi_{iB})^2\, dm$$

$$= c^2{}_{1A} + 2\, c_{1A} \sum_i c_{1B}\, \beta_{1i} + \sum_i c^2{}_{iB} = 1 \ldots (34)$$

Making further use of Eq. 27*a* the final form of the normalizing condition is

$$\omega^2{}_{1A}\, c^2{}_{1A} + \sum_i \omega^2{}_{iB}\, c^2{}_{iB} = \omega^2 \ldots\ldots\ldots\ldots\ldots\ldots (35)$$

Eqs. 33 and 35 are sufficient to determine all the values c_{1A} and c_{iB}.

In addition to the roots of Eq. 32, any multiple frequency $\omega_{kB} = \omega_{(k+1),B}$ of system B is a frequency of the original system, $\omega = \omega_{kB}$. To obtain the amplitudes of the mode corresponding to such a frequency, Eqs. 27 must be used, as demonstrated subsequently in Section 4.

Eqs. 32 to 35 enable the designer to compute the modes and frequencies of a system if the modes and frequencies of the simplified systems A and B are known. System A can always be reduced to a simple beam, the modes of which can be found by various methods. For a system B that consists of a number of ordinary beams the modes ϕ_{iB} and frequencies ω_{iB} are the total of the modes and frequencies of the separate parts of the system. If one of these beams has a certain frequency ω_i and a mode ϕ_i, system B as a whole has the same frequency $\omega_{iB} = \omega_i$. The amplitudes for the corresponding mode for any point within the beam in question are $\phi_{iB} = \phi_i$, whereas for any other point $\phi_{iB} = 0$.

If system B is not of such a simple nature but contains beams supporting one another, a preliminary computation of the modes must be made by applying the method of this paper to system B. Through repeated application of this method, one will reach a stage at which system B becomes a number of simple beams the modes of which can be found.

Discussion of the Frequency Equation.—The frequency equation (Eq. 32) can be represented graphically in a very simple and instructive manner. The abscissas in Fig. 6 are the frequencies ω and the ordinates are the values of the left-hand side of Eq. 32. For $\omega = 0$, all terms of the sum vanish; if ω approaches any one of the values ω_{kB}, one term becomes $\pm\infty$. The full curves in Fig. 6 represent the left side of Eq. 32, and the vertical dotted lines are asymptotes to these curves.

The dotted curves marked *I*, *II*, and *III* in Fig. 6 indicate the value of the right side of Eq. 32 for three different values of ω_{1A}. The quantity $\dfrac{\omega^2{}_{1A}}{\omega^2} - 1$ begins with an infinite value for $\omega = 0$, vanishes for $\omega = \omega_{1A}$, and then steadily approaches -1 for $\omega = \infty$.

The natural frequencies of the system are represented by the intersection points of the full and of the dotted curves. Fig. 6 shows that there is always one frequency that is lower than any frequency of systems A and B; and there is one frequency between each of the consecutive frequencies ω_{1B}, ω_{2B}, $\cdots$.

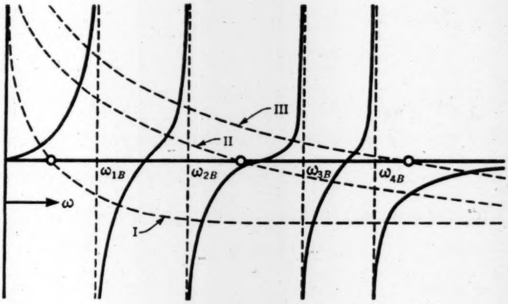

Fig. 6

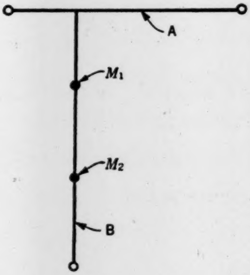

Fig. 7

If there are n frequencies, ω_{iB}, the total number of frequencies will be $n + 1$. This total number can include a trivial root $\omega = \infty$, however, which happens in the special case in which the complete system contains only masses which already form part of system B.

Fig. 7 shows an example of this condition. System B is girder B loaded with two masses. Such a system has two frequencies ω_{1B} and ω_{2B}. System A is girder A loaded with the dynamic reaction of girder B, and has one frequency ω_{1A}. Therefore, the original system should have two plus one, or three, frequencies. The system contains two masses only, however; and it has two degrees of freedom because vertical displacements only are to be counted. For mechanical reasons such a system has two natural frequencies, and no more. The frequency formula (Eq. 32) for this case, therefore, must give two proper roots. The third root must be a trivial root $\omega = \infty$ without physical meaning.

The Flooring.—Thus far only systems of beams supported by one another have been considered. The last element of a floor, however, is the flooring, which may be the open steel type, checker plating, or reinforced concrete slabs, and it is necessary to show how to introduce the flooring into the computations. Because of the small span between the secondary beams, the flooring will not deflect appreciably from beam to beam, but it must move with the supporting beams and its mass may be a considerable part of the total mass of the structure.

As a first approximation the designer could concentrate the mass of the flooring in the supporting secondary beams; however, to decide if this simplification is justified, it is necessary to make a more precise computation. If the flooring is not continuous over its supporting beams, it can be considered as a number of narrow beams spanning the secondary girders. This assumption reduces the system to a type of structure that has already been treated. In nearly all cases it will be found that the beams that replace the flooring are relatively rigid and have high natural frequencies. It is justifiable, therefore, to regard these beams as entirely rigid—that is, $I = \infty$.

Without presenting details, the result of computations, using finite difference equations, for a structure consisting of n equal secondary beams T_1, T_2, T_3, $\cdots$, T_n, with equal spacing and loading, is as follows: The values of the frequencies are

$$\omega_k = \frac{\pi^2}{l^2}\sqrt{\frac{E\,I}{m_B + \frac{2}{3}\,a\,m_s + \frac{1}{3}\,a\,m_s \cos\dfrac{k\,\pi}{n+1}}} \qquad (k = 1, 2, 3, \cdots, n)\,..(36)$$

in which m_B is the mass of the beams per unit length; m_s is the mass of the flooring per unit area; l is the span; I is the moment of inertia; and a is the spacing of the beams. The quantity $\cos\frac{k\,\pi}{n+1}$ can vary only between $+1$ and -1 and, therefore, all the frequencies must lie between the limits:

$$\frac{\pi^2}{l^2}\sqrt{\frac{E\,I}{m_B + a\,m_s}} < \omega_k < \frac{\pi^2}{l^2}\sqrt{\frac{E\,I}{m_B + \dfrac{a\,m_s}{3}}}\,\ldots\ldots\ldots\ldots(37)$$

To compare Eq. 36 with the frequency obtained if the mass of the flooring is concentrated on the beams, $m_B = m_B + a\,m_s$ and $m_s = 0$ are introduced in Eq. 36; thus,

$$\omega = \frac{\pi^2}{l^2}\sqrt{\frac{E\,I}{m_B + a\,m_s}}\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots(38)$$

The approximate value ω derived from Eq. 38 is equal to the lower limit for the exact frequencies ω_k. If $n > 2$ or if m_s is small compared to m_B, there will be little difference between the fundamental frequency ω_1 as derived from Eq. 36 and the approximate value of ω derived from Eq. 38.

Therefore, the concentration of the mass of the flooring on the beams is a good approximation for the fundamental frequency if there are more than two secondary beams. If there are two secondary beams, it is still justifiable if $a\,m_s$ is small compared with m_B; but, if higher frequencies are required, a more exact computation must be used. This conclusion remains true if the secondary beams carry unequal masses or are otherwise different.

If the secondary beams are supported by other beams forming part of a more complicated structure, the approximation can still be used as long as the frequencies which are ultimately determined remain below the lowest frequency ω of any of the secondary beams alone. On the other hand, if reliable frequencies higher than the lowest frequency are required, a more exact computation would have to be made. This condition should not occur in practical cases, however, because it would imply that the structure has a considerable number of resonant frequencies within the range of the running frequency which would make the structure unsuitable for its purpose. The difference between the approximate and the more exact solution is shown subsequently in Section 4 for a special case.

Practical Computations.—

Mass Distribution.—To obtain correct results it is of the utmost importance that the mass distribution used for the computation shall represent the actual distribution under working conditions. It would be quite erroneous to include the mass of the accidental live load, as specified for the design, in the customary manner. In most cases only the permanent load needs to be considered; but, if considerable live loads, such as those in water tanks or coal bunkers, can occur at the same time as dynamic forces generated by machines in operation, it becomes necessary to investigate two cases—the structure with live load and the structure without live load.

Another consideration is the matter of accounting for the mass of machines. A machine can usually be regarded as a rigid body supported by the structure at three or more points. Structural engineers are likely to assume that the total mass of such a machine distributes itself over the points of support in the same ratio as the static reactions. This assumption is rarely justified. A correct procedure for replacing a machine by its "dynamic reactions" at the points of support is treated in Section 3.

Number of Modes ϕ_A and ϕ_B.—As stated, it is not necessary to use all the frequencies and normal modes of the complementary systems A and B, but the higher modes can be neglected. The decision as to which modes should be retained depends on the value of the frequencies to be investigated. To be sure that a certain frequency ω_i, found by these computations, is reasonably exact, all the frequencies up to $2\,\omega_i$ should be used. It is quite obvious, furthermore, that at least one mode of each system must be considered. If system B consists of a number of separate girders, one mode of each of these should be retained.

To check the suitability of a structure for a certain frequency ω, only the resonant frequencies up to about $1.5\,\omega$ are required. Then, the limit for the frequencies of the complementary systems is $2 \times 1.5\,\omega = 3\,\omega$. Frequently this rule requires the use of only one mode of system A, and of only one mode of each of the girders forming system B.

Factor of Safety Against Resonance.—After finding the natural frequencies of a system they must be compared with the frequency of the machines carried. If the machines run at variable speed, an entire range of frequencies must be regarded as dangerous. To obtain the smallest possible vibratory deflections and to avoid resonance during the starting of the machines, the running frequency should be sufficiently lower (40% being a safe margin) than the fundamental frequency of the structure.

If this condition cannot reasonably be fulfilled, either because the running speed is high or because the span of the structure is large, the system will still prove satisfactory if the running speed is kept clear of the nearest resonant frequencies. It is essential, however, that as few frequencies as possible be below the running speed.

Concrete and Reinforced Concrete.—If the structure is concrete or if steel beams incased in concrete are used, an uncertainty is introduced because the value of E for concrete varies considerably. The only safe procedure is to determine the frequencies for limiting values of E and to regard the entire range between these frequencies as dangerous. When fixing these limits, it should be kept in mind that vibrations of structures must be restricted to small amplitudes and, therefore, to low vibratory stresses. The value of E for such low stresses is likely to be higher than the values of E used for stress computations.

Subsequent Alteration of Sections.—If the frequencies of a system are found to be unsatisfactory, the moments of inertia of some of the members are altered and the determination of the frequencies is repeated. It is not necessary to repeat the entire computation if the following property of the modes is used:

> Theorem II.—If all the moments of inertia, I, of the beams of a system are changed to αI and if the ratio α is the same for all beams and the mass distribution remains the same, the frequencies $\omega_1, \omega_2, \cdots$, of the original system are changed to $\sqrt{\alpha}\,\omega_1$, $\sqrt{\alpha}\,\omega_2, \cdots$. The numerical values of the amplitudes of the normal modes $\phi_1, \phi_2, \cdots$, corresponding to these frequencies remain entirely unaffected by such a change.

Theorem II may be applied to any system. It can be useful if changes are made in the moments of inertia which affect only the beams of one of the complementary systems. By the terms of this theorem the modes of this complementary system can be used again, and only the frequencies remain to be adjusted.

Systems That Include Statically Indeterminate Beams.—At the beginning of Section 1 the discussion was restricted to statically determinate arrangements of floors; but the method developed can be applied without modification to floors that include statically indeterminate members as long as the arrangement of the beams is such that the reactions from each beam on the others are statically determinate.

Consider the floor shown in plan in Fig. 1 and assume beam A to be part of a portal frame consisting of beam A and its supporting columns. The structure as a whole is statically indeterminate, but the reactions from the secondary beams on beam A remain statically determinate.

This type of structure can be dealt with by the method of "complementary systems" because the fundamental Eq. 10 applies. By this method it is possible to reduce the floor system gradually to one of the basic types of statically indeterminate structures, like continuous beams or portal frames. Such structures can be dealt with by known methods—as, for example, those explained in Section 3.

Statically indeterminate floors of other types require different treatment and are considered in Section 2.

2. THEORY APPLIED TO STATICALLY INDETERMINATE STRUCTURES

The procedures explained in Section 1 apply only to structures that are statically determinate or that belong to a special type of indeterminate systems mentioned at the very end of Section 1. Frequently beams are continuous because their end connections are rigid and able to transfer bending moments. To estimate the influence of rigid connections and to treat floors that contain continuous beams, or are otherwise indeterminate, the method developed in Section 1 is extended, as follows:

First, imagine that all the continuous beams are cut into simple beams, leaving a floor system of statically determinate elements. Frequencies ω_i and normal modes ϕ_i of such a system can be found by the procedure in Section 1.

Next (as in the theory of statically indeterminate structures), assume that moments X_k act on each of the points k where the continuous beams were cut. Each of the moments $X_k = 1$ acting alone on the statically determinate basic structure creates certain deflections, ψ_k. If a system is symmetrical, it is advantageous to treat symmetrical and asymmetrical modes separately. For symmetrical modes a symmetrical arrangement of moments $X_k = X'_k = 1$ can be selected as statically indeterminate values whereas for asymmetrical modes $X_k = -X'_k = 1$ is a suitable indeterminate value.

Since the deflections u of any statically indeterminate structure are equal to the deflections u_d of the statically determinate basic structure due to the same load, plus the deflections u_X due to the statically indeterminate moments X_k, the former can be expressed by the normal modes of the basic structure $u_d = \sum_i c_i \phi_i$ (see Eq. 5) and the latter are $u_X = \sum_k X_k \psi_k$. The amplitude of the normal mode, ϕ, of the statically indeterminate system can be found as the deflections u of the system due to the inertia loading $\omega^2 \phi \, dm$, from which

$$\phi = \sum_i c_i \phi_i + \sum_k X_k \psi_k \qquad (39)$$

The coefficients c_i in this expansion are defined by Eq. 6 which (after introducing $dP = \omega^2 \phi \, dm$) yields

$$c_j = \frac{\omega^2}{\omega^2_j} \int \phi \, \phi_j \, dm = \frac{\omega^2}{\omega^2_j} \sum_i c_i \int \phi_i \, \phi_j \, dm + \frac{\omega^2}{\omega^2_j} \sum_k X_k \int \psi_k \, \phi_j \, dm \qquad (40)$$

Using the orthogonality relations (Eq. 4*b* and Eq. 4*c*) the first sum on the right-hand side can be reduced to c_j. Then, replacing the dummy index j by i and introducing the abbreviation:

$$\beta_{ik} = \int \phi_i \, \psi_k \, dm \qquad (41)$$

the resulting form is

$$\left(\frac{\omega^2_i}{\omega^2} - 1\right) c_i = \sum_k \beta_{ik} X_k \ldots\ldots\ldots\ldots\ldots\ldots (42)$$

There are as many such equations as there are coefficients c_i.

Additional relations can be obtained from the theory of statically indeterminate structures. If the moments due to any loading P on a statically indeterminate structure are $\bar{M}_p$ and if the moments created by the action of any statically indeterminate moment $X_j = 1$ on the statically determinate basic structure are $\bar{m}_j$, then the condition—

$$\int \frac{\bar{M}_p \, \bar{m}_j}{E\,I} \, dx = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (43)$$

—must be satisfied. The integral has to be applied to all the beams of the system. If Eq. 43 is applied to the present system loaded with the inertia forces $\omega^2 \, \phi \, dm$, the moment $\bar{M}_p$ can be expressed by second derivative of the deflections ϕ of the system.

$$\bar{M}_p = - E\,I\,\phi'' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (44a)$$

By definition ψ_j is the deflection of the basic system due to the action of $X_j = 1$. Therefore,

$$\bar{m}_j = - E\,I\,\psi''_j \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (44b)$$

If these values are substituted for $\bar{M}_p$ and $\bar{m}_j$ in Eq. 43,

$$\int E\,I\,\phi''\,\psi''_j \, dx = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (45)$$

Substitution of the expansion (Eq. 39) in Eq. 45 yields

$$\sum_i c_i \int E\,I\,\phi''_i\,\psi''_j \, dx + \sum_k X_k \int E\,I\,\psi''_k\,\psi''_j \, dx = 0 \ldots\ldots (46)$$

The integral in the first term of Eq. 46 is the well known expression for the virtual work done by the loading $\omega^2_i \, \phi_i \, dm$, which creates the deflection ϕ_i, if the system undergoes a deflection ψ_j. According to Betti's law this virtual work must be equal to the virtual work done by the moment $X_j = 1$ if the system undergoes a deflection ϕ_i.

The virtual work done by the loads $\omega^2_i \, \phi_i \, dm$ on the deflection ψ_j is obviously $\int \omega^2_i \, \phi_i \, \psi_j \, dm$; and, with reference to Eq. 41, $\int \omega^2_i \, \phi_i \, \psi_j \, dm = \omega^2_i \, \beta_{ij}$. The virtual work done by the moments $X_j = 1$ on the deflection ϕ_i is $1 \times \gamma_{ij} = \gamma_{ij}$, in which γ_{ij} is the angle between the two tangents to the curve ϕ_i at point j. Therefore,

$$\int E\,I\,\phi''_i\,\psi''_j \, dx = \omega^2_i \, \beta_{ij} = \gamma_{ij} \ldots\ldots\ldots\ldots\ldots (47)$$

The integrals in the second term of Eq. 46 can be interpreted in a similar manner. They represent the angle α_{kj} between the two tangents to ψ_k at point j,

$$\int E\, I\, \psi''_k\, \psi''_j\, dx = \alpha_{kj} \qquad (48)$$

The value α_{kj} can easily be computed by differentiation of ψ_k. Substitution of Eqs. 47 and 48 into Eq. 46 yields

$$\sum_i \omega^2{}_i\, \beta_{ij}\, c_i + \sum_k \alpha_{jk}\, X_k = 0 \qquad (49)$$

There are as many such equations as there are statically indeterminate values of X_j. Eqs. 42 and 49 form a set of linear and homogeneous equations for the values of c_i and X_i. Such equations have solutions $c_i \neq 0$ and $X_i \neq 0$ only if the determinant Δ of their coefficients vanishes. The condition $\Delta = 0$ is therefore the frequency equation.

To obtain normalized solutions the coefficients c_i and X_i must be determined in such a way that Eq. 4*b* is satisfied. The normalizing condition can be brought into the form:

$$\sum_i c^2{}_i\, \omega^2{}_i - \sum_{j,k} \alpha_{jk}\, X_j\, X_k = \omega^2 \qquad (50)$$

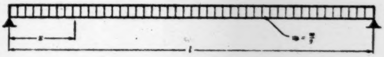

FIG. 8

To apply the formulas derived in Section 2 it is first necessary to find the values of the coefficients β_{ij} defined by Eq. 41. There would be no fundamental difficulty in computing the value of the integral in Eq. 41, but Eq. 47 gives an alternative relation which avoids the determination of any integral. From the right-hand side of Eq. 47,

$$\beta_{ij} = \frac{1}{\omega^2{}_i}\, \gamma_{ij} \qquad (51)$$

in which γ_{ij} is the angle between the two tangents of ϕ_i at point j. This angle is positive if X_j does positive work during its formation.

3. FORMULAS FOR REFERENCE

Formulas for Frequencies and Modes of Simple Beams.—A beam with distributed mass $m = \frac{w}{g}$, as shown in Fig. 8, has the frequencies:

$$\omega_i = \frac{i^2\, \pi^2}{l^2} \sqrt{\frac{E\, I}{m}} = 25{,}500\, \frac{i^2}{l^2} \sqrt{\frac{I}{w}} \qquad (52)$$

in which l is measured in feet, I, in inches^4, w, in pounds per foot, and $E = 30{,}000{,}000$ lb per in.^{2} The normal modes are

$$\phi_i = \sqrt{\frac{2}{m\, l}} \sin \frac{i\, \pi\, x}{l} \quad (i = 1, 2, 3, \cdots) \qquad (53)$$

The value of the following integral is frequently required:

$$\int_0^l m\,\frac{x}{l}\,\phi_i\,dx = (-1)^{i-1}\,\frac{\sqrt{2\,m\,l}}{i\,\pi} \qquad (54)$$

Beams or Other Structures Carrying a Small Number of Concentrated Masses. —The masses are M_1 at point 1, M_2 at point 2, etc., as shown in Fig. 9. If such a system vibrates with the circular frequency ω and the amplitudes are $y_1, y_2, \cdots$, the inertia forces acting at points 1, 2, $\cdots$ are $\omega^2 M_1 y_1$, $\omega^2 M_2 y_2$, etc. These inertia forces cause the deflections of the system; and, if the deflections due to these forces are determined, the result must be equal to the original amplitudes y_1, y_2, etc. This relationship can be used to find the natural frequencies and modes of such a system.[3]

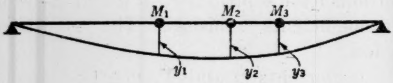

FIG. 9

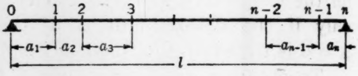

FIG. 10

If the influence numbers δ_{ik} (which indicate the deflection at any point k due to a unit load at point i) are used, the deflection y_1 due to the inertia forces can be expressed in the form:

$$y_1 = \omega^2 M_1 y_1 \delta_{11} + \omega^2 M_2 y_2 \delta_{21} + \omega^2 M_3 y_3 \delta_{31} + \cdots \qquad (55)$$

Eq. 55, and similar equations for points 2, 3, $\cdots$, can be rearranged:

$$\left.\begin{array}{l} \left(M_1\,\delta_{11} - \dfrac{1}{\omega^2}\right) y_1 + M_2\,\delta_{21}\,y_2 + \cdots = 0 \\ M_1\,\delta_{12}\,y_1 + \left(M_2\,\delta_{22} - \dfrac{1}{\omega^2}\right) y_2 + \cdots = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}\right\} \qquad (56)$$

If there are n masses, there are n such equations for the n values of y_i. Eqs. 56 are homogeneous and have only trivial solutions $y_i = 0$ unless the determinant Δ of these equations vanishes. Therefore, $\Delta = 0$ is the frequency equation:

$$\Delta \equiv \begin{vmatrix} M_1\,\delta_{11} - \dfrac{1}{\omega^2} & M_2\,\delta_{21} \cdots \\ M_1\,\delta_{12} & M_2\,\delta_{22} - \dfrac{1}{\omega^2} \cdots \\ \cdots\cdots\cdots & \cdots\cdots\cdots \end{vmatrix} = 0 \qquad (57)$$

Eqs. 57 are of the nth order in terms of $1/\omega^2$ and have n roots which need not all be different. It can be shown that all these roots $1/\omega^2$ are positive and real. Each root can be introduced in Eqs. 56 yielding a set of amplitudes y_i for each frequency ω.

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[3] "Mechanical Vibrations," by J. P. Den Hartog, McGraw-Hill Book Co., Inc., New York, N. Y., 2d Ed., 1940, p. 149.

Every set of values y_i can be multiplied by an arbitrary factor. If this factor is suitably chosen, the normalizing condition is satisfied and the amplitudes y_i become the normal mode ϕ. The normalizing condition is

$$\sum_i M_i \phi^2 = 1 \qquad (58)$$

This method is very suitable for systems with a small number of masses. If the number is high, the computations become rather involved, but approximate solutions can be derived by the matrix method.[4] The latter is not explained in this paper because such cases can be solved by the Stodola-Vianello method, which is based on the same principle. The Stodola-Vianello method is the more advantageous because it is more akin to engineers than the rather mathematical matrix method.

The Method of Stodola-Vianello.—The principle of this method is as follows:

To obtain a series of approximations of the fundamental mode assume a first approximation $y^{(1)}$ by pure guessing. Then, compute the deflections of the system due to the loading $m\, y^{(1)}$. If $y^{(1)}$ were the exact mode ϕ, or a multiple $c\,\phi$ of it, the resulting deflection $y^{(2)}$ would be equal to $\frac{1}{\omega^2}\, y^{(1)}$, because the deflections of any system due to the inertia loading $\omega^2\, m\, \phi$ corresponding to a normal mode ϕ are equal to this same mode ϕ. Since the amplitude $y^{(1)}$ is somewhat like ϕ, the amplitude $y^{(2)}$ will still be roughly similar in shape to that of $y^{(1)}$ but it will be on a different scale because of the factor $1/\omega^2$.

It can be proved that $y^{(2)}$ is nearer to the correct shape of the principal oscillation than is $y^{(1)}$. If the process is repeated, a series of approximations $y^{(1)}, y^{(2)}, y^{(3)}, \cdots$ are obtained. After a few steps these curves do not differ in shape any more—they differ only in scale and the ratios $y^{(1)}/y^{(2)}$, $y^{(2)}/y^{(3)}$, $\cdots$ are approximate values of ω^2. The most exact value is the average value of the last two steps:

$$\omega^4 = \frac{\int [y^{(n-1)}]^2\, dm}{\int [y^{(n)}]^2\, dm} \qquad (59)$$

Any method, graphical or numerical, can be used to find the consecutive deflections $y^{(1)}, y^{(2)}, y^{(3)}, \cdots$. The foregoing procedure can be applied to statically determinate as well as indeterminate structures.

One suitable method of finding the deflections of beams with either constant or variable moment of inertia I is to divide the beams in a number of short segments a (see Fig. 10) for which the moments of inertia can be assumed to be constant, and to concentrate the masses in the division points 1, 2, 3, $\cdots$. Then, the moments $\bar{M}_i$ at the division points due to the inertia loading are determined.

According to the theorem of "elastic weights" the deflections of a girder are equal to the bending moments in a beam of equal span created by the

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[4] "Mathematical Methods in Engineering," by T. von Kármán and M. A. Biot, McGraw-Hill Book Co., Inc., New York, N. Y., 1940, pp. 196–204.

"elastic weights" w_1, w_2, w_3, $\cdots$.[5] Excluding framed structures, these weights w_i are given by the formula:

$$w_i = \frac{a_i}{6\,E\,I_i}\,(\bar{M}_{i-1} + 2\,\bar{M}_i) + \frac{a_{i+1}}{6\,E\,I_{i+1}}\,(2\,\bar{M}_i + M_{i+1})\ldots\ldots(60a)$$

If the intervals a_i and the moments of inertia I_i are constant, Eq. 60*a* becomes

$$w_i = \frac{a}{6\,E\,I}\,(\bar{M}_{i-1} + 4\,\bar{M}_i + \bar{M}_{i+1})\ldots\ldots\ldots\ldots\ldots(60b)$$

If this method of "elastic weights" is used, the entire computation can be arranged in tabular form as shown subsequently in Section 4.

Since the masses M_i are now concentrated at the division points, the integrals in Eq. 59 change into sums:

$$\omega^4 = \frac{\sum M_i\,[y^{(n-1)}]^2}{\sum M_i\,[y^{(n)}]^2}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(61)$$

The amplitudes of the principal mode y are finally multiplied by a factor c and become the normal mode $\phi = c\,y$. This factor c is found from the normalizing condition:

$$\sum_i M_i\,\phi^2 = 1\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(62)$$

If a system is symmetrical, the method can be used to find the lowest symmetrical and the lowest asymmetrical modes. These two modes are obtained by starting once with a symmetrical curve $y^{(1)}$, and once with an asymmetrical curve $y^{(1)}$.

A Generalization of the Stodola-Vianello Method.—The Stodola-Vianello method can be used to find the fundamental frequency of simple beams but it is possible to extend it further and to determine higher frequencies. Making use of the fact that the fundamental mode ϕ_1 and any higher mode ϕ are orthogonal,

$$\int \phi_1\,\phi\,dm = 0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(63a)$$

the designer can find the second mode by the same procedure, if he begins with an assumed first approximation $y^{(1)}$ which is orthogonal to ϕ_1; thus,

$$\int \phi_1\,y^{(1)}\,dm = 0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(63b)$$

Functions $y^{(1)}$ which satisfy this condition can be expressed and found by the formula:

$$y^{(1)} = \bar{y} - \phi_1 \int \phi_1\,\bar{y}\,dm\ldots\ldots\ldots\ldots\ldots\ldots(64)$$

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[5] "Statically Indeterminate Stresses," by J. I. Parcel and G. A. Maney, John Wiley & Sons, Inc., New York, N. Y., 2d Ed., 1936, p. 47.

in which $\bar{y}$ is an arbitrary function. This relation can easily be checked by substituting it in Eq. 63*b*.

Beginning with a function $y^{(1)}$ according to Eq. 64 further approximations can be obtained by following the same procedure as for the first mode. If the first mode is known exactly, it would be sufficient to start with a function $y^{(1)}$ which is orthogonal to ϕ_1 and every subsequent approximation would remain orthogonal to ϕ_1; but ϕ_1 is known approximately only, and, therefore, Eq. 64 gives a function $y^{(1)}$ which is approximately orthogonal in relation to the exact amplitudes ϕ_1. Therefore, it is necessary to insure that $y^{(i)}$ remains orthogonal to ϕ_1 by applying Eq. 64 to the subsequent approximations $y^{(2)}, y^{(3)}, \cdots$.

If further modes are required the same process can be used, but the normalizing condition—

$$y = \bar{y} - \phi_1 \int \phi_1 \bar{y}\, dm - \phi_2 \int \phi_2 \bar{y}\, dm - \cdots \qquad (65)$$

—must be employed. This procedure insures that the approximations are orthogonal to all the lower modes which are already known. The convergence of the process for higher frequencies is not as good as for the fundamental one; but the first four or five frequencies can be computed with reasonable accuracy.

Replacing Rigid Bodies by Their Dynamic Reactions.—The masses supported by a beam frequently comprise machines that can be considered as rigid bodies. For computations it is desirable to replace such bodies by equivalent imaginary masses placed at the points of support. This can be done by using the concept of "Dynamic Reactions" described in Section 1.

A common case is shown in Fig. 11(*b*). The machine is supported at four points 1, 2, 3, and 4 which rest on the girders T$_1$ and T$_2$. Such an arrangement is statically indeterminate because a rigid body requires only three supports.

There is an important case in which this difficulty disappears. If axis $y-y$ is an axis of symmetry of the machine and if the amplitudes of girders T$_1$ and T$_2$ are equal, or approximately equal, each girder carries one half of the load. The mass supported by girder T$_1$ is M and equals half the total mass of the machine (see Fig. 11(*a*)); J is the polar moment of the mass M in relation to point O, which is the projection of the center of gravity on the girder axis. If the beam oscillates with amplitudes y_1 and y_2 as shown in (Fig. 11(*a*)), the movement of the machine consists of a vertical movement and of a rotation around point O. Therefore, the inertia forces are a vertical force $\omega^2 M \dfrac{l_b y_1 + l_a y_2}{l_a + l_b}$ and a couple $\omega^2 J \dfrac{y_1 - y_2}{l_a + l_b}$. Their reactions at point 1 and point 2 are

$$R_1 = \omega^2 M \frac{l_b^2 y_1 + l_a l_b y_2}{(l_a + l_b)^2} + \omega^2 J \frac{y_1 - y_2}{(l_a + l_b)^2} \qquad (66a)$$

and

$$R_2 = \omega^2 M \frac{l_a l_b y_1 + l_b^2 y_2}{(l_a + l_b)^2} - \omega^2 J \frac{y_1 - y_2}{(l_a + l_b)^2} \qquad (66b)$$

Imaginary masses at points 1 and 2 would create inertia forces:

$$R_1 = \omega^2 M_1 y_1 \quad \text{and} \quad R_2 = \omega^2 M_2 y_2 \qquad (67)$$

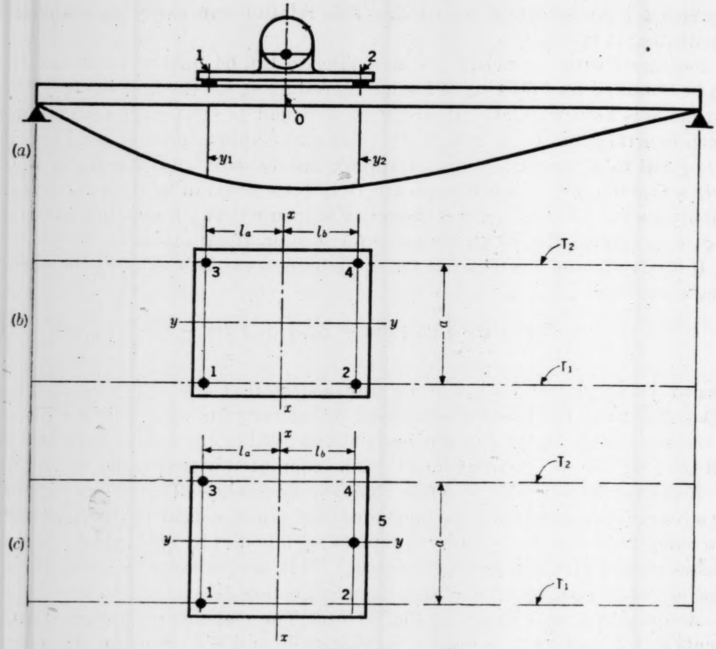

1
2
0
(a)
y_1
y_2
x
l_a
l_b
T_2
3
4
(b)
y
y
a
1
2
T_1
x
x
l_a
l_b
T_2
3
4
5
(c)
y
y
a
1
2
T_1
x

FIG. 11

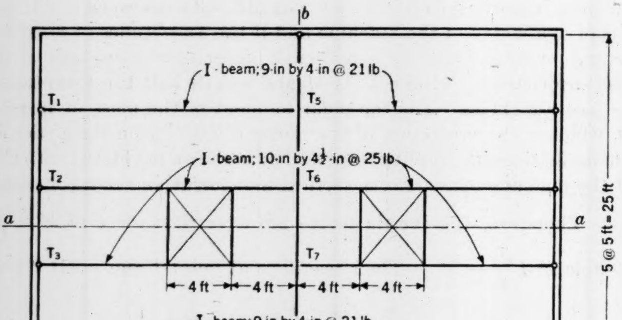

b
I - beam; 9-in by 4-in @ 21 lb
T_1
T_5
I - beam; 10-in by $4\frac{1}{2}$-in @ 25 lb
T_2
T_6
a
a
T_3
T_7
4 ft
4 ft
4 ft
4 ft
5 @ 5 ft = 25 ft
I - beam; 9-in by 4-in @ 21 lb
T_4
T_8
I - beam; 20-in by $6\frac{1}{2}$-in @ 65 lb
16 ft
16 ft
b

FIG. 12

Accordingly, the machine can be replaced by the imaginary masses:

$$M_1 = \frac{l_b^2 M + J}{(l_a + l_b)^2} + \frac{l_a l_b M - J}{(l_a + l_b)^2} \frac{y_2}{y_1} \qquad (68a)$$

and

$$M_2 = \frac{l_a^2 M + J}{(l_a + l_b)^2} + \frac{l_a l_b M - J}{(l_a + l_b)^2} \frac{y_1}{y_2} \qquad (68b)$$

These fictional masses M_1 and M_2 still depend on the ratio y_1/y_2 but they can, nevertheless, be used for frequency computations without any difficulty (see Section 4). Eqs. 68 apply for the case of equal amplitudes $y_1 = y_2$, in the form:

$$M_1 = M \frac{l_b}{l_a + l_b} \quad \text{and} \quad M_2 = M \frac{l_a}{l_a + l_b} \qquad (69)$$

Eq. 69 indicates a mass distribution according to the law of statical reactions. It is correct to use Eq. 69 only if the amplitudes of the points of support are equal or at least approximately equal.

If the machine is not symmetrical or if the girders T_1 and T_2 do not deflect equally, it is necessary to remove one of the redundant supports, analyze the remaining system, and afterward consider the effect of the reintroduction of this support by the method developed in Section 2 for statically indeterminate structures.

To demonstrate the principle, consider again the machine shown in Fig. 11(*b*). The machine still has the axis of symmetry $y-y$ but the deflections of T_1 and T_2 are assumed to be different.

The total mass of the machine is 2 M and there will be two polar moments J_x and J_y which refer to axes parallel to $x-x$ and $y-y$, respectively. In order not to lose the advantage of the symmetry but still to obtain a statically determinate case, remove the supports at points 2 and 4 and support the machine instead at point 5 halfway between points 2 and 4 by an imaginary rigid girder that rests on girders T_1 and T_2 at points 2 and 4 (see Fig. 11(*c*)).

Expressions for the fictional masses M_1, M_2, M_3, and M_4 as functions of the amplitudes y_1, y_2, y_3, and y_4 can be derived in a manner similar to that used for Eqs. 68. The frequencies and modes of the system carrying these imaginary masses can then be found by the method given in Section 1. The modes of this statically determinate basic system can then be used, as explained in Section 2, to find the frequencies and modes of the actual statically indeterminate system. The statically indeterminate values X defined in Section 2, in this case, are a couple of vertical forces acting at points 2 and 4 on the machine, and an opposite couple of equal size acting at points 2 and 4 on the girders T_1 and T_2. The value of these couples will be such that the imaginary gap between the machine and the supporting girders is closed.

It should be emphasized that, to obtain fairly exact results, it is really necessary to use the procedure indicated in this subsection. To distribute the mass M over the supports simply following the laws of static reactions and to disregard the restraint due to the indeterminate four-point support of machines would lead to considerable errors in the frequencies (particularly in the higher frequencies) except when the machines represent only a small part of the total mass of the system.

The dynamic reactions thus found are used to replace the actual masses of rigid bodies all through the computations. To determine the value of an expression such as $\int \phi^2\, dm$, the amplitudes y_i in Eqs. 68 must be interpreted to mean the values of ϕ at points i. In the case of an integral $\int \phi_A\, \phi_B\, dm$, the dynamic reactions can be computed by introducing the amplitudes of either ϕ_A or ϕ_B into these equations; but ϕ_A, or ϕ_B, must be used consistently for all dynamic reactions. The final value of the integral does not depend on which of the two is used.

4. Illustrative Example

To demonstrate the application of the theory developed in Section 1, consider a rather simple example which, nevertheless, will serve as a guide to procedure in more complicated cases. The structure is shown in Fig. 12. The 25-ft-span cross girder T_9 supports eight secondary 16-ft-span floor beams T_1 to T_8, which in turn carry open steel flooring $1\frac{1}{2}$ in. deep weighing 14 lb per sq ft. The secondary beams T_2, T_3, T_6, and T_7 carry two identical machines, each weighing 6,720 lb, including the base but without any impact allowance. The beams shown in Fig. 12 are British standard sections, and would be suitable to carry the dead load, the weight of the machines plus 50% impact, and a distributed live load of 100 lb per sq ft. It is desired to investigate whether this structure is safe against resonance if the machines run at 320 rpm.

Only the masses that are on the structure under working conditions should be taken into account. These are the masses of the flooring, of all the beams, and of the machines without impact. Under normal working conditions no appreciable live load occurs and no allowance will be made for it. For the computation of the lowest frequency of a system the distributed mass of the flooring can be assumed concentrated on the secondary floor beams (see Section 1). Making use of this simplification, the system consists of only the nine girders T_1 to T_9.

The system has two axes of symmetry $a-a$ and $b-b$ as indicated in Fig. 12. In general, computations can be simplified by considering symmetrical and asymmetrical modes separately. In this particular case it is advantageous to use the axis of symmetry $a-a$ and the modes which are symmetrical in relation to $a-a$ are considered separately from those which are asymmetrical in relation to $a-a$. For structures that do not contain continuous beams the lowest frequency corresponds, as a rule, to a symmetrical mode and, therefore, the designer needs to compute symmetrical modes only. In computations of this nature it is essential to keep the dimensions uniform. Throughout this example all distances are in inches, weights are in pounds, and time is in seconds —unless definitely stated otherwise. The acceleration of the earth in these dimensions is $g = 386$ in. per sec.^{2}

Choice of Complementary Systems.—As explained in Section 1, the system is divided into two complementary systems A and B. Girder T_9 (Fig. 12) is the only one that rests entirely on external supports. To form system B, therefore, assume girder T_9 to have $I_9 = \infty$, all other girders retaining their actual moments of inertia. System A is obtained by leaving the moment of inertia of girder T_9 at its original value, the value of I to be ∞ for all other girders. Referring to system B, the assumption $I_9 = \infty$ transforms all the girders T_1 to T_8 into simple beams on fixed supports. The normal modes of system B, therefore, comprise all the modes of the girders T_1 to T_8. These modes can be found by the methods in Section 3. The choice of complementary systems, in this case, has not been influenced by any consideration of symmetry; these systems can be used for both types of modes, symmetrical and asymmetrical.

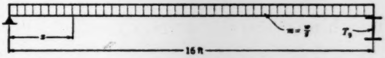


FIG. 13

The Normal Modes of System B.—System B consists of the eight secondary beams which (as a result of the simplification concentrating the mass of the flooring on the beams) oscillate independently from each other. The four girders T_1, T_4, T_5, and T_8 (see Fig. 13) are mechanically alike and carry an equal mass. Their frequencies and modes must be identical. The equally distributed load w is $5 \times 14 + 21 = 91$ lb per ft. The total mass per girder is $m\,l = \frac{w\,l}{g} = \frac{91 \times 16}{386} = 3.77\ \frac{\text{lb-sec}^2}{\text{in.}}$. The moment of inertia is $I = 81.13$ in.^{4} For $i = 1$, Eq. 52 gives the fundamental frequency of these beams as: $\omega_{1B} = \omega_{4B} = \omega_{5B} = \omega_{8B} = \frac{25{,}500}{16^2}\sqrt{\frac{81.13}{91}} = 94.0$. The amplitudes of the modes are given by Eq. 53: $\phi_{1B} = \phi_{4B} = \phi_{5B} = \phi_{8B} = \sqrt{\frac{2}{m\,l}}\sin\frac{\pi x}{l} = 0.728 \sin\frac{\pi x}{l}$. From Eq. 54, $\int_0^l m\frac{x}{l}\phi_{1B}\,dx = \int_0^l m\frac{x}{l}\phi_{4B}\,dx = \int_0^l m\frac{x}{l}\phi_{5B}\,dx = \int_0^l m\frac{x}{l}\phi_{8B}\,dx = \frac{\sqrt{2\,m\,l}}{\pi} \times \frac{\sqrt{2 \times 3.77}}{\pi} = 0.874$. The dynamic reactions of each of these beams on girder T_9 are: $M_1 = M_4 = M_5 = M_8 = \frac{m\,l}{3} = \frac{3.77}{3} = 1.26$.

To decide if higher modes of these beams are required the rule given in Section 1—that only frequencies below $3\,\omega$ need to be considered—is applied. The machines run at 320 rpm which yields $\omega = \frac{2\pi \times 320}{60} = 33.5$; and, consequently, modes corresponding to frequencies greater than $\omega = 100$ can be neglected. As the fundamental frequency of these beams is already 94, no higher modes are required, and the designer can proceed to the more complicated task of finding the modes of the girders T_2, T_3, T_6, and T_7. These latter girders are again equal in dimensions and mass and must have identical frequencies and modes.

Each machine is carried by two secondary girders. The arrangement is symmetrical, and the computation is concerned with modes that are symmetrical to axis $a-a$. Section 3 describes how such a machine should be replaced by concentrated masses; and the present case is one to which Eqs. 68 apply.

As shown in Fig. 14(a), each of these girders carries distributed load and one half of a machine. The moment of inertia of these girders is $I = 122.3$ in.[4] The distributed weight is $w = 5 \times 14 + 25 = 95$ lb per ft, the mass is $m = \frac{95}{386} = 0.246 \ \frac{\text{lb-sec}^2}{\text{in-ft}}$. The weight of each machine is 6,720 lb, of which one half is carried by each girder. The mass, therefore, is $M = \frac{6{,}720}{2 \times 386} = 8.70 \ \frac{\text{lb-sec}^2}{\text{in.}}$. To replace the machine by its dynamic reactions at the points of support A and B, the polar moment J of the machine is required. If this polar moment for the entire machine is 5,376 lb-ft², the polar moment J per beam is equal to $\frac{5{,}376}{2 \times 386} = 6.96 \ \frac{\text{lb-sec}^2\text{-ft}^2}{\text{in.}}$. Substituting these values in Eqs. 68 the dynamic reactions are

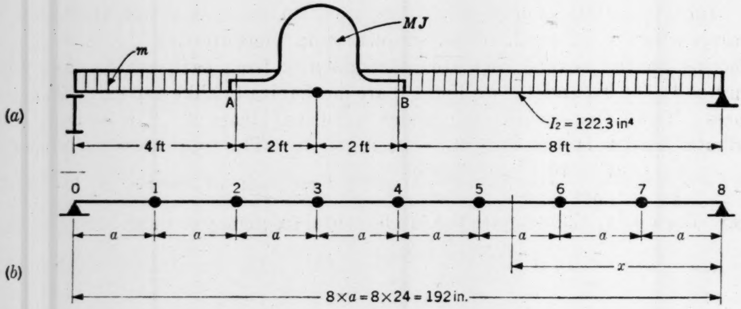


FIG. 14

$$M_A = \frac{2^2 \times 8.70 + 6.96}{4^2} + \frac{2 \times 2 \times 8.70 - 6.96}{4^2}\frac{y_B}{y_A} = 2.61 + 1.74\frac{y_B}{y_A} \quad \text{(70a)}$$

and

$$M_B = \frac{2^2 \times 8.70 + 6.96}{4^2} + \frac{2 \times 2 \times 8.70 - 6.96}{4^2}\frac{y_A}{y_B} = 2.61 + 1.74\frac{y_A}{y_B} \quad \text{(70b)}$$

Application of the Stodola-Vianello Method to the Determination of the Mode of T_2.—As shown in Fig. 14(b) the span is divided into eight equal parts and the masses are concentrated at the division points. The choice of the division points has been made in such a way that important and large concentrated masses, such as the dynamic reactions M_A and M_B coincide with division points. The masses at these points $\left(\text{in } \frac{\text{pound-second}^2}{\text{inches}}\right)$ are: $M_0 = M_8 = \frac{1}{2} \times 2 \times 0.246 = 0.25$ and $M_1 = M_3 = M_5 = M_6 = M_7 = 2 \times 0.246 = 0.49$.

Also

$$M_4 = 2 \times 0.246 + 2.61 + 1.74 \frac{y_2}{y_4} = 3.10 + 1.74 \frac{y_2}{y_4} \ldots\ldots (71a)$$

and

$$M_2 = 2 \times 0.246 + 2.61 + 1.74 \frac{y_4}{y_2} = 3.10 + 1.74 \frac{y_4}{y_2} \ldots\ldots (71b)$$

Points A and B in Fig. 14(*a*) are points 2 and 4 in Fig. 14(*b*), therefore, $y_A = y_2$ and $y_B = y_4$. As explained in Section 3 the Stodola-Vianello method makes it possible to obtain the normal modes by successive approximation. The first approximation $y^{(1)}$ is found by guessing only; then, the beam is loaded with the inertia forces corresponding to $y^{(1)}$ and the deflections are determined. These deflections are the next approximation $y^{(2)}$ of the mode; the process is then repeated as often as required.

TABLE 1.—STODOLA-VIANELLO METHOD, FIRST STEP

Point (Fig. 14)	$y^{(1)}$	Mass M	M × $[y^{(1)}]^2$	INERTIA LOADING			ELASTIC WEIGHTS			$\frac{M\,(E\,I)^2}{a^6} \times [y^{(2)}]^2$	$\omega^2\, y^{(2)}$
				$M\,y^{(1)}$	Shear V	Moment $\frac{\overline{M}}{a}$	$\frac{E\,I}{a^2}\,w$	Shear $\frac{E\,I}{a^2}\,V$	Moment $\frac{E\,I}{a^3}\,y^{(2)}$		
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
0	0	0.25	0	0		0	1.00		0	0	0
					5.998			35.412			
1	0.45	0.49	0.10	0.22		5.998	5.96		35.412	615	0.40
					5.778			29.452			
2	0.80	5.19	3.32	4.15		11.776	11.08		64.864	23,012	0.731
					1.628			18.372			
3	1.00	0.49	0.49	0.49		13.404	13.32		83.236	3,396	0.93
					1.138			5.052			
4	0.95	4.57	4.12	4.34		14.542	13.82		88.288	34,145	0.994
					−3.202			−8.768			
5	0.75	0.49	0.28	0.37		11.340	11.28		79.520	3,100	0.89
					−3.572			−20.048			
6	0.50	0.49	0.12	0.25		7.768	7.73		59.472	1,733	0.67
					−3.822			−27.778			
7	0.25	0.49	0.03	0.12		3.946	3.92		31.694	492	0.36
					−3.942			−31.698			
8	0	0.25	0	0		0.004	0.66		−0.006	0	0
Σ...	....	....	8.46	9.94	....	....	....	....	....	66,493	....

Table 1 shows the arrangement of the computations for the first step. Cols. 2, 3, and 4, Table 1, contain the values of the assumed amplitudes $y^{(1)}$, the masses M, and $M\,[y^{(1)}]^2$. To obtain the numerical values of the masses M at points 2 and 4, it must be carefully remembered that the ratios $\frac{y_4}{y_2}$ and $\frac{y_2}{y_4}$ in the expressions for these masses refer to the amplitudes just considered and that, therefore, $y_2 = 0.80$ and $y_4 = 0.95$ (see Col. 2, Table 1) which yields $M_2 = 3.10 + 1.74 \frac{0.95}{0.80} = 5.19$; and $M_4 = 3.10 + 1.74 \frac{0.80}{0.95} = 4.57$.

Cols. 5, 6, and 7 are headed "Inertia Loading" and contain the inertia load $M\,y^{(1)}$, the shear forces due to these loads, and, finally, the moments. After the reactions at point 0 are found, the shear forces are determined, one by one, by subtracting the point load from the shear in the preceding panel. Since the panel length is constant, $a = 24$ in., the values of the moment $\frac{\bar{M}}{a}$ can be found, step by step, by adding the shear force to the moment in the preceding panel point, starting with $\bar{M} = 0$ for point 0.

Cols. 8, 9, and 10, Table 1, are headed "Elastic Weights" and contain the values w_i found from Eq. 60*b*, the shear, and the moments due to the loads w_i—all with suitable multipliers. Col. 10 is the approximation $y^{(2)}$ of the fundamental mode.

Col. 11 gives the values of $M\ [y^{(2)}]^2$ which are required to find the frequency from Eq. 61: $\omega^4 = \frac{8.46 \times E^2 I^2}{66{,}493 \times a^6}$; $\omega^2 = \sqrt{\frac{8.46}{66{,}493}} \times \frac{E\,I}{a^3} = 0.01126 \times \frac{30 \times 10^6 \times 122.3}{24^3} = 2{,}990$; and $\omega = 54.7$. To obtain the values in Col. 11, the masses at points 2 and 4 must be recomputed, using the values of $y_2 = 64.864$ and $y_4 = 88.288$; thus, $M_2 = 3.10 + 1.74\,\frac{88.288}{64.864} = 5.47$; and $M_4 = 3.10 + 1.74\,\frac{64.864}{88.288} = 4.38$.

Col. 12, Table 1, finally, contains the values:

$$\omega^2\,y^{(2)} = \sqrt{\frac{8.46}{66{,}493}} \times \frac{E\,I}{a^3}\,y^{(2)} = 0.01126\,\frac{E\,I}{a^3}\,y^{(2)} \qquad (72)$$

If $y^{(1)}$ had been the correct mode, $\omega^2\,y^{(2)}$ would have been exactly equal to $y^{(1)}$, and the actual differences enable the designer to judge the degree of approximation. As this is only the first step, there are still noticeable differences.

The computation is now repeated, beginning with the values $y^{(2)}$ just found. A principal mode can always be multiplied by a factor; it is therefore permissible to start the next step with the values $\omega^2\,y^{(2)}$. This procedure is convenient because otherwise the numbers in the tables would increase after every step. It is not necessary to reproduce the computation for the further steps since the tables are quite similar to Table 1. The results of the first three steps are shown in Cols. 2, 3, and 4, Table 2.

The result $\omega^2\,y^{(3)}$ of the second step (Col. 3) differs very little from $\omega^2\,y^{(2)}$ (Col. 2) and the third step (Col. 4) produces no change. The result of the third step, therefore, is selected as final. The values in Table 2 represent the amplitudes of a principal mode and therefore contain an arbitrary factor. To find the normal mode this factor must be chosen to satisfy the normalizing condition Eq. 62:

$$\sum M\,\phi^2 = 1 \qquad (73)$$

The quantity $\sum M\,y^2 = 8.39$ has been computed in Col. 6, Table 2, and the normalizing condition will be satisfied by

$$\phi = \frac{y}{\sqrt{\sum M\,y^2}} = \frac{y}{\sqrt{8.39}} = 0.345\,y = 0.345\,\omega^2\,y^{(4)} \ldots\ldots\ldots (74)$$

Col. 9, Table 2, contains the value of $\sum M \frac{x}{l}\,\phi_{2B}$ which will be required subsequently.

The result of all these computations can be summarized as follows:

$$\omega_{2B} = \omega_{3B} = \omega_{6B} = \omega_{7B} = 54.6;$$

and

$$\sum M \frac{x}{l}\,\phi_{2B} = \sum M \frac{x}{l}\,\phi_{3B} = \sum M \frac{x}{l}\,\phi_{6B} = \sum M \frac{x}{l}\,\phi_{7B} = 2.034.$$

The exact value of the next frequency of these beams is not known without further computation, but it will be at least four times the fundamental frequency—that is, more than 200. As this value is greater than the previously explained limit of $\omega = 100$, further modes of these beams need not be considered.

TABLE 2.—STODOLA-VIANELLO METHOD, RESULT OF ADDITIONAL STEPS, AND FINAL MODE ϕ_{2B}

Line	First step $\omega^2\,y^{(2)}$	Second step $\omega^2\,y^{(3)}$	Third step $\omega^2\,y^{(4)}$	NORMAL MODE					DYNAMIC REACTION R_9		
				M	$M\,y^2$	ϕ_{2B}	$\frac{x}{l}$	$M\frac{x}{l}\phi_{2B}$	$\frac{x}{l}$	M	$M\frac{x^2}{l^2}$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
0	0	0	0	0.25	0	0	1	0	1.000	0.25	0.25
1	0.40	0.396	0.396	0.49	0.08	0.137	0.875	0.059	0.875	0.49	0.375
2	0.731	0.727	0.727	5.48	2.90	0.251	0.750	1.032	0.750	4.26	2.395
3	0.93	0.934	0.934	0.49	0.43	0.322	0.625	0.099	0.625	0.49	0.19
4	0.994	0.993	0.993	4.37	4.31	0.343	0.500	0.750	0.500	5.71	1.43
5	0.89	0.896	0.896	0.49	0.39	0.309	0.375	0.057	0.375	0.49	0.07
6	0.67	0.672	0.672	0.49	0.22	0.239	0.250	0.029	0.250	0.49	0.03
7	0.36	0.358	0.358	0.49	0.06	0.124	0.125	0.008	0.125	0.49	0.01
8	0	0	0	0.25	0	0	0	0	0	0.25	0
Σ...	....	....	....	....	8.39	....	....	2.034	....	....	4.75
ω...	54.7	54.6	54.6	...	....	....	....	....	....	....	....

The dynamic reactions of these girders on the main girder T_9 are given by the integral formula Eq. 13. For concentrated masses M this equation yields

$$\frac{1}{l^2}\int x^2\,dm \equiv \sum M \frac{x^2}{l^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (75)$$

The computation of this sum is shown in Col. 12, Table 2. In accordance with Fig. 3 the dimension x must be measured from the far end of the girder—that is, in this case from point 8, as shown in Fig. 14(b). When determining the masses M in Col. 11, Table 2, for points 2 and 4, the amplitudes y at these points are required. (Amplitudes y always refer to the amplitudes under consideration, which are, in this case, $y_2 = 0.75\, y_0$ and $y_4 = 0.50\, y_0$.)

Finally, the dynamic reactions of these girders are: $R_2 = R_3 = R_6 = R_7 = 4.75 \dfrac{\text{lb-sec}^2}{\text{in.}}$.

The Normal Modes of System A.—System A consists of girder T_9, which has a moment of inertia $I_9 = 1{,}226$ in.^{4}, and of eight imaginary beams T_1 to T_8 which are assumed to be rigid, $I = \infty$.

As explained in Section 1, such a system can be replaced by girder T_9 alone, carrying (in addition to the actual masses) further imaginary masses which are termed the dynamic reactions of the omitted girders.

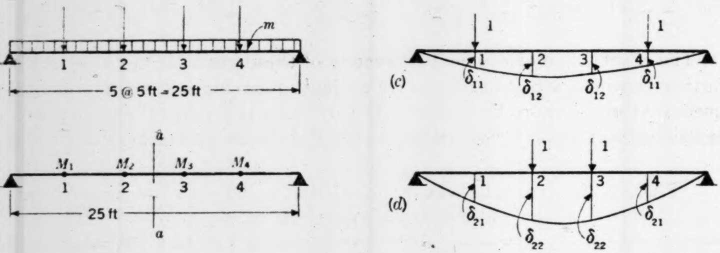


FIG. 15

Fig. 15(a) shows girder T_9 and its mass distribution. The weight of the girder, 65 lb per ft corresponds to a mass $m = \dfrac{65}{386} = 0.168 \dfrac{\text{lb-sec}^2}{\text{in-ft}}$. The additional mass at point 1 is the sum of the dynamic reactions of the beams T_1 and T_5 which have already been found equal to $R_1 = R_5 = 1.26 \dfrac{\text{lb-sec}^2}{\text{in.}}$. The additional masses at point 2 are $R_2 = R_6 = 4.75 \dfrac{\text{lb-sec}^2}{\text{in.}}$. By symmetry, the masses at points 3 and 4 are equal to those at points 2 and 1.

For the computation, concentrate the comparatively small distributed mass at points 1 to 4 and thus obtain the mass distribution shown in Fig. 15(b):

$$M_1 = M_4 = 5 \times 0.168 + 2 \times 1.26 = 3.36 \frac{\text{lb-sec}^2}{\text{in.}}$$

and

$$M_2 = M_3 = 5 \times 0.168 + 2 \times 4.75 = 10.34 \frac{\text{lb-sec}^2}{\text{in.}}$$

Because of the small number of masses the use of influence numbers (explained in Section 3) gives the quickest result. For the present only the modes that are symmetrical to axis $a-a$ are of interest; and to take full advantage of this symmetry the influence numbers δ_{ik} are defined in a special way. Numbers δ_{11} and δ_{12} are the deflections at points 1 and 2 as a result of the symmetrical arrangement of unit loads shown in Fig. 15(*c*). Because of symmetry, $\delta_{13} = \delta_{12}$ and $\delta_{14} = \delta_{11}$. Similarly δ_{21} and δ_{22} are the deflections due to the two unit loads shown in Fig. 15(*d*). The numerical values of these influence numbers are (in inches per pound): $\delta_{11} = 0.108 \times 10^{-4}$; $\delta_{12} = \delta_{21} = 0.166 \times 10^{-4}$; and $\delta_{22} = 0.274 \times 10^{-4}$.

The amplitudes of the principal modes are $y_1 = y_4$ and $y_2 = y_3$. By using the same reasoning that led to Eqs. 56 in Section 3 two equations are obtained for the amplitudes y_1 and y_2 which are identical to Eqs. 56. Substituting the numerical values for M and δ in Eqs. 56,

$$\left(0.363 \times 10^{-4} - \frac{1}{\omega^2}\right) y_1 + 1.717 \times 10^{-4}\, y_2 = 0 \qquad (76a)$$

and

$$0.558 \times 10^{-4}\, y_1 + \left(2.835 \times 10^{-4} - \frac{1}{\omega^2}\right) y_2 = 0 \qquad (76b)$$

Eqs. 76 have only the trivial solution $y_1 = y_2 = 0$ unless their determinant Δ vanishes. Therefore, $\Delta = 0$ is the frequency equation:

$$\Delta = \left(\frac{10^4}{\omega^2}\right)^2 - 3.198\,\frac{10^4}{\omega^2} + 0.071 = 0 \qquad (77)$$

There are two roots: $\frac{10^4}{\omega^2} = 3.176$ and 0.022 or $\omega = 56.1$ and $\omega = 725$. The higher value of ω is beyond the limit $\omega = 100$ and only one root remains: $\omega_A = 56.1$ and $\omega^2_A = 3{,}149$. To find the amplitudes of the normal mode, introduce this frequency in Eqs. 76 both of which give the identical result $y_1 = 0.610\, y_2$. A further relation is the normalizing condition (Eq. 58) which, for the present case, reads

$$M_1 y^2_1 + M_2 y^2_2 + M_3 y^2_3 + M_4 y^2_4 = 1 \qquad (78)$$

Substituting $y_1 = y_4 = 0.610\, y_2$ and $y_3 = y_2$, $2\,(3.36 \times 0.610^2 + 10.34)\, y^2_2 = 1$; and finally $y_2 = 0.207$ and $y_1 = 0.610 \times 0.207 = 0.127$. To express the fact that these are the amplitudes of the normal modes of system A one may write $\phi_A(1) = \phi_A(4) = 0.127$; and $\phi_A(2) = \phi_A(3) = 0.207$.

These are the values of ϕ_A at the four points where concentrated masses occur. The next step is to find the amplitudes of the mode ϕ_A for points situated along the secondary beams. Since in system A these beams are rigid, they must also remain straight. At the ends where these beams are supported by girder T_9 the amplitudes are $\phi_A = 0.127$ for girders T_1, T_4, T_5, and T_8 and $\phi_A = 0.207$ for girders T_2, T_3, T_6, and T_7; at the other ends ϕ_A must vanish. Therefore, the linear expressions are: For beams T_1, T_4, T_5, and T_8—

$$\phi_A = 0.127\,\frac{x}{l} \qquad (79a)$$

and, for beams T_2, T_3, T_6, and T_7,

$$\phi_A = 0.207 \frac{x}{l} \qquad \text{(79b)}$$

In Eqs. 79, x is the coordinate of the point considered measured from the external support of each girder.

Computation of the Coefficients β_{ik}.—The frequency equation, Eq. 32, contains the coefficients β_{ik} which are yet to be found. Eq. 26 defines these coefficients. In this example one mode ϕ_{1A} and eight modes ϕ_{kB} ($k = 1$ to 8) occur. There are, therefore, eight values of β_{1k} ($k = 1$ to 8).

The integrals in the equations for β_{ik} must be taken over all the masses of the system. Because system B consists of a number of separate beams, any value of ϕ_{kB} vanishes for all points except those lying on beam T_k. Each integral, therefore, needs to be taken over the masses of beam T_k only.

To find β_{11} the value of

$$\beta_{11} = \int_{T1} \phi_A \, \phi_{1B} \, dm \qquad \text{(80)}$$

must be found. Substituting $\phi_A = 0.127 \frac{x}{l}$ (which is valid for points on beam T_1) and remembering that $dm = m\,dx$,

$$\beta_{11} = 0.127 \int \frac{x}{l} \phi_{1B} \, dm = 0.127 \int m \frac{x}{l} \phi_{1B} \, dx \qquad \text{(81)}$$

The value of the last integral in Eq. 81 has already been found, together with the mode ϕ_{1B}, and $\beta_{11} = 0.127 \times 0.874 = 0.111$. The values of β_{1k} for girders T_4, T_5, and T_8 must be the same as those for T_1; thus, $\beta_{11} = \beta_{14} = \beta_{15} = \beta_{18} = 0.111$.

In the same manner, $\beta_{12} = \beta_{13} = \beta_{16} = \beta_{17} = \int_{T2} \phi_A \, \phi_{2B} \, dm = 0.207 \times \int_{T2} \frac{x}{l} \phi_{2B} \, dm = 0.207 \sum M \frac{x}{l} \phi_{2B} = 0.207 \times 2.034 = 0.423$.

The Frequency Equation.—If the numerical values β_{ik} and the values of the frequencies—$\omega^2_A = 3{,}149$; $\omega^2_{1B} = \omega^2_{4B} = \omega^2_{5B} = \omega^2_{8B} = 8{,}836$; $\omega^2_{2B} = \omega^2_{3B} = \omega^2_{6B} = \omega^2_{7B} = 2{,}980$—are substituted into Eq. 32, the frequency equation assumes the form:

$$\frac{0.049}{\dfrac{8{,}836}{\omega^2} - 1} + \frac{0.716}{\dfrac{2{,}980}{\omega^2} - 1} = \frac{3{,}149}{\omega^2} - 1 \qquad \text{(82)}$$

Fig. 16, with ω as variable, gives the values of the left-hand side of Eq. 82 as solid lines and the values of the right-hand side as dotted lines. The intersection points are the frequencies of the system. The roots of the frequency equation, found by trial and error, are: $\omega^2 = 1{,}654$, 7,950, and 26,000 or $\omega = 40.7$, 89.2, and 161.

It has been shown in Section 1 that Eq. 32 does not give all the frequencies of a system if some of the values ω_{sB} are alike. This occurs in this example: $\omega_{1B} = \omega_{4B} = \omega_{5B} = \omega_{8B} = 94.0$ and $\omega_{2B} = \omega_{3B} = \omega_{6B} = \omega_{7B} = 54.6$. Every multiple frequency of system B remains a frequency of the complete system and there are, therefore, two additional frequencies, $\omega = 54.6$ and $\omega = 94$.

All the frequencies found correspond to modes that are symmetrical to axis $a-a$, and these frequencies, arranged in order of increasing magnitudes, are: $\omega_1 = 40.7$, $\omega_2 = 54.6$, $\omega_3 = 89.2$, $\omega_4 = 94$, and $\omega_5 = 161$.

To enable the designer to appreciate the meaning of these results the amplitudes of the modes corresponding to these frequencies were computed. This procedure will give some enlightening information on the frequencies ω_2 and ω_4 which would otherwise not be quite clear.

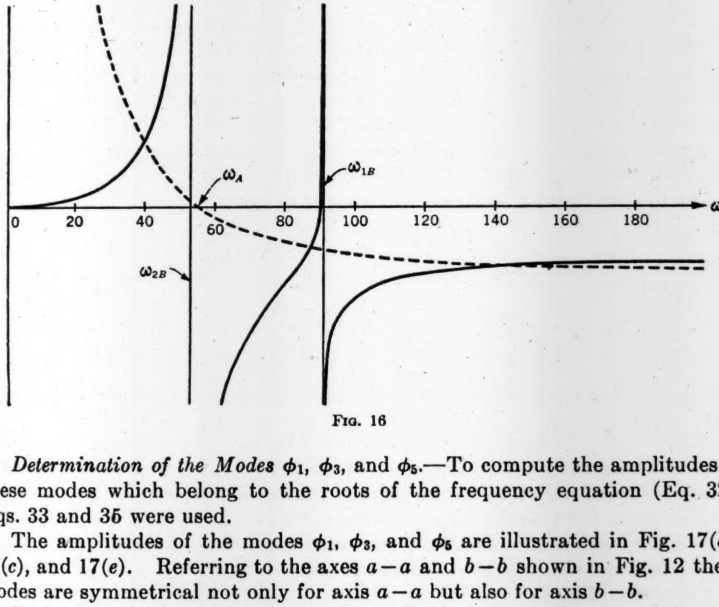


FIG. 16

Determination of the Modes ϕ_1, ϕ_3, and ϕ_5.—To compute the amplitudes of these modes which belong to the roots of the frequency equation (Eq. 32), Eqs. 33 and 35 were used.

The amplitudes of the modes ϕ_1, ϕ_3, and ϕ_5 are illustrated in Fig. 17(*a*), 17(*c*), and 17(*e*). Referring to the axes $a-a$ and $b-b$ shown in Fig. 12 these modes are symmetrical not only for axis $a-a$ but also for axis $b-b$.

Determination of the Modes ϕ_2 *and* ϕ_4.—To find the modes ϕ_2 and ϕ_4 refer to Eqs. 27 which now become

$$\left(1 - \frac{\omega^2{}_A}{\omega^2}\right) c_A + \sum_{k=1}^{8} \beta_{1k}\, c_{kB} = 0 \dots\dots\dots\dots\dots (83a)$$

and

$$\beta_{1k}\, c_A + \left(1 - \frac{\omega^2{}_{kB}}{\omega^2}\right) c_{kB} = 0 \quad (k = 1, 2, \cdots, 8) \dots\dots\dots (83b)$$

Eq. 83*b* represents eight equations because k can be any number from 1 to 8.

Investigating the root $\omega_4 = \omega_{1B}$, Eq. 83*b* gives (for $k = 1, 4, 5$, or 8) simply $\beta_{1k}\, c_A = 0$—or $c_A = 0$. For $k = 2$,

$$\beta_{12}\, c_A + \left(1 - \frac{\omega^2_{2B}}{\omega^2_{1B}}\right) c_{2B} = 0 \qquad (84)$$

and, because $c_A = 0$, $c_{2B} = 0$. Similarly, $c_{3B} = c_{6B} = c_{7B} = 0$. Introduce these results in Eq. 83*a* (remembering that $\beta_{11} = \beta_{14} = \beta_{15} = \beta_{18}$), only four terms of this equation remain; thus, $c_{1B} + c_{4B} + c_{5B} + c_{8B} = 0$. Since the mode is symmetrical to the $a-a$ axis, $c_{1B} = c_{4B}$ and $c_{5B} = c_{8B}$; or $c_{1B} + c_{5B} = 0$.

The numerical values of c_{1B} and c_{5B} are computed from the normalizing condition (Eq. 35)— $c_{1B} = c_{4B} = 0.500$ and $c_{5B} = c_{8B} = -0.500$. The amplitudes of the mode are illustrated in Fig. 17(*d*). A similar computation for $\omega_2 = \omega_{2B}$ gives the analogous result—$c_A = 0$; $c_{1B} = c_{4B} = c_{5B} = c_{8B} = 0$; $c_{2B} = c_{3B} = 0.500$; and $c_{6B} = c_{7B} = -0.500$—as illustrated in Fig. 17(*b*).

These two modes (Figs. 17(*b*) and 17(*d*)) are still symmetrical to axis $a-a$ but they are asymmetrical to axis $b-b$. The significance of these two modes now becomes quite clear; each group of secondary beams can oscillate without interference by the other beams, because the girders on both sides of T_9 move in opposition; girder T_9 remains stationary because there are no loads on girder T_9, which is due to the fact that the reactions for each pair of secondary beams T_1 and T_5, T_2 and T_6, etc., balance each other.

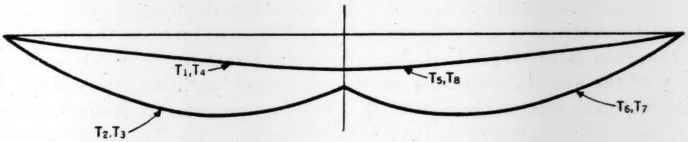
TABLE 3.—COMPARISON OF FREQUENCIES

Line	ω_1	ω_2	ω_3	ω_4	ω_5
1.......	40.7	54.6	89.2	94	161
2.......	40.7	54.6	88.7	94	158
3.......	41.0	55.4	104.5	109	166
4.......	41.0	55.4	104.3	109	162

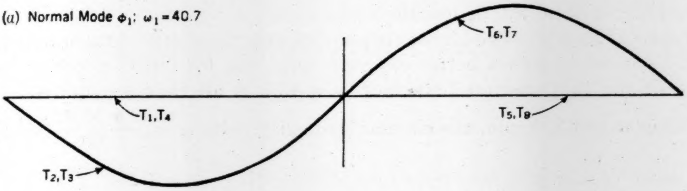
Reliability of Frequencies.—The mechanical system shown in Fig. 12 includes some distributed masses. Therefore, it has an infinite number of degrees of freedom. Such a system has an infinite number of frequencies as well. The present study has revealed only five frequencies corresponding to symmetrical modes because the frequencies of the complementary systems greater than $\omega = 100$ have been neglected, thus artificially reducing the number of degrees of freedom.

The limit $\omega = 100$ has been selected to insure that frequencies up to approximately $\omega = 50$ will be correct; greater frequencies may contain some error. For example, in Table 3 (a comparison of frequencies), line 1 shows the results of the foregoing computations; and line 2, the frequencies obtained by doubling the number of modes of the complementary systems. The first two frequencies remain entirely unaltered, which is better than could be expected.

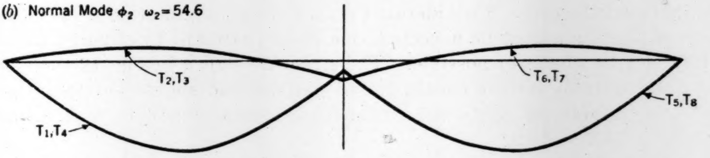
There is another simplifying assumption that affects the frequencies—the mass of the flooring has been concentrated on the secondary beams. According to the rule given in Section 1, this is a good approximation as long as the result remains below the lowest frequency of the secondary girders themselves. In the present example this lowest frequency is $\omega_{2B} = 54.6$. Therefore, ω_1 and ω_2 are good approximations, whereas the higher frequencies are not reliable. As


T1,T4
T5,T8
T6,T7
T2,T3

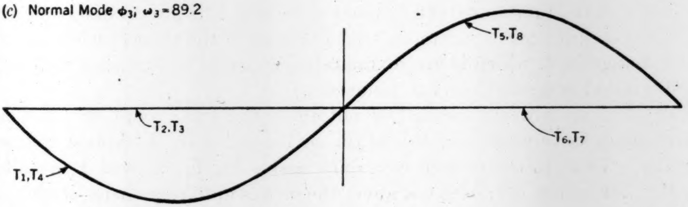
(*a*) Normal Mode ϕ_1; $\omega_1 = 40.7$


T6,T7
T1,T4
T5,T8
T2,T3

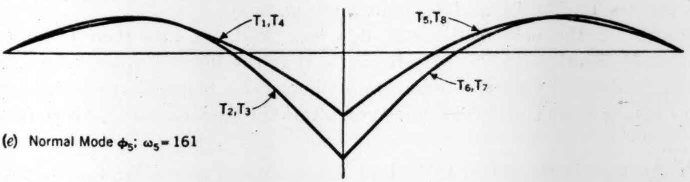
(*b*) Normal Mode ϕ_2 $\omega_2 = 54.6$


T2,T3
T6,T7
T5,T8
T1,T4

(*c*) Normal Mode ϕ_3; $\omega_3 = 89.2$


T5,T8
T2,T3
T6,T7
T1,T4

(*d*) Normal Mode ϕ_4; $\omega_4 = 94$


T1,T4
T5,T8
T6,T7
T2,T3

(*e*) Normal Mode ϕ_5; $\omega_5 = 161$

FIG. 17.—(VERTICAL SCALE OF *c*, *d*, AND *e*, REDUCED)

a check on this rule, line 3, Table 3, shows the result of a more exact computation taking into account the effect of the flooring. Line 4, gives the result obtained if the same number of modes is employed as for line 2. The more exact results in line 4 differ from line 1 only by 0.75% for the fundamental frequency and by 1.75% for the next frequency, which is quite sufficient for all practical purposes, and justifies the rules given in Section 1.

In the foregoing computations all the connections of the secondary beams have been assumed entirely hinged. In actual practice these beams will be at least partly restrained and this restraint will tend to increase the frequencies. To obtain an estimate of the effect of partial fixation, the fundamental frequency of the system with fully continuous secondary beams was determined by the method explained in Section 2. The fundamental frequency was found to be 43.8—8% higher than that without continuity. The effect of partial restraint, therefore, will be less than 8%.

Factor of Safety.—The higher frequencies and the shape of the modes have been discussed to gain a better understanding; but for practical purposes the value of the fundamental frequency $\omega = 40.7$ is all that is required. The machines run at 320 rpm, the circular frequency being $\omega = \dfrac{2\,\pi \times 320}{60} = 33.5$. The factor of safety against resonance therefore is $\dfrac{40.7}{33.5} = 1.21$. This result is not quite satisfactory. Considering that a certain margin of error in computations of this kind must be expected, one should attempt to obtain a factor of safety of 1.40 whenever possible. The structure shown in Fig. 12 therefore is not quite suitable for the contemplated purpose and should be strengthened. To secure a factor of safety of 1.40 the fundamental frequency must be raised to $1.40 \times 33.5 = 46.9$.

Alteration of the Moments of Inertia.—To find the moments of inertia required to give the fundamental frequency $\omega = 46.9$, it is not necessary to repeat the entire computation. Assuming that the mass of the structure is not altered by the change in I, which is approximately correct, it will be seen that nearly all the original computations can be used.

The most economical method of obtaining the small required increase in the frequency is to alter only the beam, or beams, with the lowest frequency ω_A or ω_B. Thus, in the present case only beams T_2, T_3, T_6, and T_7 need to be altered. This alteration does not affect the modes and frequencies of system A. Even the modes ϕ_{1B}, ϕ_{4B}, ϕ_{5B}, and ϕ_{8B} and their frequencies which originate from girders T_1, T_4, T_5, and T_8, remained unaltered.

To obtain the new modes ϕ_{2B}, ϕ_{3B}, ϕ_{6B}, and ϕ_{7B} and their frequencies, theorem II, Section 1, is used. It states that the normal modes of a system remain unaltered if all the moments of inertia are changed, proportionately, from I into $\alpha\, I$; such a change, however, alters the frequencies ω_i, which become $\sqrt{\alpha}\,\omega_i$.

If theorem II is applied to each of the separate girders T_2, T_3, T_6, and T_7, their moments of inertia are increased from I_2 to $\alpha\, I_2$. This increase alters their frequencies to $\sqrt{\alpha}\,\omega_{2B}$, $\sqrt{\alpha}\,\omega_{3B}$, $\sqrt{\alpha}\,\omega_{6B}$, and $\sqrt{\alpha}\,\omega_{7B}$, but leaves the amplitudes ϕ_{2B}, ϕ_{3B}, ϕ_{6B}, and ϕ_{7B} unchanged. As the numerical values of none of the modes have varied, the numerical values of all the coefficients β_{ik}

remain the same. Thus, the frequency equation can be written without further computations because it is only the frequencies ω_{2B}, ω_{3B}, ω_{6B}, and ω_{7B} that need to be adjusted:

$$\frac{0.049}{\frac{\omega^2{}_{1B}}{\omega^2} - 1} + \frac{0.716}{\frac{\alpha\, \omega^2{}_{2B}}{\omega^2} - 1} = \frac{\omega^2{}_A}{\omega^2} - 1 \qquad (85)$$

The effect of the change of sections on the frequency equation is only the introduction of the coefficient α. Since the desired frequency ω is known, this equation can be used in the reverse to determine the required value of α. Substituting $\omega^2 = 46.9^2 = 2{,}200$ in Eq. 85 gives $\frac{0.049}{\frac{8{,}836}{2{,}200} - 1} + \frac{0.716}{\frac{2{,}980}{2{,}200}\alpha - 1} = \frac{3{,}149}{2{,}200} - 1$

and $\alpha = 2.01$.

Originally, $I_2 = 122.3$ and now it is required that $I_2 = 122.3 \times 2.01 = 246$ in.^{3}

CONCLUSION

The method presented in this paper permits the determination of the resonant frequencies of complicated structures. It makes possible a comparison of the values of the operating frequencies of machinery with the values of resonant frequencies of a supporting floor. If any danger of resonance is found, appropriate changes can be made before the beginning of actual construction.

The author has applied the method successfully to the frequency analysis of the complicated fan floors of boilerhouses and, on one occasion, to the frequency analysis of a high radio tower on the roof of a multiple-story building.

DISCUSSION

Robert V. Hauer.[6]—This procedure for finding the natural frequencies of floor systems (or any other combination of structural members) will be of great value in all cases in which it is imperative to avoid the resonance of machines not only with the fundamental mode, but also with the higher harmonics of the structure. In buildings, however, such cases are comparatively rare. Usually, the designer will be concerned with only the gravest frequency, because the amplitudes of the higher modes are generally kept in check by the ever-present damping.

Whenever only the frequency of the fundamental mode is required, a shorter way of arriving at the solution is by equating the maximum potential and kinetic energies, as developed by Lord Rayleigh.[7] Although this method gives only approximate results (because a suitable assumption has to be made concerning the shape of the structure during vibration), it is known from all cases in which the exact and the approximate solutions could be compared that the accuracy of Lord Rayleigh's method is surprisingly high and more than sufficient for all practical purposes.

In applying the energy method to a floor system, the problem of how to correlate the shapes of the different members arises. This difficulty can be easily overcome by determining the energies throughout from the shapes of the statical deflections. Moreover, this particular choice leads to an extremely simple formula for the fundamental circular frequency[8]—namely,

$$\omega = \sqrt{g \frac{\Sigma P\, \delta}{\Sigma P\, \delta^2}} \qquad (86)$$

in which g is the acceleration of gravity; P represents the loads; and δ is the corresponding deflections. According to its derivation, Eq. 86 can be applied to any system and will appeal especially to structural engineers, because it reduces the vibration problem essentially to the routine task of computing deflections.

The writer welcomed the opportunity to check the accuracy of the energy method in the case of a floor system, for which Mr. Bleich has provided the exact solution. Applied to the numerical example of the paper, Eq. 86 yields a fundamental frequency of 41.1 per sec. This is only 1% more than the correct frequency value of 40.7 per sec—that is, practically equivalent.

As the floor layout shown in Fig. 12 is symmetrical about the axis *b–b*, it follows by inspection that the frequencies of some higher modes of the entire system must coincide with the fundamental frequencies of the individual beams T_2 (or T_3, T_6, T_7) and T_1 (or T_4, T_5, T_8). In this particular case, therefore, these higher frequencies can also be computed by Eq. 86. The results are as follows: $\omega_2 = 54.3$ (from T_2); and $\omega_4 = 94.3$ (from T_1). These values compare, respectively, with the author's values of 54.6 and 94.0.

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[6] Structural Engr., Albert Kahn, Associated Archts. and Engrs., Inc., Detroit, Mich.

[7] "Theory of Sound," by Lord Rayleigh, Macmillan & Co., New York, N. Y., 2d Ed., 1877.

[8] "Vibration Problems in Engineering," by S. Timoshenko, D. Van Nostrand Co., Inc., New York, N. Y., 2d Ed., 1937, pp. 94–95.

In the section under the heading, "Replacing Rigid Bodies by Their Dynamic Reactions," attention is called to the fact that in vibration problems a rigid body must be replaced by its dynamic reactions rather than by its statical reactions at the points of support. A similar difference occurs when the energy method is used. If S_1 and S_2 denote the statical reactions, it is seen from Fig. 11 that the kinetic energy of the rigid machine during vibration is not correctly expressed by $\frac{S_1}{2\,g}\left(\frac{dy_1}{dt}\right)^2 + \frac{S_2}{2\,g}\left(\frac{dy_2}{dt}\right)^2$, but consists of the translatory energy of the mass concentrated in the gravity center, $\frac{S_1 + S_2}{2\,g}\left[\frac{d}{dt}\left(\frac{y_1\,l_b + y_2\,l_a}{l_a + l_b}\right)\right]^2$, and the rotatory energy, $\frac{J}{2\,g}\left[\frac{d}{dt}\left(\frac{y_2 - y_1}{l_a + l_b}\right)\right]^2$. Accordingly, in the denominator of the radicand of Eq. 86 the contribution of the machine loads $S_1\,\delta^2{}_1 + S_2\,\delta^2{}_2$ should be replaced by $(S_1 + S_2)\left(\frac{\delta_1\,l_b + \delta_2\,l_a}{l_a + l_b}\right)^2 + J\left(\frac{\delta_2 - \delta_1}{l_a + l_b}\right)^2$. However, in the example considered, the writer found the influence of this correction negligible. For the frequency of the fundamental mode no change was obtained in three significant figures, whereas the frequency of the second mode increased only from 54.3 to 54.6 (thus coinciding with Mr. Bleich's result).

HANS H. BLEICH,[9] M. ASCE.—Mr. Hauer's comment that the fundamental frequency of buildings can be found by simpler methods is correct for simple structures only, and the proposed method was developed just for the cases in which these simple methods fail. The fact that the frequencies in the example in the paper can be found by the Rayleigh method should not be misconstrued; in the "Synopsis," it is stated clearly that the example is a very simple system selected only to demonstrate all the steps of the method.

To indicate the type of structure for which the method was devised, Fig. 18 shows the framing of one half of the bay of a boilerhouse to which the method was applied. There were three bays, and the column spacing was close to 60 ft in both directions. The fans had two alternate speeds, 580 rpm and 725 rpm. Structures of this type have a number of frequencies close together in the vicinity of the fundamental frequency of the main girders, for this span between 200 rpm and 300 rpm. In addition, there are higher frequencies, which had to be kept high, out of resonance with the fans. The frequency analysis showed that, in order to achieve this, 57-in.-deep plate girders had to be used for the support of the fans (see Fig. 18). These girders are of short spans and, as a result, have very low stresses—less than 3,000 lb per sq in. for static loads. All other members of the floor have the sections determined from their static loads alone.

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[9] Associate Engr., Hardesty and Hanover, Cons. Engrs., New York, N. Y.; and Lecturer in Civ Eng., Columbia Univ., New York, N. Y.

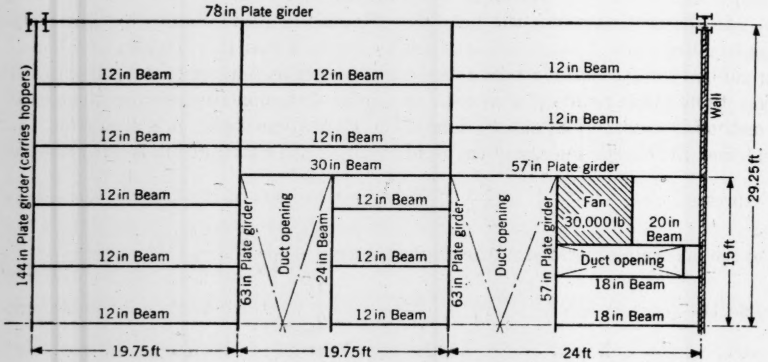
78 in Plate girder
12 in Beam
12 in Beam
12 in Beam
12 in Beam
12 in Beam
12 in Beam
30 in Beam
57 in Plate girder
12 in Beam
12 in Beam
12 in Beam
12 in Beam
12 in Beam
12 in Beam
144 in Plate girder (carries hoppers)
63 in Plate girder
Duct opening
24 in Beam
63 in Plate girder
Duct opening
57 in Plate girder
Fan 30,000 lb
20 in Beam
Duct opening
18 in Beam
18 in Beam
Wall
29.25 ft
15 ft
19.75 ft
19.75 ft
24 ft

FIG. 18

In such cases, or in similar ones, the simple method suggested by Mr. Hauer would not be sufficient. In this connection it might be mentioned that recently developed methods of earthquake analysis require the knowledge not only of fundamental modes, but also of several higher modes, which can be determined by the method presented in the paper.

AMERICAN SOCIETY OF CIVIL ENGINEERS
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TRANSACTIONS

Paper No. 2417

ROADS AND PAVEMENTS, SAMPSON NAVAL TRAINING STATION

By Jacob Feld,[1] M. ASCE

With Discussion by Messrs. P. F. Phelan, and Jacob Feld.

Synopsis

During World War II, the design and construction of temporary access and permanent roads, walks, and paved training areas was one phase of the emergency provision for the United States Navy personnel training facilities at Sampson, N. Y., on the east shore of Seneca Lake. Unfavorable soil conditions, absence of any local suitable base or aggregate materials, limited transportation facilities into the site, and unusually severe weather conditions made necessary a flexible program. The construction methods for almost 65 lane miles of roads within 6 months, with little time for preparation, and with keen competition for labor and equipment, and yet at a reasonable cost, are recorded because of the extremely low maintenance required in 4 years of concentrated use as a training station followed by 3 years of use as a temporary college. Some of the procedures deviated considerably from previously established practice and are suggested as proper methods of design and construction for roads in fine-grained plastic soil areas. Design was predicated on the availability of materials and equipment and followed the actual purchase of the materials.

Introduction

Extent of Project.—An area of 2,500 acres, adjacent to the east shore of Seneca Lake in New York, with about 5 miles of lake frontage, was to be converted into a training station for Navy personnel. The basic training areas (six in number) consisted of a rectangular 15-acre sodded drill field, with a paved drill area 50 ft wide on three sides. All six fields with barracks and service buildings were connected by paved roads and paths. Each area had an average of 1.5 miles of main road and a training strip about 0.5 mile long. A number of special service areas (for advanced training, administration, warehousing, civilian housing, and a hospital) accounted for almost as much paving as in the six training areas.

As a total, the equivalent of 51 miles of roads 22 ft wide was surfaced. The roads comprised (in million square yards): Primary road paving, 394; secondary

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Note.—Published in October, 1949, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Cons. Engr., New York, N.Y.

road paving, 158; concrete paving, 48; gravel surfacing, 17; and resurfacing existing roads, 49. About 95% of the paved area required an 8-in. stabilized gravel subbase, which with shoulders involved almost 300,000 tons of gravel. Crushed stone base was placed on all main roads. Hot lay tar concrete was used to surface all primary, secondary, and existing road areas. Penetration and seal coats required about 800,000 gal of tar, and the hot mix required 900,000 gal of tar.

Land was acquired in early June, 1942, before any surveys or plans had been executed. Construction was well under way by the end of July, and in spite of weather conditions, all road excavation and subbase were completed by December 15, with about 60% of the tar concrete in place. The station was "established" on October 17 and all areas were occupied by the end of 1942. Some 440,000 sailors received their basic training at this station.

Local Geology and Topography.—The entire work was within the strip of Hamilton shale exposure (Devonian period) (see Fig. 1). The surface soil is a Dunkirk silty clay loam (New York State Agricultural Classification), also designated as a Honeoye silty clay loam. It seemed to have only two qualities —tough gummy clay when wet, and gaseous dust when dry. Soft shale was found everywhere within a depth of 3 ft. This soil is very difficult to drain, quite unstable when wet, and readily subject to frost heave. It gives a slightly alkaline reaction. Known also as "Seneca County clay," it is considered the worst soil for highway work in New York State.

The developed land strip was about 6,000 ft wide, generally sloping uniformly toward the lake shore. Fig. 2 is a part reproduction of the Ovid quadrangle Geological Survey (United States Department of the Interior) map containing the area occupied by the Sampson Training Station. The edge 500 ft has a drop of 60 ft, the next 1,500 ft has a drop of 40 ft, and there is a drop of 100 ft in the easterly 4,000 ft. The land was divided by a series of sharply sloping gulleys, running normal to the lake shore, approximately 2,000 ft apart. Natural runoff had eroded these channels from 20 ft to 25 ft in depth; and, as far as possible, they were left undisturbed. With the watershed area limited in extent, a water level rise of almost 10 ft was measured in one gulley in less than 30 min during a thunderstorm. Runoff was almost 100%. Vegetation consisted of some tile drained grape and apple orchards, and small isolated fields of flax and buckwheat. Most of the area, however, was abandoned submarginal farmland.

Any rainfall in excess of 0.1 in. stopped rubber-tired vehicles on unpaved areas. A heavy rainfall stopped all operations for 3 days. At all other times, continuous sprinkler wagon coverage on all traveled ways was absolutely essential. During dry periods, the dust cloud was visible 10 miles away, and it was not unusual for automobiles to drive with full headlights in broad daylight.

Weather Conditions.—Local experience indicated that most of the work must be performed before the usual, heavy, early fall rains. The year 1942 did not agree with local experience, however. In July and again in August, there were 10 days in which precipitation exceeded 0.1 in. In September and October, there were only 5 such days each month. Normal construction season in this

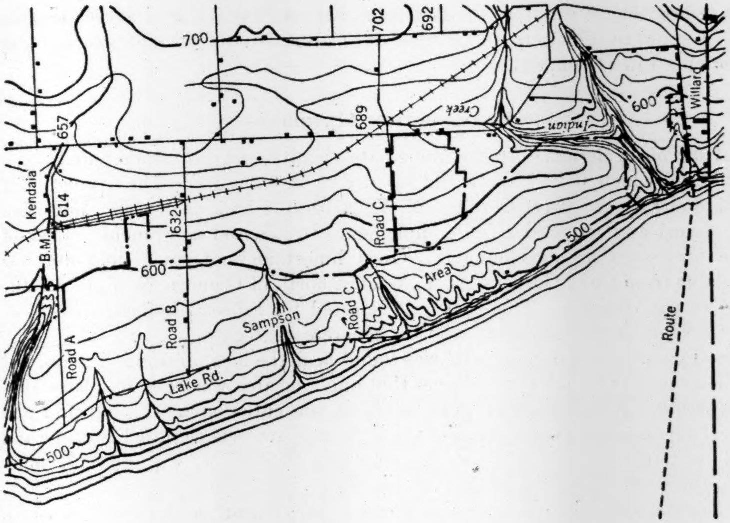
700
702
692
657
689
Kendaia
B.M.
614
632
600
Road A
Road B
Road C
Road C
Lake Rd.
Sampson
Area
500
500
Creek
Indian
600
Willard
Route

FIG. 2.—TOPOGRAPHIC MAP OF AREA

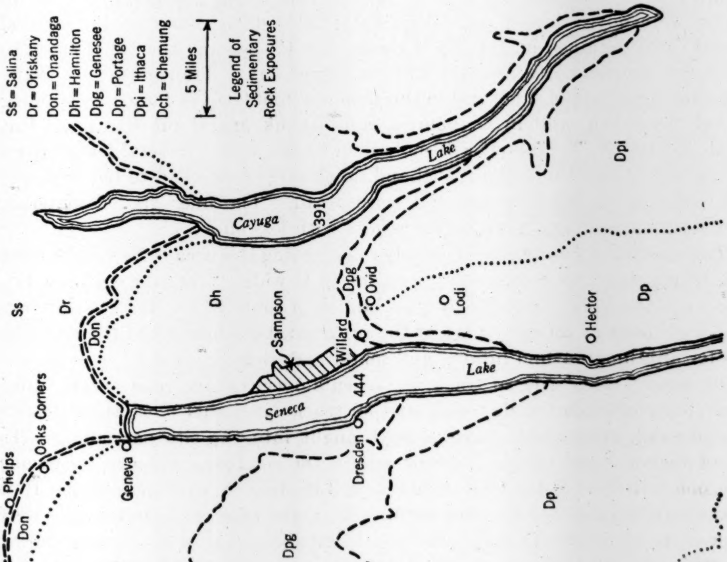
Ss = Salina
Dr = Oriskany
Don = Onandaga
Dh = Hamilton
Dpg = Genesee
Dp = Portage
Dpi = Ithaca
Dch = Chemung
5 Miles
Legend of Sedimentary Rock Exposures
Ss
Dr
Don
Dh
Dh
Dpg
Dpg
Dp
Dp
Dpi
Dch
Don
Phelps
Oaks Corners
Geneva
Dresden
Seneca
Lake
444
Sampson
Willard
Ovid
Lodi
Hector
Cayuga
Lake
391

FIG. 1.—GEOLOGICAL MAP OF AREA

area is less than 6 months; the last frost was on May 11, and winter arrived with a snowfall of 1.7 in. on October 26. Considerable subzero weather was experienced in December.

CONSTRUCTION MATERIALS

Available Materials.—In the immediate vicinity, the only stone found was a soft shale, which slaked easily and rapidly upon exposure. The nearest hard shale quarry, near Hayt Corners, N. Y. (a 10-mile haul each way), had been abandoned and required a full complement of tools and equipment. Lack of time prevented its development. Good limestone and some dolomite was available from Oaks Corners, N. Y., 6 miles north of Geneva, N. Y. (a 22-mile haul to the station). This rock was selected for railroad ballast, road base, pavement concrete aggregate, and tar concrete aggregate.

Some granular bank material was found near the local drainage gulleys, but in such small and uncertain volumes that only one area was actually placed into production. A few days of work depleted the entire area. Good sand and gravel deposits were being worked along the contact between the limestone and shale outcrops, chiefly at Oaks Corners and Phelps, N. Y., about 24 miles from the work.

The Oaks Corners pit produced a clean, fine, mason sand—double washed. The glacial deposits at Phelps gave a satisfactory run-of-bank sandy gravel suitable for road subbases, with no processing or blending. A fine sand was obtained from a deposit at Junius, N. Y., containing considerable lime intrusion from outwash seepage. Adjacent to this deposit was a sandy gravel deposit (54% retained on the No. 4 mesh, and 4% passing the No. 100 mesh) which was developed to produce 2,500 tons per day. Transportation bottlenecks and depletion of exposures in this area north of Geneva forced prospecting toward the south, and in October, a run-of-bank gravel pit was opened at North Hector, N. Y.—about a 16-mile haul to the area. Several feet of overburden was blended into the gravel and much large stone had to be "scalped" from the roads when this material was spread. The finer graded materials were more suitable and gave better results with less effort.

Transportation Facilities.—The only road feeding this area is New York route 96A, which was then a concrete pavement 16 ft wide. The lake road (see Fig. 2) was a black-top light traffic by-pass branch of route 96A. Railroad connections were built as an early part of the program, but there was little space on cars for anything beyond lumber and metal materials.

By careful scheduling of truck movements over the one road access (using about 110 trucks on the day shift and 50 trucks at night) more than 100,000 tons of sand, stone, and gravel were brought into the site per month. To permit higher speed and to increase safety, the road was widened by paving with one course of hot lay tar concrete a 3-ft strip on each side of the 16-ft concrete strip. No trucks were permitted on the lake road, and trucks were required to keep off the main road for 30 min, morning and evening, during the peaks of worker travel. Almost 10,000 people, about 90% in private cars (fully loaded), came into the station within a 30-min period at the beginning of the day shift. About 8,500 came from the north, with traffic congestion of major proportions.

Existing Roads and Temporary Facilities.—Three east and west roads were found crossing the station area (see Fig. 2). Road A was a built-up surfaced farm road, consisting of a cobblestone bed 8 in. deep over part of the width, the remainder being 6 in. of loose crushed soft shale. On this bed had been placed several layers of bituminous penetrated crushed limestone (probably a tar binder and 1 in. of unscreened stone), making up a 6-in. thickness. The total surfaced width was 16 ft, bounded by steep ditches.

Road B consisted of a 6-in. bed of soft shale with a 2-in. covering of penetrated screenings. This was also 16 ft wide. Road C was similar but the screenings had not been penetrated. Locally, these roads are known as "shuck" roads and are made by spreading crushed shale directly on the natural clay soil and letting traffic iron out the aggregate into a pavement. Later, screenings are added to fill out the ruts; and, when traffic warrants it, especially to reduce the dust in dry periods, a penetrated surface is added. The roads shed water very readily, are not slippery, and permit a 35-mile speed for all types of passenger and light truck traffic. Roads of this type were used for hauling several hundred truck loads of sod (from 3-ton loads to 5-ton loads) per day, in addition to the normal local use, with little damage to the surface of the compressed soft shale.

To widen these roads required digging new ditches and filling in the existing drainage channels. There was no time for such a major change. Ditches were cleaned and the road surfaces were scarified 1 in. deep, penetrated with grade RT-4 tar, and covered with ½ in. of stone screenings. No added surface layer could be applied because the resulting road width decrease was not considered safe.

In the middle of each road length, a traffic circle was constructed to permit reversal of direction. The radius of this circle was 30 ft. These traffic "turnarounds" provided space for trucks to pull out while waiting orders and also to unload part loads into smaller vehicles which could move on the natural soil. Eventually, after construction was almost completed, these roads were widened and resurfaced and became part of the completed road system.

Generally, permanent roads were graded and covered with a 4-in. layer of the gravel subbase for use as temporary access roads. A continuous maintenance gang of from 10 to 25 men, with a grader, a bulldozer, and 2 trucks kept all roads available for construction traffic, while also operating 3 water sprinkler wagons to keep the dust within some control.

All-weather parking and bus terminal areas were built by preparing sand-clay compacted surfaces. The procedure was to scarify the soil about 3 in. deep, cover with a 2-in. layer of fine dusty sand from the limestone outwash area, harrow and disk the mixture to a uniform color, and roll the surface. Water was added from a sprinkler truck. Control of water and of rolling was in accordance with a simple instruction: When a light private car left tire marks, more rolling was needed; when the tires raised dust, more water was needed. These parking areas were maintained by adding stone chips as low spots developed. There was no subsurface drainage and the surfaces were sloped at about 1 on 200 toward a ditch. The main parking areas were 200 ft wide, and held up remarkably well; 4 years of further use as public parking areas and bus terminals still left them in fairly good condition.

Grading Excavation.—In general all excavation and shaping of the subgrade was by tractor scrapers; fifteen such units of 12-cu-yd size, aided by 60-hp bulldozers in tough ground areas, each moved from 600 cu yd to 700 cu yd daily to build up grade and cut ditches. The work was then finished by patrol graders so that practically no handwork was needed. The road cuts were first made 5½ in. below final grade between ditch slopes and then "boxed" out 8 in. deep for the gravel subbase. For primary roads, approximately 500,000 sq yd of grading was required; and, for secondary road areas, 200,000 sq yd. Peak production occurred in the month of September when 143,000 sq yd was graded and prepared for gravel subbase. Most of the excavation volume came from the drainage ditches. This excavation was expedited to provide gravity drainage for all road excavation. To drain the box trench excavated for the road pavement bleeder trenches were cut 75 ft apart and 2 ft wide, on each side into the open ditches, and filled with broken stone. Working conditions after a rainy spell are shown by Fig. 3.

FIG. 3.—GRADING WORK AFTER A RAIN

The same labor-equipment group prepared subgrades, finished slopes of ditches, and placed the gravel subbase (from 2,000 tons per day to 4,000 tons per day) for roads and shoulders. The typical size of this group usually operating at six locations was as follows: 9 tractor scrapers, 2 bulldozers, 11 motor graders, 8 rollers, and from 150 men to 185 men including foremen.

To expedite the road preparation work, some wheel-propelled scrapers were brought to the site in October because no additional tractors were obtainable. The soil conditions made these scrapers useless since the ground was too hard to cut without the help of double bulldozers and the slightest rain eliminated all traction. The additional equipment was then used chiefly for spreading gravel from stock pile and other comparatively easy excavation.

Drainage Pipe.—Open ditches at road junctions, and across paved plazas, were connected by concrete drain pipe. The size of pipe varied from 8 in. to 48 in., the average diameter being 20½ in. Small sizes were laid with open joints and trenches were backfilled with broken stone and large gravel. The total length of all drain pipes used was 71,000 ft, with about 10,000 tons of stone fill. Maximum production in pipe laying was 25,600 ft in October, about half of which was laid in trenches dug by back hoes; the remainder was laid in sections of open ditches prepared by the road gang. In both September and November more than 15,000 ft of pipe was laid.

DESIGN PROBLEMS AND THEIR SOLUTION (SPECIFICATION REQUIREMENTS)

Pavement Design.—The general schedule for road construction planned all permanent road work for September, October, and November to permit the buildings, railroads, and subsurface utilities to have a good start in June, July, and August. Although the size of the project was almost doubled, no extra time for road construction was possible because of weather conditions. Some work was performed in December, but about a third of the final road surface was postponed to the following spring.

All pavements were to be placed above a subbase of stabilized gravel. There were three pavement classes:

a. Primary or main roads, consisting of 8 in. of gravel, 24 ft wide at the top; a 3-in. stone base, penetrated[2] with RT-12, 24 ft wide at the bottom; a 1½-in. binder course and a 1-in. wearing course of plant mixed tar concrete, 22 ft wide; and 6-in. gravel shoulders, 9 ft wide on each side.

b. Secondary roads, paved 11 ft for single-lane traffic with the six assembly areas 50 ft wide, consisting of a 6-in. gravel subbase; a prime coat (0.4 gal per sq yd) of RT-3, allowed to soak in; a tack coat (0.25 gal per sq yd) of RT-2, or RT-3; and a 1½-in. binder course and a 1-in. wearing course of plant mixed tar concrete. On the narrow roads, gravel shoulders were 5 ft wide on each side.

c. Tertiary pavement on paths and light traffic areas, consisting of a 6-in. gravel subbase; a prime coat of RT-3 (0.4 gal per sq yd); and a 1½-in. binder course (rolled with a "pup" roller) of plant mixed tar concrete.

All subgrade excavations were crowned, allowing for a final 3-in. crown on the 22-ft-wide roads. To discourage high speeds, no superelevation was provided at curves. The foregoing designs were agreed upon late in July, after alternate estimates of cost and labor requirements eliminated the possibility of concrete roads, and material controls prohibited the use of asphalt in any form.

Subbase Course.—Daily subbase gravel deliveries averaged more than 2,500 cu yd with maximum deliveries at about 4,500 cu yd per day in October, so that a stock pile of more than 32,000 tons was ready for use when weather slowed both gravel production and trucking.

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[2] "Military Roads in Forward Areas," *ASCE Manual of Engineering Practice No. 23*, New York, N. Y., 1941, p. 10, Table 1.

Bids had been asked for the following items:

1. Run-of-bank gravel, maximum size 2½ in., with more than 30% passing the ¼-in. screen;
2. Run-of-crusher limestone-dolomite, passing a 2½-in. screen to dust;
3. Run-of-crusher hard shale, passing a 2½-in. screen to dust, and of a quality that will not weather or soften when exposed to air; and
4. Slag from industrial furnaces, all passing a 2½-in. screen.

Each vender was to agree to furnish 50,000 cu yd and to deliver at a rate of 2,000 cu yd per day of 20 hours. A theoretical preferred specification was prepared as follows:

Sieve size	Passing by weight (%)
3 in.	96
2 in.	94
1½ in.	70 to 95
1 in.	55 to 85
¾ in.	50 to 80
⅜ in.	40 to 70
No. 4	30 to 60
No. 10	20 to 50
No. 40	10 to 30

The fraction passing the No. 200 mesh was specified to be between 5% and 15%, less than half of the fraction passing the No. 40 mesh. The binder includes any blended binder passing No. 40 mesh with a liquid limit less than 25 and a plasticity index of less than 6.

Actually, local field inspectors roughly checked the production at each pit and the acceptability of the gravel shipped was determined by the men placing it into the subbase. Considerable gravel from different sources was blended at the site to reduce the necessary blading and rolling, for proper compaction.

The cost varied from $1.45 per cu yd to $1.75 per cu yd delivered to the site, measured in trucks upon delivery. As the previously developed pits were depleted, prices went up to compensate for the purchase of land and for stripping and exploratory expenses. Hauling costs remained stable during the entire period at approximately 5¢ per cu-yd mile hauled, with no allowance for the return trip. Practically every available truck within 50 miles of the station was employed in this haulage, as subcontractors to the gravel venders. As an average, a ton of gravel yielded 2.67 sq yd of subbase, compacted to a thickness of 8 in. This is equivalent to a density of 125 lb per cu ft.

On main and secondary road areas, a layer of fine sand was first spread, from 1 in. to 2 in. thick, to seal the clay, before placing the gravel unless the gravel as delivered contained a surplus of fines. In the first training area built, the subgrade was covered with straw cut from the area. This straw was laid about 2 in. to 3 in. high and the gravel was dumped on it. Objections to this

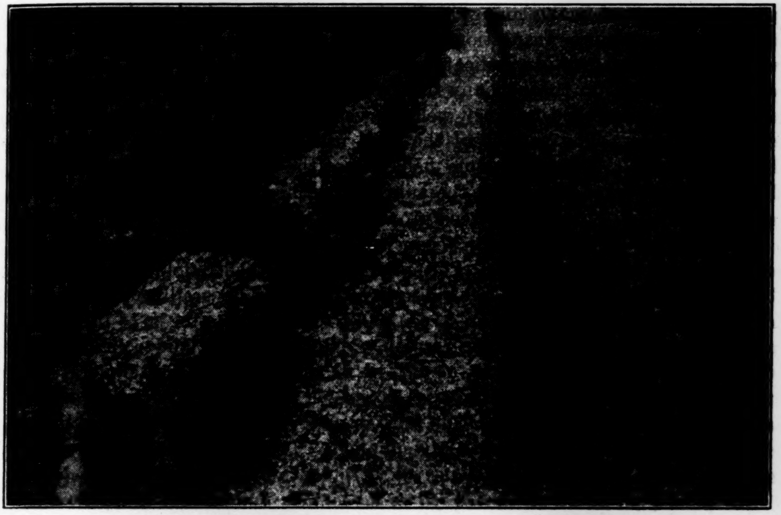

FIG. 4.—DETAIL OF SHOULDER OF MAIN CONNECTING ROAD

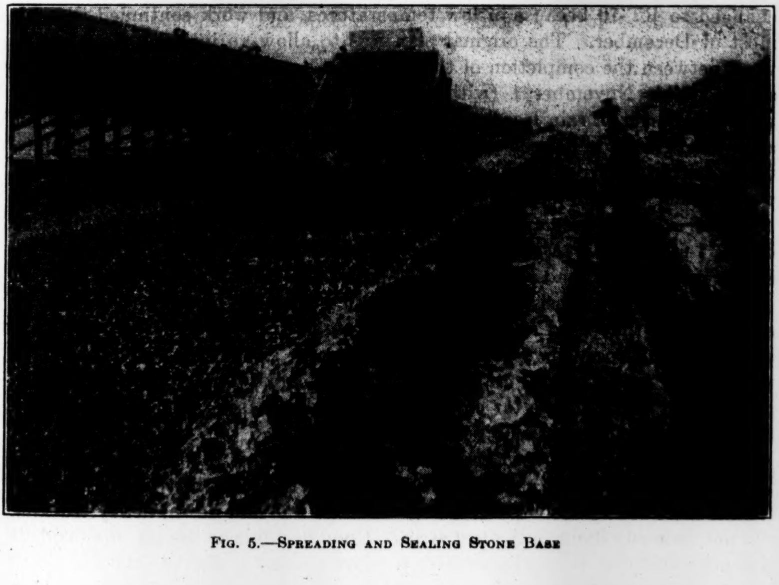

FIG. 5.—SPREADING AND SEALING STONE BASE

procedure resulted in the change to the fine run-of-bank silty sand as a screen on the clay subgrade. Inspection of one of the earliest roads built, after 1 year of use and after 3 years of use, indicated that the straw had served its purpose. No clay had worked up through any of these roads. The other roads, with the sand screen, also stood up very well; curves, grades, and crown remained in excellent shape. The volume of the sand used is included in the quantities listed as gravel. The satisfactory behavior of the subbase is partly explained by the screen layer placed on the subgrade. Shoulders of gravel remain in good shape (Fig. 4) although weed growth was not controlled.

Stone Base.—The contractor began work in the stone base and penetration on September 1, 1942 (Fig. 5). The stone used was crushed dolomite, delivered by rail and distributed from the siding by trucks. The grading specification required: 100% passing a 2¼-in. screen, 90% to 100% passing a 1⅞-in. screen, and 0% to 15% passing a 1-in. screen. After spreading and rolling to the desired crown, hot tar (RT-11 or RT-12 depending upon air temperatures) was applied by pressure distributers, 1.25 gal per sq yd. The surface was then covered with 50 lb per sq yd of from ⅛-in. stone to ½-in. stone, rolled, and broom dragged until the base stone was well chinked. This was followed by a second application of hot tar (0.50 gal per sq yd) and the surface was covered with 10 lb per sq yd to 15 lb per sq yd of screenings and broom dragged. The totals were from 330 lb to 335 lb of stone and 1.75 gal of tar per sq yd.

After a week of trial, it was found expedient to apply the stone base in two layers, which reduced the total rolling time. On November 20, the tar was changed to RT-10 because of low temperatures, and work continued through most of December. The original plan was to allow an interval of at least a week between the completion of the gravel subbase and the application of the stone. After November 1 (with freezing weather starting on October 27), with the danger of rain freezing within the gravel, the stone was applied soon after the gravel had been shaped. This required some extra stone, since traffic had usually pounded out some depressions in the gravel subbase which were refilled with stone.

Total amounts used approximated 60,000 tons of No. 3-A stone; 10,000 tons of No. 1 stone, No. 1-A stone, and No. 1-B stone; 1,000 tons of No. 3 stone for drains in shoulders; and almost 700,000 gal of tar (RT-12, RT-11, and RT-10).

A typical gang consisted of 30 men with 2 spreader boxes, 2 rollers, and 2 shapers, and they used about 400 tons of stone to cover 900 ft of road 24 ft wide. At full speed, five gangs were operating. Stone was delivered to the work by 15 trucks, and 2 cranes unloaded the cars. The stone cost 27¢, the placing cost 70¢, and the tar cost 21¢ per sq yd of base. The stone delivered to the site on cars cost $1.68 per ton.

Black-Top Pavement.—A premixed tar concrete test road section, 12 ft by 250 ft long, was laid on a gravel subbase to test the feasibility of this material for the intended traffic. Even though no base was laid down, heavy trucking did not seriously damage the test strip. Upon deciding to use tar mix (asphalt was not available), it was imperative to move a suitable plant to the site. Approximately 75,000 tons were required and the congestion of traffic on the access roads would seriously handicap any attempted delivery schedule of hot mix from locations outside the station.

Fortunately, a complete premix plant, built in 1941, and used for only 1,000 tons was located, and it was moved bodily to the site. A contractor was engaged to produce all binder and top course material at $7.65 per ton complete in place. This price covered all materials, equipment, and labor, except that a railroad siding was provided to the contractor without charge. About 20 men including truck drivers (the average haul distance was 3 miles) were employed at the plant.

The plant was an integrated mixing unit with feed bin and reciprocating feeder. Two aggregate driers, equipped with oil burners and a cyclone dust collector were set up in tandem, with hot and dust elevators. Screens in the mixing tower unit were 2 in., 5/8 in., 3/8 in., and 1/8 in. The mixer was a 4,000-lb-capacity twin pug mill, equipped with 5,000-lb aggregate batch scale and 500-lb tar bucket scale. Storage of aggregate was in a 50-ton bin with four compartments. Tar storage was in tank cars with a 10,000-gal reserve storage tank. Steam generated by a 150-hp locomotive steam boiler was used in the tar heating kettle and in the jackets surrounding the tar lines and mixer box. The mixing cycle, electrically controlled, was 45 sec, including dry and wet mix. Temperatures of the mix leaving the plant were from 180° F to 200° F for the binder and from 200° F to 230° F for the top course. Plant production averaged 100 tons of binder and 80 tons of top course per hour.

The tar concrete was mixed, laid, and rolled according to New York State Highway Specifications.[3] The specification for binder and top course were in conformity with the following limitations:

Passing screen size	Binder course (%)	Top course (%)
1½ in.	100	....
1 in.	70 to 100	....
½ in.	20 to 60	100
⅜ in.	....	85 to 100
No. 4	5 to 20	55 to 80
No. 8	0 to 5	35 to 65
No. 40	....	14 to 34
No. 100	....	6 to 20
No. 200	....	5 to 15

The aggregate sizes used in the mix were of New York State highway designation and were of the following description:

The No. 2, No. 1, and No. 1-A sizes, which were limestone, as well as the stone, sand, or limestone screenings, were shipped by rail. The sand was trucked in from a pit 22 miles distant. The proportions of each aggregate were set up to provide a dense, impervious, surface course, retaining nonskid features.

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[3] "Hot Lay Tar Concrete Pavement—Two Course Construction Mixing Method—Type 6," N. Y. State Highway Specification, Albany, N. Y., 1939.

The mixing formulas, adopted after some trial proportions were laid and modified, were as follows (Table 1):

TABLE 1.—MIXING FORMULAS FOR BLACK-TOP PAVEMENTS

Passing sieve:	Retained on sieve:	Binder to November 19		Binder After November 19		Top	
		%	Pounds	%	Pounds	%	Pounds
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
2 in.	⅝ in.	....	....	39	1,560	....	....
2 in.	¼ in.	60	2,400	....	....	....	....
¾ in.	⅜ in.	....	....	24	960	14	560
½ in.	⅜ in.	15	600	....	....	....	....
⅜ in.	⅛ in.	16	640	28	1,120	37	1,480
⅛ in.	....	6	240	6	235	42	1,680
Tar	....	3	120	3⅛	125	7	280
Total	....	....	4,000	....	4,000	....	4,000

The tar binder used was RT-12. As the weather became colder, RT-11 was substituted for RT-12. Check analyses were made daily of samples from the bins and of the mixed product. Where it was not possible, due to construction scheduling, to cover the binder course within 2 days, the binder mix (Col. 5, Table 1) was changed. This denser binder mix withstood construction traffic better than the more open mix. The better practice is to cover the binder course the same day with top mix, but in order to keep traffic flowing, and to avoid changing from binder mix to top mix at the plant during the middle of the day, the binder course was left uncovered for periods as long as 3 days or more in some few instances. When cold weather arrived all the binder was of the denser mix.

Tar mix production began on September 15 and closed down on December 9 for the winter, after 40,000 tons had been placed. An illustrated progress report of this work to this period was published[4] in January, 1943. The plant was opened again in May and completed the remaining 35,000 tons by the middle of July, 1943. At first the work progressed on a 10-hour day, 7-day-per-week basis, and changed to a 12-hr day, partly after nightfall. Only the binder mix was laid under artificial light. Production of 1,300 tons in a 12-hr day was not unusual.

The Navy's order to stop all binder placing when the temperature fell below 32° F and top course when it was below 35° F was only generally followed. The mean recorded temperatures were 54° F for October (with more than 2 in. of snow) and 41.7° F for November (with almost 5 in. of snow and 7 days with precipitation greater than 0.1 in.). With below zero weather early in December, it was impossible to continue the surfacing.

The contractor placed the tar mix using 4 paving machines to spread and finish the surface. The mix was fed from dump trucks into the hoppers of the paving machines which pushed the trucks ahead. Additional equipment used included 8 rollers, 21 trucks, 1 grader, 2 bulldozers, 1 power broom, and 1 gang broom. About 90 men were used for this work.

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[4] *Construction Methods*, January, 1943, p. 62.

Surfaces were rolled with 10-ton to 12-ton, 3-wheel rollers, or 8-ton to 10-ton tandem rollers, to produce a compression of 250 lb per in. of tread width. Surface tolerances were ¼ in. from crown template and ¼ in. from a 10-ft straightedge parallel to the axis of the road. It was pointed out by the vender in the middle of October that tar mix was being placed with air temperatures far below the 50° limit specified by the N. Y. State Highway Department and that the base was often in rather soft condition. In many cases, a 2-ton roller was used for compacting the base course to prevent undue deflections, especially when work was continued immediately after heavy rains. At times, the spreader had to be pulled forward by the trucks delivering the mix (instead of the spreader pushing the trucks) to pass through soft spots. In spite of all these conditions, the road surfaces held up very well, the gravel subbase drained out the excess moisture, and the roads were used for 4 years with practically no maintenance. An illustration of the pavement surfaces photographed in 1946 is shown in Fig. 6.

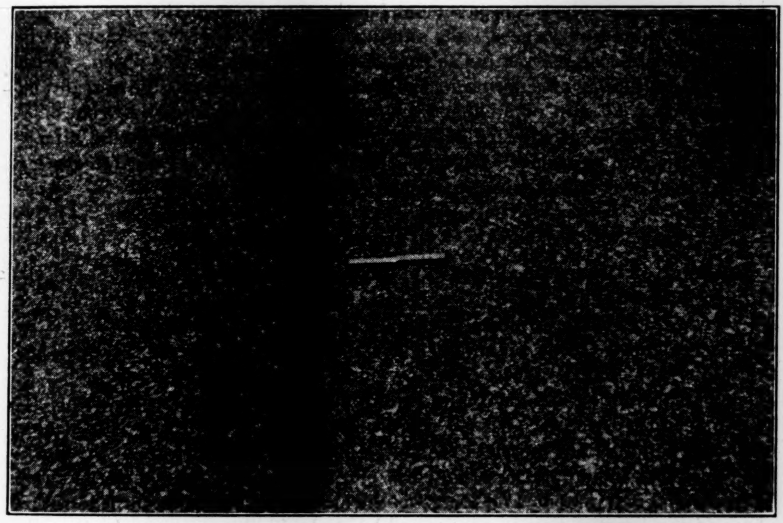

FIG. 6.—TEXTURE OF ROAD SURFACE (RIGHT) AND ASSEMBLY AREA (RULE IS 8 IN. LONG)

Theoretical yield was 7.0 sq yd per ton as compared with an actual yield of about 6.5 sq yd. The binder course of 1½ in. compacted gave 10.93 sq yd per ton and the topping course of 1 in. compacted gave 15.61 sq yd per ton. The density of the top mix, as tested from samples in place, was 141.1 lb per cu ft. The yields include the losses in material at soft spots and the filling of ruts in the stone base from traffic. The total cost of primary pavement surface was $1.08 per sq yd.

Concrete Pavement.—Concentrated trucking and vehicle servicing was expected in the warehouse and repair shop areas and pavements were designed with a concrete surface. Concrete design was specified in accordance with the N. Y. State Highway (1939) requirements for plain concrete, permitting crushed gravel as coarse aggregate. Cement content was 6 bags of Portland and 1 bag (80 lb) of natural cement per cu yd. With a water content of 5.5 gal per bag of cement, the slump was held to 2 in. When the central concrete plant was not being run to full capacity by the other operations, transit mix concrete was used for road construction. To prevent stoppage of work, a small batch bin was set up and a paver mixer, with cement in bags, was used to maintain continuous concreting. No reinforcement was used in the 6-in.-thick slab, with transverse joints filled with cane asphalt premolded strips, $\frac{3}{4}$ in. by $5\frac{1}{2}$ in., and transverse dummy joints $1\frac{1}{2}$ in. deep spaced not more than 20 ft apart.

About 50,000 sq yd of concrete pavement was placed before cold weather forced a change to black-top pavement for the remainder of the area. Water for the mix came from Seneca Lake and during the period of water main sterilization with chlorine, very rapid setting of the concrete mix was noted. With a residual of 5 ppm of chlorine in the water, concrete had to be spread and finished within 20 min of mixing, even with air temperatures between 45° F and 55° F. Little surface spalling was found in this concrete paving after 4 years of use and exposure. Paved surfaces were cured with straw covering and the work was done by an experienced concrete road gang.

PERSONNEL AND ACKNOWLEDGMENT

All work was under the direction of Capt. J. C. Gebhard, M. ASCE, (CEC), U. S. Navy, with the firm of Shreve, Lamb and Harmon associated with Gilmore D. Clarke, M. ASCE, Michael Rapuano, and Malcolm Pirnie, M. ASCE, as architect-engineer. The planning of the roads and pavements was assigned to Ralph MacDonald of Mr. Clarke's office. Contractors were the John A. Johnson Contracting Company and the Mount Vernon Contracting Company. The latter firm executed all road construction work. The Dale Engineering Company supplied the complete premix plant for the construction of the black-top pavement, including the production of all binder and top course material. The material, equipment, and labor for the stone base tar mix and penetration phase of the construction were supplied by A. L. Blades Company as the subcontractor for the Dale Company. The writer was site project manager in direct charge of the performance of all site work operations. Pavement construction planning and performance was directed by Richard Halton, superintendent of paving.

Acknowledgment is made of the able assistance of P. F. Phelan, Assoc. M. ASCE, of the Koppers Company, Incorporated, in the planning, design, and control of the tar concrete mixes.

DISCUSSION

P. F. Phelan,[5] Assoc. M. ASCE.—The design and construction of the roads and pavements at the Sampson Naval Training Station in New York are treated clearly and concisely in this paper. To those engaged in bituminous mix design, or in the inspection and control of bituminous mixing plants, it might be of interest to discuss more fully the design of the tar concrete mixtures and compare these design mixes with those actually obtained.

Based on gradations of the aggregate materials available in the stone quarry, the sand pit, and the mixing plant stock piles, the aggregates were fed into the premix plant in the proportions shown in Table 2. The batch mixing formulas, based on the gradations of materials in the hot aggregate storage bins of the plant, were shown in Table 1. Listed in Table 3 are the preliminary binder and top course design gradings, together with the actual gradation ranges and the average gradings obtained during the 1942–1943 seasons. The data in Table 3(*a*), for the binder course refer to the denser binder mix used after November 19, 1942 (see Table 1).

TABLE 2.—Aggregate Proportions at the Premix Plant

Material	Nominal size (in.)	Binder (%)	Top (%)
New York State Department of Public Works Size:			
No. 2	1 to $\frac{1}{2}$	60	....
No. 1	$\frac{1}{2}$ to $\frac{1}{4}$	20	15
No. 1A	$\frac{1}{4}$ to $\frac{1}{8}$	20	35
Stone screenings	....	....	35
Concrete sand	....	....	15
Total		100	100

Study of the data in Table 3 shows that the average mix gradings obtained in 1942 and 1943 were very close to the preliminary design gradings. It also indicates that the variations in gradation from the average were quite small, the actual gradation ranges being quite narrow. The average mix gradings, based on daily analyses of aggregate samples from the plant bins, together with the tar concrete specification limits, are plotted in Fig. 7. It is of interest to note that the average mix gradings for the two seasons are practically identical.

It is somewhat novel in the writer's experience to find a premix project of this size (75,000 tons) where the preliminary mix designs (based on samples of aggregates from the quarry and the pit stock piles) were so closely adhered to after the aggregates had been loaded into trucks or rail cars, transported to the mixing plant site, unloaded into stock piles, proportioned into the plant feeder, dried, and then screened into the various plant hot storage bins. It is also somewhat unusual to find a premix plant producing mixtures of such uniform gradation throughout a 5-month construction schedule at the rate of more than 100 tons per hr, especially where three or four aggregates are being proportioned into the plant. The results are indicative of efficient plant operation and competent inspection.

Fig. 8 indicates the type of traffic to which much of the pavement has been subjected. This view, taken when the pavement was more than 5 years old, shows practically no distress after constant subjection to heavy loads; quick starting, stopping, and turning traffic; oil, grease, and gasoline drippings; etc. The base in this area was especially soft at the time when the two courses of hot lay tar concrete were placed. Immediately in front of the warehouse doors, several cracks as wide as ½ in. and as deep as 1 in. were visible in the pavement when construction was stopped in December, 1942. The base had been too soft for adequate compaction under the roller and the mix had cracked when additional rolling was attempted after the mix had cooled. However, under traffic these "cracks" healed and have not subsequently reappeared. The writer had occasion to inspect this project in January, 1950, after 7 years, and found the roads and pavements to be in good condition, with little maintenance of any kind required. Similarly, over a large tar concrete paved area in the Navy Fire Fighters' School at Sampson, the surface was subjected to oil, grease, and gasoline spillage, water from high pressure hose lines, and intense heat from mock "shipboard" fires. Although the base was too soft at the time of construction to obtain satisfactory compaction under the roller, the mix was still in good condition after 4 years of use—and 2 years of nonuse.

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[5] Cons. Engr., Road Material Section, Tar Products Div., Koppers Co., Inc., Pittsburgh, Pa.

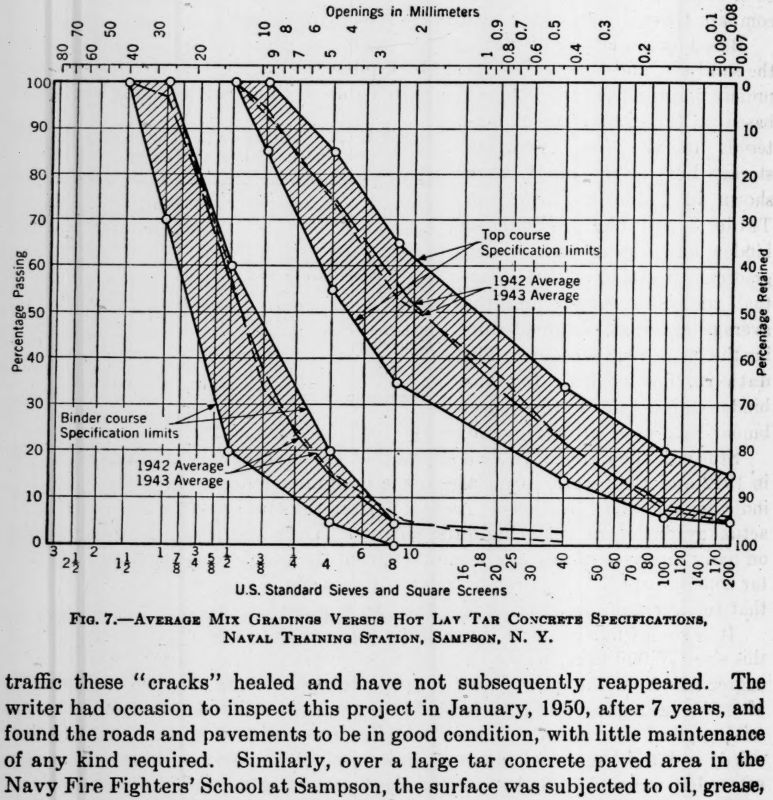

FIG. 7.—AVERAGE MIX GRADINGS VERSUS HOT LAY TAR CONCRETE SPECIFICATIONS, NAVAL TRAINING STATION, SAMPSON, N. Y.

Fig. 9 shows part of the pavement inside one of the six training areas. The primary road paving is seen in the foreground with a paved assembly area across the center of the picture—the two pavements being separated by a curb

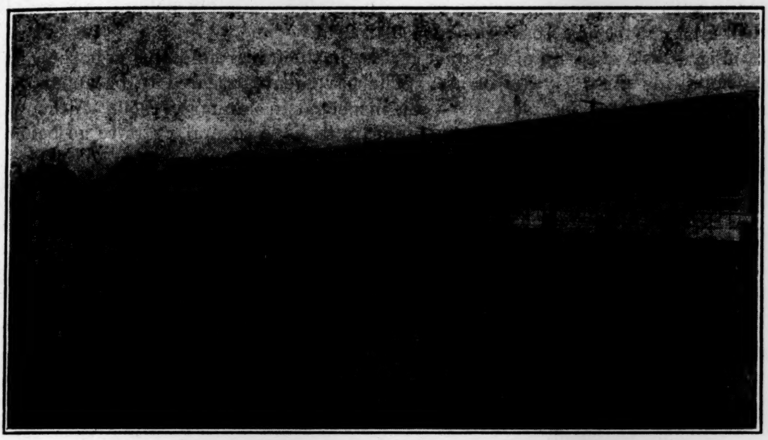

FIG. 8.—WAREHOUSE, GARAGE, AND HEAVY MAINTENANCE AREA

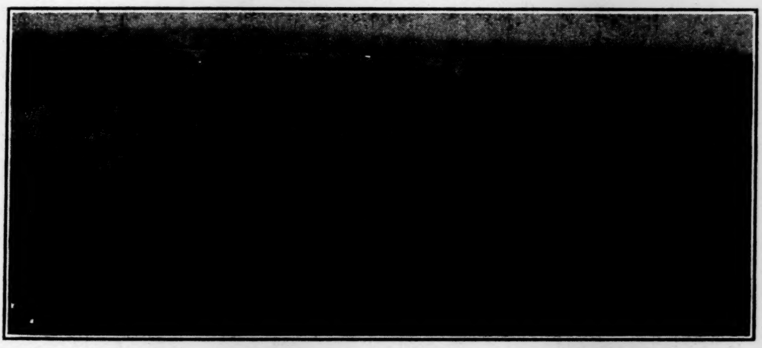

FIG. 9.—ONE OF THE SIX TRAINING AREAS

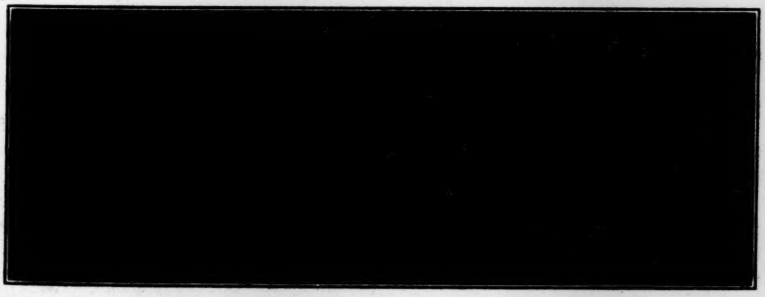

FIG. 10.—CLOSE VIEW OF PAVEMENT SURFACE IN FIG. 9

made with tar concrete surface course mix. Immediately behind this paved drill area is part of a 15-acre sodded drill field.

Fig. 10 is a close-up of the road surface in the foreground of Fig. 9, showing its condition after 5 years of use. The pavement appeared to be about the same as indicated in this view when inspected in January, 1950. Although almost 8 years old, this pavement appears to have several years to go before any appreciable maintenance will be required. If it were decided to reopen the Naval Base for full-scale operation, a surface treatment might be indicated to enrich the surface of the pavement.

TABLE 3.—TAR CONCRETE GRADATIONS (PERCENTAGE PASSING, BY WEIGHT), NAVAL TRAINING STATION, SAMPSON, N. Y.

Passing sieve:	1942			1943			Specification limits, hot lay tar concrete
	Preliminary design[a]	ACTUALLY OBTAINED[b]		Preliminary design[a]	ACTUALLY OBTAINED[b]		
		Range	Average		Range	Average	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
(*a*) BINDER COURSE							
1½ ...	100	100	....	100	100	100	100
1 ...	99	99–100	100	98	95–100	97	70–100
¾ ...	79	74-84	78	74	73–80	78	....
½ ...	53	54–68	58	53	48–60	56	20–60
⅜ ...	29	31–38	34	35	30-44	37	....
¼ ...	21	20–31	25	24	20–29	24	....
No. 4...	13	12–20	16	16	12–20	15	5–20
No. 8...	7	5–9	7	7	4–7	5	0–5
No. 10...	6	3–7	5	6	....	....	....
No. 20...	3	1–4	2	5	3–5	4	....
No. 40...	1	0–2	1	3	2–4	3	....
(*b*) TOP COURSE							
½ ...	100	100	100	100	100	100	100
⅜ ...	93	90-95	93	95	91–99	94	85–100
¼ ...	85	82 87	84	85	79–86	83	....
No. 4...	72	66-79	74	71	66–80	75	55–80
No. 8...	51	49 62	53	57	49–64	56	35–65
No. 10...	48	45 56	51	53	46–60	52	....
No. 20...	34	30–40	37	34	29–40	34	....
No. 40...	25	17–24	22	25	19–25	22	14–34
No. 60...	17	....	....	16	12–16	15	....
No. 80...	12	7–12	10	10	7–12	11	....
No. 100...	10	5–9	8	7	6–10	9	6–20
No. 200...	5	4–6	5	6	5–8	6	5–15

[a] Based on samples from quarry and plant stock piles. [b] Actual mix gradations computed from daily analyses of plant bin samples.

This paper is an excellent, complete story of the design and construction of the paving at the Sampson Naval Training Station where unfavorable soil conditions, shortage of local aggregates, and severe weather conditions were overcome to provide permanent roads, paved walks, and training areas for Navy personnel. The speed and economy with which the paving was completed are indicative of the care exercised in the planning and construction. The extremely low maintenance to date points up the adequacy of the designs adopted.

Jacob Feld,[6] **M. ASCE.**—Both the additional technical data on the actual mixes produced and the more recent physical inspections of the tar mix pavements contributed by Mr. Phelan increase the value of the paper as a possible guide to similar future projects. One of the more unusual pavements (which also was a part of this same project) consisted of some 4,000,000 sq ft of sod, which was produced off the site, cut, and laid all ready for use in 5 months. The methods employed and some indication of costs have been published elsewhere.[7]

Completion of work of such unprecedented character, when time does not permit any research to fill in gaps in information, requires the closest cooperation between technician, designer, and contractor. The writer wishes to give credit to all those who contributed to the successful accomplishments described in the paper.

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[6] Cons. Engr., New York, N. Y.

[7] "Sod," by Jacob Feld, *Roads and Streets*, April, 1948, pp. 84–86, 98, and 126.

AMERICAN SOCIETY OF CIVIL ENGINEERS
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TRANSACTIONS

Paper No. 2418

STREAM FLOW VARIABILITY

BY E. W. LANE,[1] M. ASCE, AND KAI LEI[2]

WITH DISCUSSION BY MESSRS. VEN TE CHOW, DANA M. WOOD, H. ALDEN FOSTER, C. S. OSPINA AND G. TAMA, RAPHAEL G. KAZMANN, JACK BRUIN AND H. E. HUDSON, JR., H. W. LULL, WILLIAM D. MITCHELL, C. R. OWNBEY, S. P. WING, J. E. MCKEE, G. B. SCHROYER AND E. T. SCHULEEN, AND E. W. LANE.

SYNOPSIS

An index of stream flow variability is suggested, in the form of the standard deviation of the logarithm of the stream discharge. The method of determining the index is described and the values from the flow records of a large number of streams in the eastern part of the United States were obtained. The possibilities of using this index in estimating the duration curve of the stream flow, where flow records are short or nonexistent, is indicated.

A study of the indexes, together with the conditions of the respective watersheds, demonstrated that in the northeastern quarter of the United States the geology (including soil cover) and the presence of lakes and swamps are the most important factors in stream flow variability during the greater part of the time and that the other factors have relatively smaller effect. This paper is a condensation of a more detailed paper, a copy of which is deposited in the Engineering Societies Library where it may be examined, or from which copies or microfilms may be secured.

THE PURPOSE OF A STREAM FLOW VARIABILITY INDEX

The variability of the flow of a stream has long been recognized as an important factor in the suitability of the stream for various purposes, such as water power and water supply. Hydrologists have recognized that the flow in certain streams varied greatly and that in others it was very uniform, but no generally accepted method has been evolved of indicating the degree of uniformity or of variability in a quantitative way, so that two streams could be compared with each other with respect to this characteristic. It is believed that there is need of such an index, and that if a satisfactory one is developed

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NOTE.—Published in September, 1949, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

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it will be found to have considerable usefulness in analyzing hydraulic engineering projects, especially in countries where stream flow records are meager.

By reducing the variability of flow to a single number it is possible, from a comparison of the values obtained for a large number of streams, to study the influence of the various factors such as geology, soil cover, lakes and swamps, steepness of topography, position of streams with respect to synchronization of peaks, and vegetation and rainfall, and thus to clarify greatly the fundamental relations of stream flow. It is also possible to produce synthetic duration curves with few data, which will be useful in studies of hydraulic engineering projects.

Duration Curves of Stream Discharge

Since the duration curve of stream discharge showed the extent of the variation of flow of the stream, it appeared probable that the best index of flow variability would be based upon this curve. The usual form of duration curve of stream discharge is that shown in Fig. 1, in which is plotted, on ordinary coordinates, a curve showing the percentage of the total time covered by the flow record during which the stream discharge was less (or greater) than the various discharge values. This form of curve appears to have been first developed in the United States in the late 1880's or early 1890's and has been extensively used in water power and other studies.[3],[4]

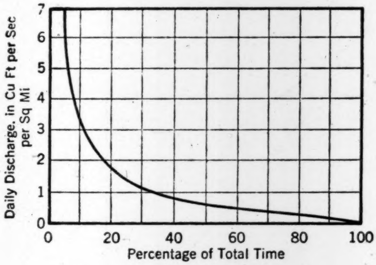

Fig. 1.—Duration Curve of Stream Discharge

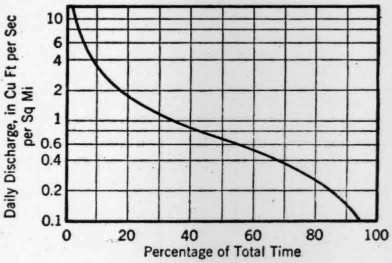

Fig. 2.—Logarithmic Plotting of a Duration Curve

When ordinary coordinates are used for duration curves as in Fig. 1, in order to fit them on a sheet of reasonable size, the discharge scale is usually such that the low flows cannot be read accurately. This can be remedied by plotting the discharge on a logarithmic scale, as shown in Fig. 2. A still better presentation has been suggested by the late Allen Hazen,[5] M. ASCE, who plotted the curves on his logarithmic probability paper and secured a curve which, for the greater part of the total time, closely followed a straight line, as shown in Fig. 3. It departs from the straight line at the ends, but for only a small part of the total time, although, because of the distortion of the scale in the probability plotting, this small part of the time covers a large proportion of the space on the graph.

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[3] "Duration Curves," by H. Alden Foster. *Transactions*, ASCE, Vol. 99, 1934, p. 1213.

[4] "Application of Duration Curves to Hydro-Electric Studies," by G. H. Hickox and G. O. Wessenauer, *ibid.*, Vol. 98, 1933, p. 1276.

[5] *Journal*, Boston Soc. of Civ. Engrs., March, 1916, p. 77, and June, 1916, pp. 299–303, and pp. 324–328.

A stream with very uniform flow will produce a curve with little slope, as shown by curve A, Fig. 3, and a variable stream will give rise to a curve of considerable slope, as shown by curve B. The steepness of the slope can be used as a measure or index of the flow variability.

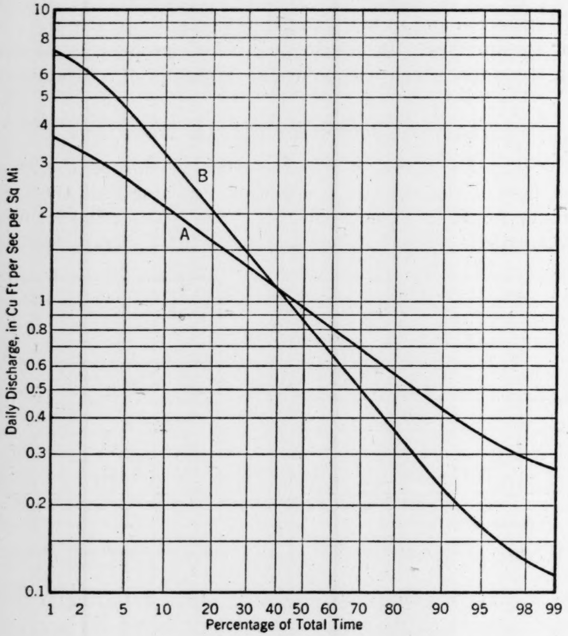

FIG. 3.—PROBABILITY PLOTTING OF A DURATION CURVE

PREVIOUS STUDIES OF DURATION CURVE PROPERTIES

Dana M. Wood,[5,6] M. ASCE, seems to have been the first to suggest that the flow characteristics of streams could be studied by comparing their duration curves. Other studies in this field were made by H. K. Barrows,[7] M. ASCE, Thorndike Saville and J. D. Watson,[8] Members, ASCE, and J. H. Morgan.[9] To a large extent the writer's idea was obtained from the work of W. C. Krumbein[10] and G. H. Otto[11] in the field of sediment particle analysis.

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[6] *Transactions*, ASCE, Vol. LXXXV, 1922, p. 106.

[7] "Water Power Engineering," by H. K. Barrows, McGraw-Hill Book Co., Inc., New York, N. Y., 1st Ed., 1927, p. 128.

[8] "An Investigation of the Flow-Duration Characteristics of North Carolina Streams," by Thorndike Saville and John Dargan Watson, *Transactions*, Am. Geophysical Union, Vol. 14, 1933, p. 406.

[9] "Flow-Duration Characteristics of Illinois Streams," by J. H. Morgan, *ibid.*, Vol. 17, 1936, p. 418.

[10] "Manual of Sedimentary Petrography," by W. C. Krumbein and E. J. Pettijohn, D. Appleton-Century Co., Inc., New York and London, 1938.

[11] "A Modified Logarithmic Probability Graph for the Interpretation of Mechanical Analysis of Sediments," by G. H. Otto, *Journal of Sedimentary Petrology*, August, 1939, p. 62.

These men applied statistical methods to the analyses of particle size data and obtained an index of size variation which has been widely adopted in the field of sediment studies. The writers have used the same procedure and have adopted the standard deviation of the logarithm of the discharge as the index of stream flow variability.

The application of statistical methods to the analysis of hydrologic data has been extensively studied by Mr. Hazen,[12,13] the late L. Standish Hall,[14] M. ASCE, H. Alden Foster,[15] M. ASCE, R. D. Goodrich,[16] M. ASCE, J. J. Slade, Jr.,[17] and L. R. Beard,[18] Jun. ASCE. Although many valuable ideas were revealed by these studies, the development of an index of stream flow variability does not appear to have been considered. The logarithmic probability paper developed by Mr. Hazen in his first paper[12] was used extensively in this study, however.

To discuss both the statistical theory underlying the stream flow variability index herein proposed and also its application to hydraulic problems would extend this paper beyond a reasonable length. Those interested in the statistical theory will find it treated at length in the papers cited herein as well as in numerous texts on statistics. In the development of this paper various phases of the mathematical theory were studied but as they proved to be unimportant to the main theme of the paper they have been omitted.

For those who are not familiar with statistical theory, it may be explained that the standard deviation is the index used by the statistician to indicate the magnitude of the variation of a quantity that fluctuates. Where the value of the index is high, the variation is great; and, where the index is low, the variation is small. Many civil engineers are familiar with this from its application in triangulation and base line measurement, where the standard deviation is known as the "root mean square error."

DETERMINING VARIABILITY INDEX OF STREAMS IN THE UNITED STATES

To develop the possibilities of an index of stream flow variability all available data on duration curves of streams in the United States east of the 100° meridian were studied. This investigation included more than 220 discharge measuring stations but they are located mostly in the northern portion of this section. These data were drawn from a large number of sources, mostly from reports by various states on the flow of their steams prepared in cooperation with the Geological Survey (U.S.G.S.) of the United States Department of the Interior. The duration curve data were based on mean daily flow. Only records of 10-year duration or more were used, and only streams on which artificial control was absent, or not an important factor, were studied.

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[12] "Storage to Be Provided in Impounding Reservoirs for Municipal Water Supply," by Allen Hazen, *Transactions*, ASCE, Vol. LXXVII, December, 1914, p. 1539.

[13] "Flood Flows," by Allen Hazen, John Wiley & Sons, Inc., New York, N. Y., 1930.

[14] "The Probable Variations in Yearly Run-Off As Determined from a Study of California Streams," by L. Standish Hall, *Transactions*, ASCE, Vol. LXXXIV, 1921, p. 191.

[15] "Theoretical Frequency Curves and Their Application to Engineering Problems," by H. Alden Foster, *ibid.*, Vol. LXXXVII, 1924, p. 142.

[16] "Straight Line Plotting of Skew Frequency Data," by R. D. Goodrich, *ibid.*, Vol. 91, 1927, p. 1.

[17] "An Asymmetric Probability Function," by J. J. Slade, Jr., *ibid.*, Vol. 101, 1936, p. 35

[18] "Statistical Analysis in Hydrology," by L. R. Beard, *ibid.*, Vol. 108, 1943, p. 1110–1160.

In the early part of these studies, graphical methods were used entirely; but it was found that, for curves giving a high index, it was not possible, by eye, to make a determination that was certainly free from considerable effects introduced by variations in personal judgment. In order to have all the data on a reasonably accurate and uniform basis, a numerical method was used. From the duration curve plotted, the values of discharge at 10% intervals from 5% to 95% of the time were read off. The logarithms of these discharges were then found and the standard deviation of these logarithms (which is the writers' variability index) was then computed by (1) taking the mean of these logarithms, (2) computing the difference of each of the ten logarithms from this mean, (3) squaring these differences, (4) taking the sum of the squares, (5) dividing it by 9, and (6) taking the square root of this value.

Factors Affecting the Variability Index

In 1936, Mr. Morgan[9] enumerated seven factors that affect the distribution of stream flow—topography, arrangement of tributaries with regard to time of concentration of surface flow, geologic structure, soil, vegetation, weather, and human developments related to flow of water. It is believed that these factors are the main ones controlling the flow, but some of them have been found to be more important than others. Since streams on which human developments have materially affected stream flow have not been included in this study, this factor does not enter. In the discussions of the results obtained in this paper, geologic structure and soils will rarely be considered separately, but will both be included under the term "geology."

The influence of these factors, as discussed in this paper, refers to the flow of the streams under the ordinary conditions and includes the effects for the greater proportion of the time. These effects may or may not be the same as the effects of the same factors on the flow of streams during large floods or in extreme droughts which usually occur less than 1% of the total time.

Results of Variability Index Determinations

The results of the variability index determinations for the 224 streams investigated, together with data on the drainage area, years of record, mean discharge, and source of the duration curve data, are presented in Table 1. These results can best be discussed by dividing them into the states in which the gaging station on the stream is located.

Variability indexes were obtained for 25 stations located in Iowa (Table 1(*a*)). The values of the variability index for the streams located principally in Iowa varied from 0.94 to 0.29. The land drained is usually rolling, without great differences in elevation, and is mostly underlain with sedimentary rocks. There are few lakes. One of the surprising results obtained was the low variability found in the regions of greatest relief. This is believed to be explained by the storage capacity available in the deeply incised porous strata of this region.

The results of the thirty-nine determinations in Wisconsin (Table 1(*b*)) follow a definite pattern. The streams in the relatively rugged region in the southwestern corner of the state gave low values of from 0.15 to 0.26, similar

TABLE 1.—VARIABILITY INDEX DETERMINATIONS FOR 224 STREAMS

Station No. (1)	River (2)	Station (3)	Drainage area (sq miles) (4)	Period of record (5)	Flow variability index (6)	Mean flow (cu ft per sec per sq mile) (7)
(*a*) Streams in Iowa						
1	Big Sioux	Akron	8.891	1935–1943	0.65	0.077
2	Nishnabotna	Hamburg[a]	2 800	1929–1942	0.58	0.238
3	Des Moines	Boone	5.490	1934–1943	0.64	0.278
4	Raccoon	Van Meter[a]	3.410	1933–1943	0.58	0.218
5	Des Moines	Des Moines	6.818	1933–1942	0.63	0.267
6	Des Moines	Ottumwa	13.200	1931–1940	0.59	0.246
7	Des Moines	Keosauqua[b]	13.900	1903–1927	0.50	0.382
8	Mississippi	Keokuk	119.000	....	0.25	....
9	Skunk	Ames	320	1934–1943	0.94	0.350
10	Skunk	Coppock	2,890	1913–1928	0.53	0.495
11	Skunk	Augusta	4.290	1915–1928	0.54	0.502
12	Iowa	Marshalltown[c]	1 500	1915–1927	0.51	0.471
13	Iowa	Iowa City[c]	3.230	1908 1927	0.49	0.457
14	Iowa	Wapello[c]	12,480	1915–1927	0.38	0.489
15	Cedar	Mitchell	845	1934–1942	0.39	0.341
16	Lime Creek	Mason City	535	1934–1943	0.63	0.343
17	Shellrock	Clarksville[c]	1.700	1915–1927	0.46	0.363
18	Cedar	Janesville[c]	1,660	1915–1927	0.38	0.407
19	Cedar	Cedar Rapids[d]	6.640	1903–1940	0.36	0.442
20	Wapsipinicon	Independence	1,060	1934–1943	0.64	0.442
21	Wapsipinicon	Dewitt	2,300	1935–1944	0.45	0.564
22	Yellow	Ion	224	1935–1944	0.35	0.553
23	Turkey	Garber	1,530	1934–1943	0.44	0.526
24	Little Maquoketa	Durango	130	1935–1944	0.39	0.579
25	Maquoketa	Maquoketa	1,550	1913–1933	0.29	0.606
(*b*) Streams in Wisconsin						
1	St. Croix	Danbury[e]	1,550	1915–1942	0.19	0.761
2	St. Croix	Grantsburg[e]	2,820	1926–1942	0.24	0.705
3	Namakagon	Trego[e]	433	1915–1927	0.14	0.933
4	St. Croix	Rush City[e]	5.120	1926–1942	0.31	0.583
5	St. Croix	St. Croix Falls[e]	5,930	1911–1942	0.30	0.566
6	Apple	Somerset[e]	550	1915–1942	0.24	0.471
7	Jump	Sheldon[e]	510	1916–1942	0.62	1.040
8	Tomahawk	Bradley[e]	422	1915–1927	0.20	1.014
9	Pine	Power Plant[e]	543	1924–1942	0.32	0.807
10	Menominee	Twin Falls[f]	1,790	1914–1929	0.27	1.002
11	Red Cedar	Menominie[e]	1,760	1914–1942	0.25	0.678
12	Chippewa	Durand[e]	9,010	1929–1942	0.29	0.879
13	Eau Claire	Augusta[e]	500	1915–1926	0.53	0.728
14	Black	Neillsville[e]	756	1915–1942	0.60	0.783
15	Rib	Rib Falls[e]	309	1926–1942	0.71	0.982
16	Eau Claire	Kelly[e]	326	1915–1942	0.38	0.819
17	Embarras	Embarras[e]	395	1920–1942	0.35	0.764
18	Wolf	Keashena Falls[e]	812	1914–1942	0.20	0.958
19	Oconto	Gillett[e]	678	1915–1942	0.26	0 891
20	Peshtigo	High Falls[e]	571	1913–1942	0.36	0.871
21	Pike	Amberg[e]	250	1915–1942	0.25	0.932
22	Menominee	Grand Rapids[f]	3,790	1911–1929	0.29	0.897
23	Black	Galesville[e]	2,120	1933–1942	0.44	0.839
24	La Crosse	West Salem[e]	412	1915–1942	0.18	0.724
25	Little La Crosse	Leon[e]	77	1935–1944	0.15	....
26	Yellow	Sprague[e]	420	1927–1940	0.65	0.540
27	Waupaca	Waupaca[e]	305	1917–1942	0.15	0.830
28	Little Wolf	Royalton[e]	485	1915–1942	0.29	0.882
29	Wolf	New London[e]	2,240	1914–1942	0.28	0.820
30	Fox	Berlin[e]	1,430	1900–1942	0.24	0.785

TABLE 1.—(*Continued*)

Station No. (1)	River (2)	Station (3)	Drainage area (sq miles) (4)	Period of record (5)	Flow variability index (6)	Mean flow (cu ft per sec per sq mile) (7)
(*b*) STREAMS IN WISCONSIN.—(*Continued*)						
31	Kickapoo	Gays Mills[e]	616	1915–1933	0.21	0.696
32	Wisconsin	Muscoda[e]	10,300	1915–1942	0.26	0.856
33	Crawfish	Milford[e]	732	1932–1942	0.57	0.399
34	Rock	Watertown[e]	971	1932–1942	0.68	0.401
35	Cedar Creek	Cedarburg[e]	113	1931–1942	0.70	0.485
36	Milwaukee	Milwaukee[e]	661	1915–1942	0.48	0.614
37	Grant	Burton[e]	257	1935–1944	0.25	....
38	Sugar	Brodhead[e]	529	1915–1942	0.24	0.655
39	Rock	Afton[e]	3,290	1914–1928	0.37	0.618
(*c*) STREAMS IN ILLINOIS						
1	Pecatonica	Freeport[b]	1,340	1914–1928	0.26	0.766
2	Fox	Algonquin[g]	1,364	1915–1933	0.39	0.614
3	Rock	Lyndon[g]	9,090	1914–1928	0.30	0.694
4	Des Plaines	Lement[g]	687	1915–1933	0.72	0.657
5	Vermilion	Streator[g]	1,080	1915–1930	0.96	0.611
6	Kankakee	Custer Park[g]	4.870	1915–1923	0.42	0.640
7	Kankakee	Momence[g]	2.340	1915–1933	0.32	0.762
8	Spoon	Seville[g]	1,600	1915–1933	0.59	0.620
9	La Moine	Ripley[g]	1,310	1922–1933	0.65	0.604
10	Sangamon	Riverton[g]	2,560	1908–1933	0.77	0.696
11	Sangamon	Monticello[g]	550	1909–1933	0.76	0.763
12	Vermilion	Danville[g]	1,280	1919–1933	0.73	0.566
13	Macoupin Creek	Kane[g]	865	1921–1933	0.85	0.721
14	South Fork	Kincaid[g]	510	1917–1933	0.89	0.824
15	Kaskaskia	Vandalia[g]	1,980	1908–1933	0.70	0.753
16	Embarras	St. Marie[g]	1,540	1910–1933	0.69	0.823
17	Little Wabash	Wilcox[g]	1,130	1915–1933	0.87	0.712
18	Skillet Fork	Wayne City[g]	475	1909–1933	1.09	0.940
19	Big Muddy	Plumfield[g]	753	1915–1933	1.01	1.021
20	Cache	Forman[g]	240	1923–1933	1.02	1.400
(*d*) STREAMS IN OHIO						
1	Maumee	Waterville[h]	6,314	1921–1940	0.79	0.671
2	Sandusky	Fremont[h]	1,248	1923–1940	0.63	0.702
3	Rocky	Berea[h]	269	1923–1935	0.81	0.948
4	Grand	Madison[h]	587	1922–1940	0.89	1.053
5	Ashtabula	Ashtabula[h]	118	1924–1940	1.17	1.144
6	Conneaut Creek	Amboy[h]	178	1922–1935	0.79	1.225
7	Maumee	Antwerp[h]	2,049	1921–1940	0.56	0.744
8	Auglaize	Defiance[h]	2,329	1915–1935	0.73	0.696
9	Auglaize	Fort Jennings[h]	333	1921–1935	0.56	0.883
10	Blanchard	Findlay[h]	343	1923–1935	0.79	0.656
11	Ottawa	Allentown[h]	168	1923–1935	0.62	0.684
12	Sandusky	Upper Sandusky[h]	299	1921–1940	0.75	0.833
13	Jerome Fork	Jeromesville[h]	120	1925–1939	0.63	0.883
14	Cuyahoga	Old Portage[h]	405	1921–1940	0.43	1.071
15	West Branch, Mahoning	Newton Falls[h]	98	1926–1939	0.55	0.939
16	Mosquito	Niles[h]	139	1929–1939	1.11	0.748
17	Mahoning	Youngstown	899	1903–1939	0.53	0.914
18	Scioto	Dublin[h]	988	1921–1939	0.69	0.784
19	Mohican	Greer[h]	942	1921–1939	0.46	0.937
20	Kokosing	Millwood[h]	472	1921–1939	0.49	1.023

TABLE 1.—(*Continued*)

Station No. (1)	River (2)	Station (3)	Drainage area (sq miles) (4)	Period of record (5)	Flow variability index (6)	Mean flow (cu ft per sec per sq mile) (7)
		(*d*) STREAMS IN OHIO.—(*Continued*)				
21	Walhonding	Nellie[h]	1,499	1910–1939	0.46	0.969
22	Tuscarawas	Newcomerstown[h]	2,439	1921–1939	0.47	1.007
23	Tuscarawas	Dover[h]	1,398	1923–1939	0.43	1.000
24	Sandy Creek	Sandyville[h]	481	1923–1939	0.41	1.065
25	Little Beaver Creek	East Liverpool[h]	505	1915–1939	0.57	1.020
26	Stillwater	Englewood[h]	646	1925–1939	0.56	0.904
27	Miami	Taylorsville[h]	1,155	1914–1939	0.54	0.846
28	Miami	Dayton[h]	2,513	1905–1939	0.44	0.921
29	Mad	Dayton[h]	632	1914–1939	0.30	1.050
30	Mad	Springfield[h]	485	1904–1939	0.29	1.053
31	Buck Creek	Springfield[h]	137	1914–1939	0.30	0.964
32	Scioto	Columbus[h]	1,624	1921–1939	0.62	0.843
33	Alum Creek	Columbus[h]	190	1923–1940	0.65	0.868
34	Big Walnut Creek	Rees[h]	544	1921–1939	0.64	0.917
35	Licking	Toboso[h]	672	1921–1939	0.50	0.992
36	Muskingum	Dresdon[h]	5,982	1921–1939	0.48	1.021
37	Muskingum	McConnelsville[h]	7,411	1921–1939	0.48	0.791
38	Twin Creek	Germantown[h]	275	1914–1939	0.61	1.045
39	Miami	Hamilton[h]	3,639	1910–1939	0.46	0.966
40	Little Miami	Milford[h]	1,195	1925–1939	0.58	1.098
41	Darby Creek	Darbyville[h]	533	1921–1939	0.65	0.818
42	Deer Creek	Williamsport[h]	331	1926–1939	0.67	0.867
43	Scioto	Chillicothe[h]	3,847	1913–1939	0.55	0.889
44	Hocking	Athens[h]	944	1915–1939	0.55	1.088
45	Whiteoak Creek	Georgetown[h]	221	1923–1935	1.00	1.150
		(*e*) STREAMS IN MAINE				
1	St. John	Fort Kent[i]	5,690	1926–1941	0.46	1.73
2	Fish	Fort Kent[i]	867	1929–1941	0.43	1.60
3	Allagash	Allagash[i]	1,250	1931–1941	0.47	1.62
4	Mattawamkeag	Mattawamkeag[i]	1,400	1913–1941	0.48	1.75
5	Piscataquis	Dover-Foxcroft[i]	286	1912–1941	0.55	2.16
6	Kennebec	Bingham[i]	2,660	1927–1941	0.47	....
7	Carrabassett	North Anson[i]	354	1925–1941	0.48	1.90
8	Sandy	Mercer[i]	514	1929–1941	0.48	1.72
9	Swift	Roxbury[i]	95	1929–1941	0.52	1.94
10	Passadumkeag	Lowell[i]	299	1915–1941	0.36	1.64
11	West Branch, Union	Amherst[i]	148	1929–1941	0.58	1.69
12	Machias	Whitneyville[i]	465	1929–1941	0.39	1.37
13	East Machias	East Machias[i]	251	1927–1941	0.46	1.98
		(*f*) STREAMS IN NORTH CAROLINA				
1	Roanoke	Old Gaston[j]	8,280	1911–1932	0.34	0.970
2	Cape Fear	Lillington[k]	3,530	1925–1935	0.59	0.876
3	French Broad	Asheville[k]	949	1925–1935	0.26	1.910
4	Broad	Boiling Springs[k]	815	1926–1935	0.25	1.727
5	Fishing Creek	Enfield[k]	462	1925–1935	0.47	0.992
6	Hiwassee	Murphy[k]	410	1925–1935	0.33	2.056
7	Deep	Ramseur[k]	343	1925–1935	0.51	0.991
8	Nottely	Ranger[k]	272	1925–1935	0.29	1.812
9	Second Broad	Cliffside[k]	230	1926–1935	0.27	1.264
10	Flat	Bahama[k]	150	1926–1935	0.62	0.893
11	Cullasaja	Cullasaja[k]	87	1925–1935	0.37	2.460

TABLE 1.—(*Continued*)

Station No. (1)	River (2)	Station (3)	Drainage area (sq miles) (4)	Period of record (5)	Flow variability index (6)	Mean flow (cu ft per sec per sq mile) (7)
		(*g*) STREAMS IN VERMONT				
1	Missisquoi	Richford[l]	479	1917–1942	0.48	....
2	Missisquoi	North Troy[l]	131	1932–1942	0.47	2.00
3	Lamoille	Johnson[l]	310	1929–1942	0.38	1.64
4	Lamoille	Cady's Falls[m]	250	1913–1923	0.38	1.73
5	Moose	St. Johnsbury[l]	126	1929–1941	0.49	....
6	Winooski	Essex Junction[n]	1,018	1909–1929	0.43	1.66
7	Winooski	Montpelier[n]	338	1909–1929	0.43	1.43
8	Otter Creek	Middlebury[l]	628	1911–1944	0.53	....
9	White	West Hartford[l]	690	1916–1942	0.45	1.67
10	Black	North Springfield[l]	158	1931–1942	0.44	....
		(*h*) STREAMS IN NEW HAMPSHIRE				
1	Connecticut	Orford[o]	3,100	1900–1915	0.40	1.712
2	East Branch of Pemigewasset	Lincoln[l]	104	1930–1942	0.42	....
3	Saco	Conway[l]	386	1929–1941	0.42	2.28
4	Pemigewasset	Plymouth[l]	622	1904–1942	0.43	....
5	Smith	Bristol[l]	85.8	1919–1942	0.48	....
6	Blackwater	Contoocook[l]	134	1919–1935	0.43	1.67
7	Contoocook	Penacook[l]	766	1930–1941	0.41	1.62
8	Ashuelot	Gilsum[l]	71.1	1923–1942	0.50	....
9	Ashuelot	Hinsdale[l]	420	1908–1942	0.45	....
10	Souhegan	Merrimack[p]	171	1910–1942	0.50	....
		(*i*) STREAMS IN MASSACHUSETTS				
1	Farmington	New Boston[l]	92	1914–1941	0.36	....
2	Middle Branch, Westfield	Goss Height[l]	52.6	1911–1941	0.58	....
3	Westfield	Knightville[p]	162	1911–1920	0.52	....
4	Manhan	Holyoke	13	1897–1910	0.51	2.03
5	Connecticut	Holyoke[p]	8,390	1880–1899	0.36	1.454
6	Connecticut	Sunderland[p]	8,000	1911–1920	0.39	....
7	Sip Pond Brook	Winchenden[l]	18.2	1917–1939	0.47	....
8	Merrimack	Lawrence[p]	4,552	1900–1920	0.37	1.430
9	Charles	Waltham[q]	227	1932–1942	0.54	....
10	Ipswich	Ipswich[l]	124	1931–1942	0.61	....
11	Moss Brook	Wendell Depot[l]	12.3	1917–1942	0.52	....
		(*j*) STREAMS IN PENNSYLVANIA				
1	Susquehanna	Harrisburg[r]	24,100	1890–1936	0.44	1.423
2	Muddy Creek	Castle Finn	133	1928–1938	0.29	1.162
3	West Conewago Creek	Manchester	510	1931–1941	0.58	1.120
4	Conodoguinet	Hogestown	470	1931–1941	0.40	1.086
5	Sherman Creek	Shermans Dale	200	1931–1941	0.50	1.265
6	Conestoga Creek	Lancaster	322	1926–1941	0.41	1.097
7	North Bald Eagle Creek	Beach Creek Station	559	1931–1941	0.40	1.172
8	Clearfield Creek	Dimeling	370	1932–1941	0.58	1.396
		(*k*) STREAMS IN KANSAS				
1	Kansas	Orden[s]	45,200	1918–1934	0.37	0.0443
2	Republican	Wakefield[s]	24,700	1919–1933	0.35	0.0397
3	Blue	Randolph[s]	9,100	1919–1934	0.39	0.132
4	Smoky Hill	Ellsworth[s]	7,580	1919–1934	0.50	0.0290
5	Solomon	Niles[s]	6,770	1920–1934	0.47	0.0435

TABLE 1.—(*Continued*)

Station No. (1)	River (2)	Station (3)	Drainage area (sq miles) (4)	Period of record (5)	Flow variability index (6)	Mean flow (cu ft per sec per sq mile) (7)
		(*k*) STREAMS IN KANSAS.—(*Continued*)				
6	Neosho	Parsons[s]	4,828	1922–1934	0.76	0.456
7	Saline	Tescott[s]	2,820	1920–1934	0.53	0.0497
8	Walnut	Winfield[s]	1,894	1922–1934	0.65	0.290
		(*l*) MISCELLANEOUS STREAMS				
1	Ohio	Paducah, Ky.	203.000	....	0.44	....
2	Colorado	Austin, Tex.	37,000	....	0.59	....
3	Tennessee	Florence, Ala.[t]	30.800	30 years	0.37	1.694
4	Coosa	Childersburg, Ala.[u]	8,390	1896–1912	0.40	1.680
5	Choccolocco Creek	Jenifer, Ala.[v]	275	16 years	0.35	....
6	Locust Creek	Trafford, Ala.[v]	622	13 years	0.72	....
7	Delaware	Port Jervis, N. Y.[w]	3,070	26 years	0.44	1.825
8	Hudson	Spier Falls, N. Y.[x]	2,777	1888–1910	0.31	....
9	Delaware	Fish Eddy, N. Y.	875	27 years	0.47	2.331
10	Delaware	Trenton, N. J.	6,800	17 years	0.35	1.690
11	Clinch	Coal Creek, Tenn.[t]	2,960	1904–1934	0.45	1.504
12	Hiwassee	Reliance, Tenn.[t]	1,220	30 years	0.31	2.190
13	Etowah	Allatoona Creek, Ga.[u]	1,100	1896–1929	0.34	2.053
14	Coosawattee	Carters, Ga.[u]	530	1896–1929	0.30	2.253
15	Roanoke	Roanoke, Va.[j]	388	1896–1931	0.39	1.030
16	Dan	South Boston, Va.[j]	2,730	1901–1931	0.30	1.050
17	Elkhorn	Waterloo, Nebr.[y]	6,390	10 years	0.28	0.108
18	Middle Loup	St. Paul, Nebr.[y]	7,320	10 years	0.19	....
19	Loup	Columbus, Nebr.[y]	14,200	10 years	0.17	0.164
20	St. Lawrence	Below Lake Ontario	286,688	1894–1925	0.045	....
21	St. Clair	Below Lake Michigan	213,941	1897–1925	0.025	....
22	St. Marys	Below Lake Superior	76,136	1897–1925	0.058	...
23	Muskegon	Croton Dam, Michigan	....	1910–1919	0.22	....

[a] "Surface Water Resources of Iowa," *Water Supply Bulletin No. 2*, Iowa Geological Survey, Iowa City, Iowa. [b] "Des Moines River, Iowa," *House Document No. 682*, 71st Cong., 3d Session, Washington, D. C., 1931. [c] "Iowa River, Iowa and Minnesota," *House Document No. 134*, 71st Cong., 2d Session, Washington, D. C., 1930. [d] "Summaries of Yearly and Flood Flow Relating to Iowa Streams," *Water Supply Bulletin No. 1*, Iowa Geological Survey, Iowa City, Iowa, 1942. [e] U. S. Geological Survey, Madison, Wis. [f] "Menominee River, Michigan and Wisconsin," *House Document No. 141*, 72d Cong., 1st Session, Washington, D. C., circa 1935. [g] "Stream Flow Data of Illinois," Dept. of Public Works, Springfield, Ill., 1937. [h] "Ohio Stream Drainage Areas and Flow Duration Tables," by C. H. Hall and C. V. Youngquist, *Bulletin No. 111*, Ohio State Univ. Experiment Station, Columbus, Ohio, May, 1942. [i] Tenth Biennial Rept, Public Utilities Comm., State of Maine, 1941–1942. [j] "Roanoke River, Va. and N. C.," *House Document No. 65*, 74 Cong., 1st Session, Washington, D. C., 1935. [k] "Discharge Records of North Carolina Streams," *Bulletin No. 39*, Dept. of Conservation and Development, North Carolina, Raleigh, N. C., 1938. [l] U. S. Geological Survey, Boston, Mass. [m] "Lamoille River, Vt," *House Document No. 145*, 72d Cong., 1st Session, Washington, D. C. [n] "Winooski River, Vt.," *House Document No. 785*, 71st Cong., 3d Session, Washington, D. C., 1931. [o] "Rainfall and Run-Off Studies," by C. E. Grunsky, *Transactions*, ASCE, Vol. LXXXV, 1922, p. 66. [p] *Journal*, Boston Soc. of Civ. Engrs., Vol. 9, 1922. [q] "Computing Run-Off from Rainfall and Other Physical Data," by Adolph F. Meyer, *Transactions*, ASCE, Vol. LXXIX, 1915, p. 1056. [r] "The Drought of 1930 in Pennsylvania," by J. W. Mangan, Div. of Hydrography, Harrisburg, Pa., 1937. [s] "Notes on the Hydrology of Kansas," *Engineering Bulletin No. 20*, Univ. of Kansas, Lawrence, Kans., 1936. [t] Tennessee Valley Authority. [u] "Alabama-Coosa Branch of Mobile River System," *House Document No. 66*, 74th Cong., 1st Session, Washington, D. C., 1935. [v] U. S. Geological Survey, Montgomery, Ala. [w] "Report of Precipitation, Flood, and Reservoir Studies," Iowa Inst. of Hydr. Research, Iowa City, Iowa. [x] Sixth Annual Rept. of the State Water Supply Comm. of New York, Albany, N. Y. [y] "Water Resources of Nebraska," Nebraska State Planning Board (revised February, 1941).

to those in the adjacent rough part of Iowa. The lake regions in the northern part of Wisconsin also produced low values ranging from 0.14 to 0.38. The great moraines of the Wisconsin glacier are largely effective in producing uniform flows, which are indicated by the low values of 0.15 to 0.35. The flat regions, without lakes or moraines, gave rise to values of 0.53 to 0.71 as in similar regions in Iowa. Large variations in value occurred within small distances, much too small to be accounted for by variations in rainfall. The only satisfactory explanation seemed to be surface storage or geological conditions.

Only a few of the 20 stations analyzed in Illinois (Table 1(*c*)) were affected by surface storage. The terminal moraines of the glaciers, however, exercised a great influence on many streams. The areas where lakes or extensive moraines were not present, but were underlain by impervious strata, produced high values of the coefficient ranging from 0.69 to 1.09. In Illinois, geology also seemed to be the principal cause, although there is a possibility that rainfall variation may have had some influence in producing the high values in the southern part of the state.

The 45 stations analyzed in Ohio (Table 1(*d*)) showed results similar to those in Iowa, Wisconsin, and Illinois. The values obtained range from 0.29 to 1.17, the latter being the highest value found in the entire study. One of the interesting developments was the differences in the indexes of the tributaries of the Miami River. Three of the main tributaries gave values ranging from 0.54 to 0.61, and one, the Mad River, had a value of 0.30. The main stream below the tributaries had values of from 0.44 to 0.46, very close to what would be obtained from a weighted mean of the tributary flows.

The low values for Mad River resulted from an unusually large area of glacial moraine in its watershed, together with porous bedrock and the bed of a large preglacial river. Except for geology, its basin does not differ noticeably from those of the other tributaries.

In Maine (Table 1(*e*)) 13 stations were found where artificial regulation was negligible, giving values ranging from 0.36 to 0.58. Lakes and swamps exercised an important influence on many of these data. The equalizing effect of a given area of lakes in this state did not appear to be so great as in Wisconsin, probably because of the more impervious soil and bedrock conditions in Maine.

There was some variation in the results obtained for the 32 stations analyzed in Vermont, New Hampshire, and Massachusetts (Tables 1(*g*), 1(*h*), and 1(*i*), respectively), the values varying from 0.36 to 0.61. Sufficient data were not available to analyze the causes of these variations.

The 11 stations studied in North Carolina (Table 1(*f*)) ranged from 0.25 to 0.62, the high values being in the flatter section of the state and the low values, in the mountains. The low mountain values probably reflect the effect of rainfall. In these mountains rain occurs very frequently, which tends to produce uniform flows.

The flow of 8 streams in the Susquehanna Valley in Pennsylvania (Table 1(*j*)) were studied because stream flow variability was known to change widely within short distances because of geological conditions. The values varied from 0.29 to 0.58, under conditions that seemed to preclude any other cause than geology.

Analysis was made of 8 streams in Kansas (Table 1(*k*)), although the rainfall on these watersheds was usually much less than that on the other streams investigated in this study, and the variability values did not differ greatly, the values ranging from 0.35 to 0.76. Some of these streams in the Great Plains extended out to, or nearly to, the Rocky Mountains. Sufficient data were not available on the geology of their basins to determine its effect on the flow variability.

Data were available from 23 miscellaneous rivers, extending from Georgia and Texas to Nebraska (Table 1(*l*)). To show the lower limit of the variability index, the flows of the St. Lawrence, St. Clair, and St. Marys rivers in the Great Lakes watershed were analyzed, giving values of 0.045, 0.025, and 0.058, respectively (based on monthly mean flows). The effect of extensive deposits of sand in the watershed in equalizing the water flow was shown by the values obtained on the Loup and Elkhorn rivers of Nebraska and the Muskegon River in Michigan, which gave values of from 0.17 to 0.28. These were among the lowest values obtained.

Use of the Variability Index

One of the principal uses of the index described in this paper is that it enables the variability of streams to be compared readily by giving numerical values to this characteristic of their flow. Usually only general terms have been used for such comparisons in the past, and only very rough comparisons were possible. The agreement of the flow duration curve with a straight line on logarithmic probability paper offers a convenient method of comparing the form of the flow duration curves.

Another use of the index is that it has revealed the effect of the several factors on the variability of the flow of streams during the greater part of the time (but not in floods or droughts). Of the seven factors listed by Mr. Morgan,[9] this study has shown that geology (including soil conditions) and surface storage in lakes and swamps are the predominant factors affecting the variation of ordinary flows in most of the streams investigated. Although data were not available to indicate conclusively how large was the effect of rainfall variability, the evidence was strong that it was usually less important than geology or surface storage.

Probably the greatest use of this index will be in developing synthetic duration curves for the analysis of hydraulic engineering problems, where the available data are meager. It can thus be used in studies of water power, water storage, and sediment production, where duration curves have been found to be useful. The method consists of:

(1) Selecting a variability coefficient depending on the characteristics of the watershed;

(2) Determining the mean annual runoff of the stream; and

(3) Drawing the line on logarithmic probability paper which is represented by the variability index and the mean annual runoff.

If stream flow records are available for watersheds in the vicinity, having similar characteristics to the one under study, the value obtained for this stream can be used. If the difference between the watersheds is small, the value desired may be obtained by applying a small correction to the known value.

If stream flow records of short duration are available, they may be used and compared with the variability values obtained from records of streams in the vicinity, both for the same and longer periods, if possible. For cases where no flow records are available, the following method has been developed from a study of all the variability index values included in Table 1 and the geological and surface storage conditions of the watersheds. The values given in Table 2 have been prepared for use in selecting a variability coefficient, assuming that the index for an average river is 0.60 and that the presence of other conditions in the watershed, indicated in Table 2, give rise to the corrections listed, which are either added to, or subtracted from, the value of 0.60. The corrections for soil and bedrock are intended to be applied only to flat or gently rolling land and not to rugged areas. The values for moraines and deep soils should be weighted according to the proportion of the watersheds covered by them. Where the soil cover is deep, the bedrock corrections should be reduced, as the influence of bedrock is less. In a terrain covered with moraines, and also containing lakes or swamps, the highest values of the corrections given should not be used.

TABLE 2.—CHARACTERISTIC VALUES OF VARIABILITY COEFFICIENT

Terrain	Description	Subtract from 0.60	Add to 0.60
Bedrock	Impervious	....	0.1 to 0.2
	Average	0.1 to 0	0 to 0.1
	Pervious	0.1 to 0.2	....
Soil, deep covering	Impervious	....	0.1 to 0.2
	Average	0.1 to 0	0 to 0.1
	Pervious	0.1 to 0.2	....
	Very pervious	0.2 to 0.3	....
	Sand hills	0.3 to 0.4	....
Relief	Flat or gently rolling	0	0
	High in impervious material	0.05 to 0.1	....
	High in average material	0.1 to 0.2	....
	High in pervious material	0.2 to 0.3	....
Moraines	Impervious	0	0
	Average	0.05 to 0.1	....
	Pervious	0.1 to 0.2	....
	Pervious, with many kettle holes	0.2 to 0.4	....
Lakes and swamps	Small areas	0.05 to 0.1	....
	Moderate areas	0.1 to 1.5	....
	Extensive areas	1.5 to 2.5	....

The streams having large subsurface storage frequently are somewhat skewed with a curve that is convex to the origin of coordinates. This gives higher flows for the low and high values than the straight line would indicate and smaller flows for the medium. As previously stated, the very high discharge values from the straight line are too high and the very low values are too low since the straight line extends to both zero and infinity.

Until considerable comparison has been made, it will not be possible to judge how far these values can be applied in regions other than the northeastern United States. In a region where data are meager, the variability values for such data as are available should be obtained, and these relations tested out on it.

In applying these values, it will be necessary to use judgment; otherwise negative values may be obtained. For example, the bedrock and relief corrections should not both be used as the bedrock corrections are included in those for relief. The correction for moraines should apply to the area covered by moraines and the soil corrections should not be applied to this area also. Where bedrock lies so deep that it is below the water table, it should not be considered.

Large watersheds will tend to have lower values than small ones, because the runoffs from rainfall from the various parts of the basin will not synchronize, and because the channel storage will tend to equalize the flow.

The method of determining the mean annual runoff of streams will depend upon the amount of data available, the more data the more accurately the runoff can be estimated. If no rainfall or stream flow data are available, the runoff can be estimated roughly from that in other localities producing similar vegetation. This method, of course, is very inexact and should be used only as a last resort. If rainfall data only are available, the runoff can be estimated by estimating the loss and subtracting it from the rainfall. The loss can be estimated from that of watersheds having similar conditions; or, if temperature records are available, by the methods of Adolph F. Meyer,[19] M. ASCE, or J. W. Johnson,[18] M. ASCE. In the eastern half of the United States, the loss may be estimated from data supplied by the U.S.G.S.[20] If a short term of runoff record is available on the watershed under study, its runoff could be compared with that for the same years of a watershed in the vicinity that had a long record and also with the runoff of this latter watershed for the total period of record.

The shape of the duration curve can be obtained by drawing a straight-line duration curve on logarithmic probability paper with a slope such that the ratio of the discharge exceeded 15.87% of the time to the discharge exceeded 50% of the time was equal to the variability index selected. The ends of this line for a short distance on each end will have to be altered to match the best possible estimates of maximum and minimum flow, since a straight line on this paper goes from infinity to zero. A considerable error in these estimates will not have much effect, as they affect only a small part of the time. The position of the curve will have to be established by trial, making the area under the curve (when plotted on coordinates similar to Fig. 1) represent the quantity of water indicated by the mean annual runoff.

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[19] "Computing Run-Off from Rainfall and Other Physical Data," by Adolph F. Meyer. *Transactions*, ASCE, Vol. LXXIX, 1915, p. 1056.

[20] "Natural Water Loss in Selected Drainage Basins," by G. R. Williams and others, *Water-Supply Paper No. 846*, U.S.G.S., Washington, D. C., 1940.

Possible Future Studies

Further studies will probably show that this variability index is directly related to such factors as the magnitude of minimum flows, of flood flows of moderate frequency, and the storage necessary to maintain various uniform flows. There is good reason to believe that the greater the variability index, the greater will be the spread between the duration curves of individual years. If this be true, a more detailed estimate could be made of the frequency of periods of various magnitude of flow than it is possible to obtain from a simple duration curve.

From a study of the relation of the watershed characteristics and the departure of some of the curves from a straight line on logarithmic probability paper, it may be possible to determine the conditions giving rise to these departures and thus enable more accurate synthetic curves to be developed than would be possible by using the straight-line method.

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DISCUSSION

Ven Te Chow.[21]—Following the computation procedure proposed in the paper, the variability index can be expressed by the formula:

$$I_v = \sqrt{\frac{\Sigma(y - \bar{y})^2}{n - 1}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

in which y is the logarithm of the selected discharge at a 10% interval of the duration curve; $\bar{y}$ is the mean of y; $y - \bar{y}$ is the deviation from the mean; and n is the number of selected discharges, or $n = 10$ in the given procedure. In a practical operation, it is seldom convenient to take the deviations from the actual mean, since such deviations usually involve decimals which are cumbersome to handle when squared. For greatest convenience (especially in machine computation), another form of Eq. 1 is obtained by utilizing the mathematical principle that the sum of the squares of deviation from the mean is equal to the sum of the squares of deviation from zero, less the product of the total and the mean. Modified slightly, this principle is expressed as

$$I_v = \sqrt{\frac{n}{n - 1}\left(\overline{y^2} - \bar{y}^2\right)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Eq. 2 is known as the Gaussian formula, which is used to estimate the standard deviation of population in statistics. In Eq. 2 $\overline{y^2}$ is the mean of the squares of y. The variability index can be defined mathematically and simply by this formula.

The numerical method of computing the variability index was primarily presented for the purpose of eliminating the effects introduced by variations in personal judgment. However, as the method used the values of discharges read from the duration curve as plotted, it did not eliminate the personal factor introduced in plotting the curve. A complete elimination of the influence of personal judgment is possible only by replacing the "eye fitting" by a mathematical process—that is, the values of $\bar{y}$ and $\overline{y^2}$ contained in Eq. 2 should be computed directly from the raw data. Such data can be grouped by the method of statistical classification to reduce the number of data handled in computation. If necessary, the computed standard deviation can be corrected by applying the Sheppard correction.

Since the mathematical part of the paper has been omitted, the method of producing a duration curve from the given values of the variability index and the mean annual runoff cannot be checked readily. However, the stipulation (under the heading, "Use of the Variability Index") that—

> "* * * the ratio of the discharge exceeded 15.87% of the time to the discharge exceeded 50% of the time was equal to the variability index selected"

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[21] Hydr. Eng. Laboratory, Univ. of Illinois, Urbana, Ill.

—would make the variability index greater than unity for most streams, because the discharge that is exceeded 15.87% of the time is usually greater than the discharge exceeded 50% of the time. Apparently this estimate does not agree with the values of the variability index listed in Table 1.

The knowledge of hydrologic statistics has been developed and extended greatly in recent years. Many investigators[17,22] have found that the distribution of most hydrologic phenomena is skew. Probability plottings of the Hazen type were also found unsatisfactory for such frequencies. The computation of asymmetric probabilities is now possible in the field of hydrology; and the development of a special probability paper[16,23] opened the way for a straight-line plotting of the duration curve. Therefore the study of stream flow would be advanced if a suitable asymmetric probability plotting could be adapted to it. The values of discharge should be plotted on a linear scale, so that the variability index would be equal to the standard deviation, in a statistical sense.

Many hydrologists like to use the coefficient of variation, C_v, instead of the standard deviation, σ, for the study of variability. The coefficient of variation is equal to the standard deviation divided by the mean, or

$$C_v = \frac{\sigma}{\bar{y}} \qquad \text{(3)}$$

For purposes of comparison, the coefficient of variation is preferable to the standard deviation, because the former, being dimensionless, eliminates the effects of the difference from the mean. This elimination is desirable, especially for stream discharge in more arid regions where the annual rainfall tends to be smaller as well as more variable.

For any statistical study, the quantity of the available data is as important as the quality. As long as the quality is kept homogeneous, the longer record, or larger number of data, would make possible more accurate estimates of the frequencies of stream flow. W. D. Potter,[24] of the Soil Conservation Service, United States Department of Agriculture, has made a probability study of rainfall and runoff. The study was confined to annual maximum values; but it is worthy to note that the frequency of distribution is a function of years of record, which is a measure of the number of data used. Since the variability index is a measure of the slope of the duration curve (a frequency curve), it should be some function of the length of record. Therefore, the variability index would be of more statistical value if the two were compared on the basis of the same length of record.

Dana M. Wood,[25] **M. ASCE.**—The selection of an analytical method depends upon the use to which the results are to be put and the time available

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[22] "Floods Estimated by Probability Method," by E. J. Gumbel, *Engineering News-Record*, June 14, 1945, p. 97.

[23] "A Simple Method of Estimating Flood Frequency," by Ralph W. Powell, *Civil Engineering*, February, 1943, p. 105.

[24] "Simplification of the Gumbel Method for Computing Probability Curves," by W. D. Potter, *No. SCS-TP-78*, Soil Conservation Service, U.S.D.A., Washington, D. C., May, 1949.

[25] Chf., Power Studies Branch, Div. of Water Control Planning, TVA, Knoxville, Tenn.

for the study. Cost is also a factor in any engineering office. Two suggestions concerning the duration curve may be of value to those desiring to extend this work.

(1) Instead of using a scale of "discharge in cubic feet per second per square mile," substituting "ratio to the average flow" for the period of record under investigation will result in relative variability curves of flow. The values of departures at 10% intervals can then be obtained similarly from this curve. This procedure eliminates the size of the drainage area and the magnitude of the runoff in any comparisons. A stream having a flow of 0.5 cu ft per sec per sq mile can be compared with one having a flow 2.5 cu ft per sec per sq mile with respect to the slopes of the duration curves—that is, the magnitude of the departures from the average, regardless of the size and other characteristics of the drainage area. The writer would like to know if this approach might in any way simplify the logarithmic calculations.

(2) The second point of importance is that the same period of years should be used when comparing different records in the same general locality. It is well known that the magnitude of the runoff varies in cycles of wet and dry years, although little enough is known about the amplitude and magnitude of the curve of variation. However, in the Tennessee basin and numerous other basins, the years from 1930 to 1948 have been generally in the "dry" part of a cycle. The previous nineteen years, from 1910 to 1929, were generally in a flat part of the cycle having normal annual variations. It is quite obvious that any analysis could lead to erronous conclusions if records in the two separate periods were compared. This point is well illustrated by the doubt regarding the analysis of records in different periods of years when attempting to determine the frequency of occurrence of floods of great magnitude.[26] However, for more distant localities, where the runoff cycles coincidentally may be radically different, further clarification and the development of methods of adjustment are needed.

This paper offers suggestions for many interesting exploratory studies, especially for the younger engineers training in these fields who are not afraid of undertaking many laborious calculations.

H. ALDEN FOSTER,[27] M. ASCE.—The authors have presented a new and interesting method for classifying the daily-flow duration curves of various streams by a single "variability index," together with the mean flow of the stream. At first glance, the variability index might seem to be related in some way to the classification of skew probability curves by the coefficient of variation and the coefficient of skew, as originally proposed by the late Allen Hazen,[12] M. ASCE, and further developed by the writer.[15] However, the latter classification was intended to be used in the study of selected samples of the original data, such as annual floods or mean annual runoff values, for which the number of items actually used was limited, so that the calculations could be performed without an excessive amount of work. To compute the coefficients of variation and skew for a daily-flow record of several years would be entirely impracticable.

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[26] "Possible and Probable Future Floods," by William P. Creager, *Civil Engineering*, November, 1939, p. 668.

[27] Prin. Associate, Parsons, Brinckerhoff, Hall & Macdonald, New York, N. Y.

Certain points related to the proposed method may need further clarification:

(1) The calculation of the variability index is independent of the unit of runoff used, whether cubic feet per second or cubic feet per second per square mile, or flow in terms of mean flow—because any change in the unit does not change the shape of the duration curve but only the relative values; and, when the curve is plotted on logarithmic probability paper, the result would be only to shift the entire curve parallel to itself.

(2) The variability index is a function only of the slope of the straight-line duration curve when plotted on logarithmic probability paper. Hence, the value of the index is not changed if the curve is shifted parallel to itself on the plotting paper.

(3) When applying the method for a study in which the data for plotting a daily-flow duration curve are incomplete, the following procedure should be followed—

(*a*) Assume a value for the variability index—I;

(*b*) Determine a value, Q_i, such that $\log Q_i = I$ (Q_i being a daily flow value, expressed in terms of the median stream flow);

(*c*) Assume a value for the median stream flow Q_{50} (the flow exceeded 50% of the time), which may be estimated from the mean flow, as explained subsequently;

(*d*) Compute the discharge exceeded 15.87% of the time as the product of Q_i and Q_{50} (derivation of this percentage value being given subsequently); and

(*e*) Then plot the duration curve as a straight line on the logarithmic probability paper, passing through the two stream flow values for 15.87% and 50% of time.

Probability Paper.—To clarify the foregoing procedure, some explanation of the method of constructing the logarithmic probability paper is necessary. As far as the writer knows, no adequate description of such a method of construction has been published. Construction of this plotting paper is based on the use of the "normal curve of error," which has the form:

$$y = \frac{h}{\sqrt{\pi}} e^{-h^2 x^2} \qquad (4)$$

and is plotted in Fig. 4(*a*). The maximum ordinate of the curve is $y_o = h/\sqrt{\pi}$. The "probable error" is $r = 0.4769/h$. The probable error is also defined by the fact that the area under the curve of error between the abscissas $-r$ and $+r$ is one half of the total area under the curve. The normal curve of error is known as a "frequency curve." If the frequency curve is integrated, the result is a "duration curve," or in the present case the curve of the "probability integral," as shown in Fig. 4(*b*). When plotted in this way, the axis of the ordinates becomes the "percentage-of-time" scale, commonly used in plotting duration curves. As shown in Fig. 4(*b*) the part of the duration curve between $x = -r$ and $x = +r$ is represented by the distance between 25% and 75% in the percentage-of-time scale.

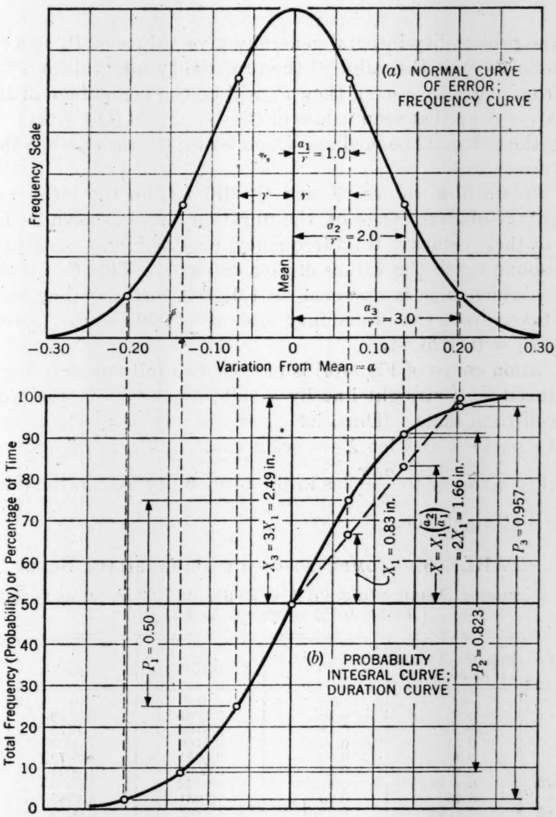

FIG. 4.—RELATION OF PROBABILITY INTEGRAL CURVE TO PROBABILITY PLOTTING PAPER

If the abscissas, *a*, of the frequency curve are expressed in terms of *r*, the area under the curve from $-a/r$ to $+a/r$ may be found from values of the probability integral, as tabulated in mathematical textbooks. This integral may be expressed as follows:

$$P_a = \frac{h}{\sqrt{\pi}} \int_{-a}^{+a} e^{-h^2x^2}\, dx = \frac{2}{\sqrt{\pi}} \int_{o}^{ah} e^{-h^2x^2}\, d(h\,x) \quad\ldots\ldots\ldots (5a)$$

or

$$P_a = \frac{2}{\sqrt{\pi}} \int_o^X e^{-x^2} dx \ldots\ldots\ldots\ldots\ldots\ldots (5b)$$

in which

$$X = a\,h = 0.4769\,\frac{a}{r} \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Tables of the probability integral generally give values of P_a as a function[28] of X or of[29] a/r. If P_a is the value of the probability integral for a "residual" of a/r (area from $-a/r$ to $+a/r$), then P_a will be the percentage-of-time interval on the duration curve between values of $(Q_{50} - a)$ and $(Q_{50} + a)$. The symbol Q_{50} denotes the value of the median, which is also the mean when the frequency curve is symmetrical.

If two stream flow values, Q_1 and Q_2, differ from the mean by a_1 and a_2, their percentage-of-time range on the duration curve (shown as P_1 and P_2 in Fig. 4(*b*)), or their values of total frequency, may be found from the probability table corresponding to the values of a_1/r and a_2/r. This fact is illustrated in Fig. 4(*b*) in which a_1/r is assumed as 1.0; the corresponding value of total frequency taken from the probability table is 0.5000 ($= P_1$). Similarly, with $a_2/r = 2.0$, $P_2 = 0.8226$, etc.

The duration curve of Fig. 4(*b*) is plotted as a full line at natural scale. It can be reduced to a straight line by a suitable change in the spacing of the percentage-of-time scale. Thus, let X_1 = scale value of P_1, in inches on the "probability paper"; and let X_2 = scale value of P_2, in inches. If X_1/X_2 is made equal to a_1/a_2, or to $\frac{a_1/r}{a_2/r}$ (as in the probability table), the duration curve will plot in a straight line.

TABLE 3.—COMPUTATION OF PROBABILITY SCALE

(Assuming Distance from 0.1% to 99 9% As 7.62 In.; from 50% to Either 0.1% or 99.9% As 3.81 In.)

Percentage of time		Relative distance from 50% line	P_a	$X = 0.4796\ a/r$	a/r	Distance from 50% line (in.)
50		0	0	0	0	0
40 or	60	0.10	0.20	0.1791	0.3756	0.31
30	70	0.20	0.40	0.3708	0.7775	0.65
25	75	0.25	0.50	0.4769	1.0000	0.83
20	80	0.30	0.60	0.5951	1.2479	1.04
10	90	0.40	0.80	0.9062	1.9001	1.58
8.87	91.13	0.4113	0.8226	0.9538	2.0000	1.66
5.0	95.0	0.45	0.90	1.1631	2.4389	2.03
2.15	97.85	0.4785	0.9570	1.4307	3.0000	2.49
2.0	98.0	0.48	0.96	1.4522	3.0451	2.53
1.0	99.0	0.49	0.98	1.6449	3.4490	2.87
0.5	99.5	0.495	0.99	1.8213	3.8190	3.18
0.1	99.9	0.499	0.998	2.1850	4.5817	3.81

For example, assume that the distance on the plotting paper between 0.1% and 99.9% is to be *d* inches, and it is desired to determine the proper scale distance between 25% and 75%: $P_{a1} = 75\% - 25\%$, or 0.50 for which

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[28] "A Short Table of Integrals," by Benjamin Osgood Peirce, Ginn & Co., New York, N. Y., 1910.

[29] "American Civil Engineers' Handbook," Thaddeus Merriman, Editor, John Wiley & Sons, Inc., New York, N. Y., 5th Ed., 1930, p. 79.

$a/r = 1.00$; and $P_{a2} = 99.9\% - 0.1\% = 0.998$, for which $a/r = 4.5817$. The scale distance from 25% to 75% will then equal $(1.00/4.5817)d$ in. Computation of other plotting positions on the probability paper is shown in Table 3.

The values of X, corresponding to P_a are taken from a published table[28] of probability integrals.

Percentage of Time for Plotting Variability Index.—When a straight line is plotted on arithmetical probability paper (arithmetical scale for ordinates), the difference between the ordinate value at any given percentage of time and that for 50% of time will be proportional to the probability integral, P_a, corresponding to the assumed percentage. Similarly, if a logarithmic scale is used for the ordinates, the corresponding difference between the logarithms will be proportional to P_a. The problem is to determine the percentage of time at which the logarithmic ordinate of the assumed straight line has a numerical value equal to the variability index of the line. This could be done

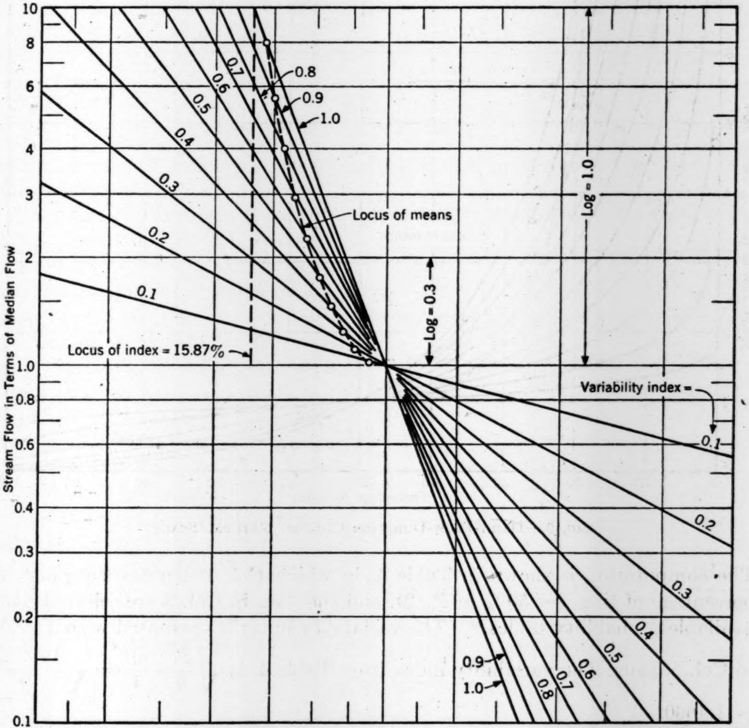

FIG. 5.—DAILY-FLOW-DURATION CURVES; LOG PROBABILITY SCALE

by scaling off the ordinates on the logarithmic probability paper at the 5%, 15%, ··· 95% points, and computing the index as explained by the authors; but a more accurate result will be obtained by assuming logarithmic ordinates proportional to the corresponding values of the probability integral, computing the corresponding values of the index, and then determining the percentage of time or probability (P_a) that would have a value of a/r equal to the index.

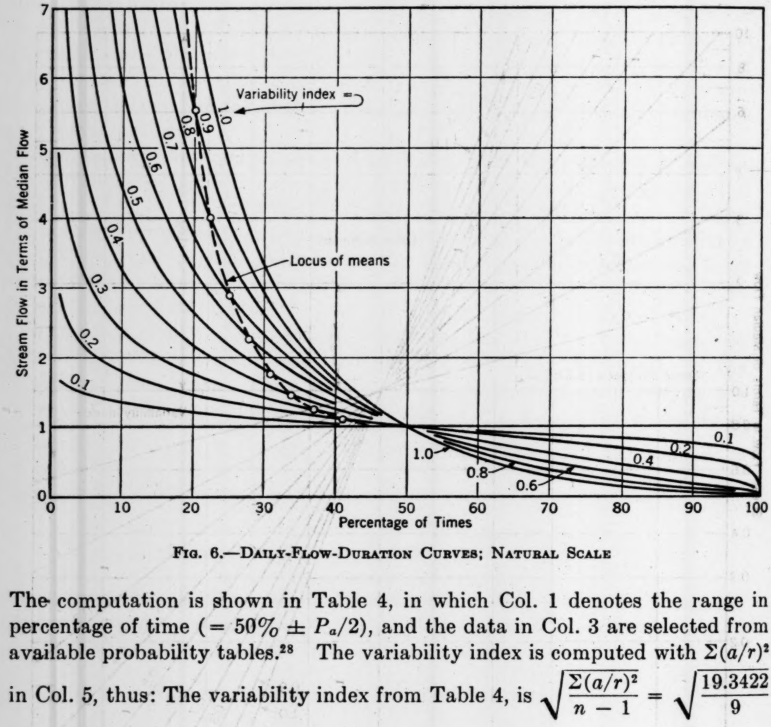

FIG. 6.—DAILY-FLOW-DURATION CURVES; NATURAL SCALE

The computation is shown in Table 4, in which Col. 1 denotes the range in percentage of time (= 50% ± $P_a/2$), and the data in Col. 3 are selected from available probability tables.[28] The variability index is computed with $\Sigma(a/r)^2$ in Col. 5, thus: The variability index from Table 4, is $\sqrt{\frac{\Sigma(a/r)^2}{n-1}} = \sqrt{\frac{19.3422}{9}} = 1.4660$.

TABLE 4.—COMPUTATION OF VARIABILITY INDEX

Percentage of time (1)	P_a (2)	$X = 0.4769 \frac{a}{r}$ (3)	$\frac{a}{r}$ (4)	$\left(\frac{a}{r}\right)^2$ (5)
5 to 95	0.90	1.1631	2.4389	5.9482
15 to 85	0.70	0.7329	1.5368	2.3617
25 to 75	0.50	0.4769	1.0000	1.0000
35 to 65	0.30	0.2725	0.5714	0.3265
45 to 55	0.10	0.0889	0.1864	0.0347
Sum..........				9.6711
Doubled............	....	....	....	19.3422

The corresponding value of X = 1.4660 times 0.4769 = 0.6991; and P_a, corresponding to $X = 0.6991$, is 0.67718. Then, the percentage of time, for the computed index, is $50 - \frac{67.718}{2} = 16.14\%$. The discrepancy between this value and 15.87% as given by the authors may be due to the omission of the decimals.

TABLE 5.—CHECK COMPUTATION OF VARIABILITY INDEX

Percentage of time	Q	Log Q	(Log Q)^2
5	6.65	0.8228	0.67700
15	3.30	0.5185	0.26884
25	2.17	0.3365	0.11323
35	1.56	0.1931	0.03728
45	1.165	0.0663	0.00440
Σ(log Q)^2	....	....	1.10075
Doubled	....	....	2.20150

A series of duration curves with values of the variability index from 0.1 to 1.0 is shown in Fig. 5. For convenience in plotting, the 50% or median stream flow value is assumed as 1.0 (logarithm = 0.0). For an index of 1.0, the line will pass through a value of 10.0 (logarithm = 1.0) at 15.87% of time. The other lines will have arithmetical ordinates, at the same percentage of time, equal to their respective index values.

To show the true relative shape of these duration curves, they have been replotted in Fig. 6 on arithmetical scales. A check of the accuracy of this method is given in Table 5, in which the values of flow (Q) were scaled from the line plotted for the index = 0.50. As in the case of Table 4, the variability index is $\sqrt{\frac{2.20150}{9}} = 0.495$.

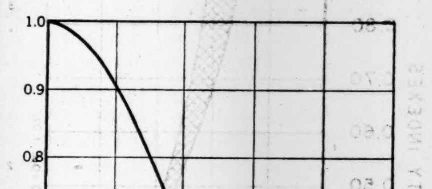

FIG. 7

Ratio of Median to Mean Value of Stream Flow.—Unless the duration curve of stream flow has been actually calculated, there is no method of determining the median value from the original records, although the mean stream flow is generally known or may be estimated. The method of constructing the duration curve proposed by the authors requires the use of the median, and consequently it is necessary to establish a relation between these two values for various values of the index. The writer has attempted to establish such a relation by arithmetical integration of the plotted curves. The results are shown in Fig. 7 which also indicates the percentage of time at which the duration curve crosses the mean ordinate.

C. S. OSPINA[30] AND G. TAMA,[31] JUN. ASCE.—The "variability index" offers an excellent method for the hydrologist to compare conditions in different watersheds. Its computation is simple and it does quite accurately represent the greater part of the flow variations. To facilitate plotting adjusted curves defined by the mean flow and the variability index, the writers suggest the construction of a graph which shows the percentage of time during which the mean flow is exceeded, as a function of the variability index (Fig. 8). For indexes greater than 0.35 only a range can be indicated, with sufficient precision for practical purposes, because the average is affected by high discharges occurring during a small percentage of the time. Fig. 9 shows the adjusted duration curves used, which are straight lines on logarithmic probability paper.

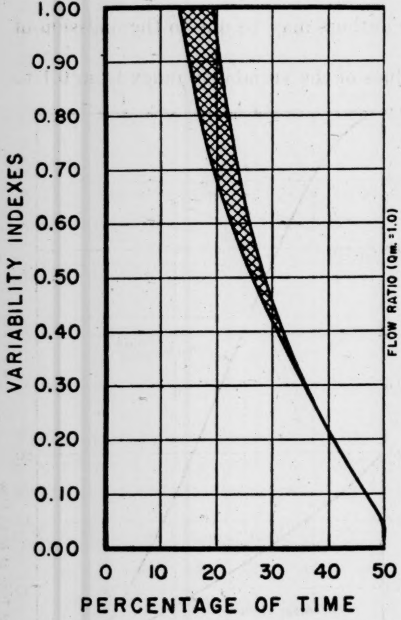

FIG. 8.—ADJUSTED UNIT DURATION CURVES; PERCENTAGE OF TIME DURING WHICH MEAN FLOW IS EXCEEDED

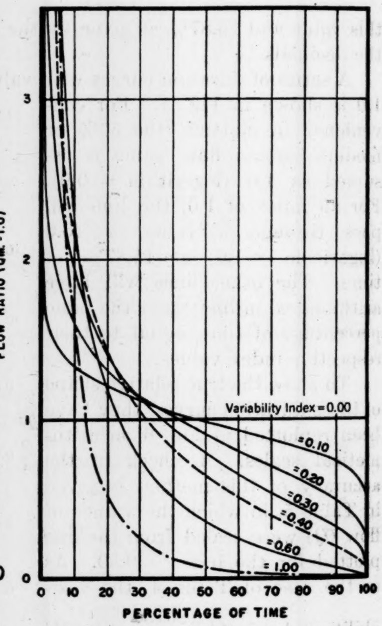

FIG. 9.—ADJUSTED DURATION CURVES FOR DIFFERENT VARIABILITY INDEXES

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[30] Partner, Olarte, Ospina, Arias and Payán, Engrs., Cali, Colombia.

[31] Associate, Olarte, Ospina, Arias and Payán, Engrs., Cali, Colombia.

The published data reveal no definite relation between mean unit flow (cubic feet per square mile) and the variability index, except for a certain similarity in rivers situated in the same part of the United States. It is noted that more than half of the 224 stations analyzed have drainage areas of less than 1,000 sq miles, and that the authors' conclusions in regard to the variability index incline to refer to small rivers. Both common sense and study of the tables confirm a definite influence of the size of the watershed on the index. The values in Table 6 have been deduced from the data presented by the authors. It is apparent that extremely high indexes occur only in small watersheds and that there is a general tendency toward lower values as the drainage area increases.

TABLE 6.—VARIABILITY INDEXES

Drainage area (square miles)	No. of stations	Most frequent	Maximum	Minimum
0.1 to 1,000.....	127	0.40 to 0.50	1.17	0.14
1,000 to 5,000...	59	0.40 to 0.50	0.96	0.18
5,000 +........	37	0.30 to 0.40	0.79	0.025

For the Cauca basin in Western Colombia the following information can be given: Flow variations on the main river and most of the tributaries follow the same trend, regardless of the size of the watershed. An index of 0.23 is typical. The mean annual runoff is exceeded during 40% of the time and it averages 1.8 cu ft per sec per sq mile. In contrast to the uniformity of the variability index the unit runoff varies considerably even in watersheds with similar rainfall characteristics. The total drainage area at the lower end of the so-called Cauca Valley is 9,000 sq miles, the greater part of which is located in rugged country. Soils vary from impervious silts to pervious alluvial deposits in the flat regions; in most of the mountainous regions the soils are impervious with grass, bush, and forest cover. There are two dry seasons and two rainy seasons and the annual rainfall is about 40 in.

After years of work in irrigation and power planning, the writers were surprised to find a variability index so low in comparison to that for streams in the United States.

The authors recommend their procedure for studies where the available hydrological records are meager and they rightly state some of the factors which limit its use. The annual mean flow is to be determined from short-term flow data, from comparison with other hydrological records and drainage basins, from temperature records, etc. The selection of the variability index is to be governed by geological conditions and surface storage only, whereas rainfall distribution and intensity, vegetation cover, and arrangement of tributaries are acknowledged to be of influence to an unknown degree.

Actually, the number of factors is such that no accurate predictions can be made unless flow records exist covering a certain period. To the writers' knowledge, even short-term discharge data are likely to be more useful than coefficients derived from conditions in other regions. Attention is called to the fact that geologic conditions may be defined for restricted areas, yet in large basins the problem will become rather complex. Preliminary estimates of stream flow in undeveloped countries are hampered by deficient hydrological records and by little or no information as to the area and general features of the drainage basin. The first guess will reflect personal judgment, whatever estimating method is used. As investigations proceed, flow records from newly established gaging stations will provide more reliable information. Older hydrological records, if existent, may be used to establish correlation. Unit duration curves, which are based on mean flow and have the advantage of being dimensionless, as well as chronological records from other rivers, are valuable to determine flow variations (variability index), annual mean discharge, and duration of droughts on the river studied. Mass curves of rainfall will be found useful to analyze the general trend. The accuracy of stream flow records should be checked, since systematic errors in high discharges, due to deficient section rating, will affect the variability index.

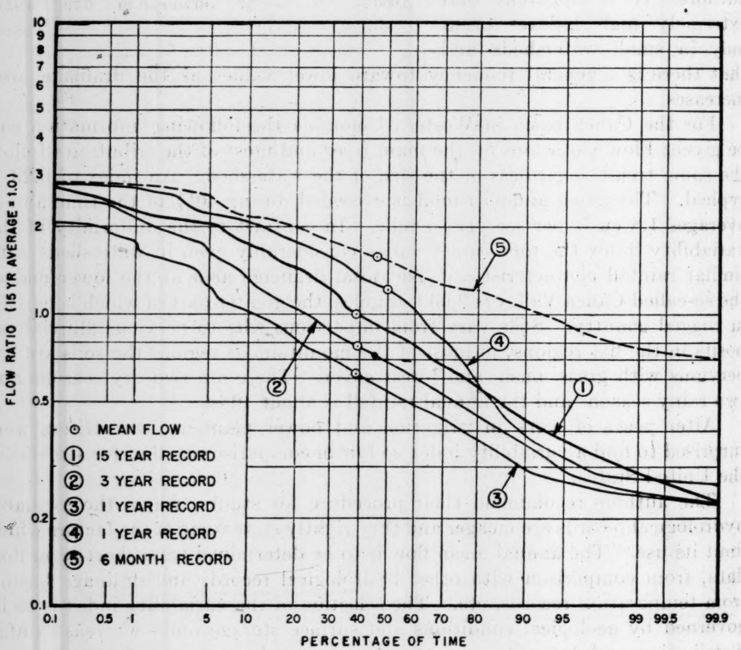

FIG. 10.—VARIATION OF DURATION CURVES FOR DIFFERENT PERIODS ON THE CAUCA RIVER AT CALI, COLOMBIA

The adjustment of short-term records to fit average conditions requires experience, since both mean flow and variability index change with the length of the period considered. Fig. 10 shows the actual duration curves of the same river (Cauca) for different periods, related to the 15-year average of $Q_M = 1$. The mean flow of the other periods is indicated by circles. It is evident that the "index" method of adjusting duration curves to straight lines on logarithmic probability paper should be applied only to long-term records. It is necessary to examine thoroughly all pertinent relations until a representative duration curve can be plotted.

Finally, it should be remembered that most hydraulic studies will require an estimate of critical conditions during a period of severe drought. These are not fully defined by a duration curve representing a long-term record, and they have to be determined by a design hydrograph based on actual discharge data and adjusted to the most critical conditions.

The writers' experience on the hydrological studies in the Cauca Valley shows that, when only short-term records are available, the most difficult task is the determination of the minimum flow hydrograph—which represents not only the minimum flow itself but also the hydrograph for the entire critical dry season. This hydrograph is of the greatest importance in planning irrigation and power projects where good storage reservoir sites are not available. The ratio of minimum flow to average flow for a period of from 3 to 15 years varies from 1:2.3 to 1:5.8 for the tributaries of the Cauca River, all with practically equal variability indexes. The Cauca itself has a ratio of 1:3.6. On the other hand, the characteristics of the dry season hydrograph vary very widely from one river to the other.

All this indicates the complexities and uncertainties of hydrological extrapolations. The writers consider that the greatest usefulness of the variability index, where few records are available, is to help develop the judgment of the engineer and thus aid him to make a better guess of the mean and the minimum flows.

The principal value of the paper is considered to be the development of a useful statistical investigating procedure and the analysis of a considerable volume of flow data from North American rivers. It is then hoped that other hydrologists will contribute information from different regions and nations in order to further the study of the factors which affect the index.

Raphael G. Kazmann,[32] **Assoc. M. ASCE.**—The authors would do a great service if they would enlarge on the use of their variability index. For example, the method depends on the proper selection of the variability index as outlined in Table 2. This table translates qualitative descriptions of geology, hydrology, and topography into quantitative terms. Apparently a rather thorough geologic and topographic survey of the drainage area is needed, together with a considerable amount of soil sampling and testing if comparable results are to be obtained by independent observers.

What permeabilities and what percentages of the drainage area are meant by the various subheadings under "bedrock and soil?" What slopes and percentages of drainage area are referred to under "relief?" Likewise what, quantitatively, is the average permeability of moraines? Under "lakes and swamps" what percentages of drainage area correspond to "small," "moderate," and "large?"

Then, there is the problem of determining the mean annual runoff of the stream. Using the method given by Don Johnstone, Assoc. M. ASCE, and

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[32] Hydrologic Engr., Ranney Method Water Supplies, Columbus, Ohio.

William Perry Cross,[33] M. ASCE, the length of stream flow record needed to estimate the mean annual flow to various degrees of accuracy, for three Ohio streams, is shown in Table 7. These streams have been gaged continuously since 1923. On an ungaged stream, how many years of gaging do the authors suggest, and what degree of accuracy is desired, in determining the mean annual runoff for use with Table 2?

TABLE 7.—LENGTH OF RECORD NEEDED

River	Location	Drainage area (sq miles)	LENGTH OF RECORD, IN YEARS REQUIRED TO ESTIMATE MEAN ANNUAL FLOW WITHIN:			
			10%	15%	20%	25%
Licking . . .	Toboso, Ohio	672	54	25	15	10
Scioto	Chillicothe, Ohio	3,847	62	29	17	12
Mad	Springfield, Ohio	485	53	25	15	10

The writer would like to withhold judgment on the proposed method of developing flow duration curves until studies are made on watersheds whose physical characteristics are known and where long-term gaging records are available. The magnitude of deviation between the computed duration curves and the actual duration curves would show conclusively how applicable Table 2 is to field conditions. Naturally, all the suggested studies would have to be made without reference to actual long-term gaging records if valid comparisons are to result.

JACK BRUIN[34] AND H. E. HUDSON, JR.,[35] M. ASCE.—The proposed method of evaluating stream flow variability is obviously useful in applying data from streams with long periods of record to comparable streams for which short periods of record exist.

There have been a number of developments of small Illinois watersheds for municipal supply purposes. It is expected that there will be an increase in the use of smaller watersheds for impounded supplies to minimize sediment damage and spillway costs. Periods of zero flow exceeding 5% of the time are often encountered in such streams, making the method of this paper inapplicable. Similar situations are found in streams flowing on highly permeable valley fills and in arid regions where rainfall is infrequent. The method is worthy of modification to make it applicable to intermittent streams.

An attempt was made to compare the stream flow in Money Creek above Lake Bloomington with other gage records in central Illinois. For this purpose stream flow variability indexes were computed. An immediate difficulty made itself evident in that Money Creek had a zero flow 13.54% of the 12-year period of record. The variability index became infinity.

An inspection was made of the stream flow records from Illinois gaging stations, and it was found that this problem existed for five watersheds in the group from 45 sq miles to 500 sq miles. The largest of the gaged areas having zero flow more than 5% of the time is Mazon River, which drains a watershed only a small part of which is morainic, the remainder being largely lacustrine.[36] Money Creek drains a watershed of 45 sq miles, which lies between two moraines of Wisconsin age. The following approximate distribution of subsoil permeability[37] exists in the two watersheds (areas in square miles):

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[33] "Elements of Applied Hydrology," by Don Johnstone and William Perry Cross, Ronald Press Co., New York, N. Y., 1948.

[34] Eng. Asst., Eng. Sub-Div., State Water Survey Div., Dept. of Registration and Education, Urbana, Ill.

[35] Head, Eng. Sub-Div., State Water Survey Div., Dept. of Registration and Education, Urbana, Ill.

Permeability	Mazon River	Money Creek
Slowly to very slowly	173	0
Moderately slow	212	7
Moderate	85	38
Total	470	45

The Lake Bloomington watershed is somewhat representative of the average subsoil permeability in Illinois, whereas the Mazon River subsoils are considerably less permeable than most subsoils in the state. Zero flow was reported for Mazon River during 7.06% of the period of record, which is 9 years. In this period the runoff from streams in the same region was slightly above normal. Station discharge duration curves for Money Creek above Lake Bloomington and Mazon River at Coal City, Ill., computed from records gathered through cooperation with the United States Geological Survey, are shown in Fig. 11.

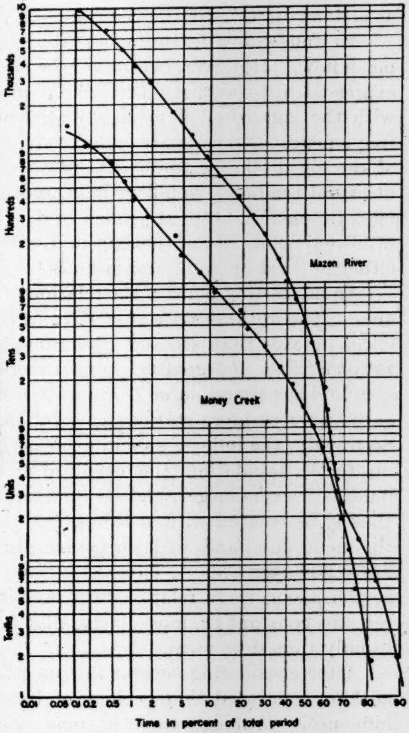

FIG. 11.—DISCHARGE DURATION CURVE, MONEY CREEK AND MAZON RIVER IN ILLINOIS

In view of the inapplicability of the authors' method to streams in which there is zero flow more than 5% of the time, it is suggested that the variability index be based on the period during which flow exists. Calculated values for the streams discussed herein, using the percentage values and procedure proposed by the authors are as follows:

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[36] "Physiographic Divisions of Illinois," by M. M. Leighton, George E. Ekblaw, and Leland Horberg, *The Journal of Geology*, January, 1948, p. 16.

[37] "Principal Soil Association Areas of Illinois," Dept. of Agronomy, Univ. of Illinois Agri. Experiment Station, Urbana, Ill., May, 1948.

Description	Mazon River	Money Creek
Variability index	1.27	0.94
Percentage of time of zero flow	7.06	13.54

This index would be accompanied by a statement of the proportion of time during which there is zero flow.

H. W. Lull.[38]—A particularly interesting feature of this ingenious index of variability, not pointed out by the authors, is its comparability in meaning to a coefficient of variation—that is, it represents not variation, as such, but variation in respect to the mean. Since a constant logarithmic deviation corresponds to a constant percentage change, computing the standard deviation of the logarithms of the discharge gives an index of deviations in percentage of the mean, as is done directly by the coefficient of variation.

An undesirable feature of the index is that it cannot be related directly to mean flow. However, because of the normality of the distribution of discharge, expressed in logarithms, throughout most of the range, the index when used with the logarithm of mean flow provides estimates of discharge at various frequencies. No estimates, obviously, could be made for the curvilinear parts of the log discharge frequency relationship. In this connection, perhaps the utility of the index would be increased if estimates of discharge at the 1% and 99% frequencies were appended to it. These values, by indicating the nature of the curvilinearity of the extremes, would be of importance to those interested either in flood or in minimum flows.

An interesting point is the relation of the variability indexes to the drainage area. The authors state that streams from large watersheds will tend to have lower indexes than streams from small watersheds because of less synchronization of flow and greater channel storage. Surprisingly, a tabulation of the area and the variation of 22 streams, listed by the authors (each of which were gaged at 2 or more stations), reveal that this is not the case (see Table 8). Seemingly, the indexes should decrease in the downstream direction. According to the tabulation, this occurred on 11 streams, whereas on 8 streams the indexes became larger, and on 3 streams (Table 8(*a*)) they remained constant. Closely associated with a decrease in the variability indexes is an increase in the mean flow; and, with an increase in the indexes, there is a decrease in the mean flow (see Cols. 4, Table 8). Since the indexes reflect variation in respect to the mean, these relationships permit the speculation that either variation remains constant or, more likely, that it changes in the same direction but less rapidly than does mean flow.

After considering several factors which affect stream flow variability, the authors concluded that geology and soils, and surface storage had the major influence. Although there is some evidence to support this conclusion, it is hardly sufficient to warrant the finality the authors give to it. In a study of this kind, it is extremely doubtful if factors having an important bearing on stream flow variability can be isolated.

One reason for this doubt is the complication posed by the wide range of drainage areas. On small drainages, vegetation and weather may be important factors, local occurrence of high intensity rainfall or droughts, or density of vegetation, having a greater effect on variability than they would have on larger areas. Geology and soils could well be the most influential factor on larger areas possessing a variety of different types of vegetation. Similarly, a small area possessing a uniformity of topography would produce stream flow of an entirely different character than larger drainages including a diversity of topographical features. Thus, it is difficult to select factors of major importance when watersheds of different sizes are considered.

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[38] Forester, Div. of Forest Influences, Forest Service, U. S. Dept. of Agriculture, Washington, D. C.

TABLE 8.—RELATION OF STREAM FLOW VARIATION TO SIZE OF DRAINAGE AREA

Station[a] (1)	Drainage area[b] (2)	Index[c] (3)	Mean flow[d] (4)
(*a*) VARIATION REMAINS CONSTANT WITH INCREASE IN AREA			
Muskingum River, Ohio (Table 1(*d*))			
36	5,982	0.48	1.021
37	7,411	0.48	0.791
Lamoille River, Vermont (Table 1(*g*))			
4	250	0.38	1.73
3	310	0.38	1.64
Winooski River, Vermont (Table 1(*g*))			
7	338	0.43	1.43
6	1,018	0.43	1.66
(*b*) VARIATION DECREASES WITH INCREASE IN AREA			
Des Moines River, Iowa (Table 1(*a*))			
3	5,490	0.64	0.278
5	6,818	0.63	0.267
6	13,200	0.59	0.246
7	13,900	0.50	0.382
Skunk River, Iowa (Table 1(*a*))			
9	320	0.94	0.350
10	2,890	0.53	0.495
11	4,290	0.54	0.502
Iowa River, Iowa (Table 1(*a*))			
12	1,500	0.51	0.471
13	3,230	0.49	0.457
14	12,480	0.38	0.489
Cedar River, Iowa (Table 1(*a*))			
15	845	0.39	0.341
18	1,660	0.38	0.407
19	6,640	0.36	0.442
(*b*) VARIATION DECREASES WITH INCREASE IN AREA (*Continued*)			
Wapsipinicon River, Iowa (Table 1(*a*))			
20	1,060	0.64	0.442
21	2,300	0.45	0.564
Sandusky River, Ohio (Table 1(*d*))			
12	299	0.75	0.833
2	1,248	0.63	0.702
Scioto River, Ohio (Table 1(*d*))			
18	988	0.69	0.784
32	1,624	0.62	0.843
43	3,847	0.55	0.889
Miami River, Ohio (Table 1(*d*))			
27	1,155	0.54	0.846
28	2,513	0.44	0.921
39	3,639	0.46	0.966
Ashuelot River, New Hampshire (Table 1(*h*))			
8	71.1	0.50	....
9	420	0.45	....
Connecticut River, Massachusetts (Table 1(*i*))			
6	8,000	0.39	....
5	8,390	0.36	1.454
Delaware River, New York and New Jersey (Table 1(*l*))			
9	875	0.47	2.331
7	3,070	0.44	1.825
10	6,800	0.35	1.690
(*c*) VARIATION INCREASES WITH INCREASE IN AREA			
St. Croix River, Wisconsin (Table 1(*b*))			
1	1,550	0.19	0.761
2	2,820	0.24	0.705
4	5,120	0.31	0.583
5	5,930	0.30	0.566
Wolf River, Wisconsin (Table 1(*b*))			
18	812	0.20	0.958
29	2,240	0.28	0.820
Kankakee River, Illinois (Table 1(*c*))			
7	2,340	0.32	0.762
6	4,870	0.42	0.640
Sangamon River, Illinois (Table 1(*c*))			
11	550	0.76	0.763
10	2,560	0.77	0.696
Maumee River, Ohio (Table 1(*d*))			
7	2,049	0.56	0.744
1	6,314	0.79	0.671
Auglaize River, Ohio (Table 1(*d*))			
9	333	0.56	0.883
8	2,329	0.73	0.696
Tuscarawas River, Ohio (Table 1(*d*))			
23	1,398	0.43	1.000
22	2,439	0.47	1.007
Mad River, Ohio (Table 1(*d*))			
30	485	0.29	1.053
29	632	0.30	1.050

[a] Cols. 1, 2, and 3, Table 1. [b] Col. 4, Table 1. [c] Col. 6, Table 1. [d] Col. 7, Table 1.

Furthermore, unless each of the several factors is given equal consideration, it is doubtful if any conclusion is justified as to relative importance. At the present time, lack of data on all factors would prohibit such an analysis; but it may be that, if the authors had given more attention to factors considered of minor influence, a different conclusion would have been reached as to relative importance of the factors.

For instance, from the authors' conclusions the inference can be made that vegetation is not an influential factor. A comparison of the variation of streams from forested and nonforested areas of similar size in Iowa and Wisconsin does not support this view (see Table 9). Furthermore, it should be emphasized that the relatively small variation from areas of greatest relief, in Iowa and North Carolina, is perhaps due more to the fact that they are mainly forest covered than to the reasons given by the authors. If weather were truly considered a factor, perhaps the authors should have considered the effect of the accumulation and rapid melting of snow on the variability in the more northern states. In addition, the factors deemed important by the authors, when considered singly and in respect to the area involved, are not of major importance. Surface storage can obviously be an important factor only in the relatively limited areas in which it exists. Geology and soils were found to be important in certain areas, but not in enough areas to justify selection as a major factor.

TABLE 9.—COMPARISON OF VARIABILITY INDEXES OF FORESTED AND NONFORESTED DRAINAGES

FORESTED AREAS (SQ MILES)			NONFORESTED AREAS (SQ MILES)		
River	Area	Index	River	Area	Index
(a) IOWA					
Little Maquoketa	130	0.39	Skunk	320	0.94
Yellow	224	0.35	Lime Creek	535	0.63
Turkey	1,530	0.44	Iowa	1,500	0.51
			Shellrock	1,700	0.46
			Cedar	1,660	0.38
			Wapsipinicon	1,060	0.64
(b) WISCONSIN					
Apple	550	0.24	Crawfish	732	0.57
Jump	510	0.62	Rock	971	0.68
Pine	543	0.32	Milwaukee	661	0.48
Eau Claire	500	0.53	Sugar	529	0.24
Black	756	0.60			
Wolf	812	0.20			
Oconto	678	0.26			
Peshtigo	571	0.36			
Mean variability (Table *(a)*)		0.39	Mean variability (Table *(a)*)		0.49

Perhaps a more reasonable conclusion would be that the relative importance of the factors cannot be determined for as large an area as the one concerned. All the factors may be considered as having a bearing on variability; each is present to a degree in all drainages; in certain drainages one or two factors may be of particular importance because they are present in sufficient intensity to mask the effect of the other factors. However, because of the variability of the factors over the particular area, in addition to the variation in size of the drainages which tends to intensity the effect of the more variable factors on the smaller areas, and the fact that evidence is not available to permit a thorough analysis of the effect of each factor, the data that the authors present do not allow an evaluation of the relative importance of the factors affecting stream flow variability.

William D. Mitchell,[39] **M. ASCE.**—The authors have made a sound and comprehensive presentation of stream flow duration. Their use of logarithmic probability scales is particularly commendable.

Instances are rare indeed in which the logarithmic probability plot of the flow duration curve will yield a straight line throughout the entire range of flow. For such rare cases, the curve is adequately described by the two parameters suggested by the authors—namely, the variability index and the mean discharge. As they have pointed out, the position of the curve must be established by trial, but that task is not particularly difficult. On the other hand, in all cases with which the writer is familiar, there is substantial curvature near one or both ends of the logarithmic probability plot so that the two parameters are not adequate to define the curve. This fact has been recognized by the authors, as evidenced by their statement (under the heading, "Use of the Variability Index"): "The ends of this line for a short distance on each end will have to be altered to match the best possible estimates of maximum and minimum flow * * *."

A project on which the writer is engaged involves the flow duration curves for 15 of the Illinois stations used by the authors. The basic data differ in only one major respect—namely, the records have been extended to include all complete water years through 1945. One of the stations, Big Muddy River at Plumfield, Ill., has been selected to show the relation between the synthetic curve obtained by the authors' method, and the actual curve derived from the station data.

In the illustration the synthetic development assumed that the variability index and the mean discharge per square mile (q) were known to be, as given in Table 1, Cols. 6 and 7, 1.01 and 1.021, respectively. From published data,[40] it was determined that the extremes of discharge prior to September 30, 1933, were 16,300 cu ft per sec and 0 cu ft per sec (the equivalent of 21.6 cu ft per sec per sq mile and 0 cu ft per sec per sq mile) and it was inferred that the time of no flow was about 9 days in 19 years, or a little more than 0.1%.

The resulting synthetic curve appears as the continuous line in Fig. 12. In determining the slope of this line, it was necessary to modify the statement of the authors (under the heading, "Use of the Variability Index"):

> "The shape of the duration curve can be obtained by drawing a straight-line duration curve on logarithmic probability paper with a slope such that the ratio of the discharge exceeded 15.87% of the time to the discharge exceeded 50% of the time was equal to the variability index selected."

Since the selected index is 1.01, it appears that a line of such slope would be nearly horizontal, and not at all representative of the flow duration curve. Obviously, the ratio of the discharges should be equal to the antilogarithm of the selected index, or, in this case, 10.23. The line was drawn on this slope and, on the first trial, was made to pass through the point given by J. H. Morgan[41]—q = 1.01 and percentage of time equals 25.8. The data on extreme discharges offered no reason for curvature in the lower end of the line, but the upper end obviously must become asymptotic to the horizontal $q = 21.6$, and the line was drawn accordingly. Step integration of the area under this line yielded a mean discharge of 1.25 cu ft per sec per sq mile, greatly in excess of the assumed value of 1.02 cu ft per sec per sq mile, which indicated that the curve must be moved leftward to the position as shown.

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[39] Hydr. Engr., U. S. Geological Survey, Champaign, Ill.

[40] "Surface Water Supply of the United States, 1933," *Water-Supply Paper No. 745*, U.S.G.S., Washington, D. C., 1935, p. 213.

[41] "Flow-Duration Characteristics of Illinois Streams," by J. H. Morgan, *Transactions*, Am. Geophysical Union, Vol. 17, 1936, p. 419.

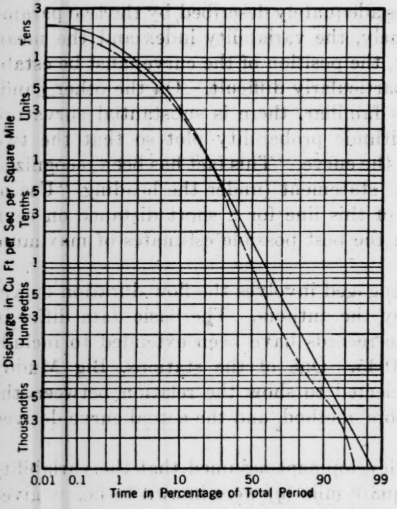

FIG. 12.—DURATION CURVES FOR DAILY FLOWS OF BIG MUDDY RIVER AT PLUMFIELD, ILL.

For comparison, the duration curve as developed by the writer, based on actual records for the 35 water years, from 1909 to 1912 and from 1915 to 1945, is shown as the dashed line in Fig. 12. As indicated in Table 10, the variability index for the longer period is 1.08, whereas the mean discharge is 0.891 cu ft per sec per sq mile. The duration of the mean discharge is 23.5% of the total time.

The decrease in mean discharge, due to the use of additional record, is largely responsible for the downward shift from the synthetic to the actual curve. On a percentage basis this shift is a maximum at about the 70% duration point, where the synthetic curve indicates a discharge of more than double that of the actual curve. Had this particular point been the subject of investigation, the use of the synthetic curve could not be recommended. At other points, particularly in the region of the mean discharge, the agreement between the two curves is very satisfactory. It should be noted that the assumed shape of the synthetic curve has been helpful in obtaining the good agreement in the region near the mean discharge. Since the parameters for the synthetic curve are based on the same data as used by Mr. Morgan, a properly shaped duration curve for the years from 1915 to 1933 would have passed through the point given by him; that is, $q = 1.01$ and percentage of time equals 25.8. However, since the shape of the curve was only approximated by the synthetic procedure, it was necessary to compensate for the inaccuracy by shifting the synthetic curve to pass through the point $q = 1.01$ and the percentage of time equals 21.8. It is believed that the improvement thus obtained is only coincidental; in other cases the shift necessary to compensate for inaccurate shape may be such as to move the synthetic curve away from the actual curve in the region of the mean discharge.

Furthermore, the curvature in the upper part of Fig. 12 is opposite to that mentioned by the authors (under the heading, "Use of the Variability Index").

This is due to the fact that Big Muddy River is bordered by extensive flood plains which, by their storage effects, materially reduce the magnitude of the high discharges.

The foregoing example benefits, of course, from the fact that very pertinent values were assumed for the index and the mean discharge. Had it been necessary to estimate these values from data given in Table 2 or from records of near-by watersheds, any less appropriate estimates would have been reflected in the synthetic curve.

TABLE 10.—CHANGE IN VARIABILITY INDEX AND MEAN DISCHARGE AT SELECTED SITES, DUE TO ADDITION TO LENGTH OF RECORD; STREAMS IN ILLINOIS (SEE TABLE 1(*c*))

Station No.	River	Station	Variability Index			Mean Discharge		
			Lane-Lei	Mitchell	Ratio	Lane-Lei	Mitchell	Ratio
1	Pecatonica	Freeport	0.26	0.30	1.15	0.766	0.677	0.88
2	Fox	Algonquin	0.39	0.42	1.08	0.614	0.560	0.91
3	Rock	Lyndon (Como)	0.30	0.32	1.07	0.694	0.597	0.86
7	Kankakee	Momence	0.32	0.30	0.94	0.762	0.747	0.98
8	Spoon	Seville	0.59	0.63	1.07	0.620	0.615	0.99
9	La Moine	Ripley	0.65	0.71	1.09	0.604	0.583	0.97
10	Sangamon	Riverton	0.77	0.75	0.97	0.696	0.678	0.97
11	Sangamon	Monticello	0.76	0.76	1.00	0.763	0.727	0.95
12	Vermilion	Danville	0.73	0.70	0.96	0.566	0.691	1.22
13	Macoupin Creek	Kane	0.85	0.86	1.01	0.721	0.738	1.02
14	South Fork	Kincaid	0.89	0.94	1.06	0.824	0.878	1.07
15	Kaskaskia	Vandalia	0.70	0.74	1.06	0.753	0.717	0.95
18	Skillet Fork	Wayne City	1.09	1.10	1.01	0.940	0.813	0.86
19	Big Muddy	Plumfield	1.01	1.08	1.07	1.021	0.891	0.87
20	Cache	Forman	1.02	1.07	1.05	1.400	1.23	0.88

The writer has had no opportunity to test the data given in Table 2. Some information may be provided, however, on the variation of the index at a given point on a given stream, as a result of variation in the period of record. As stated, new duration curves have been prepared for 15 of the stations described by the authors. These new curves include data for all complete water years prior to 1946. New values of the mean discharge and of the variability index, computed by the authors' method, are shown in Table 10. At certain stations, the index value has changed appreciably, and, in general, the index value has increased while the mean discharge has decreased. These fifteen cases are too few to warrant firm conclusions as to trends; but they do raise a question as to whether an increase in the value of the variability index may not be expected with an increase in the length of record.

C. R. OWNBEY,[42] JUN. ASCE.—This paper should help to stimulate the interest of hydraulic engineers in the possibilities of a powerful tool for the evaluation of hydrologic data, which heretofore has not received the attention it deserves. With some notable exceptions, hydraulic engineers have not kept pace with, or taken proper advantage of, the developments in the mathematical science of statistics. The authors have introduced, to the analysis of stream flow data, one of the characterizing numbers used by statisticians as a measure of the nature of the distribution of data.

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[42] Asst. Prof. of Civ. Eng., Univ. of Tennessee, Knoxville, Tenn.

In introducing their variability index (see under the heading, "Previous Studies of Duration Curve Properties") they state that:

> "Many civil engineers are familiar with this from its application in triangulation and base line measurement, where the standard deviation is known as the 'root mean square error.'"

They might also have stated another analogy that is perhaps even more familiar to civil engineers—that of the mathematical and physical similarity to the moment of inertia and the radius of gyration of areas. In characterizing stream flow data, the engineer is interested in the spread, or dispersion, of values, rather than in the probable error; and this is essentially what the radius of gyration measures for the cross-sectional area of a column or other structural member.

Without having seen the original, more detailed paper filed in the Engineering Societies Library, the writer believes that this paper has suffered in condensation from the omission of background material. The explanation of the method of constructing a synthetic frequency curve from the variability index and the mean flow, in particular, is not clear.

It is wondered why the authors chose to depart from the accepted definition of "standard deviation" in defining their variability index. The defining formula for standard deviation is

$$\sigma = \sqrt{\frac{\Sigma(X - \bar{X})^2}{N}} \qquad \text{.......................(7)}$$

in which X is an individual variate; $\bar{X}$ is the mean value; and N is the total number of variates in the population group, which, in the case of daily stream flows, would mean the number of days from minus to plus eternity. When the calculation of standard deviation is based on a "sample" consisting of n days of record, a fair approximation is obtained by applying a correction factor $\sqrt{n/n - 1}$, making Eq. 7 equal to

$$\sigma = \sqrt{\frac{n}{n - 1}}\sqrt{\frac{\Sigma(X - \bar{X})^2}{n}} \qquad \text{..................(8)}$$

The precision of the number so obtained increases with the number of days, n, used in the calculation, and it will be noted that the correction factor approaches unity as n increases.

In effect, the authors have applied the correction factor in Eq. 8 by using the divisor 9 for their calculations based on the 10 values picked from a frequency curve, which is an overcorrection (as will be explained).

The use of 10 numbers picked from a graph is questionable. It would seem better to group the raw data into equal class intervals and to use midvalues of these intervals, with corresponding frequencies, for the calculation of the index. These intervals should be chosen with regard for proper statistical representation of the data, but would surely be such as to give more than 10 groups in this case. At the same time, the arithmetical simplifications made possible by the use of equally-spaced intervals would reduce the labor involved to a minimum. It is doubtful that the calculation (suitably made) of the index from the raw data would require any more time than a computation by the method of picking 10 points from the duration curve as described by the authors, especially in view of the time required to arrange and plot 3,650 values of daily flow for a 10-year period of record.

Referring again to the use of 9 as a divisor in calculating the "variability index," this would be the proper number to use if the index were based on only 10 values of raw data, except that, if the index had been based on only 10 days of record, it would be statistically worthless. The authors have stated that the minimum period used in the compilation of their tables was 10 years, which for daily flows means a minimum of 3,650 variates. Would it not have been more nearly correct to divide by 10, the number so obtained being theoretically subject to a correction factor equal to $\sqrt{3{,}650/3{,}649}$, which is negligible?

The principal value of probability paper in these studies would seem to be that it provides a ready, but rough, method of determining whether the data of the problems at hand follow a curve called the normal curve of distribution, with its related probability law—the proof being that points plotted on this paper yield a straight line. Apparently the authors concluded that the actual values of stream flow do not follow the normal law, but that the logarithms of these values do follow it—hence, the use of logarithms rather than actual stream flow values in calculating the index.

In any problem for which it can be assumed that the variates (whether raw data or their logarithms) follow the normal distribution curve, it is a simple matter to compute values for a frequency curve with percentage of time as its argument (called a flow-duration curve by hydraulic engineers). For this purpose it is necessary to have only the mean value and the standard deviation (known or estimated from tables such as those presented in this paper) and a table of solutions of the probability integral, readily found in references. However, one characteristic of the normal distribution is equality of the mean and the median. For stream flow it would be necessary to assume that the mean flow was the same as the flow equaled or exceeded 50% of the time, and this is not usually found to be true. Perhaps the authors' method of constructing a synthetic frequency curve is applicable to skew frequencies. If so, it is hoped that further explanation will be given in the closing discussion.

The numerical value of the standard deviation as applied to the flow of a stream reflects not only the variability of the flow but also its magnitude. The great value of data such as presented in this paper should be in the possible use of a stream as a "model," from which to construct or check a flow-duration curve for some comparable stream for which data are scarce. To eliminate the effect of size from such a comparison, it seems that the ratio of the standard deviation to the mean flow (the coefficient of variation) should be used, rather than the standard deviation itself. It should be understood, of course, that in order for 2 streams to be comparable, their sizes could not be greatly different—certainly not of the order of scale ratios commonly used for other purposes in hydraulics.

S. P. WING,[43] M. ASCE.—In a different connection, John S. Cotton,[44] M. ASCE, emphasized the fact that the determination of reservoir capacity needs "* * * as much or more attention than most other current problems * * *" to which engineers habitually give their attention. The paper provides additional techniques through which such capacity studies can be made. Nearly all countries possess rainfall information from which reasonable estimates of mean annual runoff are possible; but knowledge of the variation of the daily stream flow that determines reservoir capacity is much more limited. This situation is particularly true in underdeveloped countries where knowledge of minimum stream flow is especially needed now because of current emphasis on wide-scale river basin planning. The paper deals with two problems: First, the development of an adequate index through which daily stream flow variation can be expressed; and, second, the determination of the factors from a knowledge of which its quantitative value can be estimated.

The utility of arithmetic averages for summarizing a mass of data is commonplace. Hydrologists guide their judgment through their knowledge of mean runoff per square mile, mean rainfall, mean flood discharges, and so forth. Knowledge of these mean values helps prevent mistakes when the engineer is dealing with unreasonable observations or fragmentary data. On the other hand, acceptable data on the mean values of stream flow variation are few, perhaps because of the lack of a generally accepted variation coefficient through which past studies could have been summarized. In placing before the profession their logarithmic "flow variability index," and in giving its value for some 200 American streams as computed from hundreds of thousands of observations, the authors provide hydrologists with type values for streams whose physical characteristics are well known and hence provide the engineer with a "judgment" background.

Long ago Francis Galton[45] pointed out that if, instead of characterizing the frequency distributions of sets of observations of natural phenomena such as runoff, rainfall, and so forth by standard deviations, they be characterized by the standard deviation of the logarithm of the observations, a much better and more useful variation index would be obtained, and one better fitting the observed distributions. Luigi Gherardelli[46] in Italy and Aime Coutagne[47] in France early in the 1930's utilized his suggestions in their studies of runoff variations. In a brilliant paper classifying streams not only by their indexes of variation with respect to daily stream flow as in the present paper, but also by a series of other variation indexes to take account of seasonal and subsoil seepage characteristics, Mr. Coutagne presents values for the Lane-Lei variation coefficient (in terms of its reciprocal) for a number of European streams. His values are very similar to those in the paper but his conclusions are different.

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[43] Civ. Engr., Washington, D. C.

[44] Discussion by John S. Cotton of "Multiple-Purpose Reservoirs: A Symposium," *Transactions*, ASCE, Vol. 115, 1950, p. 900.

[45] "Statistics by Intercomparison, with Remarks on the Law of Frequency of Error," by Francis Galton, *The London Edinburgh and Dublin Philosophical Magazine and Journal of Science*, 4th Series, Vol. XLIX, January to June, 1875, p. 33.

[46] "Au Sujet de Quelques Récentes Formules Statistiques Concernant les Déterminations Hydrologiques," by Luigi Gherardelli, *Rapports Bulletin No. 21*, Assn. Internationale d'Hydrologie Scientifique, Rome, Italy. 1934.

[47] "Classification des Cours D'Eau D'Après Les Méthodcs Statistiques," by Aime Coutagne, Pithiviers, Imprimerie des Caisses d'Épargne, Grenoble, France, 1935.

Presumably Mr. Coutagne utilized standard methods similar to those presented by the American Society of Testing Materials[48] in computing the variation index, procedures which the writer would recommend. It is hoped, in their closure, the authors will clarify their own methods by illustrating with an example, as it is important that their method be fully understood by those utilizing their work and the present procedures are not clear.

A couple of illustrations of the physical meaning of the variation index should help expand its use. A stream whose logarithmic variability is 0.6 (the typical stream of Table 2) has an antilogarithm of 4. The theory of statistics indicates that a stream with this standard deviation of its logarithms of flow has two thirds of its daily flows within the range of one fourth to four times the yearly daily average. Average flow here means the flow corresponding to the antilogarithm of the average logarithm, a value of the order of 0.6 times the ordinary arithmetic mean flow. Such a stream for 83% of the time would develop 25% of its potential power without storage. The wide range in the suitability of the streams for power developments from a purely hydrological viewpoint is indicated by the wide range of the authors' indexes, from 1.2 to 0.2. These, having antilogarithms of 16 and 1.6, mean that for 83% of the time these streams had flows that were greater than 6% and 63% of the yearly average.

The second part of the paper concerns a discussion of the factors on which day-to-day stream flow variation depends. In the "Synopsis," it is stated:

> "* * * in the northeastern quarter of the United States the geology (including soil cover) and the presence of lakes and swamps are the most important factors in stream flow variability during the greater part of the time and * * * other factors have relatively smaller effect."

In view of the many varying characteristics included in the watersheds of the stream cited in the paper (such as rainfall, drainage area, geography, and culture—all of which are commonly thought to effect the uniformity of stream flow), it is regretted that the paper does not contain supporting analysis for this conclusion. In Table 2 the authors propose that the variation index for an unfamiliar stream be estimated by utilizing the average variation index of 0.60 typical of their data and then modifying it to values of from 0.7 to 0.2 (a ratio of 3.5 to 1) purely on the basis of effects due to varying geological factors—that is, percentages of glacial moraine and bed rock, or varying percentages of lakes or swamps. Other factors are given no consideration. The writer proposes briefly to examine whether other factors may not be of equal importance and demonstrated better.

Rainfall.—That varying rainfall causes a varying runoff is almost too well known to require comment. Correlations greater than 0.90 are commonly found when comparing these two variables whether on the basis of daily, weekly, monthly, or annual flows. If this is true a glance at typical rainfall variation indexes should give a guide to the variation to be expected of runoff. From Mr. Coutagne's data the following daily rainfall variation indexes are presented for certain European stations:

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[48] "Manual on Presentation of Data," Committee E-1, A.S.T.M., Philadelphia, Pa., April, 1943.

Station	Index
Lyon, France	0.71
Marseille, France	0.57
Brest, France	0.55
The Alps	0.42

Rainfall at Lyon is 1.7 times as variable as at an Alps station, and, presumably, streams having similar drainage basins in the two locations would have runoffs varying in a similar manner. The writer suggests that, as more daily rainfall indexes become available from locations in the tropics to the thunderstorm belts of the arid west, the range of rainfall variation will be found quite comparable to the range in stream flow variation presented by the authors.

Altitude.—That varying altitude affects the runoff variability index is suggested by the authors in commenting on their North Carolina data. There they found that their index varied from 0.62 to 0.25 (a ratio of 2.5 to 1) as they went from the plains to the mountains and ascribed it to the more uniform mountain rainfall. That this effect exists is also shown in Fig. 13, from data supplied by Winters P. Nakamura.[49] Here in terms of the arithmetic deviation to which, roughly, the authors' index is proportional, the annual rainfall variation index changes from 50 to 20, a ratio of 2.5 to 1 as the altitude becomes greater.

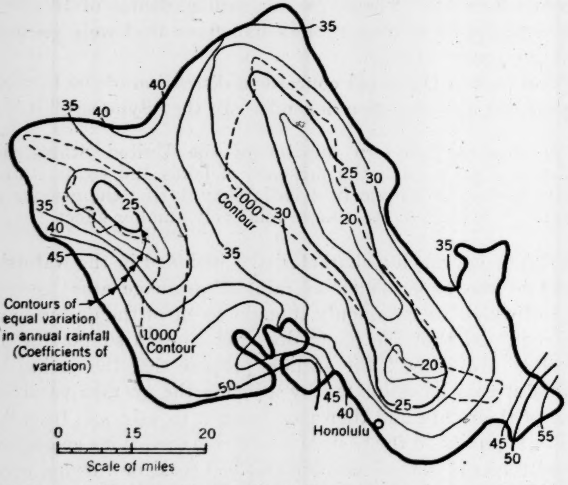
35
40
35
40
25
30
35
25
1000 Contour
30
20
35
40
45
35
Contours of equal variation in annual rainfall (Coefficients of variation)
1000 Contour
20
50
45
40
25
Honolulu
0
15
20
Scale of miles
45
50
55

FIG. 13.—ANNUAL RAINFALL BECOMES MORE UNIFORM WITH INCREASE IN ELEVATION

Area.—The effect of the area of watershed on the uniformity of its drainage is likewise common knowledge. Big rivers tend to be much less variable than small flash streams. The authors' data, when plotted against drainage area, indicate that, as the size of the watershed increases from 100 sq miles to 10,000 sq miles, the variation index is reduced on the average from 0.76 to 0.36—a reduction to 2 to 1. Mr. Coutagne's data, shown as follows, indicate similar reductions, the ratio being 2.5 to 1:

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[49] "Rainfall Coefficient of Variation," by Winters P. Nakamura, *Monthly Weather Review*, December, 1933, pp. 354–360.

Stream	Area (sq miles)	Coefficient
Reno, Italy	300	0.83
Arno, Italy	2,500	0.82
Tiber, Italy	5,000	0.72
Rhone, Lyon, France	5,000	0.33
Rhone, Beaucaire, France	20,000	0.30
Danube, Germany	25,000	0.29
Nile, Egypt (compiled by writer)	....	0.40

To conclude, the foregoing data, limited as they are suggest that variations in rainfall, altitude, or area may each have effects on the "flow variability index" comparable to that ascribed by the authors to "geology." The writer feels that, before the authors' conclusion that "geology" is a principal factor in runoff variation can be accepted, correlation studies must be made to eliminate the effects of variable rainfall, elevation, and area as just pointed out. Although some engineers, lacking other data, may wish to accept the authors' recommendations and estimate unknown stream flow variations following the authors' recommendations of Table 2, the writer believes that, for an unknown stream, there will be less error if the flow variability index be assumed similar to the rainfall variability index, measured or estimated, or if, on the basis of observed physical characteristics of a drainage basis, a "type" variation index be chosen from Table 1.

J. E. McKee,[50] M. ASCE.—It is gratifying to note that Messrs. Lane and Lei have applied standard statistical procedures to stream flow data and have thus analyzed all reliable records of streams east of the 100° meridian. Too often, civil engineers are prone to develop methods and procedures for analyzing data with utter disregard for established statistical parameters which are quite applicable; and as a result they are plagued with hybrid indexes, coefficients, averages, and ratios. The authors have attempted wisely, to avoid such hybrid parameters by using the well-established probability analysis; but (as will be shown) they have apparently overlooked a significant feature of their "variability index."

By its definition and by its method of computation, the variability index must be a logarithm. It is defined by the authors as the standard deviation of the logarithms of the discharge values (X) at 10% intervals, as taken from the duration curve. As such, it is nothing more than the logarithm of the geometric standard deviation, since

$$\log \sigma_g = \sqrt{\frac{\Sigma(\log X - \log M_g)^2}{N - 1}} \qquad (9)$$

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[50] Associate Prof. of San. Eng., California Inst. of Technology, Pasadena, Calif.

in which X is a value read from the duration curve; M_g is the geometric mean, since $\log M_g$ is the arithmetic mean of the $\log X$ values; N is the number of observations (in this case 10); and σ_g is the geometric standard deviation. In the form of its logarithm, the geometric standard deviation has little or no actual significance, other than as another index to be carried in the minds of engineers. As the antilogarithm of the variability index, however, the geometric standard deviation proper has a definite meaning and can be applied directly to stream flow data.

Inasmuch as the duration curves of stream flow data that plot as straight lines on logarithmic probability paper, between limits of 5% and 95%, may be considered as approximately geometrically normal, the geometric standard deviation, σ_g, is the accepted statistical measure of dispersion (or variability) of such data. It has a physical significance in that, when multiplied by, or divided into, the geometric mean (or 50% value in this case), it results in direct values having the same dimensions as the mean. The product and quotient so obtained represent the values at 84.13% and 15.87%, respectively. Thus, if the geometric mean of certain runoff data is expressed as 1.20 cu ft per sec per sq mile and σ_g is found to be 2.20, then the runoff may be expected to be less than 1.20×2.20 or 2.64 cu ft per sec per sq mile for 84.13% of the time and less than $\frac{1.20}{2.20}$ or 0.545 cu ft per sec per sq mile for 15.87% of the time. The variability index used by the authors, which would be $\log \sigma_g$, or 0.34 in this example, has no such ready application. Few human minds are accustomed to thinking logarithmically, and, hence, few engineers can envision what 0.34 means in this instance.

Although the variability index is definitely established as a logarithm by its definition and by its magnitude in the various tables and examples, and although the characteristic values of the variability coefficient given in Table 2 are logarithms, the authors were apparently thinking of the antilogarithm of their index, or the true geometric standard deviation, when they stated (under the heading, "Use of the Variability Index"):

> "The shape of the duration curve can be obtained by drawing a straight-line duration curve on logarithmic probability paper with a slope such that the ratio of the discharge exceeded 15.87% of the time to the discharge exceeded 50% of the time was equal to the variability index selected."

The quoted statement does not hold for the logarithms used in this paper as the variability index. It holds only for the true geometric standard deviation.

The writer concurs with the authors in their reasoning that a measure of variability of generalized stream flow data is needed and he feels indebted to them for the mass of data that has been thus compiled, analyzed, and compared. He recommends, however, that the variability index be called by its established statistical name—the logarithm of the geometric standard deviation—and that this variability be expressed directly rather than as a logarithm.

G. B. Schroyer,[51] Jun. ASCE, and E. T. Schuleen,[52] M. ASCE.—The authors are to be commended on their excellent attempt to devise a stream flow variability index. An accurate appraisal of flow variability, by some kind of rational numerical coefficient, has long been needed in hydrologic investigation, and any step in that direction is salutary. Further commendation is warranted for the authors' astuteness in utilizing, as a basis for their mathematical analysis, the earlier work of the sedimentary geologists, especially W. C. Krumbein, F. J. Pettijohn, and G. H. Otto, as applied in the field of sediment particle analysis. It is often more difficult to visualize the adaptability of work in a foreign field to a given problem than to devise a method of attack on the problem from fundamental concepts of physics and engineering. Similarity of approach to two problems is strikingly apparent only when the two

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[51] Design Engr., Safe Harbor Water Power Corp., Baltimore, Md.

[52] Hydrographic Engr., Pennsylvania Water & Power Co., Conestoga, Pa.

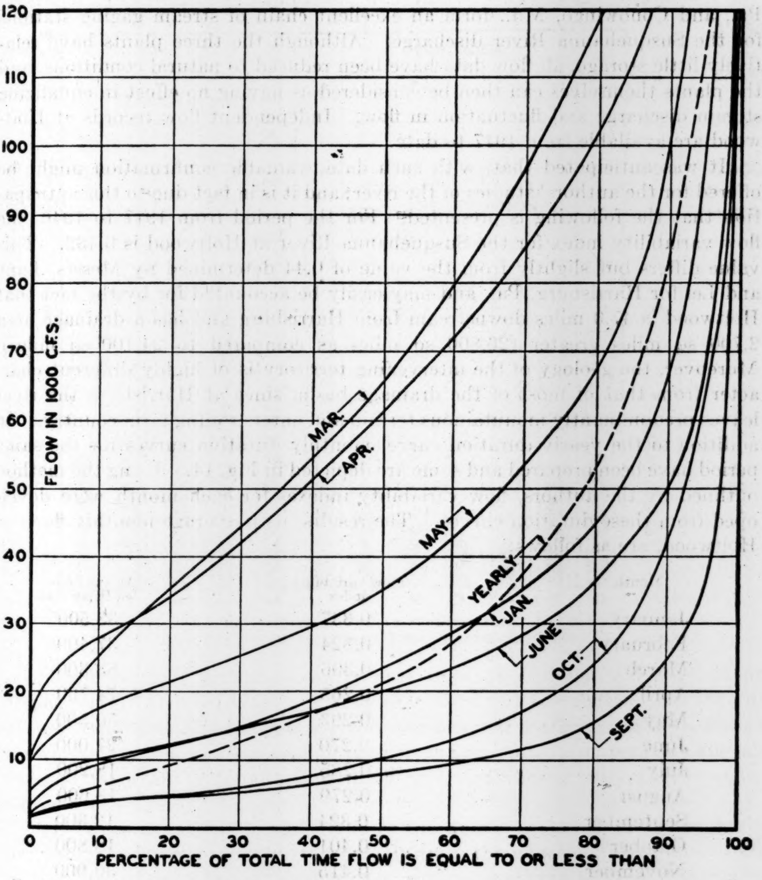

Fig. 14.—Typical Monthly Duration Curves

concepts have been reduced to dimensionless partial differential equations, and a relationship of the factors involved is at an uncomplicated mathematical level. However, as in the case of any new theories, all concepts that are included must be explained.

The three run-of-river hydroelectric plants at Safe Harbor, Pa., Holtwood, Pa., and Conowingo, Md., form an excellent chain of stream gaging stations for the Susquehanna River discharge. Although the three plants have relatively little storage, all flow data have been reduced to natural conditions, and the plants themselves can then be considered as having no effect in equalizing stream discharge and fluctuation in flow. Independent flow records at Holtwood are available from 1917 to date.

It was anticipated that, with such data, valuable confirmation might be offered for the authors' studies of the river; and it is in fact due to this anticipation that the following is presented. For the period from 1917 to 1946, the flow variability index for the Susquehanna River at Holtwood is 0.432. This value differs but slightly from the value of 0.44 determined by Messrs. Lane and Lei for Harrisburg, Pa., and may easily be accounted for by the fact that Holtwood is 47.0 miles downstream from Harrisburg and has a drainage area 2,700 sq miles greater (26,800 sq miles as compared to 24,100 sq miles). Moreover, the geology of the intervening territory is of highly different character from that of most of the drainage basin, since at Harrisburg the river leaves predominantly mountainous terrain and enters rolling farm country. In addition to the yearly duration curve, monthly duration curves for the same period have been prepared and some are depicted in Fig. 14. Using the method outlined by the authors, flow variability indexes for each month were developed from these duration curves. The results, with average monthly flows at Holtwood, are as follows:

Month	Flow variability index	Average flow (cu ft per sec)
January	0.337	33,500
February	0.324	37,400
March	0.306	83,300
April	0.265	74,700
May	0.293	50,500
June	0.270	27,000
July	0.285	18,200
August	0.279	13,600
September	0.324	12,300
October	0.401	18,800
November	0.415	30,900
December	0.376	33,300
Yearly	0.432	36,100

These indexes vary from 0.265 for April to 0.415 for November; they present an unusual condition, although entirely possible in this instance, wherein the annual variability index seemingly bears no relation to the monthly indexes.

Since it can logically be assumed that the geological structure, soil, topography, and the arrangement of tributaries, with regard to time of concentration of surface flow, are constant for any particular river, the variations in the monthly indexes must arise from one or more other variables. An effort was made to see if such variation could be accounted for by some function of the weather. The monthly variability indexes are plotted in Fig. 15 showing average temperature, precipitation, and runoff. The temperature and precipitation averages are based on records from 33 stations and 58 stations, respectively. It is strikingly evident that no direct correlation between any of these quantities and the flow variability exists; nor is any correlation between the variability index and the runoff coefficient apparent. Temperature appears to be an important factor affecting runoff, and vegetation density would roughly follow the temperature curve.

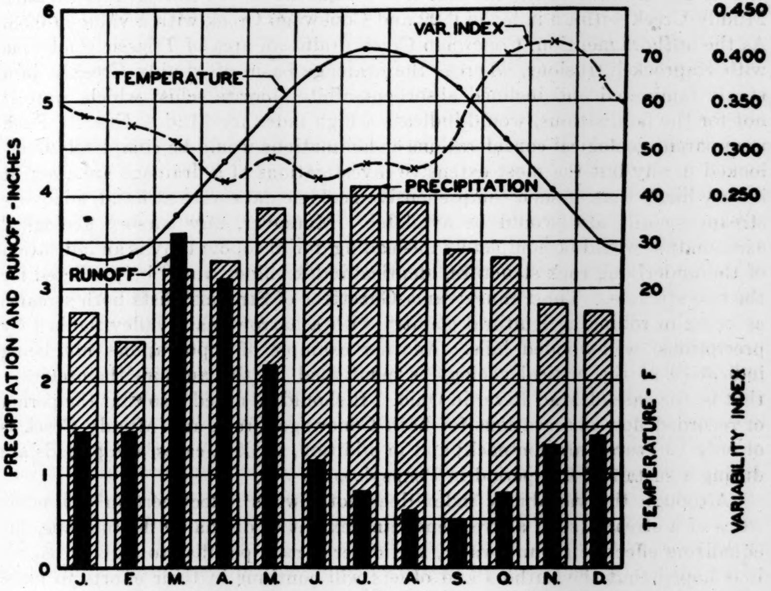

FIG. 15.—MONTHLY PRECIPITATION, RUNOFF, TEMPERATURE, AND FLOW VARIABILITY INDEX FOR THE SUSQUEHANNA RIVER ABOVE HOLTWOOD, PA.

The wide variations in the shape of the duration curves, and, hence, the variability indexes, probably stem largely from the effects of snow and frozen ground in delaying runoff during the winter months and from the higher percentages of flow contributed by ground water during the late summer and fall months. Although the deviations in the monthly variability indexes neither prove nor disprove the authors' contention that geology and topography are the principal factors involved, they act as a warning against the indiscriminate use of indexes to develop duration curves. The foregoing comparison of the relatively constant indexes for April, May, and June, with the widely different duration curves for these months in Fig. 14, illustrates the degree of error possible.

It has been mentioned by Messrs. Lane and Lei that the topography and geology of Pennsylvania are highly diversified, and that extremely varied flow indexes might be expected on individual streams. It should be pointed out that, solely because of this variance, any attempt to predict a variability factor for a given stream would meet with difficulty. As an example of the influence of geology, the authors cite the case of two adjacent small Pennsylvania streams, Muddy Creek with an index of 0.29 and Conewago Creek with a value of 0.58. As the authors mention, Conewago Creek drains an area of Triassic sandstone with traprock intrusions, whereas the drainage basin of Muddy Creek is in a highly laminated and inclined absorbent albite chlorite schist which, were it not for the laminations, would indicate a high index for Muddy Creek. Such unapparent geological considerations as laminations would be completely overlooked in any but the most extensive investigations of a drainage area, and it is very likely that, if such comprehensive geologic data were at hand, adequate stream records also would be available. Moreover, any cursory geological examination would be completely misleading, for, if topography is an indication of the underlying rock structure, identical index values would be predicted for the two streams. The Pennsylvania Gazetteer of Streams[53] lists both streams as being in rolling agricultural country with a narrow main valley flanked by precipitous, well-wooded hills. Another example of topographies not being indicative of the critical geology is mentioned by the authors themselves—that is, the case of Mad River in Ohio. It should be noted also that the period of recorded flow observations for both Conewago Creek and Muddy Creek is of only 10 years duration each; and in addition, in both cases, the periods are during a sustained dry period in Pennsylvania.

Although the variability index does not always reflect certain characteristics of a stream (such as frequency and extent of discharge fluctuations, the equalizing effect of ground water, and other factors peculiar to a given basin), it is hoped that the authors and others will continue in their efforts to place stream flow variation more nearly on an absolute and mathematical basis.

E. W. LANE,[54] M. ASCE.—It is not possible for Mr. Lei to join the writer in preparing the closing discussion of this paper because of the disrupted communication between China and the United States. Although it is believed that he would concur in most, if not all the following remarks, the writer alone is responsible for them.

The large number of constructive criticisms presented by the various discussers has added greatly to the value of the paper and they are appreciated. They fall into two general classes: (1) Those dealing with the statistical aspects of the paper and the mathematics of probability paper, and (2) those dealing with the hydrological aspects and applications of the variability indexes.

Considering first the discussions in class (1), it is advisable to give the reasons for adopting the method of handling the flow of data used in the paper. The data used were drawn from a large number of sources—and were given in four different forms: (1) Plotted duration curves; (2) discharges exceeded during certain selected percentages of time; (3) percentages of time that the discharge exceeds certain selected values; and (4) mean daily discharges for each day covered by the record. To use all these data, a method which would handle data expressed in any of these forms had to be selected. Since future users of the suggested variability index will also encounter data in all these forms, in order that they may all use the same procedure, it is necessary that the method selected be applicable to all types of data. The method of preparing a duration curve, and of reading values from it at certain selected typical points, was the only process the writer and Mr. Lei could devise which met these requirements.

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[53] "Water Resources Inventory Report: Gazetteer of Streams, Pt. III," Water Supply Comm. of Pennsylvania, Harrisburg, Pa., 1917.

[54] Hydr. Consultant on Chf. Engr.'s Staff, Bureau of Reclamation, Denver, Colo.

In making duration-of-flow studies from records of mean daily flow, the shortest method seems to be to divide the entire range of the stream flow into an adequate number of partial ranges and to go through the daily records, tallying the number of days found in each partial range. From the numbers shown by the tallies of these partial ranges the number of days with discharges below the upper limit of each partial range can be easily obtained by adding the tallies in that range to those in all partial ranges covering lower discharges. The writer and Mr. Lei used this method of obtaining the data for plotting the duration curves, where the data used were in the form of mean daily discharges; they believe that any other method is unnecessarily laborious.

Mr. Ownbey expressed the opinion that in computing the index the sum of the squares of the differences of the logarithms from the mean of the logarithms should be divided by 10 instead of by 9, as was done in the paper. Before selecting 9, the writer and Mr. Lei consulted several people who are well versed in statistical methods, in addition to studying the methods of handling similar data recommended by W. C. Krumbein. Still others who are well acquainted with statistical theory have discussed this paper without raising this question; therefore, it seems likely that the use of 9 by the writer and Mr. Lei is sound. If it should not be sound, however, it would have no effect on the practical usefulness of the index developed in the paper, which differs by a constant ratio from the one which Mr. Ownbey would obtain.

Mr. McKee suggests that the geometric standard deviation would be a better value to use for the variability index than the value adopted in the paper. Although the use of the geometric standard deviation as an index would have the advantage of more easy designation, and, therefore, might be more acceptable to statisticians, it would have no practical advantages over the one in the paper, which is the antilog of the one Mr. McKee suggests.

The use of the coefficient of variation, and the use of discharges expressed as ratios to the mean flow have been suggested as possible improvements on the index offered in the paper. One of the advantages of the index used by the writer and Mr. Lei is that, by using the logarithms, the index obtained gives an expression of the relative variability from the mean, and, therefore, accomplishes the same purpose as the coefficient of variation. Converting the discharges to values expressing their magnitudes in terms of the mean and then determining the variability index would result in exactly the same value as was obtained by the method given in the paper. One of the great advantages of the index used by the writer and Mr. Lei is that it is independent of the discharge units used, as Mr. Foster has shown. The values obtained are the same as long as the flows are expressed in units of volume per unit of time.

The writer cannot agree with Mr. Chow's contention that the variability index is a function of the length of the record, when the time is expressed as a percentage of the total time of flow. Of course, there are differences for the duration curves of different years, but this is a chance and not a functional variation.

Several discussers have called attention to an error in the paper (under the heading, "Use of the Variability Index") which resulted from the omission of the word "antilogarithm." The correct statement is:

> "The shape of the duration curve can be obtained by drawing a straight-line duration curve on logarithmic probability paper with a slope such that the ratio of the discharge exceeded 15.87% of the time to the discharge exceeded 50% of the time is equal to the antilogarithm of the variability index selected."

The method given by Mr. Foster for obtaining the predicted flow duration curve from the estimated value of the variability index, and the curve showing the ratio of the median to the mean discharge for various values of the variability index, as given by Mr. Foster and also Messrs. Ospina and Tama, will be of considerable assistance in applying the methods presented in the paper. The explanation of the method of constructing logarithmic probability paper offered by Mr. Foster will be found of great assistance to those who may have occasion to make up such forms.

Turning now to the hydrological aspects and applications of the variability index, Mr. Wing's discussion of the history of the idea of this index is of great interest. As he has stated, its roots go back over seven decades to Francis Galton, and the basic idea was developed by Aime Coutagne about a decade before the independent development of it by the writer and Mr. Lei. Mr. Coutagne expressed his idea in the form of a uniformity index—the higher his value the more uniform the flow—whereas the writer and Mr. Lei developed a variability index, the reciprocal of that developed by Mr. Coutagne, with the higher values allocated to the more variable stream flows.

Some difference of opinion developed in the discussions regarding the variation of the index values with the drainage area. Mr. Lull presents data indicating very little decrease in the variability with drainage area; and Mr. Wing and Messrs. Ospina and Tama believe that the variability decreases appreciably with drainage area. The view that the index decreases with drainage area was given in the paper. However, since the decrease is of a low order (a decrease of perhaps about 50% in index value for an increase of drainage area of 10,000%), it might be argued that the increase was negligible for many cases.

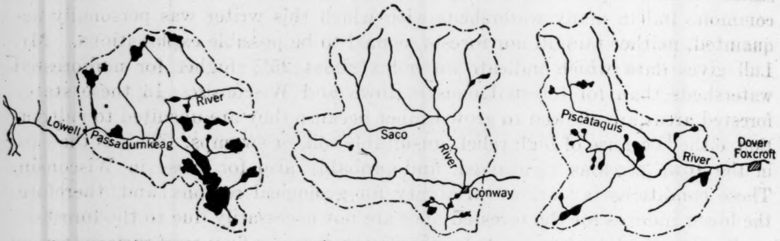
It is regretted that the space available did not permit a more complete justification of the contention that geology and surface storage were the most important factors in controlling flow variability within the area covered by the data submitted. Had the maps showing the variability indexes and their relations to local geology and the discussions of watershed conditions which were a part of the comprehensive paper been published in full, it is believed that this point would have been much more clearly demonstrated. Other factors that have been suggested to account for the variation were rainfall variation and forests. In regions of high relief, rainfall variations within short distances can be appreciable, but it is difficult to believe that such can be the case in flat states such as Iowa, Wisconsin, Illinois, and Ohio. In these states variations of 100% or more in the index within short distances were common, and in many watersheds with which this writer was personally acquainted, neither rainfall nor forests seemed to be possible explanations. Mr. Lull gives data which indicate an index about 25% higher for nonforested watersheds than for forested ones in Iowa and Wisconsin. In these states forested areas are allowed to grow timber because they are unsuited to cultivation, either because of high relief, unsuitable soil, or swamps. This is the case in the Iowa streams mentioned, and probably also for those in Wisconsin. These conditions favor low variability for geological reasons, and, therefore, the lower indexes for the forested ones are not necessarily due to the forests.

Perhaps the best arguments for the preponderant influence of surface storage are the very low values of the St. Lawrence, St. Clair, and St. Marys rivers of the Great Lakes system, which are much lower than any others obtained. Strong arguments for geology being very important in the area covered by the records are the low flows of the Kansas and Nebraska streams. Although no quantitative data on the rainfall variability in these Great Plains states are available, there is every reason to believe that it is less uniform than in the states farther east. The watersheds of these streams are nearly treeless. In spite of both these factors, these Great Plains streams give low indexes.

It should have been emphasized more forcefully perhaps that the conclusions of the paper were confined to the areas covered by the flow records available, and that, until data from other regions are available, it is not possible to determine how applicable they are elsewhere. With the exception of some streams in North Carolina and perhaps in southern Illinois, no cases were found which seemed to indicate a rainfall variability effect. However, this writer does not claim that rainfall effects are everywhere generally negligible, but only that they were generally negligible in the area covered by the records. In fact, where there are great differences in rainfall variability there are certain to be differences in flow variability. The flow variability is the result of the interaction of rainfall variability and natural storage, either above ground or below ground; the flow variability varies in the same direction as the rainfall variability and in an inverse direction from the natural storage. It is possible that the low values obtained by Messrs. Ospina and Tama for the streams of the Cauca basin in Western Colombia were caused largely by a comparatively uniform rainfall.

Messrs. Bruin and Hudson have mentioned the necessity of expressing the time in terms of the percentages of time of flow rather than as the percentage of total elapsed time. In all the flows used by the writer and Mr. Lei, these were the same; but in small streams, where the flow ceases a part of the time, it is necessary to use only the time of flow.

Mr. Kazmann has called attention to the lack of exactness in the terms given in Table 2 for permeability, relief, and areas of lakes and swamps. Greater exactness would be desirable, but the labor involved in working up the data on the individual watersheds, in quantitative terms on which generalizations could be based, would have been prohibitive, even if the data had been available. In most cases this was not true. For classifying the area of lakes and swamps, Fig. 16 will serve as a guide in handling this variable.



(a) LARGE LAKE AREAS (b) SMALL LAKE AREAS (c) SOME LAKE AREAS

FIG. 16

Messrs. Schroyer and Schuleen have shown the great differences in variability indexes of flow for the Susquehanna River in the different months of the year. This serves to emphasize the necessity of having the data cover one or more complete cycles, such as years, in which the conditions are much the same, rather than to compare conditions which are consistently different, such as those in one month of the year as compared to those in another month.

In judging the geological and surface characteristics influencing flow variability in drainage areas for which geological data are not available, air photographs or airplane reconnaissance would be very valuable. Considerable work has been done toward interpreting these conditions from air photographs,[55,56,57,58] which would repay study by those concerned.

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[55] "Airphoto Interpretation of Engineering Sites and Materials," by Jean E. Hittle, *Photogrammetric Engineering*, December, 1949, pp. 589–603.

[56] "Aerial Photographs and Their Applications," by Harold T. U. Smith, D. Appleton-Century Co., Inc., New York, N. Y., 1943.

[57] "The Origin, Distribution, and Airphoto Identification of United States Soils," by D. S. Jenkins, D. J. Belcher, L. E. Gregg, and K. B. Woods, *Technical Development Report No. 52*, Civil Aeronautics Administration, Washington, D. C., 1946.

[58] "Aerial Photographs Used for an Engineering Evaluation of Soil Materials," by R. E. Frost and K. B. Woods, *Proceedings*, 2d International Conference on Soil Mechanics and Foundation Eng., Rotterdam, Vol. I, 1948, pp. 324–330.

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END RESTRAINTS ON TRUSS MEMBERS

BY HAROLD E. WESSMAN,[1] M. ASCE, AND THOMAS C. KAVANAGH,[2] ASSOC. M. ASCE

WITH DISCUSSION BY MESSRS. JACK R. BENJAMIN, GEORGE WINTER, ABRAHAM SLAVIN, J. EDMUND FITZGERALD, JOSEPH S. NEWELL, CHARLES W. DOHN, AND HAROLD E. WESSMAN AND THOMAS C. KAVANAGH.

SYNOPSIS

One of the problems of current interest to structural engineers is the behavior of compression members that are parts of rigidly connected structural frames and are consequently subjected to varying end restraints. A method is presented herein for the calculation of these end restraints and the effective lengths of compression members under different load conditions. It is demonstrated that the end restraints decrease and that the effective lengths increase as loads are increased, until the frame buckles. It is shown that when a truss buckles all members tend to fail simultaneously.

The procedure is an extension of the moment distribution method introduced by E. E. Lundquist, Assoc. M. ASCE, for analyzing the stability of a structural frame. The relation between the series criterion and the stiffness criterion for testing stability is restated. General equations are then formulated for the stiffness of an axially loaded bar with the far end elastically restrained. The general equation for the instability of a bar with known elastic restraints is also derived in terms of the moment distribution method. The procedure is illustrated by the calculations for a specific truss.

Finally, an important fact which is not well understood is presented—namely, that for compression members in steel trusses with the usual range of slenderness ratios, the buckling load coincides practically with the yield point of the steel regardless of end restraints. Hence any elaborate analysis of stability and end restraints is unwarranted for a steel truss.

The behavior of a compression member or column as a part of a riveted or welded truss or rigid frame is a subject of current interest to a growing number

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of structural engineers. Current design practice is not very consistent in the treatment of restraints existing at the ends of a column by virtue of its more-or-less rigid connection with other members of varying stiffness. Some widely accepted column formulas contain factors for end conditions, whereas other equally accepted formulas do not. Most practicing engineers have little knowledge of these restraints, beyond the usual textbook statements that a pin-ended column corresponds to a length reduction factor of $k = 1$, whereas a fixed-ended column corresponds to $k = \frac{1}{2}$, in the Euler formula for the critical load P_{cr} in the elastic range:

$$P_{cr} = \frac{\pi^2 E I}{(k L)^2} \dots\dots\dots (1)$$

in which k is a factor depending on the degree of restraint at the ends; E is the modulus of elasticity; I is the moment of inertia of the column cross section; and L is the length of the column.

In the elastic region, consequently, a fixed column has four times as much capacity with respect to buckling as does the same pin-ended column. This condition is not true, however, in the inelastic range. The compression member usually encountered in civil engineering practice has such a low slenderness ratio that it acts in the inelastic range as load is increased and approaches the critical value. Eq. 1 then becomes:

$$P_{cr} = \frac{\pi^2 E' I}{(k L)^2} \dots\dots\dots (2)$$

in which E', the effective modulus, replaces E, the elastic modulus. The tangent modulus is now generally accepted by aeronautical engineers as the effective modulus[3] for columns. It has also been accepted by the Column Research Council of Engineering Foundation. For structural steel, E' gradually decreases from 30×10^6 lb per sq in. to zero at the yield point, a value that may be termed a "ceiling" for the critical average stress. As the slenderness ratio is decreased, the Euler load curves based on the tangent modulus gradually converge toward this ceiling. It will be found then that the buckling load for a fixed-end column may be only slightly greater than that for the same column with pinned ends. The ratio is no longer 4; it decreases rapidly to values less than 2 and very close to 1.

In the actual column as it exists in a framework, the ends are neither pinned nor fixed; in other words, the end restraints are neither zero nor infinity, but have some intermediate values best characterized by rotational springs having spring constants s (see Fig. 1). In this type of column, the critical buckling load will have a value which is between that for the pinned column (the lower limit) and that for the fixed column (the upper limit). The reader should bear in mind that these two limits are not far apart for columns with low slenderness ratios. Moreover, for structural steel, they are very close to the yield point of the steel.

It is not generally recognized that, as loads are increased on a framework, the end restraints acting on a compression member in the framework do not remain constant. They may vary between almost absolute fixation and almost perfect freedom, depending on the changes in stiffness of the abutting members. In fact, the abutting members, instead of creating an end moment on the compression member as it starts to buckle, may have moments presented to them by the buckling member. This change of restraint, as a framework is loaded, is the reason why it is difficult to test an isolated column in a testing machine and to duplicate the variable end conditions obtaining during the test of a framework.

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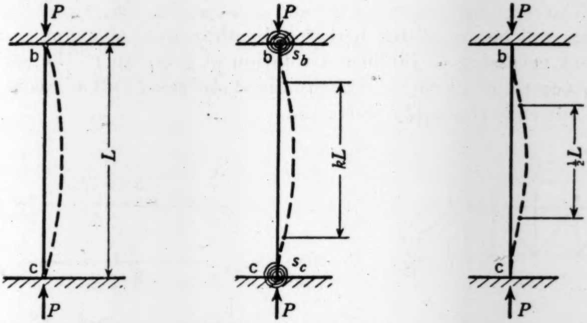
[3] "Inelastic Column Theory," by F. R. Shanley, *Journal of the Aeronautical Sciences*, May, 1947, p. 261.



(*a*) PINNED ($k=1$ AND $s=0$) (*b*) ELASTICALLY RESTRAINED ($1 \geq k \geq \frac{1}{2}$ AND $0 \leq s \leq \infty$) (*c*) FIXED ($k=\frac{1}{2}$ AND $s=\infty$)

FIG. 1

The theoretical procedures for determining critical buckling loads for trusses are well established. One excellent method which is widely known among American aeronautical engineers is that developed by Mr. Lundquist.[4] This method is an adaptation of the moment distribution method developed by Hardy Cross,[5] Hon. M. ASCE. In the procedure of convergence to an accurate answer, stiffnesses of members are usually based on the condition that the far end is temporarily fixed at any one stage in the process. The actual end restraints acting on any member for any particular load condition are not usually evaluated in the Lundquist method. It is possible, however, to extend the procedure to determine the actual magnitude of the restraints and then to evaluate the effective length of a member corresponding to a particular load condition. There is some question as to the significance of "effective length" in column design formulas, inasmuch as the effective length changes as loads are increased. That question, however, will not be discussed in detail in this paper. The Lundquist method will now be restated briefly and then extended to the calculation of actual end restraints and effective lengths.

In stability calculations employing the moment distribution procedure, the stiffness and carry-over factor depend not only upon $E' I/L$ but upon the magnitude and sense of the axial load in the member. Evidently, an axial tensile load will tend to increase the stiffness of a member, whereas an axial compressive load will reduce its stiffness. The expressions for stiffness and carry-over factor for axially loaded bars are well known, and are reproduced in the Appendix.

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[4] "Stability of Structural Members Under Axial Load," by Eugene E. Lundquist, *Technical Note No. 617*, National Advisory Committee for Aeronautics, Washington, D. C., October, 1937.

[5] "Analysis of Continuous Frames by Distributing Fixed-End Moments," by Hardy Cross, *Transactions*, ASCE, Vol. 96, 1932, p. 1.

The stiffness of a bar on unyielding supports at each end is defined in the Lundquist method as the moment at one end necessary to produce a rotation of $\frac{1}{4}$ radian at that end. The stiffness obviously depends on the restraint at the far end, and three cases are considered: S is the stiffness, far end fixed; S' is the stiffness, far end elastically restrained; and S'' is the stiffness, far end pinned. The restraint created at the end of a member by an abutting member or members will be designated hereafter by the spring constant s, defined as the moment necessary to produce a rotation of $\frac{1}{4}$ radian in the spring. Accordingly, s corresponds to $\Sigma S'$, the combined stiffness of all adjacent members with their far ends elastically restrained.

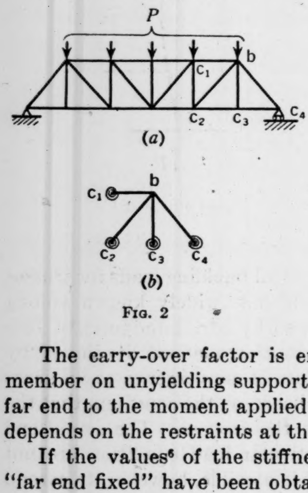

FIG. 2

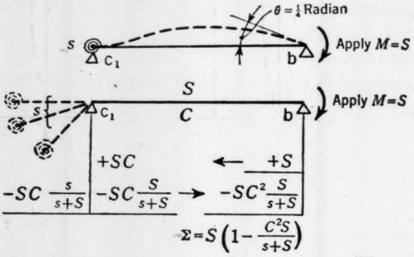

FIG. 3

The carry-over factor is employed in its usual sense with relation to a member on unyielding supports, as the ratio of the moment developed at the far end to the moment applied at the near end. Its value (C, C', or C'') also depends on the restraints at the far end.

If the values[6] of the stiffness and carry-over factors for the condition of "far end fixed" have been obtained, it is possible, by direct moment distribution, to obtain the factors for the other end conditions as well. Thus, for the stiffness S' of the bar bc₁ of the truss in Fig. 2, isolate the bar as shown in Fig. 3, where the bars at the left of c₁ have been denoted by an equivalent spring of total stiffness s. From the simple moment distribution shown, and the definitions of S' and C',

$$S' = S - \frac{C^2 S^2}{s + S} = S\left(1 - \frac{C^2 S}{s + S}\right) \ldots\ldots\ldots\ldots\ldots (3)$$

and

$$C' = \frac{C}{1 + \frac{S}{s}(1 - C^2)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

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[6] "Extended Tables of Stiffness and Carry-Over Factor for Structural Members Under Axial Load," by Eugene E. Lundquist and W. D. Kroll, *Wartime Report L-255* (ARR 4B 24), National Advisory Committee for Aeronautics, Washington, D. C., February, 1944.

If, now, a wire model of a rigid-jointed truss (Fig. 2) is visualized as being in stable equilibrium under the action of some external force system P, and if to any joint b of this truss an arbitrary small rotation θ_b is applied and held by hand, the structure will assume a new configuration in which each joint of the truss will be rotated through some angle that will be a multiple of θ_b, and the magnitude of which will depend on the stiffness of every member of the truss. If, as was assumed, the truss is in stable equilibrium, all such rotations will be finite, and on release of the joint, the truss will snap back to its original position.

As the load system P is increased, resulting in larger loads in the members of the truss, the stiffness of the individual members will be decreased or increased depending on whether they are in compression or in tension, respectively, and the resultant total stiffness of each joint will vary with increasing load. For a joint dominated by compression members, the total stiffness will decrease with increasing loads P, and the application of the small arbitrary rotation θ_b will cause increasingly larger rotations (or, in the model, the springiness or snapback of the joint on release will diminish continuously). Obviously, the criterion for stability is that the rotations $\left(\theta = \frac{M}{\Sigma S'}\right)$ at all joints be finite; this leads to the condition for neutral stability which Mr. Lundquist termed as the "stiffness criterion":

$$\Sigma S'_{bc} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

or

$$\Sigma S'_{bc} = \Sigma \left[S_{bc} \left(1 - \frac{C^2{}_{bc}\, S_{bc}}{s_c + S_{bc}} \right) \right] = 0 \ldots\ldots\ldots\ldots (6)$$

The stiffness criterion is usually difficult to apply, because the values of s are unknown, as a result of the unknown end restraints. It is applied indirectly, however, in the following manner: Eq. 3 states that the stiffness S' of a bar bc is equal to the stiffness S minus a correction, $\frac{C^2\,S^2}{s + S}$, the latter being physically the moment carried back to (or the unbalanced moment at) joint b after the far end c has been balanced. It must be kept in mind that an initial moment of S was applied to bar bc at joint b, after which joint b was released and reclamped. If a number of members bc meet at joint b (Fig. 2(*b*)), all the S'_{bc}-values can be obtained by applying a moment ΣS_{bc} to the entire joint b, releasing and reclamping the joint. Then the process of moment distribution is applied to all the remaining joints only, using a sufficient number of cycles to insure accuracy. Finally, the moment values that have been carried back to joint *b* for each individual member meeting at b are summed. This summation for each member is the term $\frac{C^2\,S^2}{s + S}$.

In practice, it is simpler to apply a unit moment to joint b rather than a moment of ΣS_{bc}. Accordingly, the initial moment at the end b of a member bc₁ is $\frac{S_{bc1}}{\Sigma S_{bc}}$ (see Fig. 3). The total moment carried back, designated as r_1 to conform to the Lundquist notation, is then

$$r_1 = \frac{C^2_{bc1}\,S^2_{bc1}}{s_{c1} + S_{bc1}}\,\frac{1}{\Sigma S_{bc}} \qquad (7)$$

The moment carried back to end b of member bc₂ is r_2. The total for all the members meeting at b is termed r and it is equal to $r_1 + r_2 + r_3 + \cdots$. It is the total unbalanced moment at joint b after all the other joints have been successively released, balanced, and clamped for several cycles of moment distribution. From Eq. 7,

$$r_1\,\Sigma S_{bc} = \frac{C^2_{bc1}\,S^2_{bc1}}{s_{c1} + S_{bc1}} \qquad (8)$$

Comparing Eq. 8 with Eq. 3, the latter can be restated as follows for the members meeting at joint b:

$$\left.\begin{aligned} S'_{bc1} &= S_{bc1} - r_1\,\Sigma S_{bc} \\ S'_{bc2} &= S_{bc2} - r_2\,\Sigma S_{bc} \\ S'_{bc3} &= S_{bc3} - r_3\,\Sigma S_{bc} \\ &\cdots\cdots \end{aligned}\right\} \qquad (9)$$

Thus, an alternative to Eq. 3 is available for obtaining the stiffness at one end of a member with the far end elastically restrained. In the process of moment distribution, r_1, r_2, etc., are obtained directly for the members at joint b rather than the spring restraints s_{c1}, s_{c2}, etc., in Eq. 3. Once the corrected values of stiffness are obtained for the ends of the members adjacent to joint b, the stiffnesses at both ends of the remaining truss members can be obtained by repeated application of Eq. 3, solving either for an unknown S' in terms of the s at the far end, or for an unknown s in terms of a known S'.

The total stiffness at joint b with the far ends restrained elastically is the summation of Eqs. 9, or

$$\Sigma S'_{bc} = (1 - r)\,\Sigma S_{bc} \qquad (10)$$

At the critical or buckling load, Eq. 10 is set equal to zero, according to the stiffness criterion, and this immediately gives the very important condition for neutral stability, described by Mr. Lundquist as the "series criterion":

$$r = 1 \qquad (11)$$

The series criterion derives its name from the fact that the unbalanced moment at the joint is initially 1; after one release and reclamping, it is r; after the second release and reclamping it is r^2; etc. The final moment is then $1 + r + r^2 + \cdots$, which is a geometric series whose sum is $\frac{1}{1 - r}$; this moment is finite (and the structure is stable) as long as $r < 1$.

In determining the critical buckling load for a truss, the series criterion is more readily applied than is the stiffness criterion. With a given load condition acting on the truss, a unit moment is applied to any joint of the truss.

The joint is released, then clamped, and the total moment r, carried back to the joint through its abutting members, is calculated after all the remaining joints of the truss have been balanced by the application of several cycles of the moment distribution process. If $r < 1$, the truss is stable under the given load condition. The loading is increased and the procedure repeated, resulting in a new value for r. The determination of the buckling load for the truss—that is, the load at which $r = 1$—will require the analysis of a number of loading conditions, the number depending on the degree of accuracy sought.

The exact stiffness and the spring restraint for each end of any member for any one load condition, however, are quickly determined from Eqs. 9 and 3 after the values r_1, r_2, r_3, etc., have been found in determining r at a particular joint.

It is perhaps symbolic of the universal adaptability of the moment distribution method that it also affords a procedure for determining the value of the length reduction factor k from the two end restraints on a column. Considering the elastically restrained column bc (Fig. 1), and applying the stiffness criterion to joint b,

$$\Sigma S'_b = s_b + S'_{bc} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (12a)$$

from which, using Eq. 3,

$$s_b + S_{bc} - \frac{C^2_{bc}\, S^2_{bc}}{s_c + S_{bc}} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (12b)$$

Eq. 12*b* simplifies to

$$(s_b + s_c) + \frac{s_b\, s_c}{S_{bc}} + S_{bc}\,(1 - C^2_{bc}) = 0 \ldots\ldots\ldots\ldots (12c)$$

The last term on the left-hand side of Eq. 12*c* is S''_{bc}, however, which is the stiffness with the far end pinned. The final result is

$$(s_b + s_c) + \frac{s_b\, s_c}{S_{bc}} + S''_{bc} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (13a)$$

For ease in application, Eq. 13*a* is rewritten in terms of the nondimensional restraint coefficients:

$$\left(\frac{s_b}{E'\,I/L} + \frac{s_c}{E'\,I/L}\right) + \frac{\dfrac{s_b}{E'\,I/L} \times \dfrac{s_c}{E'\,I/L}}{\dfrac{S_{bc}}{E'\,I/L}} + \frac{S''_{bc}}{E'\,I/L} = 0 \ldots . (13b)$$

Eq. 13*b* can be solved easily by trial from the tables[6] previously mentioned, since these tables give corresponding values of $\frac{S}{E'\,I/L}$ and $\frac{S''}{E'\,I/L}$ for values of L/j, in which $j = \sqrt{\frac{E'\,I}{P}}$. When the proper value of L/j has been found which satisfies Eq. 13*b*, the value of k is determined from Eq. 2 as

$$k = \frac{\pi}{L/j} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

A trial solution is necessary because Eq. 13*a* is fundamentally a transcendental equation, since the functions S and S'' are trigonometric.[7] A convenient chart which gives a direct solution has been published by N. J. Hoff,[8] and is reproduced in Fig. 4, with the same notation as in this paper. Eq. 13*a* is merely one form of the fundamental equation of an elastically restrained bar, which was probably first stated by H. Zimmermann.[9] W. R. Osgood,[10] M. ASCE, was perhaps the first to note the equivalence between Mr. Zimmermann's basic equation and the moment distribution concepts employing stiffness and carry-over factors.

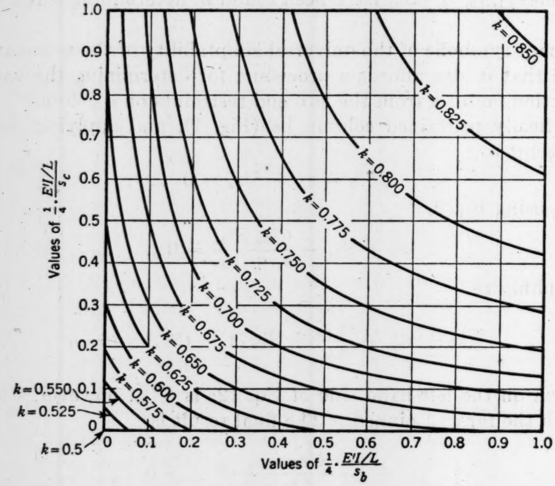

FIG. 4.—EFFECTIVE LENGTH FACTORS k FOR COLUMNS WITH END RESTRAINTS s_b AND s_c

The stability criteria first stated by Mr. Lundquist were given rigorous proof by Mr. Hoff [11,12] employing energy methods. By this latter procedure, Mr. Hoff was able to demonstrate that, for a truss to be stable, not only must the moment distribution converge to finite values, but the final values must be unique (that is independent of the order of balancing).

The procedure of finding end restraints and effective lengths will now be illustrated for the truss shown in Fig. 5(*a*). This truss is a "textbook" illus-

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[7] "Principal Effects of Axial Load on Moment Distribution Analysis of Rigid Structures," by B. W. James, *Technical Note No. 534*, National Advisory Committee for Aeronautics, Washington, D. C., July, 1935.

[8] "Elastically Encastred Struts," by N. J. Hoff, *Journal*, Royal Aeronautical Soc., September, 1936, p. 663.

[9] "Knickfestigkeit der Stabverbindungen," by H. Zimmermann, W. Ernst & Sohn, Berlin, 1925.

[10] Discussion by W. R. Osgood of "Principles of Moment Distribution Applied to Stability of Structural Members," by E. E. Lundquist, *Proceedings*, 5th International Cong. for Applied Mechanics, Cambridge, Mass., 1938, pp. 149-150.

[11] "Stable and Unstable Equilibrium of Plane Frameworks," by N. J. Hoff, *Journal of the Aeronautical Sciences*, January, 1941, p. 115.

[12] "Stress Analysis of Aircraft Frameworks," by N. J. Hoff, *Journal*, Royal Aeronautical Soc., July, 1941, p. 241.

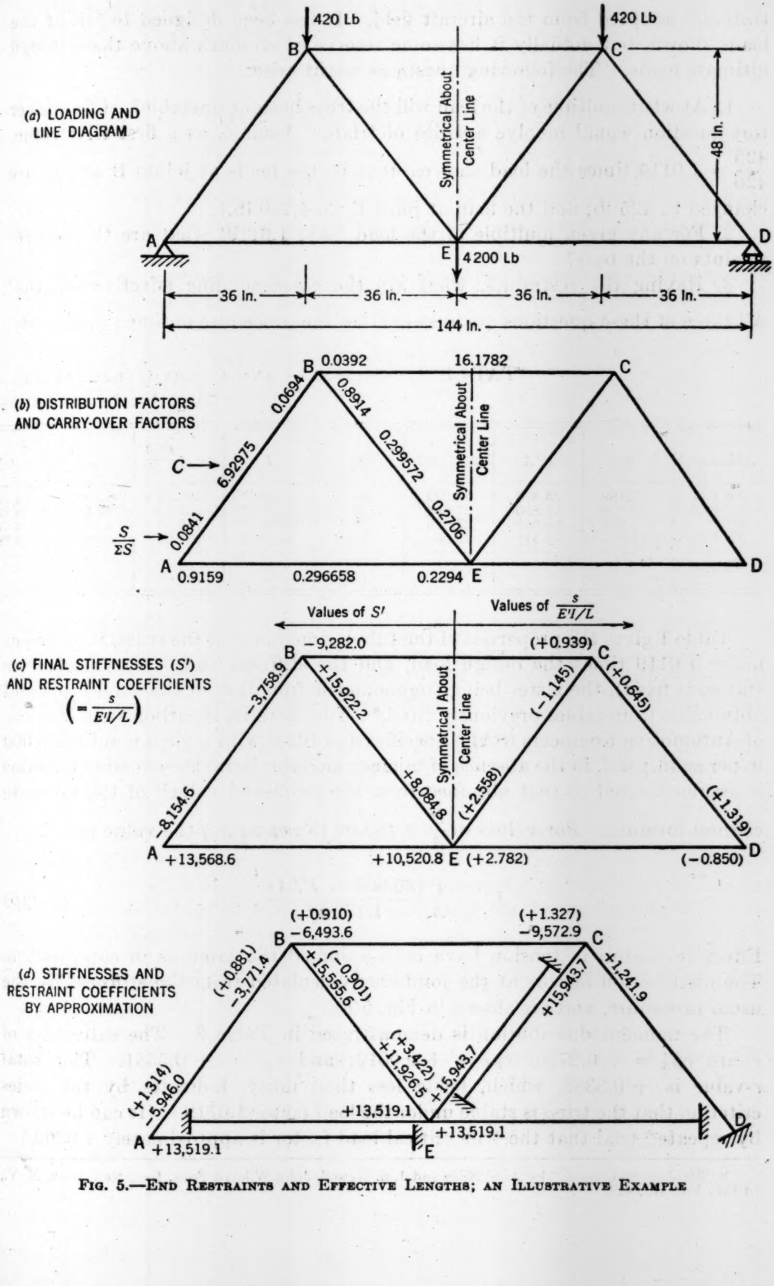
420 Lb
420 Lb
B
C
(a) LOADING AND LINE DIAGRAM
Symmetrical About Center Line
48 In.
A
D
E
4 200 Lb
36 In.
36 In.
36 In.
36 In.
144 In.
(b) DISTRIBUTION FACTORS AND CARRY-OVER FACTORS
B
0.0392
16.1782
C
0.0694
0.8914
C →
6.92975
0.299572
Symmetrical About Center Line
S/ΣS →
0.0841
0.2706
A
0.9159
0.296658
0.2294
E
D
Values of S'
Values of s/(E'I/L)
(c) FINAL STIFFNESSES (S') AND RESTRAINT COEFFICIENTS (= s/(E'I/L))
B
−9,282.0
(+0.939)
C
−3,758.9
+15,922.2
(−1.145)
(+0.645)
Symmetrical About Center Line
+8,084.8
(+2.558)
−8,154.6
(+1.319)
A
+13,568.6
+10,520.8
E
(+2.782)
(−0.850)
D
(d) STIFFNESSES AND RESTRAINT COEFFICIENTS BY APPROXIMATION
(+0.910)
B
−6,493.6
(+1.327)
−9,572.9
C
(+0.881)
−3,771.4
+15,555.6
(−0.901)
+15,943.7
+1,241.9
(+3.422)
+11,926.5
+15,943.7
(+1.314)
−5,946.0
+13,519.1
A
+13,519.1
+13,519.1
E
D

Fig. 5.—End Restraints and Effective Lengths; an Illustrative Example

tration,[13] adopted from the aircraft field. It has been designed to fail at the loads shown, but actually it has some reserve of strength above these design ultimate loads. The following questions might arise:

1. At what multiple of the load will the truss become unstable? (To answer this question would involve a series of trials. Assume, as a first trial value, $\frac{425}{420} = 1.0119$ times the load shown—that is, the loads at joints B and C are changed to 425 lb; and the load at joint E, to 4,250 lb.)

2. For any given multiple of the load (say, 1.0119) what are the end restraints on the bars?

3. Having the restraints, what are the corresponding effective lengths?

All three of these questions are answered by the procedure outlined previously.

TABLE 1.—STIFFNESS AND CARRY-OVER FACTORS FOR
(ALL UNITS IN TERMS OF

Member	P	P/A	$E'/1{,}000$	L	I	$j^2 = \frac{E'I}{P}$	j	L/j
AB CD	−3,189	23.870	25,022	60	0.02467	193.57	13.913	4.313
BE EC	+2.656	19,880	27,695	60	0.02467	257.24	16.039	3.741
BC	−3,506	18,962	27,920	72	0.03339	265.90	16.306	4.416
AE ED	+1,913	14,319	28,000	72	0.02467	361.09	19.002	3.789
Totals	....	....	....	....	....	....	....	....

Table 1 gives the properties of the tubular members of the truss, the stresses under 1.0119 times the design load, and the stiffness and carry-over factors (far ends fixed), the latter being trigonometric functions of the axial loads and obtainable from tables previously cited.[6] The material is carbon steel (Society of Automotive Engineers (SAE) Specification 1025) with a yield point of 36,000 lb per sq in.; and, in the absence of tangent modulus data, the effective modulus is assumed equal to that obtained from the published value[13] of the ultimate column formula. For values of $\frac{P}{A} \geqq 18{,}000$ lb per sq in., the value is

$$E' = \frac{P}{A} \frac{(36{,}000 - P/A)}{1.172\,\pi^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (15)$$

Effective moduli in tension have been assumed the same as in compression. The distribution factors at the joints are calculated from the stiffness by the usual procedure, and are shown in Fig. 5(*b*).

The moment distribution is demonstrated in Table 2. The subvalues of r are $r_{BA} = +0.2796$; $r_{BE} = +0.0012$; and $r_{BC} = +0.5581$. The total r-value is $+0.8389$, which, being less than unity, indicates by the series criterion that the truss is stable under the load factor 1.0119. (It can be shown by repeated trial that the true critical load factor is approximately 1.0309.)

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[13] "Airplane Structures," by A. S. Niles and J. S. Newell, John Wiley & Sons, Inc., New York, N. Y., 3d Ed., Vol. II, 1943.

The stiffnesses S' of the members at joint B are computed immediately from Eq. 9. Thus, in inch-pounds,

$$\begin{aligned} S'_{ba} &= 1{,}241.9 - 0.2796 \times 17{,}885.6 = - 3{,}758.9 \\ S'_{be} &= 15{,}943.7 - 0.0012 \times 17{,}885.6 = + 15{,}922.2 \\ S'_{bc} &= \phantom{15{,}}700.0 - 0.5581 \times 17{,}885.6 = - 9{,}282.0 \\ \Sigma S'_{b} &= \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots = + 2{,}881.3 > 0 \end{aligned}$$

Only now is the analyst in a position to use the stiffness criterion, although this is obviously not necessary, since the stability has already been checked by the series criterion.

For a simple symmetrical truss of this type, and for most simplified groupings, the stiffnesses of the remaining members can now be obtained merely by repeated application of Eq. 3, solving either for an unknown S'-value in terms of the spring constant s at the far end, or for an unknown value of s at the far end in terms of the known stiffness S'. The final values are shown in Fig. 5(c). Typical computations are now given:

TRUSS MEMBERS SUBJECTED TO 1.0119 DESIGN LOAD
INCHES AND POUNDS)

$\dfrac{SL}{E'I}$	$\dfrac{E'I}{L}$	S	ΣS_A	ΣS_B	ΣS_E	C	Member
0.12071	10,288	1,241.9	1,241.9	1,241.9	....	6.92975	AB CD
1.40017	11,387	15,943.7	....	15,943.7	31,887.4	0.299572	BE EC
0.05406	12,948	700.0	....	700.0	....	16.1782	BC
1.40912	9,594	13,519.1	13,519.1	....	27,038.2	0.296658	AE ED
....	....	....	14,761.0	17,885.6	58,925.6	....	

$$S'_{ab} = 1{,}241.9 \left(1 - \frac{6.92975^2 \times 1{,}241.9}{6{,}640.2 + 1{,}241.9}\right) = -\ 8{,}154.6 \text{ in-lb}$$

In this calculation 6,640.2 is the spring restraint s_{ba}, acting at end B of member AB. It is the sum of the stiffnesses S' of the two adjacent members BE and BC.

Now consider member AE. From the preceding calculation for S'_{ab}, the value of s_{ae} is equal to $-$ 8,154.6. This value might also be stated as a restraint coefficient as follows:

$$\text{Restraint coefficient} = \left(\frac{s}{E'I/L}\right)_{ae} = -\frac{8{,}154.6}{9{,}594} = -\ 0.850$$

The value for S'_{ae} is readily obtained from the following computation, inasmuch as $S'_{ae} = s_{ab}$, the spring restraint acting at end A of member AB′:

$$-\ 3{,}758.9 = 1{,}241.9 \left(1 - \frac{6.92975^2 \times 1{,}241.9}{s_{ab} + 1{,}241.9}\right)$$

from which

$$s_{ab} = S'_{ae} = +\ 13{,}568.6$$

and

$$S'_{ea} = 13{,}519.1 \left(1 - \frac{0.296658^2 \times 13{,}519.1}{-8{,}154.6 + 13{,}519.1}\right) = +10{,}520.8$$

These computations illustrate the procedure for a simple, symmetrical truss. Final stiffness values and restraint coefficients are shown in Fig. 5(*c*).

For more complicated and extended frameworks, Eqs. 3 and 9 may not suffice to give all the values of S'. In this case one may resort to the final moments in the bars at each joint obtained from the calculations leading to one determination of r, together with Eqs. 9 and 3 or 4. This procedure can be demonstrated on the truss of Fig. 5(*a*) if the symmetry is disregarded and the stiffness values are computed from joint B to joint C, to joint D, and to joint E. From the values of S' at joint B, determined as shown previously,

TABLE 2.—MOMENT DISTRIBUTION FOR

Cycle	Step	JOINT A			JOINT B				JOINT	
		AB	AE	Unbalanced moment M	BA	BE	BC	Unbalanced moment M	CB	CE
	$S/(\Sigma S)$.....	0.0841	0.9159		0.0694	0.8914	0.0392		0.0392	0.8914
	C.........	6.92975	0.296658		6.92975	0.299572	16.1782		16.1782	0.299572
1	Balanced ..				−0.0694	−0.8914	−0.0392	+1.0000		
	Carry-over.	−0.4809							−0.6342	
2	Balanced ..	+0.0404	+0.4405	−0.4809					+0.0249	+0.5653
	Carry-over.	0	+0.0182		+0.2803	+0.0217	+0.4022		0	+0.0217
3	Balanced ..	−0.0015	−0.0167	+0.0182				+0.7042	−0.0009	−0.0193
	Carry-over.	0	−0.0204		−0.0106	−0.0243	−0.0138		0	−0.0243
4	Balanced ..	+0.0017	+0.0187	−0.0204				+0.6555	+0.0083	+0.1896
	Carry-over.	0	+0.0067		+0.0119	+0.0080	+0.1349			
(Cycles 5 through 17 omitted)										
18	Balanced ..	0	0	0				+0.8388	0	+0.0001
	Carry-over.	0	0		0	+0.0001	0		0	0
19	Balanced ..	0	0	0				+0.8389	0	0
Summation.....		−0.4406	+0.4406	0	+0.2796	+0.0012	+0.5581	+0.8389	−0.5997	+0.7858
Unbalanced moments		....	....	....	r_{BA}	r_{BE}	r_{BC}	r	....	....

Eq. 3 gives $S'_{cb} = -9{,}271$. Eq. 3 can also be used to solve for the restraint s_{cb} since the value of S'_{bc} is known. Thus:

$$S'_{bc} = -9{,}282.0 = 700.0 \left(1 - \frac{16.1782^2 \times 700.0}{s_{cb} + 700.0}\right)$$

Solving,

$$s_{cb} = +12{,}148.9$$

—but this restraint at joint C on member CB is provided by the sum of the stiffnesses of the members CD and CE. In other words,

$$S'_{cd} + S'_{ce} = +12{,}148.9$$

Inasmuch as a moment coming from joint B to joint C is distributed at joint C to members CD and CE in proportion to their stiffnesses S', the final moments in Table 2 at ends C of CD and CE provide the distribution ratio. Hence:

$$S'_{cd} = +\,12{,}148.9 \times \frac{-\,0.1861}{+\,0.7858 - 0.1861} = -\,3{,}770$$

$$S'_{ce} = +\,12{,}148.9 \times \frac{+\,0.7858}{+\,0.7858 - 0.1861} = +\,15{,}919$$

This process could be continued until all the stiffnesses S' are determined. The foregoing values should (and do) check those previously obtained, within the limits of accuracy of the computation used.

Load Factor 1.0119 (See Fig. 5)

C		Joint D			Joint E					Cycle
CD	Unbalanced moment M	DC	DE	Unbalanced moment M	EA	EB	EC	ED	Unbalanced moment M	
0.0694 6.92975		0.0841 6.92975	0.9159 0.296658		0.2294 0.296658	0.2706 0.299572	0.2706 0.299572	0.2294 0.296658		
						−0.2670				1
+0.0440	−0.6342				+0.0612	+0.0723	+0.0723	+0.0612	−0.2670	2
0		+0.3050	+0.0182		+0.1307	0	+0.1693	0		
−0.0015	+0.0217	−0.0272	−0.2960	+0.3232	−0.0688	−0.0182	−0.0812	−0.0688	+0.3000	3
−0.1884		−0.0104	−0.0204		−0.0049	0	−0.0058	−0.0878		
−0.0148	−0.2127	+0.0026	+0.0282	−0.0308	+0.0226	+0.0267	+0.0267	+0.0226	−0.0985	4
(Cycles 5 through 17 omitted)										
0	−0.0001	0	0	0	0	0	0	0	0	18
0		0	0		0	0	0	0		
0	0	0	0	0	0	0	0	0	0	19
−0.1861	0	+0.3878	−0.3878	0	+0.1342	−0.26245	+0.23975	−0.1115	0	Σ
....	....	....	....	....	....	....	....	....	....	M

It should be emphasized that the method of the preceding paragraph is largely academic, however, since in practice, even for the simple truss shown in Fig. 5(*a*), one complete moment distribution is too long for the purposes intended, and small simplified groupings are employed. It is not proposed to explore the merits of any particular group arrangements (a comprehensive study of which has been made by A. Slavin,[14] M. ASCE, at New York University, New York, N. Y.), other than to note as an example an arrangement for which the results are available in the foregoing computations.

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[14] "Stability Studies of Structural Frames," by A. Slavin, thesis presented to New York Univ., New York, N. Y., in May, 1948, in partial fulfilment of the requirements for the degree of Doctor of Engineering Science.

If one were interested, say, in the restraints on the bars immediately contiguous to joint B of Fig. 5(*a*), they could be approximated by stopping the moment distribution at the first cycle of carry backs to joint B, giving an approximate series of subvalues of r equal to $+0.2803$, $+0.0217$, and $+0.4022$ for BA, BE, and BC, respectively (see Table 2)—or a total value of $r = +0.7042$. Physically such a grouping corresponds to fixing the joints twice removed from the joint at which the unit moment is applied (see Fig. 5(*d*)), and such a grouping generally gives a value of the critical load of a steel truss within 1% or 2% of the true value. The corresponding values of the stiffnesses S' and the restraint coefficients $\frac{s}{E' I/L}$ are given in Fig. 5(*d*)).

With the values of the restraints known, it is a simple matter to calculate the reduced length by Eq. 13*b* and Eq. 14 or by Fig. 4. Applying Eq. 13*b* to member AB, for example, and using the exact values of the restraint factors found in Fig. 5(*c*),

$$(+\,0.645 + 1.319) + \frac{(0.645 \times 1.319)}{\dfrac{S}{E' I/L}} + \frac{S''}{E' I/L} = 0 \ldots\ldots (16)$$

It is not difficult to determine from tables[6] that Eq. 16 is satisfied by the S-values and S''-values for $L/j = 4.508$. From Eq. 14, $k = \frac{\pi}{4.508} = 0.697$—which is the effective length factor for bar AB with end restraints corresponding to the stresses in Table 1. These stresses are for a loading condition below the buckling load for the truss. If the same end restraints continued to act on AB, and if the effective modulus E' in Table 1 remained unchanged, the critical load for bar AB could be calculated from Eq. 2, and the result would be 3,480 lb. It would be the buckling load for an isolated bar AB with end restraints and E' as indicated. The value has little significance, however, because member AB is a part of the truss of Fig. 5(*a*). As the loads are increased on the truss, the end restraints and the values of E' decrease and the effective length increases. A study of the change in the value of the length factor k for the compression members AB and BC of this particular truss, over a range of multiples of the design loading, is shown in Fig. 6. As the loading increases, the value of k as determined from Eqs. 13*a* and 14 increases. The reduced length $k\,L$ varies from $0.649\,L$ to $0.716\,L$ for member AB, whereas for member BC it varies from $0.632\,L$ to $0.702\,L$. If the stress (3,247 lb) in member AB, due to the buckling load of the truss, is used in Eq. 2, and that equation is solved for k, it will be found that k for member AB practically checks the value 0.716 (see

TABLE 3.—TYPICAL VALUES OF k AND r

Load factor	RESTRAINT FACTOR[a] k		Moment r
	AB	BC	
0	0.649	0.632	0.0867
0.5	0.652	0.641	0.1593
1.0119	0.697	0.694	0.8389
1.0310	0.716	0.702	1.0061

[a] Computed from Eqs. 13*b* and 14.

Table 3). On the other hand, a similar calculation for member BC gives a value of k slightly greater than the value 0.702.

This fact would seem to indicate that member AB is the first member of the truss to fail and that the failure of member BC follows immediately thereafter. Actually, all the truss members, tension members as well as compression members, reach neutral equilibrium simultaneously, even though one member may be weaker than other members when considered alone. An extremely precise calculation of end restraints, effective lengths, and instability loads would confirm this conclusion. In fact, the phenomenon of truss buckling is a beautiful illustration to support the theory of limit design. A truss with rigid connections does not fail as a whole by primary instability until every reserve of that element of strength associated with bending resistance is drained from the stronger members to assist the weaker members.

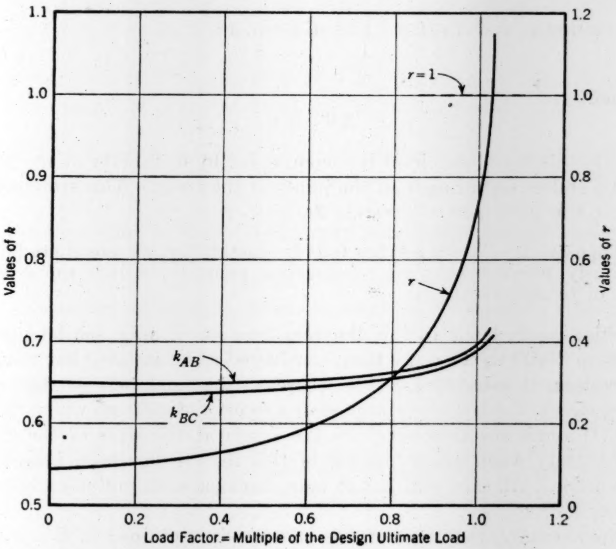

FIG. 6.—PLOT OF k AND r VERSUS LOAD FACTOR

As an illustration, the following theorem relating to the stiffness criterion may be postulated:

> Theorem 1. If the total stiffness of any joint b of a truss becomes zero ($\Sigma S'_b = 0$) under a given loading condition, then the total stiffness for any other joint c also becomes zero ($\Sigma S'_c = 0$) under the same loading condition.

This theorem is intuitively true because, if $\Sigma S'_b = 0$ (that is, joint b is unstable), and $\Sigma S'_c > 0$ (that is, joint c is stable), joint c would have an unused reserve of stiffness which it would then dispatch to joint b through the medium of the truss framework to assist the latter joint and thus to prevent its instability. Mathematically, the theorem is proved as follows: Referring to Fig. 1(*b*) and applying Eq. 5 to end b:

$$\Sigma S'_b = s_b + S'_{bc} = 0 \qquad (17a)$$

Hence,

$$s_b = -S'_{bc} \qquad (17b)$$

—but, from Eq. 3,

$$S'_{bc} = S_{bc}\left(1 - \frac{C^2_{bc}\,S_{bc}}{s_c + S_{bc}}\right) \qquad (18a)$$

and

$$S'_{cb} = S_{bc}\left(1 - \frac{C^2_{bc}\,S_{bc}}{s_b + S_{bc}}\right) \qquad (18b)$$

Eliminating s_b and S'_{bc} from Eqs. 17*b* and 18,

$$S'_{cb} = -s_c \qquad (19a)$$

Consequently,

$$\Sigma S'_c = 0 \qquad (19b)$$

Thus instability at one joint is accompanied by instability at the neighboring joints, and so on through all the joints of the truss. This statement leads directly to the statement of theorem 2:

> Theorem 2. When a truss fails by instability, all members fail simultaneously, whether they are compression members, axially unloaded members, or tension members.

Intuition again leads to this theorem because, if only one member has a tendency to "fail" in the sense that, considered as an isolated bar with known end restraints, it satisfied Eq. 13*a* whereas other members of the truss still remained stable, the latter would possess a reserve of stiffness which they could dispatch through the framework to the joints at the ends of the distressed member. Only when every last bit of this reserve has been exhausted will collapse occur. All bars will fail at once, because each individually will have no further reserve.

Mathematically, the proof of this theorem is contained in Eq. 13*a*, which was derived from the equation $\Sigma S' = 0$. If the truss is unstable, $\Sigma S' = 0$ at every joint (theorem 1), and Eq. 13*a* is satisfied by every bar entering into every joint.

Theorem 2 introduces the new and perhaps startling concept that an axially unloaded bar in a rigidly connected truss, or a bar with an axial tensile load, can be unstable. Eq. 13*a*, which was set up in a perfectly general way irrespective of the presence of tensile or compressive stresses, applies for all types of members (with the understanding that different tabular values of S and S'' apply for tensile and compressive stresses). For tensile, or axially unloaded bars, S and S'' are always positive; Eq. 13*a* will thus never be satisfied if the restraints s are both positive, as is self-evident; hence, tensile or axially unloaded bars in a truss which is on the point of buckling are actually subjected to negative end restraints with values that satisfy Eq. 13*a*.

Thus the meaning of the instability of an unloaded bar or a tensile member is clear. A tensile or axially unloaded bar will become unstable if negative restraints exist at one or both ends, with magnitudes such as to satisfy Eq. 13*a*. Generally the presence of restraint increases stability. Conversely, absence of restraint decreases stability. There is no physical reason why the lower limit of restraint must be zero; it can also extend into the negative domain.

If the foregoing concept is understood, the often-appearing statement to the effect that the entire truss fails at once is given powerful meaning and corroboration, because every joint tends toward infinite rotation, and every member (considered as an isolated bar with its appropriate end restraints) becomes unstable. From this point of view it is then incorrect of course to state that one member fails first by primary instability, regardless of whether or not it has been underdesigned. It must be kept in mind that secondary or localized failure is not being discussed. Actually, such failure would generally occur before the truss failed as a whole.

This "one hoss shay" action of a truss at failure due to primary instability takes place regardless of the make-up of the members, and consequently does not constitute a criterion for economy of the design. There is no question but that the individual members of a truss do contribute in different degrees toward producing failure. Any compression member that would fail if considered pin ended must draw restraint from the adjacent bars; a weakly designed compression member might indeed tend to produce failure by draining off the restraints from an otherwise well-designed framework. Conversely diagonal compression members designed for part-span loadings help to forestall early failure under full-span loadings, by virtue of their added stiffness under the full-span loadings.

Since the equivalence of the series criterion and the stiffness criterion for stability has already been established, a corollary to theorem 1 may be of some interest:

> Theorem 3. If the series factor, r, becomes 1 for a unit moment applied at any joint b of a truss under a given external load system, r would also equal 1 if the unit moment were applied at any other joint c of the truss under the same external load system.

A graph of r versus the load factor for a unit moment applied at joint B of the truss in Fig. 5(*a*) is shown in Fig. 6. At the critical load factor of 1.0309, $r = 1$. If the unit moment had been applied at joint E instead, r would again reach the value of 1 at the load factor 1.0309. For load factors below this value, however, the r-curve would have different values from the curve shown in Fig. 6. As a matter of practical interest it is to be noted that the joint with the largest r-value for any load factor tends to be the "weakest" joint in the sense that, according to Eq. 10, it is losing its stiffness at the fastest rate.

It is not important to compute effective lengths for any load condition other than the critical load for a truss. Such a procedure was followed in this paper to illustrate one method of calculation and, what is more significant, to show how the effective length varies as the loading is increased. If one has determined the buckling load for a truss, there is absolutely no necessity for calculating end restraints and then applying Eqs. 13*b* and 14, or for using curves such as those in Fig. 4 to obtain the length factor *k*. In the light of theorem 2, the Euler equation with the reduced modulus, Eq. 2, is sufficient, and much easier to apply, in securing the value of *k* for each compression member in the truss. Even if approximate methods are used in obtaining the critical load (methods[14] employing small groups of truss members, in conjunction with the Lundquist-Southwell[15] procedure of extrapolation), the values of the effective lengths at failure may be determined closely enough for all practical purposes from Eq. 2.

The investigation of approximate methods for determining truss buckling loads is beyond the scope of this paper. It will be of some interest however to know that additional investigation has demonstrated clearly that the buckling loads for steel building and bridge trusses, with members having slenderness ratios in accord with current design practice, are so close to the loads corresponding to the yield stress of the steel employed, that there is absolutely no need for buckling load analyses. The yield strength is the ceiling regardless of whether ordinary structural steel, low alloy steel, or silicon steel is used. This conclusion is not true, however, for structural aluminum alloys.

The development of economical design procedures, based on current knowledge of buckling phenomena, is also a matter that cannot be discussed in detail in this paper. It is sufficient to state that design procedures should reflect conditions that exist at ultimate loads, rather than conditions existing at working loads. Otherwise, factors of safety may be quite misleading.

The method presented in this paper is not the only procedure for evaluating end restraints. Classical methods, such as those of H. Zimmermann[9] (who first introduced the idea of spring constants or "Spannziffer") and W. Prager,[16] involve the solution of simultaneous equations containing the restraints as unknowns. An iteration procedure devised by P. T. Hsu[17] involves the assumption and successive refinement of end restraints, and the results in this paper check those obtained on the same problem by Mr. Hsu. A complete historical statement of the various methods relating to restraints and stability analysis of trusses has been compiled.[18]

As presented in this paper, the method is applicable only to trusses, where the effect of joint translation is small. It should be noted that the possibility of appreciable joint translation such as would exist in rigid frames, not only complicates the analysis but widens the range of possible values of *k*. Although the lower limit of *k* remains at 0.5 (fixed-ended column), the upper limit is increased to $k = 2$ (cantilevered column). The range of loads at which buckling may take place may be increased considerably.

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[15] "A Method for Estimating the Critical Buckling Load for Structural Members," by Eugene E. Lundquist, *Technical Note No. 717*, National Advisory Committee for Aeronautics, Washington, D. C., July, 1939.

[16] "The Buckling of an Elastically Encastred Strut," by W. Prager, *Journal*, Royal Aeronautical Soc., November, 1936, p. 833.

[17] "Elastic Stability of Members in Trusses," by P. T. Hsu, thesis presented to Cornell Univ., Ithaca, N. Y., in May, 1947, in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

[18] "Instability of Plane Truss Frameworks," by T. C. Kavanagh, thesis presented to New York Univ., New York, N. Y., in April, 1948, in partial fulfilment of the requirements for the degree of Doctor of Engineering Science.

ACKNOWLEDGMENTS

The paper is part of a series of studies of the behavior of compression members made while both writers were at New York University.

APPENDIX

B. W. James[7] has demonstrated that, for a prismatic bar of length L, moment of inertia I (about the centroidal axis normal to the plane of bending), and effective modulus of elasticity E', and acted on by an axial load P, the stiffness and carry-over factor are as follows:

Carry-over factor—

$$C = \frac{\alpha}{2\,\beta} \quad \text{(20a)}$$

Stiffness, far end pinned—

$$S'' = \frac{3}{4\,\beta} \quad \text{(20b)}$$

Stiffness, far end fixed—

$$S = \frac{3\,\beta}{4\,\beta^2 - \alpha^2} \quad \text{(20c)}$$

in which, for compressive loads P,

$$\alpha = 6\,\frac{\frac{L}{j}\csc\frac{L}{j} - 1}{\left(\frac{L}{j}\right)^2} \quad \text{(21a)}$$

and

$$\beta = 3\,\frac{1 - \frac{L}{j}\cot\frac{L}{j}}{\left(\frac{L}{j}\right)^2} \quad \text{(21b)}$$

whereas, for tensile loads P,

$$\alpha = 6\,\frac{\frac{L}{j}\,\mathrm{csch}\,\frac{L}{j} - 1}{-\left(\frac{L}{j}\right)^2} \quad \text{(22a)}$$

and

$$\beta = 3\,\frac{1 - \frac{L}{j}\coth\frac{L}{j}}{-\left(\frac{L}{j}\right)^2} \quad \text{(22b)}$$

For both tensile and compressive loads P,

$$\frac{L}{j} = \frac{L}{\sqrt{\frac{E'I}{|P|}}} \quad \text{(23)}$$

DISCUSSION

Jack R. Benjamin,[19] Assoc. M. ASCE.—For their clear presentation of an exceedingly complex subject the authors are to be complimented. Although ultimate failure by instability is the criterion stipulated in the paper, applications to cases where unit stress controls, rather than instability, are important. The unit stress limit merely provides an alternate failure criterion.

The variation in stiffness and carry-over factor for columns under loading as contrasted to beams under loading is important in the design of building frames. In building frames the column loads are almost independent of girder moments so that a single adjustment in stiffness and carry-over factor is sufficient in an elastic analysis.

Two factors are neglected by the authors—influence of secondary stress moments and moments caused by initial curvature. Secondary stress influences will affect the buckling load of the entire truss. In addition, although the initial curvature of truss members does not influence stiffness or carry-over factors, it does add initial joint moments to the moment distribution procedure. These moments are approximately proportional to the axial loads and can be of the order of magnitude of 50% of the allowable moment in the member when initial curvatures specified by the current tolerances of the American Institute of Steel Construction are used. In an elastic analysis, these curvature moments can be significant. Whether or not these two influences reduce the buckling load of a truss a significant amount deserves investigation.

George Winter,[20] M. ASCE.—Methods of investigating the buckling stability of structural frameworks are basically of two types: (*a*) "Classical methods" of solving the simultaneous transcendental equations which determine the buckling load; and (*b*) numerical methods of successive approximation. A very interesting historical review of the former was reported by Mr. Kavanagh[18] in 1948. For customary trusses with a considerable number of bars the classical methods are too cumbersome mathematically. The two numerical methods most promising for practical use are: (1) The Lundquist-Hoff[4,11,12] adaptation of the moment distribution approach, on which the present paper is based, and (2) the end restraint method developed by P. T. Hsu.[17] Method (2) has been published as part of an extensive study on buckling of trusses and rigid frames by the writer and his collaborators.[21] The contribution of the present paper consists in showing that the same end restraints which are found by method (2) can also be calculated from method (1). With regard to the implications of these various approaches the following comments can be made:

(1) In the Lundquist-Hoff method, ably restated by the authors, the stability of the truss is investigated at successively increasing loads until a load

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[19] Asst. Prof. of Civ. Eng., Stanford Univ., Stanford University, Calif.

[20] Prof. and Head, Dept. of Structural Eng., Cornell Univ., Ithaca, N. Y

[21] "Buckling of Trusses and Rigid Frames," by George Winter, P. T. Hsu, B. Koo, and M. H. Loh, *Bulletin No. 36,* Cornell Univ. Eng. Experiment Station, Ithaca, N. Y., April, 1948.

is reached for which the truss proves unstable. This procedure brackets the unknown exact critical load between the highest investigated load which proved stable and that load which showed instability. This bracketed range can be narrowed only by further time-consuming trials. The disadvantage of this approach is that each successive trial gives merely a new lower limit for the critical load so that the next trial load must be guessed, and may prove to be too high or too low by a considerable amount. (There are ways to reduce the required number of trials in the Lundquist-Hoff method; since no mention of them was made by the authors, they will not be discussed here.) In connection with this method the determination of restraints and effective lengths is wholly superfluous, as, in fact, is stated by the authors. After showing how to compute these quantities, the authors make no practical use of either of them.

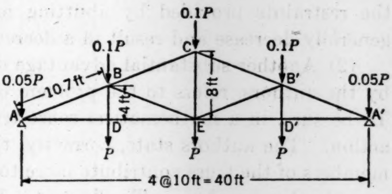

FIG. 7.—ILLUSTRATIVE EXAMPLE

In contrast, the end restraint approach[17,21] gives a method of determining restraints and effective lengths directly without a detour through moment distribution. These quantities are then used to calculate critical loads directly. Here, too, the truss is investigated under successively increasing loads until the critical load is bracketed sufficiently closely. However, the advantage of this method is that each successive step gives both an upper limit and a lower limit for the critical load which reduces considerably the required number of trials. This fact is easily illustrated by the example of the roof truss with heavy, suspended ceiling analyzed in detail by the writer and his collaborators,[22] as shown in Fig. 7.

For a trial load $P = 38$ kips, using end restraints, pertinent information was obtained for the three compression members, as listed in Table 4, in which k is the effective length coefficient; N_{38} is the compression force in a member caused by the loading $P = 38$ kips; N_{cr} is the critical load of the individual member for restraints determined for $P = 38$ kips; and $P_{N(cr)}$ is the loading P that would produce a bar force N_{cr} (for example, to have 177.9 kips in bar AB, a loading of $P = 40$ kips must be applied). Two facts are demonstrated by Table 4:

TABLE 4.—ROOF TRUSS ANALYSIS

Member	k	N_{38}	N_{cr}	$P_{N(cr)}$
AB........	0.720	169.0	177.9	40.0
BC........	0.885	112.2	117.8	39.8
BE........	0.685	55.2	74.1	51.0

(*a*) The truss is stable at $P = 38$ kips since the critical bar forces are larger than those caused by this loading or, in other words, all $P_{N(cr)}$-values are larger than 38 kips. The latter value, therefore, is a lower limit for the exact critical load.

(*b*) On the other hand, the smallest of the $P_{N(cr)}$-values—that is, 39.8 kips—is an upper limit for the true critical load because, with increasing load, restraints decrease or (at best) remain practically constant so that, in turn, effective lengths increase, unless they also stay constant (see Fig. 6). Thus, with a load larger than 38 kips, say, 39 kips, all N_{cr}-values will be found smaller or, at best, equal to the tabulated values. Hence the corresponding $P_{N(cr)}$-values cannot exceed those found for 38 kips.

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[22] "Buckling of Trusses and Rigid Frames," by George Winter, P. T. Hsu, B. Koo, and M. H. Loh, *Bulletin No. 36*, Cornell Univ. Eng. Experiment Station, Ithaca, N. Y., April, 1948, pp. 45–40.

This particular trial, therefore, was found to bracket the exact critical load P between 38 kips and 39.8 kips, that is, within a 5% interval. For practical purposes, no further trial would be needed.

The reason for condition (*b*) is that with increasing axial load the effective rigidities of the compression members decrease. On the other hand, the rigidities of the tension members increase, but at a much slower rate and with generally negligible effect. Consequently, with increasing load on the truss the restraints provided by abutting members to the compression members generally decrease and result in a decreasing or, at best, a constant N_{cr}.

(2) Another substantial advantage of the restraint method, not mentioned by the authors, refers to the problem of design rather than that of analysis. To be sure, in a mathematical sense trusses fail as units by "one hoss shay" action. The authors state, correctly, that despite this action some "weaker" members of the truss contribute more to limiting its critical load than do other, more husky members. To illustrate: If, in the foregoing truss example, the critical load of slightly more than 38 kips was found insufficient for the given, actual design loads, the truss would have to be redesigned and strengthened. Theoretically the strengthening of any member will strengthen the entire truss. However, it is more effective to strengthen some members than others and the designer will want to change these members for maximum economy. The tabulated values show immediately that for this particular truss the members AB and BC should be strengthened, whereas substituting a heavier member for BE would have a small effect on the carrying capacity. This fact is evident because, for members AB and BC, the N_{cr}-values exceed the actual bar forces N_{38} only slightly, whereas the difference between these values exceeds 30% for member BE. The latter, therefore, has excess strength which is drawn upon by members AB and BC to supplement their own insufficient rigidity. The determination of such "weakest members" rather than the determination of the "weakest joints" discussed by the authors is important since it is not a joint but a member that must be redesigned for the desired effect. The determination of these "weakest members" is an automatic feature of the end restraint method and requires no additional computation.

(3) In some structures, particularly those subject to fatigue, the determination of actual maximum stresses at loads below the critical is important, and governs design. These stresses will exceed the value of P/A if lateral loads, eccentricities, or other imperfections induce bending. Such calculations are greatly facilitated by determination of end restraints for these subcritical loads. Corresponding methods (including tables and charts) were reported in 1948[21] as a further development of the end restraint method. A chart of the general type of Fig. 4[8] is also essential in connection with end restraint methods.

However, negative as well as positive restraints can, and do, occur; and the chart developed at Cornell University, in Ithaca, N. Y., includes the range of such negative restraints, in contrast to Fig. 4.

(4) As stated by the authors, the methods in the paper apply only to frames without joint translation. The problem of effective length or of actual critical load, however, is of particular importance for rigid frames not restrained against sidesway. Fig. 8[21] shows the tremendous influence of sidesway restraint on the critical load of simple portal frames, for which

$$P_{cr} = \frac{\pi^2 E I}{(k\, l)^2} \qquad (24)$$

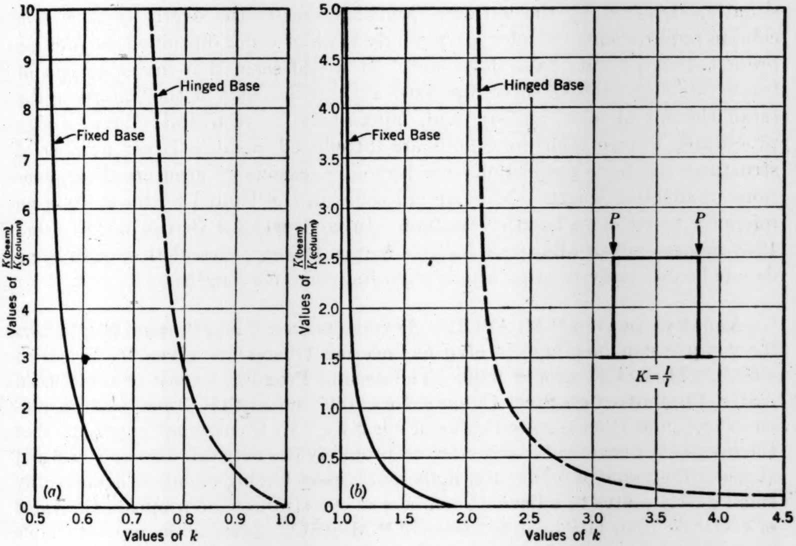

FIG. 8.—EFFECTIVE LENGTH COEFFICIENTS k FOR PORTAL FRAMES
(a) Portal Frame, Sidesway Prevented
(b) Portal Frame, Sidesway Permitted

If such restraint is provided, effective length coefficients do not exceed 0.7 and 1.0 for the cases of fixed base or hinged base, respectively; without such restraint the corresponding maxima are 2.0 and ∞, respectively. The upper limit of $k = 2$ advanced by the authors in their last paragraph is seen to be correct, therefore, only for frames with completely fixed bases. It will be exceeded considerably for frames hinged or only partly restrained at the footings. The latter condition is true of most soils. Since buckling loads are inversely proportional to k^2, consideration of sidesway buckling is of paramount importance for rigid frames.

(5) In one of their concluding statements the authors seem to indicate that effective length determinations are superfluous for ordinary steel trusses and that compression members of such trusses can be designed on the basis of yield strength without regard to buckling. Some skepticism concerning this contention seems justified. As an example, at the critical load, values of P/A for the roof truss in Fig. 7—with slenderness ratios varying from 145 to 175 (that is, within the range allowed by current codes)—range from approximately 16 kips per sq in. to 25 kips per sq in. This range is considerably lower than the yield point of 33 kips per sq in. for mild steel, not to mention the still higher yield points that apply to the alloy steels mentioned by the authors. To be sure, current truss practice usually results in smaller slenderness values than those of this example; and, therefore, the design approaches more closely the situation depicted by the authors—probably due to the overly conservative column requirements in codes, particularly in older codes during whose lifetime present design habits have developed. It would seem that investigations of frame buckling such as those discussed are justified only if they promote the establishment of more realistic and, most likely, more liberal column design procedures. Once such procedures are introduced, a much larger number of structures will have proportions which require reasonably accurate determinations of effective length. Most design codes in the United States contain no reference to effective length variation. In contrast, the British code[23] dated 1948 makes explicit allowance for this factor, fifteen pages of that code being devoted exclusively to stipulations regarding effective lengths.

ABRAHAM SLAVIN,[24] M. ASCE.—Average values of length reduction factors for compression members in civil engineering trusses are given by Friedrich Bleich,[25] M. ASCE, as $k = 0.80$. The Second Progress Report of the Special ASCE Committee on Steel Column Research[26] notes that from tests on pin-ended columns the suggested value of k is 0.78. E. H. Salmon[27] mentions that the k-value for practical end conditions probably lies between 0.56 and 1.00 and suggests the value of 0.78. For light roof trusses having members attached by small gusset plates to adjacent members of less stiffness, he proposes the value of $k = 1.0$. The late Leon S. Moisseiff,[28] M. ASCE, gave, as the nearest simple equivalent to an actual compression member in civil engineering structures, a column with partly restrained ends for which $k = 0.75$, or the average of the theoretical value 0.50 for fixed ends and the theoretical value 1.0 for pinned ends in an ideal column. A. S. Niles, Assoc. M. ASCE, advises that for airplane trusses of welded jointed tubular sections, designed by ultimate load formulas, the value of $k = 0.70$ in the prime compression member (highest value of L/j at critical load) was practically substantiated by static tests to failure made at an army airfield in United States during 1920 and later. This value is practically equal to that theoretically derived for a compression member having one end restrained and the other end pinned, although both ends of the truss column were actually subject to an intermediate degree of restraint. J. S. Newell reports that, for most of the airplane truss frames he analyzed, the critical load corresponded with the prime compression member having a k-value of about 0.70.

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[23] "The Structural Use of Steel in Buildings," *British Standard Code of Practice, CP 113*, London, 1948.

[24] Cons. Civ. Engr. and Architect, New York, N. Y.; and with Dept. of Civ. Eng., Polytechnic Inst. of Brooklyn, Brooklyn, N. Y.

[25] "Die Knickfestigkeit Elasticher Stabverbindungen," by Friedrich Bleich, *Der Eisenbau*, April, 1919, p. 83.

[26] "Steel Column Research: Second Progress Report of the Special Committee," *Transactions*, ASCE, Vol. 95, 1931, p. 1201.

[27] "Materials and Structures," by E. H. Salmon, Longmans, Green & Co., London and New York, Vol. 2, 1937.

[28] "Design Specifications for Bridges and Structures of Aluminum Alloy 27 S-T," by Leon S. Moisseiff, Aluminum Co. of America, Pittsburgh, Pa., Revised Ed., March, 1940.

The foregoing results are only a few of those from numerous reports regarding restraint coefficients. Perhaps the difficulty, if any, is that many of the published column formulas do not indicate the type of end restraint, and it is assumed that the column formulas are for the practical compression members which may have varying degrees of end restraint. In fact, the column formula specified by the American Institute of Steel Construction does not indicate the type of end restraints; and, from discussion with well-known structural research engineers, the general assumption is made that the formula applies to a practical column with no definite or theoretical value for the end restraints.

In the fourth paragraph of their paper, the authors state:

> "It is not generally recognized that, as loads are increased on a framework, the end restraints acting on a compression member in the framework do not remain constant."

From the experience of the writer he believes that these facts are fairly widely recognized. Computed restraint factors from the stability studies of the plane trusses of three types[14] are given in Table 5. The k-values were obtained by the simple method of applying the Euler formula with the reduced modulus.

The airplane truss in Table 5, designed by ultimate load formulas, is the same as that used by the authors (except for engineering notation) and by Messrs. Niles and Newell.[29] The building truss by E. J. Squire, M. ASCE, and the highway truss,[14] both designed by working load formulas, are included as technical supporting data for values of k, and as an illustration of the relation of critical stresses to yield point values. For the Howe type of building truss (studies 8 and 9, Table 5(*b*)), the interior web members are reversed in direction to parallel the end posts. In this discussion, E_u refers to the effective modulus evaluated from the basic column formula; E_T is the tangent modulus evaluated from a typical stress-strain curve for structural steel; E_D is the double modulus; and E is the elastic modulus. In the case of the airplane truss, E may be used instead of E_u with an increase in value of the critical load of only 1.5%.

For both studies of the airplane truss, k may be taken as practically equal to 0.7. The first four studies of the building truss, and the first study of the highway truss, on the basis of the conservative modulus E_u, indicate a k-value close to 0.78, which has been suggested instead of unity for pin-ended columns in civil engineering structures to allow for the friction at the pin joints. For

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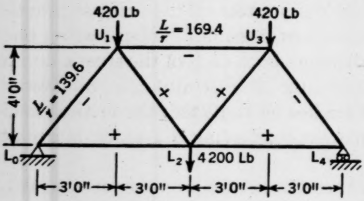
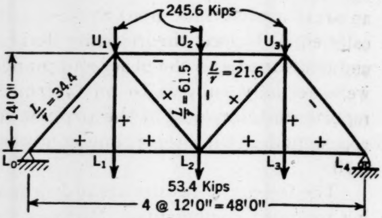
[29] "Airplane Structures," by A. S. Niles and J. S. Newell, John Wiley & Sons, Inc., New York, N. Y., 3d Ed., Vol. II, 1943, p. 306.

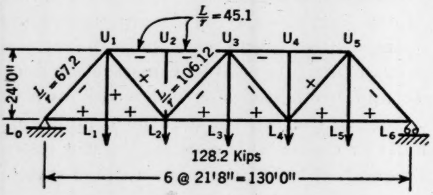
TABLE 5.—CRITICAL VALUES FOR TRUSSES [a]


420 Lb
420 Lb
U_1
$\frac{L}{r} = 169.4$
U_3
4'0''
$\frac{L}{r} = 139.6$
L_0
L_2
4200 Lb
L_4
3'0''
3'0''
3'0''
3'0''

(*a*) AIRPLANE TRUSS


245.6 Kips
U_1
U_2
U_3
14'0''
$\frac{L}{r} = 34.4$
$\frac{L}{r} = 61.5$
$\frac{L}{r} = 21.6$
L_0
L_1
L_2
L_3
L_4
53.4 Kips
4 @ 12'0'' = 48'0''

(*b*) BUILDING TRUSS, ($\frac{L}{r}$ Values for Study No. 1)


$\frac{L}{r} = 45.1$
U_1
U_2
U_3
U_4
U_5
24'0''
$\frac{L}{r} = 67.2$
$\frac{L}{r} = 106.12$
L_0
L_1
L_2
L_3
L_4
L_5
L_6
128.2 Kips
6 @ 21'8'' = 130'0''

(*c*) HIGHWAY TRUSS

Study	Design formula	Yield point[b]	$\overline{E}$	Critical load factor	CRITICAL VALUES FOR MEMBER: L_0U_1		U_1U_2[c]	
					Stress[b]	*k*	Stress[b]	*k*
	(*a*) AIRPLANE TRUSS							
1	Ultimate...........	36,000	E_u	1.0310	24,307	0.715	19,320[d]	0.704[d]
2	Ultimate...........	36,000	E	1.0452	24,648	0.758	19,589[d]	0.701[d]
	(*b*) BUILDING TRUSS							
1	Old AREA[e].........	36,000	E_u	2.5553	35,035	0.834	35,681[d]	0.765[d]
2	Old AREA[e].........	40,000	E_u	2.8372	38,900	0.829	39,617[d]	0.781[d]
3	AISC[f].............	36,000	E_u	2.1509	34,969[d]	0.766[d]	34,294	1.532
4	AISC[f].............	40,000	E_u	2.3868	38,805[d]	0.768[d]	38,055	1.524
5	Old AREA[e].........	36,000	E_T	2.57791	35.343	1.530	35,998[d]	0.572[d]
6	AISC[f].............	36,000	E_T	2.21109	35,948[d]	0 715[d]	35,254	2.185
7	AISC[f].............	33,000	E_T	2.0272	32,960[d]	0.704[d]	32,321	2.228
8	AISC[f].............	36,000	E_u	2.1344	34,701[d]	0.860[d]	35,072	1.064
9	AISC[f].............	40,000	E_u	2.3680	38,499[d]	0.861[d]	38,911	1.073
	(*c*) HIGHWAY TRUSS							
1	AREA (1943)[e]......	36,000	E_u	2.3983	32,209	0.847	34,536[d]	0.786[d]
2	AREA (1943)[e]......	33,000	E_T	2.2865	30,708	1.150	32,926[d]	0.703[d]

[a] All Warren type trusses except studies 8 and 9, Table 5(*b*), which are Howe type trusses. [b] Pounds per square inch. [c] In the airplane truss, Table 5(*a*), this becomes member U_1U_3. [d] Values for the member with the highest ratio L/j. [e] Old specifications of the American Railway Engineering Association, as distinguished from the 1943 specifications. [f] American Institute of Steel Construction.

building trusses of the Howe type (studies 8 and 9), on the basis of E_u, $k = 0.86$. For the studies using E_T, with the exception of study 5, the k-values are about 0.7, or the same as those evaluated for the airplane truss with the modulus concept of E_u. The restraint coefficients will vary with loading, but the value at the critical load is of interest. All the data in Table 5 were evaluated and compiled prior to, and independently of, the preparation of the paper by the authors and were reported to them as part of the group research work.

In presenting Eq. 2, the authors mention that the tangent modulus is now generally accepted by aeronautical engineers and that it has also been accepted by the Column Research Council of the Engineering Foundation. However, in the computations for the airplane truss, the authors (in applying Eq. 15), as well as the writer, used the reduced modulus E_u. Since the effective modulus value is important for computing the critical loads and the corresponding restraint coefficients for the compression members in the framework, comment regarding the evaluation of the reduced modulus is warranted. The value of E_u is lower than the value of the tangent modulus and is intended to replace the Engesser and the von Kármán formulas that make no allowance for the effect of the practical column as compared to that of the corresponding ideal column. With a low value of the reduced modulus, the numerical value of L/j is increased and the computed theoretical critical load is a conservative value. This method of computing the reduced modulus is given by Messrs. Niles and Newell.[30] The report of the Army-Navy-Civil Committee on Aircraft Design Criteria[31] also states that the modified Euler formula does not have much practical importance in determining the short column curve, but that it is of practical interest in connection with the determination of the effective modulus that can be used to compute instability stresses. N. J. Hoff[32] mentions that the specification of a short column formula establishes a uniquely determined connection between the reduced modulus and the elastic modulus. W. Prager,[16] Eugene E. Lundquist, Assoc. M. ASCE, and Claude M. Fligg,[33] and Mr. Moisseiff and Frederick Lienhard,[34] M. ASCE, also have used E_u.

Recommendations for the tangent modulus by F. R. Shanley[3] and O. H. Basquin[35] are based on the ideal unit columns. The recommendations by Mr. Shanley are essentially a verification of the results of prior laboratory work on ideal columns by the Aluminum Research Laboratories[36,37] at New Kensington, Pa. K. Borkmann[38] suggests an expedient estimate of the effective modulus on the basis of a linear relation between the proportional limit and the yield point. The writer[14] proposes an effective modulus based on a linear relation between seven tenths of the yield point and the yield point, which is intended for structural steels. If seven tenths of the yield point value is above the known value of the proportional limit, it does not apply and in its place the Borkmann modulus is suggested. It is the considered judgment of the writer that allowance should be made in the modulus value for the difference between the practical column and the ideal column.

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[30] "Airplane Structures," by A. S. Niles and J. S. Newell, John Wiley & Sons, Inc., New York, N. Y., 3d Ed., Vol. I, 1943, pp. 332–334.

[31] "Strength of Aircraft Elements," *ANC-5*, Army-Navy-Civil Committee on Aircraft Design Criteria, Revised Ed., Washington, D. C., December, 1942 (as amended August, 1946).

[32] "A Note on Inelastic Buckling," by N. J. Hoff, *Journal of the Aeronautical Sciences*, April, 1944, p. 164.

[33] "A Theory for Primary Failure of Straight Centrally Loaded Columns," by Eugene E. Lundquist and Claude N. Fligg, *Technical Report No. 582*, National Advisory Committee for Aeronautics, Washington, D. C., 1937.

[34] "Theory of Elastic Stability Applied to Structural Design," by Leon S. Moisseiff and Frederick Lienhard, *Transactions*, ASCE, Vol. 106, 1931, pp. 1052–1091.

[35] "Tangent Modulus and the Strength of Steel Columns in Tests," by O. H. Basquin, *Journal of Research*, National Bureau of Standards, September, 1924, p. 381.

[36] "Column Strength of Various Aluminum Alloys," by R. L. Templin, R. G. Sturm, E. C. Hartmann, and M. Holt, *Technical Paper No. 1*, Aluminum Co. of America, Pittsburgh, Pa., 1938.

[37] "Typical Tensile and Compressive Stress-Strain Curves for Aluminum Alloy 24 S-T, Alclad 24 S-T, 24 S-RT, and Alclad 24 S-RT Products," by R. L. Templin, E. C. Hartmann, and D. A. Paul, *Technical Paper No. 6*, Aluminum Co. of America, Pittsburgh, Pa., 1942.

[38] "Charts for Checking the Stability of Compression Members in Trusses," by K. Borkmann, *Technical Memorandum No. 800*, National Advisory Committee for Aeronautics, Washington, D. C., July, 1936.

In the derivation of their formulas, the authors use a spring constant s, and M is the applied moment equal to S, the stiffness for the far end fixed. Since a unit moment is applied for each loading state to determine the unbalanced moment or stability factor r, the assumption that $M = S$ is a limited condition.

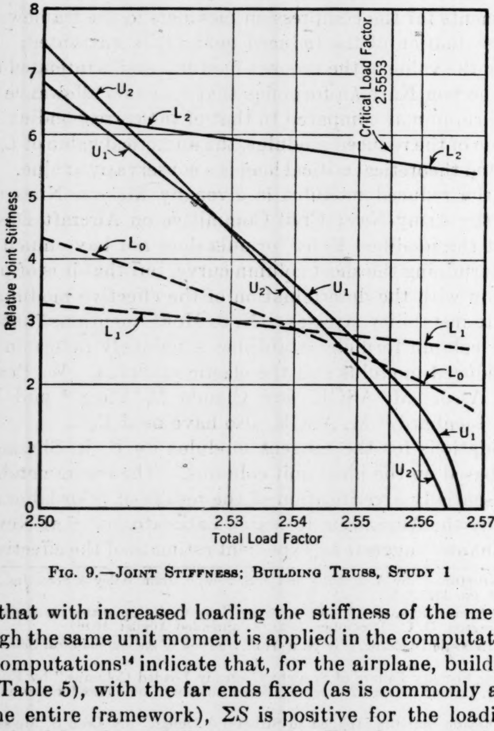

FIG. 9.—JOINT STIFFNESS; BUILDING TRUSS, STUDY 1

It is evident that with increased loading the stiffness of the members will decrease, although the same unit moment is applied in the computation procedure. The writer's computations[14] indicate that, for the airplane, building, and highway trusses (Table 5), with the far ends fixed (as is commonly assumed in the analysis of the entire framework), ΣS is positive for the loading above the critical determined by the stability $r = 1.0$. For the trusses in Table 5, the difference in critical load on the basis of the stiffness criterion $\Sigma S = 0$ and the series criterion $r = 1.0$ is about 17% for the airplane truss, about 0.4% for the building truss, and about 1.4% for the highway truss.

For the building truss in Table 5(b), the relative stiffness of the joints under increase in loading is shown in Fig. 9. Joint U_2 has the highest relative stiffness at the design load up to a load factor of about 2.52, beyond which, with load increase, it becomes the joint of least stiffness. Thus, the weakest joint is not the one having the least stiffness at the design load but rather the joint having the greatest rate of loss of stiffness with loading increase. The authors have also verified this statement with reference to the airplane truss. Since the joint stiffness is the sum of the stiffness of all members entering the truss joint, the data in Fig. 9, which are related to the truss in Table 5(*b*), indicate that the stocky compression members at joint U_2, which have high unit stress, lose their contributing stiffness at the greatest rate because of the buckling tendency under increased loading; and, in turn, their respective restraint values are also reduced. The total stiffness of each joint has been found to decrease in relation to the prevalence of compression members composing the joint, and not in proportion to the increase in loading.

TABLE 6.—COMPARISON OF FIXED-ENDED COLUMNS AND PIN-ENDED COLUMNS FOR THE PRIMARY COMPRESSION TRUSS MEMBERS HAVING THE HIGHEST RATIOS *L/j* AT CRITICAL LOADS

(Yield Point, 36 Kips per Sq In.)

(*a*) MEMBER U_1U_2, BUILDING TRUSS				(*b*) MEMBER U_1U_2, HIGHWAY TRUSS			
Load factor	Stress (lb per sq in.)	Ratio, *L/j*	Percentage range[a]	Load factor	Stress (lb per sq in.)	Ratio, *L/j*	Percentage range[a]
2.539	35,455	3.142	0.0	2.335	33.625	3.142	0.0
2.555[b]	35.680	4.108	+0.63	2.398[b]	34.536	3.977	+2.70
2.568	35,865	6.288	+1.14	2.460	35,425	6.372	+5.35

[a] Percentage range in load factor. [b] Critical load factor.

In the paragraph presenting Eq. 2, the authors claim that the buckling load in civil engineering trusses for a fixed-end column may be only slightly greater than that for the same column with pinned ends. The data in Table 6 from studies 1 of the trusses in Tables 5(*b*) and 5(*c*) cast some light on this question. The member having the highest ratio of *L/j* at the critical load, on the basis of E_u, is considered.

In stability computations, $L/j = \pi$ for pinned-end columns, and $L/j = 2\pi$ for theoretical fixed-end columns. A narrow range in load factors between these two conditions of end restraint is indicated for the cited typical civil engineering trusses. However, as noted by Mr. Hoff,[12] in actual frameworks the end fixity is elastic and is not perfectly rigid, and the upper limit of $L/j < 2\pi$ for a compression member with both ends rigidly fixed cannot be reached. The first and last values of the data for each truss in Table 6 may be used to approximate the buckling load, from which *k*-values may be estimated, since there is a small range in load factors between the upper and lower values of *L/j*. The data in Table 6 also indicate that the critical compressive unit stresses are close to the yield point. This reasoning does not apply to the airplane truss in Table 5(*a*), in the same sense that it applies to the airplane truss in

Fig. 5, because the compression members have high slenderness ratios and they fail at unit stresses considerably below the yield point.

In Fig. 6, the plot of r versus the load factor for the airplane truss applies to a unit moment at joint U_1. With application of a unit moment at joint L_0, the r-values are lower up to the equivalent critical load value at $r = 1.0$ and then are higher. For verification of the equivalence of the critical load, where $r = 1.0$, the unit moment should be applied separately to each joint at which compression members enter. It will be observed that the critical load, where $r = 1.0$, is the same regardless of the joint at which the unit moment is applied, as discussed by the authors in theorem 3. It is evident, however, that, for a truss frame in which all members are interdependent, at the critical load the frame fails as a unit regardless of the reserve strength in any members, since the continuity of the frame is destroyed.

The authors credit Mr. Lundquist[4] for the stability criteria and Mr. Hoff[11,12] for his contribution. The analysis method should be called the Lundquist-Hoff method. Although the work by Mr. Lundquist is shown as applicable to member groups of a truss and although it may be applied to the entire truss frame, the procedure by Mr. Hoff is definitely applicable as a valid solution for the entire framework. In addition, credit is due to Messrs. Niles and Newell[29] for their moment distribution method which modifies the usual procedure. In the joint at which the unit moment is applied, the carry-over values are held against further distribution, which thus materially shortens the computations for the stability factor r.

The authors note that, for compression members in steel trusses, with the usual range of slenderness ratios, the buckling load practically coincides with the yield point of the steel regardless of the end restraints; and therefore any elaborate analysis of stability and end restraint is unwarranted for a steel truss. It is well known that steel (whether carbon steel, silicon steel, or nickel steel) has a well-defined yield point and that the elastic modulus E is the same for all regardless of the different values for proportional limit and yield point. However, the statement by the authors should be qualified to apply to civil engineering structures designed on the basis of working load formulas. It does not apply to steel trusses used in airplanes, which are designed by ultimate load formulas and have slender compression members that fail at stresses considerably below the yield point.

The data in Tables 5 and 6 indicate definitely that, for civil engineering trusses of structural steel, the critical compressive unit stresses are practically at the yield point. Therefore it is evident that, for these trusses, the necessary stability analysis for the critical load is futile. This does not apply to the airplane truss (as in Table 5(a)), because the critical compressive unit stresses are considerably below the proportional limit. From a plot of column curves for structural carbon steel, for both pinned and fixed restraints,[14,18] it is readily observed that (in the usual range of L/r-ratios for civil engineering sections) the critical unit stress is close to the yield point. The relation between the buckling unit stress of a unit structural steel column (of the usual stocky section in civil engineering structures) and the yield point of the material has been discussed for many years. The suggestion has been made that the buckling unit stress be approximated at nine tenths of the yield point for columns in civil engineering structures. It is also known that a column, as part of a framework, is subject to conditions different from those in the member acting as a unit.

Prior investigators, particularly in the field of aeronautic structures, have indicated that, when a steel frame is being loaded, its compression members will drain possible reserve strength from the adjacent members. This fact is mentioned by the authors and has been independently verified by the writer.[14] Since previous research reveals that, for a unit column of the civil engineering type, the buckling unit stress is close to the yield point, by intuition it follows that the buckling unit stress of a column in a framework must be practically at the yield point because of the stress contributed by the possible reserve strength in the adjacent members. In fact, in discussions with a number of structural analysts the same general opinion was expressed to the writer although it was not based on stability calculations. However, this relation does not apply to the airplane truss example, used by the authors and the writer, and it should be noted that stability analysis is required for such structures. The critical load factor for the airplane truss cited is 1.0309 in the paper and 1.0310 according to the writer—indicating a 3% safety factor on the basis of design load. It is evident therefore that a stability analysis is required for the airplane truss.

The authors state that it is advisable to use small simplified member groups of the truss instead of the entire framework to determine the critical load, and that the illustrative example is from an arrangement proposed by the writer.[14] The grouping for the airplane truss is shown in Fig. 10. Because of the limited size of the truss all the members are employed when the terminals are twice removed from the joint where the unit moment is applied. For simplicity in calculation fixed ends are used throughout. The grouping in Fig. 10 was recommended to Mr. Kavanagh because it was found applicable to both aeronautical and civil engineering trusses. The calculations for r are simple, and satisfactory values for k result in close to the true values of the critical load. Based on the generalized application of the Saint Venant theory, it is evident that, the farther the terminals are from the joint where the unit moment is applied, the closer the results will come to the true answer. Furthermore, the use of a grouping with terminals twice removed from both ends of a compression member requires considerable calculations with different r-values at each end, up to the value of the equivalence r. In this case, there will be approximately the same variation from the true critical load as for the grouping in Fig. 10. With terminals twice removed, contribution to the stability limit is allowed from the twice removed tension members and from any possible

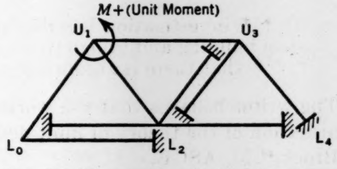

FIG. 10.—MEMBER GROUP, AIRPLANE TRUSS

low-stressed compression members. For the grouping in Fig. 10, the operation is applicable only to the members that enter the joint where the unit moment is applied, because no moment is carried back from the twice removed terminals to the joints once removed. Denoting the stiffness ratio of the members as S_r, instead of as $K/\Sigma K$; and, with C as the carry-over factor, the unbalanced moment r for a unit moment at joint U_1 in Fig. 10 is equal to

$$r = \Sigma \left[\begin{array}{l} (S_r\ \overline{U_1L_0})\ (S_r\ \overline{L_0U_1})\ C^2 \\ (S_r\ \overline{U_1U_3})\ (S_r\ \overline{U_3U_1})\ C^2 \\ (S_r\ \overline{U_1L_2})\ (S_r\ \overline{L_2U_1})\ C^2 \end{array} \right] \ldots\ldots\ldots\ldots\ldots (25)$$

Thus, a simple calculation instead of a lengthy moment distribution is available for satisfactory values of r.

The developments by the authors, and this discussion, are both based on planar trusses, with buckling within the plane, assuming the effect of joint translation to be negligible.

J. Edmund Fitzgerald,[39] Jun. ASCE.—An original contribution to, and a survey of, the various methods of determining truss capacities are presented in this paper. It is to be regretted, however, that the authors did not include an example of an actual steel truss, preferably indeterminant, in order that the reader might be enabled to judge their method as applied to commonly encountered design problems.

The paper derives its value from the authors' presentation of a method of handling trusses with rigid joints whose $\frac{L}{i}$-values are greater than those normally encountered in practice, since for the usual truss the authors state (third paragraph following theorem 3):

> "* * * investigation has demonstrated clearly that the buckling loads for steel building and bridge trusses * * * are so close to * * * the yield stress * * * that there is absolutely no need for buckling load analyses."

The writer believes that the worth of the paper lies in the mathematical confirmation of the theory of limit design as originally presented by J. A. Van den Broek,[40] M. ASCE.

As will be shown, the writer disagrees with the limitation set by the authors and believes that there is no need of a buckling analysis for trusses with a slenderness ratio greater than those commonly encountered in practice. As a case in question, consider the truss shown in the example in Fig. 5. This truss is unquestionably in the slender range since the values of L/i for its members vary between $L/i = 140$ and $L/i = 170$.

In spite of the mathematically correct statement by the authors that "* * * when a truss buckles all members tend to fail simultaneously" (which would be difficult indeed to demonstrate in the field or in a laboratory), the writer will employ dictum (3), from Mr. Van den Broek's paper to solve the truss in Fig. 5.

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[39] Asst Prof. of Civ Eng., North Dakota State College, Fargo N. Dak.

[40] "Theory of Limit Design," by J. A. Van den Broek, *Transactions*, ASCE, Vol. 105, 1940, pp. 638-661.

In dictum (3) it is stated:

> "When in a * * * n-fold redundant structure, n redundants are stressed to * * * their critical buckling load, the deformations involved are of the order of magnitude of elastic deformations until an $(n + 1)$th member has reached * * * critical buckling capacity."

In order to provide for the redundants that are produced by the moment restraints at the various joints, the analyst may use the common design practice of assuming the effective lengths of the members as $L' = 0.75\ L$—that is, $K = 0.75$. For members with $\frac{L}{i}$-ratios between 100 and 200, this assumption is borne out quite well. For $\frac{L}{i}$-ratios less than 100, using the yield stress will give excellent results. This is in agreement with the authors' recomendations. For L/i greater than 200, the structure will probably be poorly designed. Thus, the assumption of 0 redundants is incorporated in the effective length and the designer may proceed to the analysis by limit design.

Since the truss shown has 0 redundants, $n + 1 = 1$, and the problem resolves itself into finding the weakest member—that is, in the authors' terms, finding the member which approaches failure the fastest (thus decreasing the reserve of strength in the structure to zero) and produces incipient failure.

Considering the truss as loaded with a load P at joint E and with loads $0.1\ P$ at joints B and C the individual bar stresses are determined by statics. The entire calculation is shown in Table 7.

The values for the ultimate or buckling unit load were taken by interpolation from data published by Mr. Van den Broek. For $\frac{L}{i}$-values of 100 or so, the $\frac{L'}{i}$-ratio will give an ultimate unit load approximately equal to the yield point value.[41] The areas of the members and other pertinent data can be taken from the original Van den Broek paper.

The authors' overload factor of 1.03 represents a 1% difference from the result in Table 7. Actually, the variation of the physical and geometric constants of the truss itself is likely to exceed any error introduced by choosing a L'-value that is in error. The authors have found that the correct L'-value at failure is about 0.702. This represents a difference of almost 7%.

TABLE 7.—ANALYSIS OF THE TRUSS IN FIG. 5

Member	Stress	$\frac{L}{i}$	$\frac{L'}{i}$	Ultimate strength	Load	P	Overload factor
AB,CD.........	$-0.750\ P$	140	105	24,000	3,200	4.270	1.017 governs
BE,EC.........	$+0.625\ P$	140	....	36,000	4.800	7,740	1.840
BC............	$-0.875\ P$	170	127	19.000	3.520	5.620	1.340
AE, ED........	$+0.450\ P$	167	....	36.000	4.800	10,670	2.540

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[41] "Theory of Limit Design," by J. A. Van den Broek, John Wiley & Sons, Inc., New York, N. Y., 1948, p. 85, Fig. 51.

The tentative conclusion to be drawn from all this is that the theory of limit design may be used for the analysis and design of rigid trusses of any reasonable degree of slenderness ratios with the ensuing results in close agreement with those obtained by the more laborious, but mathematically more rigorous, method of this paper. From the designer's point of view, the paper should serve as another confirmation of the validity and universality of the theory of limit design. It should be mentioned that the authors' use of the ultimate load for design purposes rather than the allowable stress concept shows again how rapidly this newer and infinitely more logical procedure is taking hold. Let it be hoped that, in time, the building codes will recognize it at least as an alternate method of design.

The statement (second paragraph beyond Eq. 2)—

> "* * * the abutting members, instead of creating an end moment on the compression member as it starts to buckle, may have moments presented to them by the buckling member"

—seems to state the same thing twice—action and reaction, etc. Actually, the statement could be clarified to explain that the abutting members, instead of impressing a moment on the compression members of one sense before buckling occurs, which tends to buckle the column, will impress a moment of opposite sense on the column after it buckles, which tends to reduce the degree of buckling. This is the same phenomenon which occurs as reversed eccentricity when testing a flat-ended eccentric column.

In the "Synopsis," the authors make a statement which seems to persist in any discussion about columns. It represents a confusion of physical effect that even the Column Research Council has been guilty of on past occasions. To state that "* * * end restraints decrease and that the effective lengths increase as loads are increased * * *" is simply to state the same physical effect in two ways. The stiffness of the column may be determined by using a fixed value of end restraint, say, $S = 0$, and considering the effective length, L', to vary; or the effective length may be assumed constant, say, $L' = L$, and the degree of end restraint may be considered to vary. The stiffness effect in the writer's example was provided for by assuming $S = 0$ and $K = 0.75$. This was done to use the results of tests on pinned-end columns in the selection of buckling unit loads. It certainly represents a needless complication to consider both the variation in K and S since they are directly related;[42] thus:

$$K = L' = \frac{2}{3}\left(\frac{S + 3\,I/L}{S + 2\,I/L}\right) \qquad (26)$$

The writer should like to congratulate the authors on their clarity of presentation, and hopes that in the closing discussion there will be an example of the type mentioned at the beginning of this discussion.

Joseph S. Newell.[43]—The authors have summarized, clearly and concisely, the state of the art of stability determination as it is currently applied to trusses. Their excellent paper leads the reader to the conclusion that stability analyses of frames or trusses involve computations which, although not difficult, are tedious. It certainly indicates that further research is desirable to develop simpler methods of evaluating stiffness coefficients for various members or various joints. Table 2, which is used in determining such coefficients for trial loads 1.0119 times those used in the basic example,[13] is by no means short, yet it omits thirteen of the nineteen cycles actually made. Even so, it yields little definite information beyond the fact that the criterion shows the structure under investigation to be stable at this load and, hence, that the computations must be repeated for at least one increase in load if the limit of stability is to be determined exactly.

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[42] "Interrelation of Certain Structural Concepts," by Camillo Weiss, *Transactions*, ASCE, Vol. 111, 1946, p. 391.

[43] Prof. of Aeronautical Structural Eng., Massachusetts Inst. of Technology, Cambridge, Mass.

It is no reflection on the merit of the paper that so much work produces so little actual information. It is, rather, an indication of the present state of the art, an indication that great rewards may be had from studies that will enable analysts to approximate critical loads or stiffness coefficients by less tedious methods.

Perhaps the conclusion—that for compression members in steel trusses with the usual range of slenderness ratios the buckling stress coincides practically with the yield point of the steel regardless of end restraints—will eliminate the need for stability studies for many bridge and building structures. It will not do so in aircraft design, however, or in any structure that includes members subject to local buckling at stresses below the yield point of the material. These constitute an important fraction of the structures subject to stability investigations.

What should be regarded as the yield point? J. A. Van den Broek,[44] M. ASCE, has shown test results which indicate the peak of the curve before the "drop-of-the-beam" value to be the desired yield. Many engineers may accept this value, or the drop-of-the-beam value for mild steels, but what is to be used for alloy steels or aluminum alloys whose yield stresses are established arbitrarily?

Theorems 1, 2, and 3 cannot be stated with sufficient emphasis. They present points which many engineers have never considered, and few have analyzed to the extent that they clearly understand the action of a truss when it reaches a condition of instability. Whether the "series" or the "stiffness" criterion be used, it is important to realize that in a truss which has become unstable the reserve strength and stiffness in every member and joint have been exhausted, the unstable members undergo deflections, and the unstable joints undergo rotations, which are not proportional to the loads on the structure. The authors are to be congratulated on the clarity of their discussion of these points.

Now that they have summarized the long and tedious procedures in current use, it is hoped that the authors, or others inspired by their work, can evolve approximate methods which, although they may not eliminate a final "series-criterion" analysis, will reduce the preliminary or trial-and-error computations through which the critical load is estimated. The current procedure is not difficult to understand, but it is far too tedious to be practicable for routine application to structures having a large number of members.

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[44] "Euler's Classic Paper 'On the Strength of Columns,' " by J. A. Van den Broek, *American Journal of Physics*, July–August, 1947, p. 309.

Charles W. Dohn,[45] **Jun. ASCE.**—It is not entirely clear to the writer as to what constitutes "primary instability" of trusses and he would appreciate a "word picture" illustration or further explanatory examples. When a truss fails by primary instability do the individual members buckle and tear, and why should this be any different from what is commonly defined by the terms "ultimate load" and "final failure?" Are primary instability and critical load to be thought of as precise terms defining or predicting the exact loading point at which the compression members of a truss will buckle (taking into account relaxation in end restraint with increase in load); and are these terms applicable only where the effect of joint translation is small?

Compression members for trusses are commonly designed more or less as free-ended columns according to empirical formulas. This supposedly proportions them to carry a definite load with reasonably known safety. When a truss fails as a result of overload, the exact point and method of failure ordinarily would be impossible to predict, in most cases. However it seems safe to assume that approximately the same factor of safety that applied to the design of the individual members would apply to the truss as a whole. The authors' investigations verify this assumption by a wide margin for trusses in which the effect of joint translation is small and compression members buckle as their end restraints become less and less with increasing load. The authors state that they have found that the limiting load coincides practically with the yield point of the steel regardless of end restraints. Cannot truss compression members be thus designed for a uniform allowable stress of the same value as that used for tension members, plus a minimum l/r, and thereby dispense with column formulas for truss compression member design? It should be borne in mind that compression members cannot be reasonably designed to have the same ultimate strength as tension members in any case. Such methods of design might be limited to trusses with small joint translations. How may designers simply determine when a truss has joint translations small enough to include such a method of design? Could a ratio of truss deflection divided by truss span or truss depth over truss span, or some similar ratio, be set up to define such limits, and what would be some values for such a ratio for small, medium, large, and excessive joint translations? Could such methods of design be practicable?

In discussing theorem 3, the authors state: "* * * the method is applicable only to trusses, where the effect of joint translation is small"; and, in discussing theorem 2: "* * * secondary or localized failure is not being discussed. Actually, such failure would generally occur before the truss failed as a whole." It would appear that joint translation and secondary stresses are likely to be effective factors in a truss with rigid connections and reserve "elements of strength associated with bending resistance." Failure of connections would generally precede buckling of compression members. This can be deduced by logic. Before a compression member will fail by buckling, it must deflect more than the full elastic range. In approaching this condition the joint displacements will be such that the elastic range of the joint material will be exceeded and the joint will yield. The joint thus functions in a somewhat unpredictable manner, following the magnitude and direction of the secondary stresses, and in the manner of framing compression members. Although it might be argued that such yielding relieves the secondary stresses, nevertheless, it places the truss one step closer to complete failure inasmuch as the restraining effects of the joints might be changed and the compression members are likely to be more easily subject to failure by buckling. Removal of the restraining effects of joints by secondary stress failure will involve joint rotation; and, since the direction which the buckle assumes is immaterial, it will naturally form in the direction of the greatest rotation caused by the secondary stress failure. Thus, the Euler formula factor, *k*, might even be assumed to have a value greater than 2—if indeed this factor might still be considered applicable. The factor of safety of the truss as a whole may, therefore, be somewhat less than the factor of safety for individual members in the case of a truss with large joint translations. This suggests that trusses might be grouped into three classifications: (1) Trusses with small joint translations which may be designed for uniform allowable stress; (2) trusses with medium joint translations which may be designed with the aid of ordinary column formulas; and (3) trusses with large joint translations which may be designed as in class (2), but with an increased safety factor varying with joint translation. On the other hand, all compression members of a truss might be designed based on a given allowable stress with a deducting factor to vary directly with joint translation or truss deflection divided by truss span, or truss depth over truss span, including perhaps a factor based on maximum, minimum, or average lengths of members. A fairly low minimum slenderness ratio would be specified.

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[45] Engr., New York, N. Y.

As a result of further considerations it might be possible to vary stability and obtain a wider range of elastic action preceding failure up to the limits defined by primary instability by using different materials. For instance, joint material might be made of aluminum, with compression members of steel. Such joints might be simple aluminum blocks of standard dimension with an insulative cover, and with holes for bolts or rivets. In an aluminum truss, short steel cover plates might be added to compression members at points of critical buckling. Trusses so built and designed might serve to be doubly useful as stiffening trusses for suspension bridges where the dampening value of a truss built of materials with different physical properties can be utilized to counteract aerodynamic instability.

Can continuous trusses and a continuous tied arch truss, such as those used in the Dubuque (Iowa) crossing[46] of the Mississippi River, be properly considered in the category of steel trusses with the usual range of slenderness ratios? Could not rotation resulting from secondary stress failure be added to changes in end restraint, with increase in load to cause buckling of extremely long members carrying heavy loads? Such buckling might occur at the supports of continuous trusses, particularly in trusses that deflect considerably.

The writer would be most happy to go on designing usual trusses with usual slenderness ratios and usual allowances for secondary stress. He is most grateful to the authors for giving pertinent factors concerning compression members and buckling loads as related to end restraints on truss members.

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[46] Discussion by Charles W. Dohn of "Mississippi River Bridge at Dubuque, Iowa," by R. N. Bergendoff and Josef Sorkin, *Transactions*, ASCE, Vol. 114, 1949, p. 1299.

He feels that most designers might find primary instability of academic interest but difficult to study and to apply to actual truss design problems unless the concept leads to definite recommendations in the form of design specifications.

HAROLD E. WESSMAN,[47] M. ASCE, AND THOMAS C. KAVANAGH,[48] ASSOC. M. ASCE.—The contributions of the discussers are appreciated. The writers are particularly indebted to Mr. Newell for focusing attention on theorems 1, 2, and 3. From a philosophical viewpoint, these theorems are extremely important. Mr. Winter, on the other hand, overlooked this and devoted his primary attention to advancing the argument that the end restraint method developed by P. T. Hsu[17] is superior to the extension of the Lundquist-Hoff method presented in the paper. The writers do not accept this argument, and will refute it presently; but, lest the general reader be led astray by a discussion of techniques, the writers will state at this point their frank opinion that there is no advantage in using either one of the methods as presented. Quoting from Mr. Newell's discussion, both are: "* * * far too tedious to be practicable for routine application to structures having a large number of members."

The paper high lights the need for a simple approximate method for determining the stability limit for trusses. Mr. Slavin has presented one method of simplifying the solution. It involves the use of groups of members into which the truss may be resolved. These groups, in which external ends of boundary members twice removed from a reference joint are fixed, are quite satisfactory for determining the buckling load within close limits. However, this approximate solution is not enough of a simplification for design office practice. Further simplifications, such as those presented by one of the writers (Mr. Kavanagh) in his research thesis[18] presented to New York University, in New York, N. Y., offer the most promise. The most important of these is the procedure which determines the critical loading from one trial load, which the writers feel goes a long way toward meeting the objectives set up by Mr. Newell. This procedure will be outlined briefly.

Prior to any actual analysis, the limiting range of the buckling load should be established. The following theorem[18] will prove helpful:

> Theorem 4. For a rigid-jointed truss framework, in the range of loading between that multiple of load which first produces $L/j = \pi$ in any compression member, and that which first produces $L/j = 2\pi$ in any compression member, there can be at most one critical load.

Accordingly, once the range of loading limits is known (and it is easily computed), the trial-and-error procedure which characterizes the stability analysis can begin at any loading in this range without fear of encountering higher buckling loads or missing the lowest buckling load. (The exception to this theorem is the pin-ended strut, which has a buckling mode at $L/j = \pi$ and another at $L/j = 2\pi$, but this case is so easily recognized that it does not impair the general applicability of theorem 4.) This range of possible limits is often so very narrow as to make any further formal analysis unnecessary.

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[48] Prof. of Civ. Eng., Pennsylvania State College, State College, Pa.

Of equal importance, the preliminary survey undertaken to set the range of buckling values gives, at once, an indication of the "weakest member" as well as of the "weakest joint." It also identifies those compression members which require restraint (so-called "buckling" members) and those compression members which actually have a reserve of strength to offer to "weaker" members. At the end of his formal analysis, Mr. Winter discusses facts which most analysts recognize in the basic procedure of setting up any problem.

An analysis of a great number of different types of practical trusses has indicated that satisfactory approximations can be attained by considering groups of members, rather than the truss as a whole. From a practical point of view, however, the time required to analyze a truss, by groups extending more than one joint from any reference member or joint, is usually excessive. Of all the group patterns studied, that group based on fixed (or pinned) joints once removed from the ends of each "buckling" compression member has been found to give the most satisfactory approximation, and at the same time to offer a close approach to the physical action of a restrained member, which would be desired from a design viewpoint. The use of fixed terminals is not essential to this procedure, but it has been found entirely satisfactory, even though slightly on the unsafe side, and has the added advantage of eliminating the need for one set of tabular stiffnesses (S''). Pin ends can be employed if desired.

A noticeable disadvantage of this grouping lies in the fact that all adjacent members are considered as restraining members, even though some might themselves be distressed compressive members (buckling members) incapable of offering restraint. An improvement in accuracy in all cases is afforded by a modification of the grouping by more closely estimating the true restraints. One selects each buckling compression member ($L/j > \pi$) and considers it restrained at each end by a proportion of the stiffness of all adjacent members (far ends fixed), excluding other adjacent buckling members. The total stiffness of the restraints is thus divided sensibly among those distressed members actually requiring support, in line with the true physical action of the structure. As an example, the airplane truss would be divided into groups centering on the buckling members AB, BC, and CD, as shown in Fig. 11.

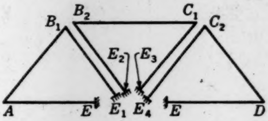

FIG. 11

Although the proportionment (for example, of the stiffness of member BE between buckling members AB and BC) by a formula may be fairly accurate, it has been found that any reasonable proportionment may be assumed. The error in the proportionment will be evident from the presence of different r-values when a unit moment is applied at joint B in the group AB_1E_1, and another at joint B in the group $E_2B_2C_1E_3$. This error may be readily corrected, for it has been observed that the geometric mean of all the r-values at a joint is usually close to the true r-value for the joint in the truss as a whole. This observation reaffirms an observation made by the French investigator, Henri

Rivière,[49] who used the geometric mean successfully in a stability problem of slightly different character.

The importance of the foregoing method of grouping lies in the fact that the restraints on any buckling compressive member for a trial loading in the range of possible values determined by theorem 4 are primarily tension members whose stiffness changes very slowly and which may be considered constant within the range of analysis. Accordingly, it is immediately possible to give an approximate value of the true buckling load of the truss, once the results (or even some preliminary data) are available from one trial load. It is immaterial which technique is employed to jump from the trial load to the estimated critical load; the transition can be accomplished by the Zimmermann formula (Eq. 13*b*), or by a modification of the Lundquist procedure, or by other methods. The important fact is that the particular grouping allows for a substantially constant set of restraints on each compression member within the range of analysis defined by theorem 4, and for the determination of an approximate critical load within 1% or 2% of the true value.

For example, applying the data for trial load factor 1.0119 to the grouping of Fig. 11, and arbitrarily dividing the stiffness of member BE equally between members AB and BC, one obtains the calculation in Table 8 from simply using the Zimmermann formula (Eq. 13*b*) or the chart of Fig. 4. The inequality of the load factors in Table 8 indicates the error in the assumed proportionment of the stiffness of member BE. Obviously, member BC requires more restraint than member AB, and a more judicious proportionment might have been assumed, particularly since it is known in advance that BC is the weakest member.

TABLE 8.—ANALYSIS OF FIG. 11 USING DATA FOR ONE LOAD FACTOR (1.0119)

Member (Fig. 11)	Total stiffness, s	$\frac{E' I}{4 L}$	Restraint factor, k	Critical load, P_{cr} (lb)	Load factor	Geometric mean
AB—						
s_A........	13,519.1	0.1903	0.685	3,607	1.145	1.028
s_B........	7,971.8	0.3226				
BC—						
s_B, s_C.......	7,971.8	0.4061	0.745	3,198	0.923	

The estimated critical load factor is $\sqrt{1.145 \times 0.923} = 1.028$—only 0.3% less than the true value.

Up to this date, the method requiring the least number of trial loads was the Lundquist adaptation of the Southwell procedure[15] for determining critical loads in columns from test data. This technique required a minimum of three trial loads and, in all problems worked by the writers, proved far too time consuming.

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[49] "Calcul des Poutres Comprimées Encastrées Elastiquement à leurs Extremités," by Henri Rivière, *L'Aerotechnik* (supplement to *L'Aeronautique*), No. 200, January, 1936, pp. 7-12.

Previously in this closing discussion, the writers have emphasized the philosophical significance of theorems 1, 2, and 3. From a practical viewpoint, however, the most important conclusion is presented in the statement in the paper (three paragraphs beyond theorem 3) that:

> "* * * additional investigation has demonstrated clearly that the buckling loads for steel building and bridge trusses, with members having slenderness ratios in accord with current design practice, are so close to the loads corresponding to the yield stress of the steel employed, that there is absolutely no need for buckling load analyses."

In this connection, Mr. Winter states that: "Some skepticism concerning this contention seems justified." To illustrate his point, he uses a roof truss with slenderness ratios varying from 145 to 175. Such a truss is decidedly not typical of current design practice. The main compression chords in most bridge trusses have slenderness ratios under 75. The slenderness ratios of main compression members in roof trusses may be a little higher, but it is most unusual to find chords in roof trusses with slenderness ratios greater than 100.

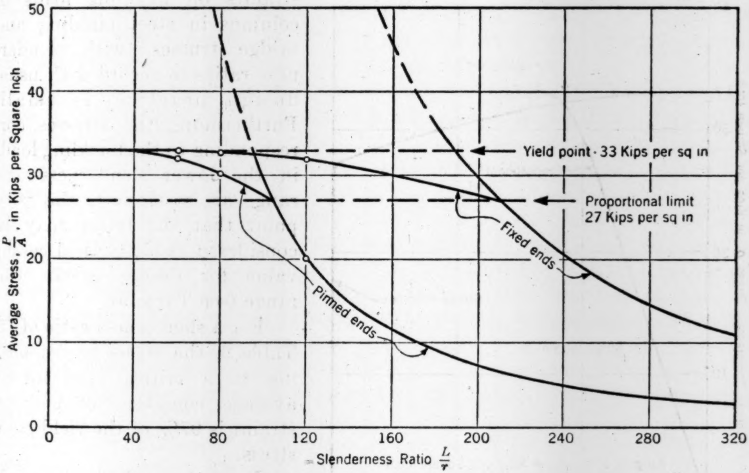

FIG. 12.—COLUMN CURVES FOR A-7 STEEL

Compression diagonals near the centers of trusses often have slenderness ratios in the range of from 100 to 120, but these diagonals seldom dominate the stability of the truss. In most cases, their working stresses for full-span loadings are relatively low. They are governed by part-span live loads which do not result in maximum stresses in adjacent chord members.

Fig. 12 demonstrates why the buckling load is so close to the yield strength for most steel members. This figure gives the theoretical buckling curves for ideal straight columns of an A-7 steel, plotted from the Euler formula (Eq. 2) in which E', the effective modulus, is the tangent modulus:

$$\frac{P}{A} = \frac{\pi^2 E'}{(k\,L/r)^2} \tag{27}$$

The tangent modulus values are shown in the curves in Fig. 13 for a stress-strain diagram which is based on assumed values for the proportional limit of 27 kips per sq in., at a strain of 0.0009 in. per in., and a yield point of 33 kips per sq in. (the minimum specified for A-7 steel) at a strain of 0.0013 in. per in., with a parabolic variation between these points. The modulus shown in Fig. 13 is the tangent modulus.

One of the curves (Fig. 12) is plotted for a column with pinned ends; the other, for a column with fixed ends. Obviously, for a column with an intermediate degree of restraint, the column curve will be represented by a line lying somewhere between the two limiting curves shown. For slenderness ratios less than 100, all three curves are close to one another and also close to the yield point value. This illustrates graphically that the effect of end restraints on buckling loads of columns in steel building and bridge trusses, with slenderness ratios in accord with usual design practice, is small. Furthermore, the stresses corresponding to the buckling loads in the lower slenderness-ratio range are so close to the yield point that the latter may be considered as a practical ceiling value for design within this range (see Table 9).

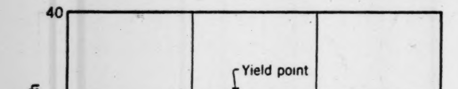

FIG. 13.—STRESS-STRAIN AND STRESS-MODULUS CURVES FOR A STRUCTURAL CARBON STEEL (YIELD POINT, 33 KIPS PER SQ IN.)

For a slenderness ratio of 75, Table 9, the stress corresponding to a critical load for an average condition of end restraint is 97% of the yield point stress.

It is true that the yield point value is slightly on the unsafe side. On the other hand, although the specified yield point for A-7 steel is 33,000 lb per sq in., the actual yield point in many steel samples generally exceeds 33,000. Working stresses are based on the minimum value.

The writers believe that it is academic to make involved stability analyses for steel trusses of structural carbon steel. The same conclusion holds for most silicon steel and low alloy steel trusses. However, as Mr. Newell notes in his discussion, this conclusion is not applicable for the type of trusses encountered in aircraft design using aluminum alloys whose yield stresses are established arbitrarily.

Mr. Benjamin states that: "Two factors are neglected by the authors—influence of secondary stress moments and moments caused by initial curvature." The discussion does not indicate clearly what is meant by the "influence of secondary stress moments." The moments associated with end restraints are those which are created by rigid connections. They are the moments that cause secondary stresses. It is doubtful if secondary moments are of much significance in members with such a large slenderness ratio that failure would be caused by primary instability rather than local yielding due to localized stress intensity. John I. Parcel, M. ASCE, and Eldred B. Murer,[50] Assoc. M. ASCE, indicate that: "* * * the ultimate strength is practically unaffected, even by high secondary stresses * * *."

TABLE 9.—CRITICAL LOAD RATIOS

Stress ratios	SLENDERNESS RATIOS			
	50	75	100	125
$\dfrac{P_{cr}\text{ (for fixed ends)}}{P_{cr}\text{ (for pinned ends)}}$	1.01	1.06	1.17	1.67
$\dfrac{\text{Average } P_{cr}/A}{P/A\text{ at yield point}}$	0.99	0.97	0.91	0.77

Mr. Fitzgerald notes his belief that the worth of the paper lies in the mathematical confirmation of the theory of limit design. As noted by the writers (first paragraph following Eq. 16): "* * * the phenomenon of truss buckling is a beautiful illustration to support the theory of limit design." Mr. Fitzgerald furthermore presents the view (which is quite opposite from that presented by Mr. Winter) that there is no need of a buckling analysis for trusses with a slenderness ratio greater than those commonly encountered in practice. Mr. Fitzgerald's conclusion evidently was based on the fact that the usual design assumption of an average value of k (namely, 0.75) was not far from the true k-values obtained in the analysis. The writers wish to emphasize, however, that this would not have been true had the tension members been designed, say, as flat bars capable of offering little flexural restraint.

Concerning the arguments advanced by Mr. Winter in favor of the Hsu method of calculating end restraints by iteration, as compared with the procedure using moment distribution, it can be shown in general[18] that any method of indeterminate analysis may be employed for stability analysis, provided the effect of axial loads is introduced. The moment distribution concepts, already widely known and used, require no new theory, no modification of physical ideas, no special graphs, formulas, or charts or other aids for their use in stability analyses. (Tables of modified stiffness and carry-over factors are now standard tools for aeronautical engineers in beam-column work, and even civil engineering texts are beginning to include these data.) On the other hand, many methods of indeterminate analysis based on "fixation factors," "degree of fixity," "degree of restraint," etc., have been advanced in the past;[9,51,52,53,54,55,56] yet it is doubtful if these can be said to have gained widespread usage.

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[50] "Effect of Secondary Stresses Upon Ultimate Strength," by John I. Parcel and Eldred B. Murer, *Transactions*, ASCE, Vol. 101, 1936, p. 289.

The compactness, neatness, and, above all, the self-contained nature of the moment distribution process are factors which, in the writers' opinion, make this seeming "detour" a far smoother road to travel. Once the moment distribution has been set up (and all methods require about the same amount of preliminary setup work), the computation flows by simple arithmetical methods to a solution, without distraction from auxiliary equations or graphical aids. The fact that the entire method centers around a single all-inclusive factor, r, upon which one focuses his entire attention during the course of the analysis, makes this technique a far more unified and powerful tool.

Although the two methods under discussion appear to employ different techniques, they are fundamentally very closely related, and any investigator faced with the need for extended analyses would be amply repaid by a study of the method which Mr. Winter and his associates have so thoroughly explored. It is not correct to conclude that a moment distribution solution for one trial loading gives only the lower limit of the critical load. Aside from the fact that the need for bracketing the critical load is eliminated by the approximation procedure previously described (which yields the critical load immediately), bracketing can be accomplished by moment distribution in a number of ways, thus—

(*a*) Apply a unit moment at the end of the "weakest" member (usually the "weakest" joint), and from the r-values, determine restraints and upper and lower limits of P_{cr} by a procedure analogous to that indicated by Mr. Winter; or

(*b*) Using a purely moment distribution approach, "force" the total stiffness ($\Sigma S'$) of the weakest joint to zero by allowing only the "weakest" member to vary. Thus, in the airplane truss, using the data previously calculated for $\Sigma S'_b$, one may write immediately: $S_{bc} - 0.7192\ (17{,}185.6 + S_{bc}) = -12{,}163.3$. Consequently: $S_{bc} = 665$; $\frac{S\,L}{E'\,I} = 0.0513$; and $L/j = 4.562$ for member BC.

This procedure gives the upper bound of the critical load factor as 1.10. As stated, however, these bracketing procedures may be dispensed with if the direct approximation procedure is utilized.

Mr. Dohn states that:

> "* * * most designers might find primary instability of academic interest but difficult to study and to apply to actual truss design problems unless the concept leads to definite recommendations in the form of design specifications."

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[51] "Statically Indeterminate Frameworks," by Thomas F. Hickerson, Univ. of North Carolina Press, Chapel Hill, N. C., 3d Ed., 1949.

[52] "Analysis of Continuous Frames by the Method of Restraining Stiffnesses," by Earle B. Russell, Ellison and Russell, San Francisco, Calif., 1934.

[53] "Continuous Beam Structures; a Degree of Fixity Method and the Method of Moment Distribution," by Eric Shepley, Concrete Publications, Ltd., London, 1942.

[54] "Rigid Frames," by L. T. Evans, Edwards Bros., Ann Arbor, Mich., 1938.

[55] "Neue Statik der Tragwerke aus Biegesteifen Stäben," by Max Mayer, Bauwelt, Berlin, 1937.

[56] "Allgemeine Theorie des Elastisch Eingespannten Balkens," by M. Ritter, *Publications*, International Assn. for Bridge and Structural Eng., Vol. II, 1933, p. 290.

It is hoped that the studies made by the writers, and those which are being made elsewhere under the egis of the Column Research Council, will eventually lead to definite recommendations which will be incorporated in design specifications and which will reflect, more scientifically, a knowledge of the behavior of compression members in the different types of structures in which they are used.

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AERODYNAMIC THEORY OF BRIDGE OSCILLATIONS

By D. B. Steinman,[1] M. ASCE

With Discussion by Messrs. David J. Peery, Alexander Klemin, Abraham Slavin, Charles W. Dohn, Boris A. Bakhmeteff, F. J. Maher and J. B. Eades, Edmund Pinney, Harold A. Thomas, Elmo G. Peterson, William Allan, Paul Lieber, S. K. Ghaswala, Peter L. Tea, Edward Adams Richardson, Hans H. Bleich, F. B. Farquharson, George S. Vincent, and D. B. Steinman.

Synopsis

General equations are derived, expressing the aerodynamic forces of lift and torque on any oscillating section of two independent degrees of freedom. The parameters representing the distinctive aerodynamic characteristics of the section and the coefficients representing the variation with wind velocity are derived from static wind-tunnel tests on straight and curved section models. The effects of vertical velocity, angular displacement, and angular velocity are included, as well as the effects of the angle of incidence. The complete expressions for the aerodynamic forces are equated to the corresponding expressions for the dynamic forces acting on the section, including elastic (and gravitational) restoring force, inertia, and damping. The resulting general equations yield solutions for all practical problems of vertical, torsional, and coupled oscillations, including analytical determination and prediction of frequencies, critical velocities, instability response (rate of amplification), and amplitude response.

Numerical examples are given, to illustrate the practical application of the derived relations and to confirm the validity of the underlying theory. Graphic methods are presented to facilitate application and visualization.

Introduction

A primary requisite for a comprehensive solution of all problems in aerodynamic oscillations is the writing of the general "aerodynamic" equations for the forces (L and M) acting on any section, together with the general "dynamic" equations for the same forces. Once these general equations are correctly set up, the rest is simple algebra—to determine frequencies, critical

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Note.—Published in October, 1949, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Cons. Engr., New York, N. Y.

velocities, amplification or damping, limiting (maximum or minimum) amplitudes, amplitude response curves, and other relations for vertical, angular, and coupled oscillations.

For the complete solution, the aerodynamic equations must include the effects of vertical velocity, angular position, and angular velocity of the section.

The dynamic equations must include the elastic (and gravitational) resistance to displacement, the inertia resistance to acceleration, and the damping force or resistance (in phase with the velocity).

Theoretical equations for the limiting case of a flat plate or streamline airfoil have been derived by Theodore Theodorsen,[2] R. Kassner and H. Fingado,[3] H. Glauert,[4,5,6] H. G. Küssner,[7] Herbert Wagner,[8] Theodor von Kármán, M. ASCE, and J. M. Burgers,[9] and others, using the methods of classical aerodynamic theory (based on potential flow and the application of the Kutta condition).

For the more general case of bridge sections and other nonstreamline sections for which the Kutta condition is not applicable, it appears impossible to write the aerodynamic equations by purely theoretical derivation. Some of the necessary coefficients must be supplied by experiment.

Consequently, for a practically feasible solution to cover all cases, it is necessary to deduce rationally the correct general form of the aerodynamic equations, with the unknown coefficients or parameters (representing the distinctive characteristics of different sections) to be supplied experimentally.

Since only the aerodynamic characteristics of the section need to be determined experimentally, full-scale models incorporating the dynamic characteristics of the complete structure are unnecessary; simple section models suffice. Moreover, to the extent that static wind-tunnel tests reveal the distinctive aerodynamic characteristics of a section, oscillating model tests are not required. The thesis that the distinctive aerodynamic characteristics of oscillating sections can be mathematically predicted from the known or measurable aerodynamic characteristics of stationary section models has now received cumulative and consistent confirmation.

Notation.—The letter symbols used in this paper follow customary notation as far as practicable.

Vertical vectors are measured positive upward; angular vectors are measured positive clockwise; wind velocity is to the right, and wind inclination is measured positive upward.

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[2] "General Theory of Aerodynamic Instability and the Mechanism of Flutter," by Theodore Theodorsen, *Technical Report No. 496*, National Advisory Committee for Aeronautics, Washington, D. C., 1935.

[3] "Das ebene Problem der Flügelschwingung," by R. Kassner and H. Fingado, *Luftfahrtforschung*, November 20, 1936, pp. 374–387. ("The Two-Dimensional Problem of Wing Vibration," by R. Kassner and H. Fingado, *Journal*, Royal Aeronautical Soc., Vol. 41, 1937, pp. 921–944.)

[4] "The Accelerated Motion of a Cylindrical Body Through a Fluid," by H. Glauert, *Reports and Memoranda No. 1215*, British Aeronautical Research Committee, 1928–1929, Vol. 1, January, 1929, pp. 118–127.

[5] "The Lift and Pitching Moment of an Aerofoil Due to a Uniform Angular Velocity of Pitch," by H. Glauert, *Reports and Memoranda No. 1216*, British Aeronautical Research Committee, 1928–1929, Vol. 2, January, 1929, pp. 636–644.

[6] "The Force and Moment on an Oscillating Aerofoil," by H. Glauert, *Reports and Memoranda No. 1242*, British Aeronautical Research Committee, 1929–1930, Vol. 2, March, 1929, pp. 742–758.

[7] "Zusammenfassender Bericht über den instationären Auftrieb von Flügeln," by H. G. Küssner, *Luftfahrtforschung*, December 20, 1936, pp. 410–424.

[8] "Ueber die Entstehung des dynamischen Auftriebes von Tragflügeln," by Herbert Wagner, *Zeitschrift für angewandte Mathematik und Mechanik*, February, 1925, pp. 17–36.

[9] "General Aerodynamic Theory—Perfect Fluids," by Theodor von Kármán and J. M. Burgers in "Aerodynamic Theory" (W. F. Durand, editor), Springer, Berlin, 1935, Vol. 2.

As used in this paper, b is the chord or width of the section (in feet), and the dynamic parameters, w, m, and K, are per full width.

Numeral subscripts distinguish corresponding parameters for vertical and torsional oscillations: ω_1, K_1, δ_1, g_1, s_1 (and s_3) for vertical oscillations; ω_2, K_2, δ_2, g_2, s_2 (and s_4) for torsional oscillations.

USE OF STATIONARY SECTION MODELS

Use of Curved Models.—By a principle of relativity, the motion of the section (regarded as an immersed boundary surface) may be translated into a relative tilt or distortion of the flow lines in the fluid, or into a contrary relative tilt or distortion of the boundary surface, so that in either case the boundary surface may then be treated as stationary.

For parallel vertical motion of the section with downward velocity v, the graphic representation (Fig. 1) consists of the simple composition of horizontal flow lines with spacing $1/V$ and relative (upward) vertical flow lines with spacing $1/v$. By this graphic composition, the horizontal flow lines in the fluid are simply tilted up to a positive angle of incidence v/V. Instead of tilting the horizontal flow lines upward, an equivalent procedure is to tilt the section downward the same angle v/V; this is also given (Fig. 1) by the graphic composition of the same superimposed flow lines with the direction of the vertical lines reversed. Accordingly:

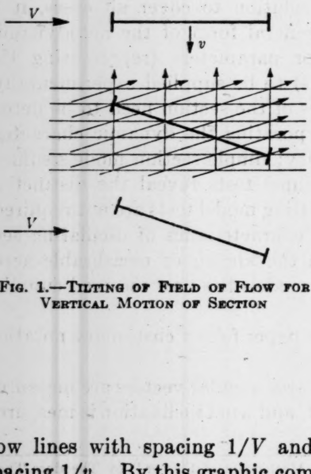

FIG. 1.—TILTING OF FIELD OF FLOW FOR VERTICAL MOTION OF SECTION

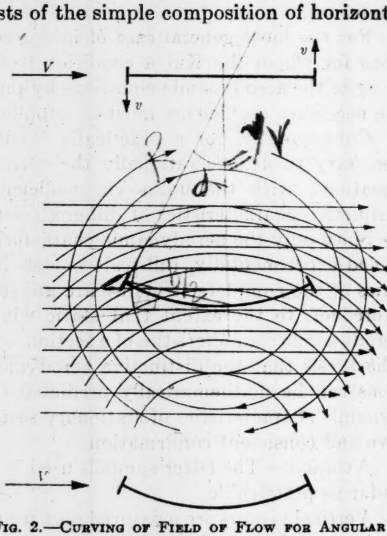

FIG. 2.—CURVING OF FIELD OF FLOW FOR ANGULAR MOTION OF SECTION

> Analogy 1.—The tilting of the stationary model in a wind tunnel takes the place of a vertical velocity of the section.

When the same procedure is applied to angular velocity of the section, the graphic representation (Fig. 2) consists of the simple composition of the horizontal flow lines with spacing $1/V$ and a series of concentric circular flow lines with graduated spacing $\frac{1}{2}\, b/(c\, v)$—in which v is the linear downward velocity at the leading edge and c is the distance of any flow filament from the center. This graphic composition yields a series of concentric circular flow lines, convex upward. The angular velocity of the section is thus translated into a curved field of flow, as in an arching wind tunnel, with the immersed section model kept stationary and unchanged. Instead of curving the field of flow, an equivalent procedure is to curve the immersed section model an equal and opposite amount. This curvature is also given (Fig. 2) by the graphic composition of the same superimposed flow lines with the direction of the rotational flow lines reversed. The equivalent warping of the immersed section model is symmetrical about the center of angular rotation, with horizontal tangent at this center and with slope change v/V at each end; the radius of curvature is $\frac{1}{2}\, b\, V/v$. Accordingly:

> Analogy 2.—The curvature of the stationary model in a wind tunnel takes the place of an angular velocity of the section.

With a curved section model, as in the more familiar tilted section model, the added angle of incidence at the leading edge remains $-v/V$.

For wind-tunnel duplication of this condition, the section model must be warped to the circular (or parabolic) curvature indicated above, with horizontal tangent at the center of rotation and the curvature changed for different desired values of the slope β at the leading edge (angle of incidence at the leading edge equivalent to angular velocity of section). This curvature is called the "dynamic camber."

The first wind-tunnel tests on curved models of bridge sections were made in 1947 at the Virginia Polytechnic Institute at Blacksburg by F. J. Maher, in cooperation with the writer. The results yield the needed additional data for the analysis of torsional instability.

In the absence of such tests on curved models, the analysis for the effect of angular velocity has previously been based on the static torque graphs obtained from straight section models (simply tilted instead of curved). Qualitatively, the results and conclusions should be substantially unchanged, since the major disturbance of the incident stream flow is determined by the effective angle of incidence at the leading girder. Quantitatively, the indicated correction from straight section models to curved section models is in the direction of reducing stability torque or intensifying instability torque.

It should be noted that the straight, tilted section model has a dual significance. It represents the effect of angular position, α, also the identical effect of vertical velocity, v. Accordingly, in aerodynamic formulas, the two are interchangeable; the conversion relation is simply $\alpha = -v/V$. The two contributions may be bracketed, as $(\alpha - v/V)$, with the same factors and coefficients applied to both.

The curved section model represents solely the effect of angular velocity, $\dot{\alpha}$. The conversion relation is $\beta = -v/V = -b\,\dot{\alpha}/(2\,V)$.

For ordinary values of β, it is sufficiently accurate to use a circular or parabolic arc for the curvature of the model. For higher values of β, the curve is more accurately a catenary, to keep the elements of the section width unchanged in length. For the limiting case of $V = 0$, yielding $\beta = -90°$, the catenary becomes a folded straight line (zero span, and sag $= b/2$).

Curvature of the section model in static wind-tunnel tests, as the equivalent of angular velocity of the section, is (for this application) a new concept introduced by the writer. A search of aerodynamic literature reveals that a similar idea has been applied in the study of curvilinear flight of airships.[10] The same idea, but intended to be applied analytically instead of experimentally, is also briefly suggested by Messrs. Kassner and Fingado,[3] who state that one of the terms of the lift force due to angular velocity of an airfoil "could be deduced by replacing the rotating airfoil by a correspondingly cambered airfoil." This relationship is also confirmed by comparing the respective aerodynamic formulas.[9] A simple analytic derivation yields a parabola for the dynamic camber.

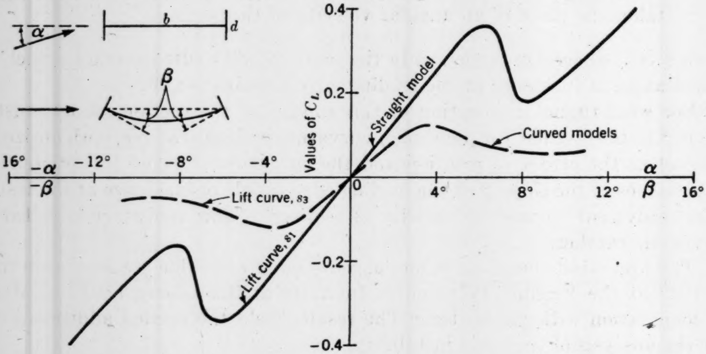

FIG. 3.—STATIC LIFT GRAPHS FOR STRAIGHT AND CURVED MODELS OF AN H-SECTION ($d/b = 0.20$)

Lift and Torque Graphs from Curved Models.—Fig. 3 shows the static lift graphs obtained from straight and curved section models, respectively, for an H-section of section ratio $d/b = 0.20$ (approximately the section of the original Tacoma (Wash.) bridge). The respective slopes are $s_1 = +4.00$ for the straight model (corrected for drag) and $s_3 = +2.00$ (approximately) for the curved model. Accordingly, for this section, the vertical lift resultant produced by a given β (at the leading edge) due to angular velocity is approximately one half the vertical lift resultant produced by an equal α due to vertical velocity or to angular position ($q_3 = s_3/s_1 = 0.50$).

Fig. 4 shows the static moment graphs obtained from the same straight and curved section models. The respective slopes are $s_2 = -0.50$ for the straight model and $s_4 = -1.00$ for the curved model (corrected for drag). Accordingly, for this section, the aerodynamic instability torque produced by β (at the leading edge) is twice the aerodynamic instability torque produced by an equal α due to vertical velocity or to angular position ($q_4 = s_4/s_2 = 2.0$).

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[10] "Method of Curved Models and Its Application to the Study of Curvilinear Flight of Airships," by G. A. Gourjienko, *Technical Memorandum No. 829* (Pt. 1) and *Technical Memorandum No. 830* (Pt. 2), National Advisory Committee for Aeronautics, Washington, D. C., 1937.

Since the lift is halved and the torque is doubled, the eccentricity of the resultant (or center of pressure) on the curved model is four times as great as on the straight model, the respective values for this section being $e_4 = s_4/s_3 = -0.50$ (placing the resultant at the leeward edge of the section) and $e_2 = s_2/s_1 = -0.125$. Accordingly, the ratio M/L for angular velocity is not the same as for vertical velocity or angular position. The resultant center of pressure does not remain stationary (along the section) during a cycle of angular oscillation, nor during a cycle of coupled vertical and angular oscillations.

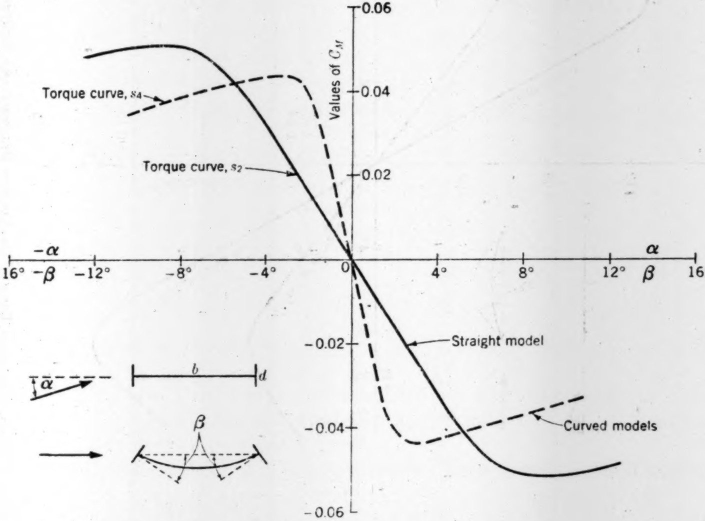

FIG. 4.—STATIC TORQUE GRAPHS FOR STRAIGHT AND CURVED MODELS OF AN H-SECTION ($d/b = 0.20$)

As shown by the positive value of s_1 and the negative values of s_2 and s_4, the section $d/b = 0.20$ is vertically stable and torsionally unstable. For such sections, the curved model yields increased negative slope of torque graph ($q_4 > 1$), representing intensified torsional instability.

TABLE 1.—GRAPH SLOPES FROM STRAIGHT AND CURVED SECTION MODELS

Section (d/b)	s_1	s_2	s_3	s_4	e_2	e_4	q_3	q_4
0 (theoretical)........	6.28	1.57	−3.14	0	0.25	0	−0.50	0
0.02 (test)...........	5.25	1.10	−1.75	0.55	0.21	−0.31	−0.33	0.50
0.20 (test)...........	4.00	−0.50	2.00	−1.00	−0.125	−0.50	0.50	2.00

For torsionally stable sections, as typified by a flat plate, the curved model yields a reversed slope of lift graph (Fig. 5) and a reduced slope of torque graph (Fig. 6). For comparison, the slopes and their respective ratios, for $d/b = 0$ (ideal flat plate), $d/b = 0.02$ (flat plate tested), and $d/b = 0.20$, are collated in Table 1.

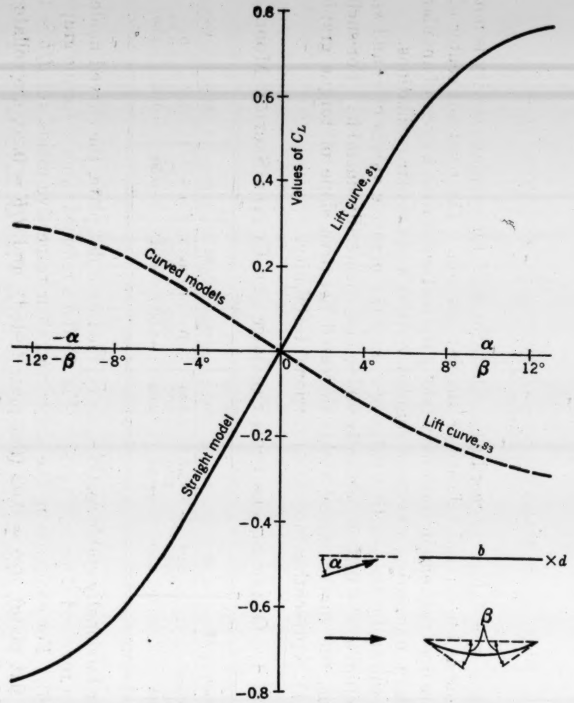

FIG. 5.—STATIC LIFT GRAPHS FOR STRAIGHT AND CURVED MODELS OF A FLAT PLATE ($d/b = 0.02$)

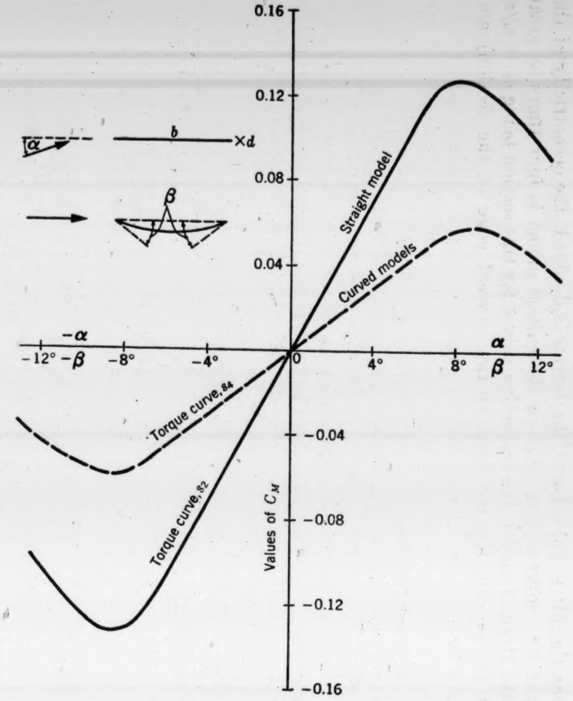

FIG. 6.—STATIC TORQUE GRAPHS FOR STRAIGHT AND CURVED MODELS OF A FLAT PLATE ($d/b = 0.02$)

GENERAL AERODYNAMIC EQUATIONS

For Oscillating Sections.—For the derivation of the following aerodynamic equations, let:

α_i = initial angle of incidence (for the section at rest);
L_i = initial static lift at α_i;
M_i = initial static torque at α_i;
η = vertical displacement;
$\dot{\eta} = d\eta/dt$ = vertical velocity;
$\ddot{\eta} = d^2\eta/dt^2$ = vertical acceleration;
α = angular displacement;
$\dot{\alpha} = d\alpha/dt$ = angular velocity; and
$\ddot{\alpha} = d^2\alpha/dt^2$ = angular acceleration.

Then the general aerodynamic force equations for a moving or oscillating section are

$$L = L_i + L(\alpha) + L(\dot{\eta}) + L(\dot{\alpha}) + L(\ddot{\eta}) + L(\ddot{\alpha}) \qquad (1a)$$

and

$$M = M_i + M(\alpha) + M(\dot{\eta}) + M(\dot{\alpha}) + M(\ddot{\eta}) + M(\ddot{\alpha}) \qquad (1b)$$

in which $L(\alpha)$, $L(\dot{\eta})$, ..., denote the contributions to L due to α, $\dot{\eta}$, ...; and $M(\alpha)$, $M(\dot{\eta})$, ... denote the corresponding contributions to M. All moments are taken about the midpoint of the width b.

The angular displacement α is measured from the initial angle of incidence α_i. The slopes s_1 and s_2 are measured at α_i (for small displacements).

For vertically symmetrical sections, as a flat plate or an H-section, in horizontal wind, $\alpha_i = 0$; $L_i = 0$; and $M_i = 0$.

The ordinates of the static lift and torque graphs are C_L and C_M, respectively, defined by: $L = (\frac{1}{2}\,\rho\, V^2\, b)\, C_L$; and $M = (\frac{1}{2}\,\rho\, V^2\, b^2)\, C_M$.

The slopes of the static lift and torque graphs are as follows:

s_1 = slope of static lift graph, straight model = $\partial C_L/\partial\alpha$
s_2 = slope of static torque graph, straight model = $\partial C_M/\partial\alpha$
s_3 = slope of static lift graph, curved model = $\partial C_L/\partial\beta$
s_4 = slope of static torque graph, curved model = $\partial C_M/\partial\beta$

The contributions due to angular displacement α will then be

$$C_L = s_1\,\alpha \qquad \text{and} \qquad C_M = s_2\,\alpha$$

Similarly, the contributions due to vertical velocity $\dot{\eta}$ will be

$$C_L = s_1\,(-\,\dot{\eta}/V) \qquad \text{and} \qquad C_M = s_2\,(-\,\dot{\eta}/V)$$

since α and $(-\,\dot{\eta}/V)$ are equivalent in aerodynamic effect.

Similarly, from the slopes s_3 and s_4 of the static lift and torque graphs obtained from the curved model, the contributions due to angular velocity $\dot{\alpha}$ will be

$$C_L = s_3\,\beta \qquad \text{and} \qquad C_M = s_4\,\beta$$

in which $\beta = -\frac{1}{2} b\, \dot{\alpha}/V$. For this case, however, one correction is required. Referring to Fig. 2, the curved model, held stationary in horizontal wind, is equivalent to a straight model held stationary in an opposite curving airflow, or to a straight model moving in an opposite curvilinear path in a straight airflow. In each case the radius of curvature is $R = V/\dot{\alpha}$. This introduces a radial acceleration or centrifugal force involving the mass m_a of the cylinder of air enveloping the section. This radial acceleration is $V^2/R = V\,\dot{\alpha}$, and the resulting inertia force is $-m_a V \dot{\alpha}$. Since this is an extra force created by the curvilinear flow or motion represented by the curved model, it must be subtracted (algebraically) from the lift L given by the curved model. It does not affect the moment $M(\dot{\alpha})$, since this lift correction is centered at the midpoint $(b/2)$. Consequently the sole effect of this correction for centrifugal force is to augment $L(\dot{\alpha})$ by $\Delta L\,(\dot{\alpha}) = m_a\, V\, \dot{\alpha}$.

The contributions due to vertical acceleration $\ddot{\eta}$ are

$$L(\ddot{\eta}) = -m_a\,\ddot{\eta} \qquad \text{and} \qquad M(\ddot{\eta}) = 0$$

The contributions due to angular acceleration $\ddot{\alpha}$ are

$$L(\ddot{\alpha}) = 0 \qquad \text{and} \qquad M(\ddot{\alpha}) = -I_a\,\ddot{\alpha}$$

in which I_a is the effective polar moment of inertia of the enveloping cylinder of air.

Collecting the component values of L and M and substituting in Eqs. 1*a* and 1*b*,

$$L = L_i + \tfrac{1}{2}\rho\, V^2 b \left[C_1 s_1 \left(\alpha - \frac{\dot{\eta}}{V} \right) - C_3 s_3 \left(\frac{b\,\dot{\alpha}}{2\,V} \right) \right] + m_a\,(V\,\dot{\alpha} - \ddot{\eta}) \quad ..(2a)$$

and

$$M = M_i + \tfrac{1}{2}\rho\, V^2 b^2 \left[C_2 s_2 \left(\alpha - \frac{\dot{\eta}}{V} \right) - C_4 s_4 \left(\frac{b\,\dot{\alpha}}{2\,V} \right) \right] - I_a\,\ddot{\alpha} \quad ..(2b)$$

in which C_1, C_2, C_3, and C_4 are phase-correction factors for the effect of time lag between any change in attitude or motion of the section and the resulting pressures across the width of the section. The inertia forces are instantaneous with the accelerations that produce them; hence, the terms containing m_a and I_a contain no phase-lag factors. Similarly initial or constant contributions, like L_i and M_i, contain no C-factors.

Eqs. 2 are of fundamental importance. They are the pivotal formulas of the analysis presented. Theories for evaluating the coefficients C_1, C_2, C_3, and C_4 vary; but, whatever evaluation of those coefficients is adopted, the basic validity of the general equations (Eqs. 2) is an independent proposition.

The applicability of Eqs. 2 is without restriction as to the form of the section and without restriction as to the nature of the motion or oscillation of the section (uniform, irregular, or harmonic). Moreover, the section may be horizontally stationary in a steady wind (as in the case of a bridge) or moving horizontally relative to the wind (as in the case of an airplane); V is the relative wind velocity.

In the case of harmonic oscillations, the coefficients C_1, C_2, C_3, and C_4, are functions of the velocity ratio $\frac{V}{N\,b}\left(\text{or of } k = \frac{\omega\, b}{2\,V} = \frac{\pi\, N\, b}{V}\right)$. In such case, the C-coefficients are of the vector form, $C = F - i\,G$, representing an angle of lag equal to $\tan^{-1} G/F$. As derived subsequently, the values of F and G may be calculated and plotted from the pressure distribution graphs for the section (F_1, F_2, G_1, and G_2 from the straight model; F_3, F_4, G_3, and G_4, from the curved model).

In Eqs. 2 all the terms and parameters within the brackets are dimensionless. The outside terms may also be written in terms of $\rho\, V^2\, b$ or $\rho\, V^2\, b^2$ and dimensionless factors.

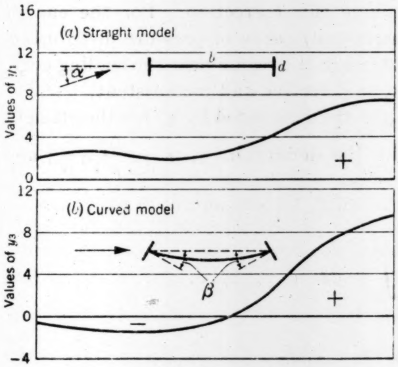

FIG. 7.—PRESSURE DISTRIBUTION GRAPHS FOR STRAIGHT AND CURVED MODELS OF AN H-SECTION ($d/b = 0.20$)

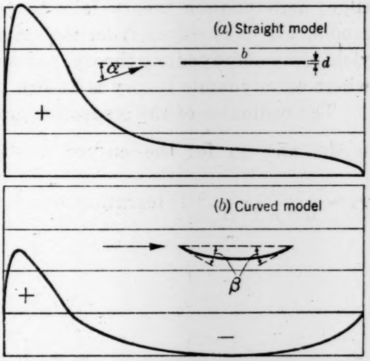

FIG. 8.—PRESSURE DISTRIBUTION GRAPHS FOR STRAIGHT AND CURVED MODELS OF A FLAT PLATE ($d/b = 0.02$)

For a thin flat plate ($d/b = 0$), the mass of the enveloping cylinder of air is $m_a = \frac{\pi}{4}\,\rho\, b^2$. For deeper sections (H, deck, or through sections), the corresponding indicated value is $m_a = \frac{\pi}{4}\,\rho\, r_1\, b^2$, in which $r_1 = 1 + (d/b)^2$, which reduces to $r_1 = 1$ for the thin plate.

By aerodynamic theory, for a thin flat plate, $I_a = \frac{b^2}{32}\, m_a = \frac{\pi}{128}\,\rho\, b^4$. This value of I_a is one fourth of the polar moment of inertia of the air cylinder, or one half of the moment of inertia about the vertical axis. For deeper sections, with sufficient accuracy, also, $I_a = \frac{b^2}{32}\, m_a = \frac{\pi}{128}\,\rho\, r_1\, b^4$. (This last formula considers that the effective radius of gyration is based on b and not on $\sqrt{r_1}\, b$.)

Because the mass m_a of the enveloping cylinder of air is small in comparison with the mass m of the bridge section, the terms representing the effects of acceleration ($\ddot{\eta}$ and $\ddot{\alpha}$) may be omitted. The correction (if desired) is easily made by simply including m_a and I_a of the circumscribing cylinder of air in the values of m and I, respectively. This implies that the frequencies N_1 and N_2 are the natural frequencies in still air, not in a vacuum. The fractional additive corrections to m and I are given by $\frac{m_a}{m} = \frac{\pi}{4} \mu r_1 \approx 0.02$; and $\frac{I_a}{I} = \frac{\pi}{32} \mu \frac{r_1}{a_1} \approx 0.005$.

Pressure Distribution Graphs.—Fig. 7 shows typical pressure distribution graphs for a straight and a curved model, respectively, for the section $d/b = 0.20$, as obtained by manometer readings in static wind-tunnel tests. Fig. 8 shows the corresponding pressure distribution graphs for a flat plate ($d/b = 0.02$). Fig. 8(a) is similar to the curve of pressure distribution by aerodynamic theory,[2] except for the correction to zero pressure at the leading edge; aerodynamic theory fails to provide this correction. For the curved model, Fig. 8(b) differs from the (symmetrical) curve of pressure distribution yielded by aerodynamic theory, particularly in the region near the leading edge, where aerodynamic theory is known to be defective and inconsistent.

The ordinates of the respective graphs are designated by y_1 for the straight model and y_3 for the curved model. By definition: $y_1 = \frac{\Delta p}{0.5\, \rho\, V^2\, \alpha}$; and $y_3 = \frac{\Delta p}{0.5\, \rho\, V^2\, \beta}$. Integrating across the width b (or from $x = 0$ to 1),

$$s_1 = \int_0^1 y_1\, dx \ldots\ldots (3a)$$

$$s_2 = \int_0^1 y_1 \left(\tfrac{1}{2} - x\right) dx \ldots\ldots (3b)$$

$$s_3 = \int_0^1 y_3\, dx \ldots\ldots (3c)$$

and

$$s_4 = \int_0^1 y_3 \left(\tfrac{1}{2} - x\right) dx \ldots\ldots (3d)$$

These values of s_1, s_2, s_3, and s_4, refer to the stationary section, and should be identical with the slopes of the static lift and torque graphs (as in Figs. 3 and 4, or Figs. 5 and 6).

Effect of Phase Lag Across the Width of the Section.—Let ϕ_1 = total phase lag across the width b; ϕ = phase lag at any element dx of the width b, and x = abscissa along the width b (taken as unity). Then, in the case of harmonic oscillations of frequency N,

$$\phi_1 = \frac{2\, \pi\, N\, b}{V} = \frac{\omega\, b}{V} = 2\, k \ldots\ldots (4a)$$

and

$$\phi = x\, \phi_1 = 2\, k\, x \ldots\ldots (4b)$$

Each pressure ordinate, y_1 or y_3, lags an amount ϕ behind the respective displacement or velocity vector that produces it. Hence, for the oscillating section, the static ordinates y_1 and y_3 are replaced by the lagging values—$y_1\,e^{-i\phi} = y_1\,(\cos\phi - i\sin\phi)$; and $y_3\,e^{-i\phi} = y_3\,(\cos\phi - i\sin\phi)$—and the static values s_1, s_2, s_3, and s_4 are replaced by the modified lagging values:

$$C_1\,s_1 = \int_0^1 y_1\,e^{-i\phi}\,dx \qquad (4c)$$

$$C_2\,s_2 = \int_0^1 y_1\,e^{-i\phi}\,(\tfrac{1}{2} - x)\,dx \qquad (4d)$$

$$C_3\,s_3 = \int_0^1 y_3\,e^{-i\phi}\,dx \qquad (4e)$$

and

$$C_4\,s_4 = \int_0^1 y_3\,e^{-i\phi}\,(\tfrac{1}{2} - x)\,dx \qquad (4f)$$

For each force or moment vector $C\,s$, $F\,s$ is the cosine component and $-\,i\,G\,s$ is the sine component. The angle of lag of the resultant force or moment vector is $\tan^{-1}(G/F)$. Accordingly, for the force vectors,

$$F_1 = \frac{\int_0^1 y_1\cos\phi\,dx}{\int_0^1 y_1\,dx\ (= s_1)};\qquad G_1 = \frac{\int_0^1 y_1\sin\phi\,dx}{\int_0^1 y_1\,dx\ (= s_1)};\qquad C_1 = F_1 - i\,G_1 \qquad (5a)$$

$$F_3 = \frac{\int_0^1 y_3\cos\phi\,dx}{\int_0^1 y_3\,dx\ (= s_3)};\qquad G_3 = \frac{\int_0^1 y_3\sin\phi\,dx}{\int_0^1 y_3\,dx\ (= s_3)};\qquad C_3 = F_3 - i\,G_3 \qquad (5b)$$

and, for the moment vectors,

$$F_2 = \frac{\int_0^1 y_1\,(\tfrac{1}{2} - x)\cos\phi\,dx}{\int_0^1 y_1\,(\tfrac{1}{2} - x)\,dx\ (= s_2)};\ G_2 = \frac{\int_0^1 y_1\,(\tfrac{1}{2} - x)\sin\phi\,dx}{\int_0^1 y_1\,(\tfrac{1}{2} - x)\,dx\ (= s_2)};\ C_2 = F_2 - i\,G_2. \qquad (5c)$$

$$F_4 = \frac{\int_0^1 y_3\,(\tfrac{1}{2} - x)\cos\phi\,dx}{\int_0^1 y_3\,(\tfrac{1}{2} - x)\,dx\ (= s_4)};\ G_4 = \frac{\int_0^1 y_3\,(\tfrac{1}{2} - x)\sin\phi\,dx}{\int_0^1 y_3\,(\tfrac{1}{2} - x)\,dx\ (= s_4)};\ C_4 = F_4 - i\,G_4. \qquad (5d)$$

In Eqs. 5 the denominators (by Eqs. 3) represent s_1, s_2, s_3, and s_4 and the numerators represent $F_1 s_1$, $F_2 s_2$, $F_3 s_3$, and $F_4 s_4$, and $G_1 s_1$, $G_2 s_2$, $G_3 s_3$, and $G_4 s_4$. For all cases of F, G, and C,

$$\text{Limit}_{k=0} F = 1 \qquad \text{limit}_{k=0} G = 0 \qquad \text{limit}_{k=0} C = 1$$

For infinite V or zero N,

$$\frac{V}{N\,b} = \infty \qquad \phi = 0 \qquad k = 0$$

Hence, $F = 1$; $G = 0$; $C = 1$; and $C\,s = s$. All phase difference vanishes, and the forces and moments acting on the section coincide in phase with their respective causative displacement or velocity vectors, and coincide in magnitude with the values given directly by the static lift and torque graphs (for the straight and curved models) without any correction for phase lag.

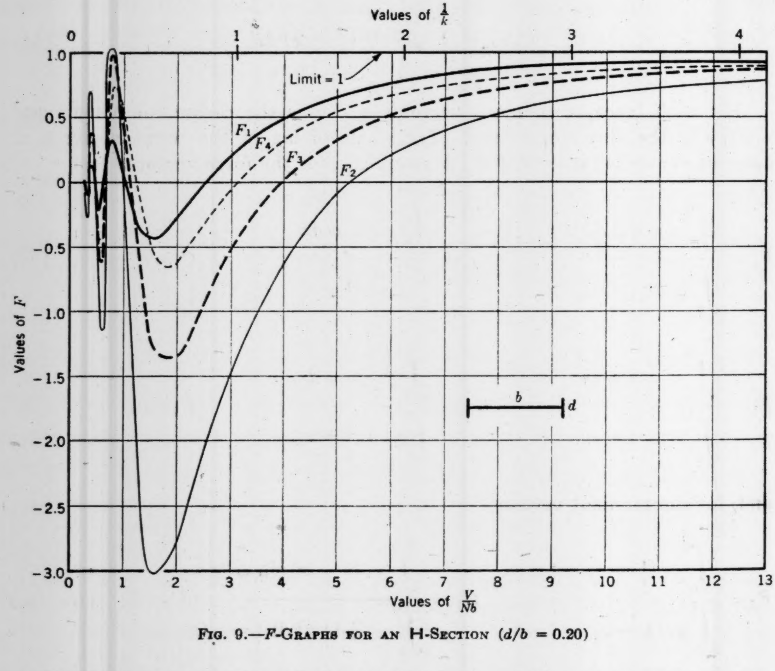

FIG. 9.—*F*-GRAPHS FOR AN H-SECTION ($d/b = 0.20$)

The *F*-graphs and *G*-graphs for the H-section of $d/b = 0.20$, computed by Eqs. 5 from the pressure distribution graphs of Fig. 7, are plotted in Figs. 9 and 10; and the *F*-graphs and *G*-graphs for the flat plate of $d/b = 0.02$, computed by Eqs. 5 from the pressure distribution graphs of Fig. 8, are plotted in Figs. 11 and 12.

General Dynamic Equations

For Oscillating Sections.—The general dynamic equations for a symmetrical oscillating section are

$$L = (L_i) + m\,\ddot{\eta} + c_1\,\dot{\eta} + K_1\,\eta \qquad (6a)$$

and

$$M = (M_i) + I\,\ddot{\alpha} + c_2\,\dot{\alpha} + K_2\,\alpha \qquad (6b)$$

in which η and α are measured from the initial static η_i and α_i, K_1, and K_2 are the spring constants or restoring-force coefficients, and c_1 and c_2 are the damping coefficients. The terms L_i and M_i are canceled by the identical terms in the aerodynamic equations. For the uncommon case of unsymmetrical bridge sections, Eqs. 6 require modification by including the lateral eccentricity of the center of gravity and of the stiffness axis.

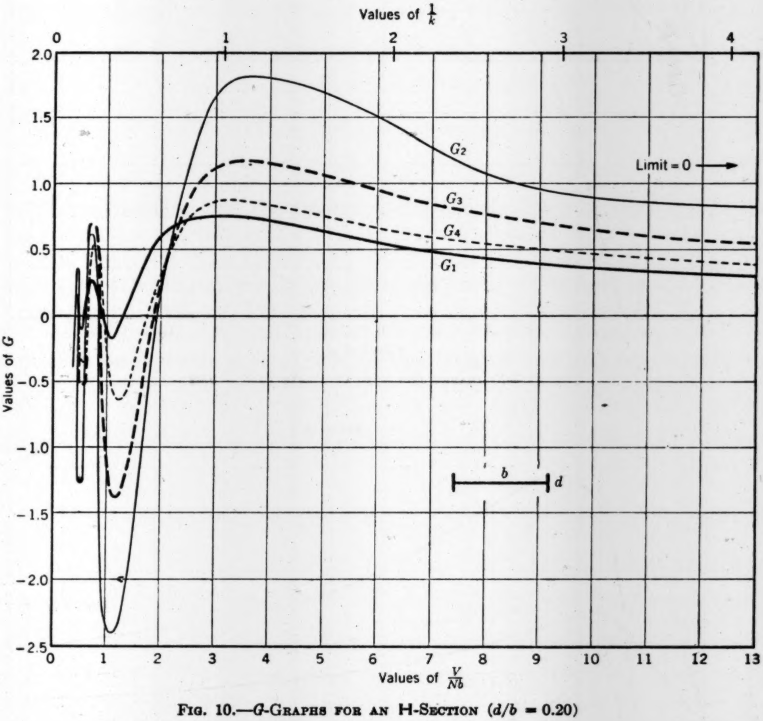

Fig. 10.—G-Graphs for an H-Section ($d/b = 0.20$)

If η_o and α_o are the vertical and angular amplitudes, respectively,

$$\eta = \eta_o\, e^{i(\omega t + \psi)} \qquad (7a)$$

and

$$\alpha = \alpha_o\, e^{i\omega t} \qquad (7b)$$

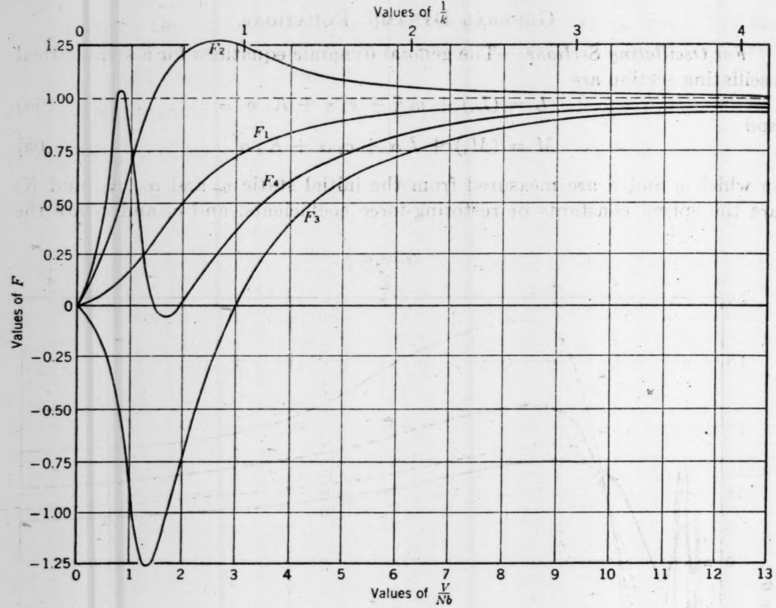

FIG. 11.—*F*-GRAPHS FOR A FLAT PLATE ($d/b = 0.02$)

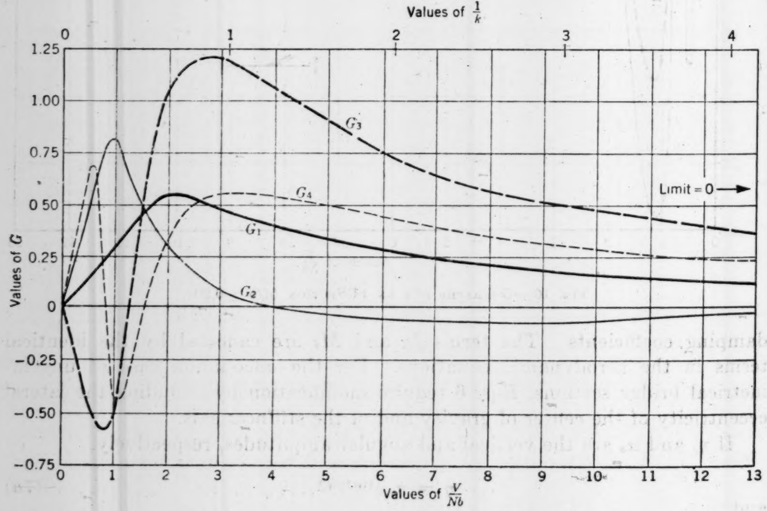

FIG. 12.—*G*-GRAPHS FOR A FLAT PLATE ($d/b = 0.02$)

in which ψ is the phase difference (angle of lead of the vertical oscillations) in coupled vertical and angular oscillations. Eqs. 7 yield

$$\dot{\eta} = i\,\omega\,\eta \qquad \ddot{\eta} = -\,\omega^2\,\eta$$

$$\dot{\alpha} = i\,\omega\,\alpha \qquad \ddot{\alpha} = -\,\omega^2\,\alpha$$

On substituting,

$$K_1 = m\,\omega^2{}_1 \qquad \frac{c_1}{m\,\omega} = g_1 = \frac{\delta_1}{\pi}$$

$$K_2 = I\,\omega^2{}_2 \qquad \frac{c_2}{I\,\omega} = g_2 = \frac{\delta_2}{\pi}$$

Eqs. 6 reduce to

$$L = L_i + m\,\omega^2 \left[\left(\frac{\omega_1}{\omega}\right)^2 + i\,g_1 - 1\right] \eta \ldots\ldots\ldots\ldots(8a)$$

and

$$M = M_i + I\,\omega^2 \left[\left(\frac{\omega_2}{\omega}\right)^2 + i\,g_2 - 1\right] \alpha \ldots\ldots\ldots\ldots(8b)$$

The structural damping (within a cycle of oscillation) can be represented as a force in phase with the velocity but of a magnitude proportional to the restoring force (corresponding to ω^2). Mr. Theodorsen and I. E. Garrick[11] suggest the form: $m\,\omega^2{}_1\,(1 + i\,g_1)\,\eta$ and $I\,\omega^2{}_2\,(1 + i\,g_2)\,\alpha$, but this would not yield $g_1 = \delta_1/\pi$ and $g_2 = \delta_2/\pi$ for the general case. The form used in Eqs. 8 meets the indicated requirements.

Combined Force Equations for Oscillating Sections.—Equating the aerodynamic forces given by Eqs. 2 with the dynamic resisting forces given by Eqs. 8, the general force equations for an oscillating section are obtained; thus,

$$m\,\omega^2 \left[\left(\frac{\omega_1}{\omega}\right)^2 + i\,g_1 - 1\right] \eta = \tfrac{1}{2}\,\rho\,V^2\,b \left[C_1\,s_1 \left(\alpha - \frac{\dot{\eta}}{V}\right) - C_3\,s_3 \left(\frac{b}{2\,V}\,\dot{\alpha}\right)\right] + m_a\,V\,\dot{\alpha} \ldots(9a)$$

and

$$I\,\omega^2 \left[\left(\frac{\omega_2}{\omega}\right)^2 + i\,g_2 - 1\right] \alpha = \tfrac{1}{2}\,\rho\,V^2\,b^2 \left[C_2\,s_2 \left(\alpha - \frac{\dot{\eta}}{V}\right) - C_4\,s_4 \left(\frac{b}{2\,V}\,\dot{\alpha}\right)\right] \ldots(9b)$$

The $\ddot{\eta}$-terms and $\ddot{\alpha}$-terms are canceled by including m_a and I_a in m and I.

On substituting, $\mu = \rho\,b^2/m$; $I = m\,r^2$; $\dot{\eta} = i\,\omega\,\eta$; $k = \omega\,b/(2\,V)$; $a_1 = 4\,r^2/b^2$; and $\dot{\alpha} = i\,\omega\,\alpha$—Eqs. 9 reduce to

$$\frac{4}{\mu}\left[\left(\frac{\omega_1}{\omega}\right)^2 + i\,g_1 - 1\right] \frac{2\,\eta}{b} = \left[\frac{1}{k^2}\,C_1\,s_1\,\alpha - \frac{i}{k}\,C_1\,s_1\,\frac{2\,\eta}{b} + \frac{i}{k}\,(\pi\,r_1 - C_3\,s_3)\,\alpha\right] = \frac{C_L}{k^2} \ldots(10a)$$

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[11] "Mechanism of Flutter: A Theoretical and Experimental Investigation of the Flutter Problem," by Theodore Theodorsen and I. E. Garrick, *Technical Report No. 685*, National Advisory Committee for Aeronautics, Washington, D. C., 1940.

and

$$\frac{2\,a_1}{\mu}\left[\left(\frac{\omega_2}{\omega}\right)^2 + i\,g_2 - 1\right]\alpha = \left[\frac{1}{k^2}\,C_2\,s_2\,\alpha - \frac{i}{k}\,C_2\,s_2\,\frac{2\,\eta}{b} - \frac{i}{k}\,C_4\,s_4\,\alpha\right] = \frac{C_M}{k^2} \quad \text{..(10b)}$$

All the terms are in dimensionless form.

Eqs. 10 are the key equations for solving the various problems of aerodynamic oscillations.

Initial Static Lift and Torque.—The initial static values of lift and torque (L_i and M_i) at zero amplitude are given by the ordinates $(C_L)_i$ and $(C_M)_i$ of the straight model static lift and torque graphs, respectively, at the angle of incidence α_i. This angle of incidence is given by

$$\alpha_i = \alpha_w + \alpha_d \quad \text{......................(11a)}$$

in which α_w is the angle of inclination of the wind and α_d is the static pitch or static angular displacement produced by M_i. By definition,

$$L_i = \tfrac{1}{2}\,\rho\,V^2\,b\,(C_L)_i = K_1\,\eta_d \quad \text{.................(11b)}$$

$$M_i = \tfrac{1}{2}\,\rho\,V^2\,b^2\,(C_M)_i = K_2\,\alpha_d \quad \text{................(11c)}$$

and

$$(C_L)_i = (C_L)_w + s_1\,\alpha_d \quad \text{...................(11d)}$$

$$(C_M)_i = (C_M)_w + s_2\,\alpha_d \quad \text{...................(11e)}$$

From Eqs. 11*c* and 11*e*,

$$\alpha_d = \frac{(C_M)_w}{K_2/(\frac{1}{2}\,\rho\,V^2\,b^2) - s_2} \quad \text{...................(11f)}$$

By Eqs. 11 a positive initial torque M_i augments the apparent angle of incidence α_w by the amount α_d to yield the corrected angle of incidence α_i. This α_d-correction, commonly overlooked, should be considered in experimental determinations of critical angles of incidence. The K_1-relations and K_2-relations in Eqs. 11 are written for a section model (uniform L, M, η, and α). For a bridge or a scale model of a bridge, α_d has sinusoidal variation and the mean value of α_d is reduced. Consequently, section models and complete scale models yield different apparent values of critical angles of incidence. Inconsistent results, inadequately explained by the induced angle correction for aspect ratio, may be explained by the α_d-correction.

The initial torsion α_d and its accompanying vertical deflection η_d are functions of V and α_w. This explains how a change in velocity or in inclination of the wind may initiate the oscillations of a structure.

The values of L_i and M_i are aerostatic forces and are balanced by the equilibrating deflections and restoring forces ($K_1\,\eta_d$ and $K_2\,\alpha_d$) evoked in the structure. Their sole influence in aerodynamic relations is in determining the true angle of incidence α_i from which the angular displacements α are measured and at which the slopes s_1 and s_2 are taken.

Angle of Incidence: Numerical Example.—Given: $\alpha_w = +5°$; $(C_M)_w = +0.070$; $s_2 = +1.00$; $K_2 = 60{,}000$ ft-lb; $b = 50$; and $V = 100$ (= 68 miles per hr). Then $\frac{1}{2}\rho V^2 b^2 = 30{,}000$ ft-lb. By Eq. 11*f*, $\alpha_d = +0.07 = +4°$. Hence, $\alpha_i = 5° + 4° = +9°$. The initial static torque, in this case, increases the angle of incidence from $\alpha_w = +5°$ to $\alpha_i = +9°$; half of the increase is due to the positive slope, s_2. The static torque is thereby increased from $M_w = 2{,}100$ ft-lb to $M_i = 4{,}200$ ft-lb. (All moments are "moments per foot of length.")

VERTICAL INSTABILITY

Vertical Oscillations.—For pure vertical oscillations (single degree of freedom, or torsional oscillations either not initiated or damped out), with $\alpha = 0$ and $\dot{\alpha} = 0$, Eq. 10*a* reduces to

$$\frac{4}{\mu}\left[\left(\frac{\omega_1}{\omega}\right)^2 + i\,g_1 - 1\right] = -\frac{i}{k}(F_1 - i\,G_1)\,s_1 \qquad (12)$$

The imaginary terms furnish the equation: $\frac{4}{\mu}g_1 = -\frac{1}{k}F_1\,s_1$; whence,

$$\delta_1 = -\frac{\mu}{4}(F_1\,s_1)\frac{V}{N\,b} \qquad (13)$$

in which $\mu = \rho\,b^2/m \approx 0.0766\,b^2/w$. Eq. 13 is identical with the equations previously published for δ in vertical oscillations,[12,13,14,15] with F_1 written for the identical β-function, and with w, m, and K now taken per full width. The real terms of Eq. 12 furnish the equation: $\frac{4}{\mu}\left[\left(\frac{\omega_1}{\omega}\right)^2 - 1\right] = -\frac{1}{k}(G_1\,s_1)$, whence,

$$\left(\frac{N_1}{N}\right)^2 = \left(\frac{\omega_1}{\omega}\right)^2 = 1 - \frac{\mu}{4\,\pi}(G_1\,s_1)\frac{V}{N\,b} \qquad (14)$$

Eq. 14 gives the modified frequency (N or ω) of aerodynamic vertical oscillations in terms of the natural frequency (N_1 or ω_1) at $V = 0$. For positive $G_1\,s_1$, $N > N_1$. This modification of vertical frequency increases with $V/(N\,b)$, but is only a small percentage change—usually negligible.

Vertical Instability: Numerical Examples.—For an H-section of $d/b = 0.20$, with $b^2/w = 0.268$ (hence $\mu = 0.0206$), and $s_1 = +4.00$, and using the ordinates of the F_1-graph presented in Fig. 9, Eq. 13 yields the vertical instability graph plotted in Fig. 13. The highest instability range (at zero structural damping) extends from $V/(N\,b) = 1.08$ to 2.57, with lower ("subharmonic") instability ranges at $V/(N\,b) = 0.50$ to 0.68, 0.33 to 0.40, etc. (In the original Tacoma Narrows Bridge, with $d/b = 0.205$, the corresponding ranges of re-

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[12] "Rigidity and Aerodynamic Stability of Suspension Bridges," by D. B. Steinman, *Transactions*, ASCE, Vol. 110, 1945, p. 450, Eqs. 29.

[13] *Ibid.*, p. 558, Eq. 158*a*.

[14] *Ibid.*, p. 563, Eq. 164*a*.

[15] "Simple Model Tests Predict Aerodynamic Characteristics of Bridges," by D. B. Steinman, *Civil Engineering*, January, 1947, p. 21, Eqs. 2 and 3.

corded oscillations were $V/(N\,b) = 0.93$ to 2.35, 0.53 to 0.89, 0.37 to 0.43, etc., representing substantial agreement subject to admitted inaccuracy in the original observations.) As shown by the graph in Fig. 13, this section is a vertically "stable" section—that is, one that is stable in the high velocity (or "catastrophic") range, as predicted from the positive slope of the lift graph (Fig. 3). The maximum value of δ (the logarithmic increment) occurs at $V/(N\,b) = 1.7$ (= 6 miles per hr for the original Tacoma bridge) and amounts to 0.0152; all ordinates of the δ-graph would be proportionately increased for a higher value of b^2/w.

At $V/(N\,b) = 1.7$, representing maximum vertical instability for this section, the graph recorded in Fig. 10 gives $G_1 = +\,0.46$. With these values, Eq. 14 yields $N = 1.0026\,N_1$, an almost negligible increase above the natural frequency N_1.

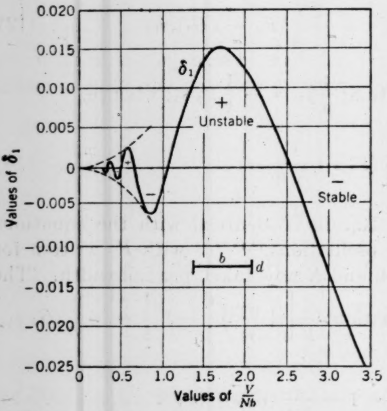

FIG. 13.—VERTICAL INSTABILITY GRAPH FOR AN H-SECTION ($d/b = 0.20$)

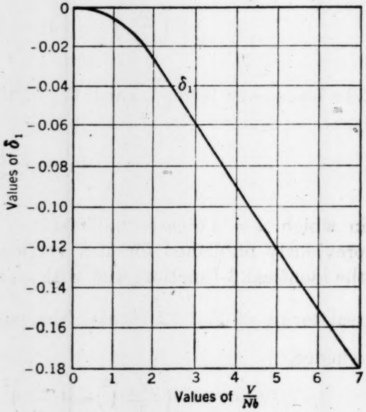

FIG. 14.—VERTICAL STABILITY GRAPH FOR A FLAT PLATE ($d/b = 0.02$)

For a flat plate of $d/b = 0.02$, with $s_1 = +\,5.25$ and $b^2/w = 0.268$, and using the ordinates of the F_1-graph represented in Fig. 11, Eq. 13 yields the vertical instability graph plotted in Fig. 14. With all ordinates negative, this graph shows vertical stability at all wind velocities.

TORSIONAL INSTABILITY

Torsional Oscillations.—For pure angular oscillations (single degree of freedom, or vertical oscillations either not initiated or damped out), with $\dot{\eta} = 0$, Eq. 10*b* reduces to

$$\frac{2\,a_1}{\mu}\left[\left(\frac{\omega_2}{\omega}\right)^2 + i\,g_2 - 1\right] = \frac{1}{k^2}\,C_2\,s_2 - \frac{i}{k}\,C_4\,s_4 \ldots\ldots\ldots (15)$$

The imaginary terms furnish the equation: $\frac{2\,a_1}{\mu}\,g_2 = -\,\frac{1}{k^2}\,G_2\,s_2 - \frac{1}{k}\,F_4\,s_4$;

whence,

$$\delta_2 = -\frac{\mu}{2\,a_1}\left(\frac{G_2}{k}\,s_2 + F_4\,s_4\right)\frac{V}{N\,b}\ldots\ldots\ldots\ldots\ldots(16)$$

in which $\mu = \rho\,b^2/m \approx 0.0766\,b^2/w$, and $a_1 = 4\,r^2/b^2$.

Eq. 16 is identical with the equations previously published for δ in torsional oscillations,[16,17,18,19] except that ($\beta\,s_2$) is now represented by the amplified expression: $\beta\,s_2 = \left(\frac{G_2}{k}\,s_2 + F_4\,s_4\right)$; in which β was the prior symbol for the phase-lag factor. This last expression is significant. The first term represents the integrated effect of angular position, a contribution previously overlooked or dismissed as apparently canceling out over the cycle. The second term represents the integrated effect of angular velocity, but is now based on static wind-tunnel tests on curved section models. At high wind velocities, although G_2 approaches zero as a limit, $\text{limit}_{k-0}\,\frac{G_2}{k} \geqq 2$; and $\text{limit}_{k-0}\,F_4 = 1$. Hence, for sections yielding $s_4 \approx 2\,s_2$, $\text{limit}_{k-0}\,\beta\,s_2 = \left(\frac{G_2}{k}\,s_2 + F_4\,s_4\right) \geqq 4\,s_2$.

It was previously known[20] that a value of $\beta \leqq 4$ at high wind velocities was needed in order to check the known test results on torsional oscillations. The foregoing amplification of the theory supplies this missing factor of 4, and yields a perfect check on the recorded results of oscillating model tests.

The real terms of Eq. 15 furnish the equation: $\frac{2\,a_1}{\mu}\left[\left(\frac{\omega_2}{\omega}\right)^2 - 1\right] = \frac{1}{k^2}\,F_2\,s_2 - \frac{1}{k}\,G_4\,s_4$; whence,

$$\left(\frac{N_2}{N}\right)^2 = \left(\frac{\omega_2}{\omega}\right)^2 = 1 + \frac{\mu}{2\,\pi\,a_1}\left(\frac{F_2}{k}\,s_2 - G_4\,s_4\right)\frac{V}{N\,b}\ldots\ldots(17)$$

As a rule, for negative $F_2\,s_2$ (torsional instability), $N > N_2$. At high wind velocities, this increase in torsional frequency may be substantial. (In catastrophic vertical oscillations, the corresponding effect would be a small reduction in frequency.)

Torsional Instability: Numerical Examples.—For an H-section of $d/b = 0.20$—with $s_2 = -0.50$, $s_4 = -1.00$, $b^2/w = 0.238$, and $a_1 = 0.476$, and using the F_4-values and G_2-values represented in Figs. 9 and 10, respectively—Eq. 16 yields the torsional instability graph plotted in Fig. 15. This graph shows catastrophic instability starting at a critical velocity ratio of $V/(N\,b) = 2.50$. For comparison, Fig. 15 also shows the torsional instability graph recorded by

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[16] "Rigidity and Aerodynamic Stability of Suspension Bridges," by D. B. Steinman, *Transactions*, ASCE, Vol. 110, 1945, p. 460, Eqs. 51.

[17] *Ibid.*, p. 559, Eq. 160*a*.

[18] *Ibid.*, p. 563, Eq. 164*b*.

[19] "Simple Model Tests Predict Aerodynamic Characteristics of Bridges," by D. B. Steinman, *Civil Engineering*, January, 1947, p. 21, Eqs. 4 and 5.

[20] "Rigidity and Aerodynamic Stability of Suspension Bridges," by D. B. Steinman, *Transactions*, ASCE, Vol. 110, 1945, pp. 463–465.

Louis G. Dunn[21] in 1941 from oscillating tests on a section model of the same d/b and b^2/w. With an indicated vertical shift of the experimental graph to correct for a residual positive damping that was overlooked by the investigators,[22] the two graphs compared in Fig. 15 are found to be almost identical. Without the indicated correction, Mr. Dunn's graph showed a critical $V/(N\,b)$ of 3.33, which is now known to have been too high. More recent tests (1945–1946) on oscillating models of the Tacoma section yield values of critical $V/(N\,b)$ of 2.50 (at the California Institute of Technology in Pasadena) and from 2.43 to 2.56 (at the University of Washington in Seattle), confirming the correctness of the theoretical graph in Fig. 15 and the validity of the theory represented by Eq. 16. The striking agreement of the slopes of the two graphs in Fig. 15 is further confirmation of the correctness of the theory here presented. Without the angular position term and the use of the curved model for the angular velocity term in Eq. 16, the slopes and the ordinates of the computed δ-graph would be only one fourth as large. If instability graphs can be predicted scientifically, as in Fig. 15, without expensive and time-consuming oscillating model tests, the theory is justified.

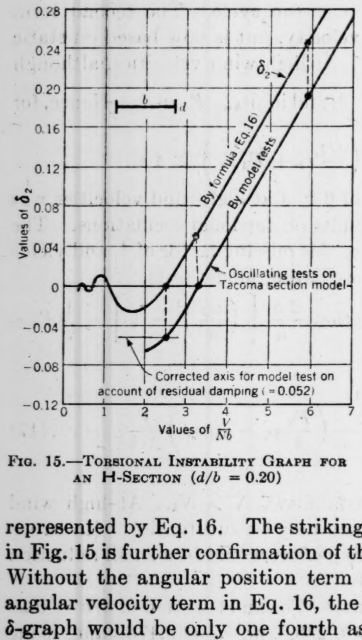

FIG. 15.—TORSIONAL INSTABILITY GRAPH FOR AN H-SECTION ($d/b = 0.20$)

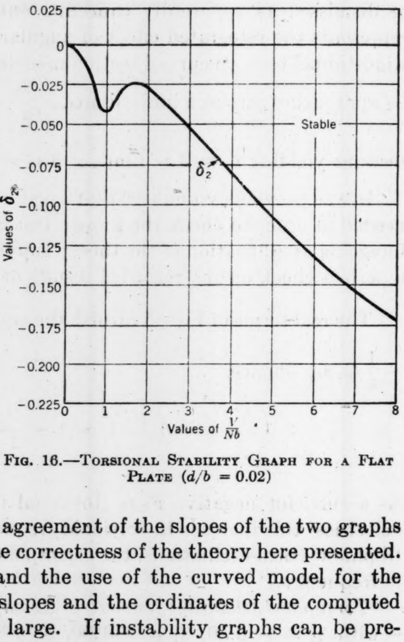

FIG. 16.—TORSIONAL STABILITY GRAPH FOR A FLAT PLATE ($d/b = 0.02$)

The indicated critical $V/(N\,b) = 2.50$, applied to the original Tacoma bridge, represents a critical wind velocity of only 13.3 miles per hr (at zero damping) for catastrophic torsional instability. Such earlier torsional oscillations of the span in the first asymmetric mode were prevented by the midspan stays until the latter failed.

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[21] "The Failure of the Tacoma Narrows Bridge," a report to the Hon. John M. Carmody, Administrator of the Federal Works Agency, Washington, D. C., March 28, 1941, by a Board of Engineers consisting of O. H. Ammann, Theodor von Kármán, and Glenn B. Woodruff, Appendix VIII, Fig. VIII-2.

[22] "Rigidity and Aerodynamic Stability of Suspension Bridges," by D. B. Steinman. *Transactions*, ASCE, Vol. 110, 1945, p. 569.

At $V/(N\,b) = 7.5$ (corresponding to the wind velocity of 40 miles per hr that destroyed the Tacoma span), the graphs in Figs. 9 and 10 give $F_2 = +\,0.48$ and $G_4 = +\,0.59$. With these values and $b^2/w = 0.268$ (or $\mu = 0.0206$) and $a_1 = 0.476$, Eq. 17 yields $N = 0.9995\,N_2$, which is an almost imperceptible decrease below the natural frequency N_2. For the same section, at $V/(N\,b) = 18.75$ (corresponding to 100 miles per hr for the same assumed value of N), $F_2 = +\,0.91$ and $G_4 = +\,0.26$, and Eq. 17 yields $N = 1.21\,N_2$, which is a 21% increase above the natural frequency.

For a flat plate of $d/b = 0.02$, using the F_4-graphs and G_2-graphs represented in Figs. 11 and 12, with $s_2 = +\,1.10$, $s_4 = +\,0.55$, $b^2/w = 0.268$, and $a_1 = 0.333$, Eq. 16 yields the torsional instability graph plotted in Fig. 16. With all ordinates negative, this graph shows torsional stability at all wind velocities.

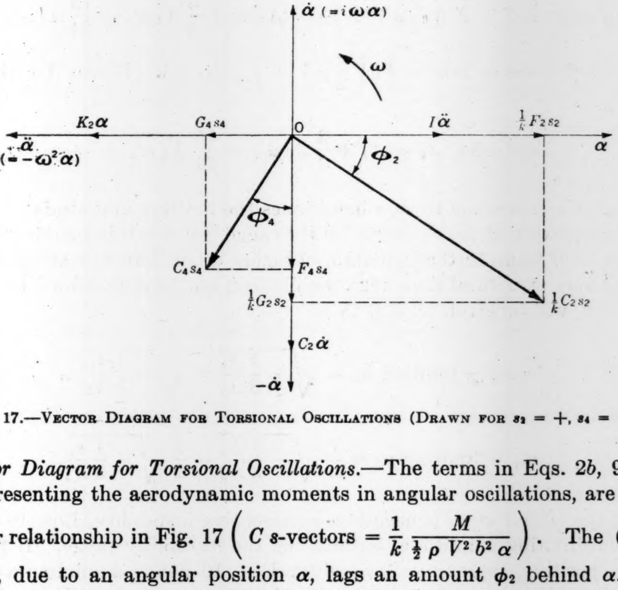

FIG. 17.—VECTOR DIAGRAM FOR TORSIONAL OSCILLATIONS (DRAWN FOR $s_2 = +$, $s_4 = +$)

Vector Diagram for Torsional Oscillations.—The terms in Eqs. 2*b*, 9*b*, and 10*b*, representing the aerodynamic moments in angular oscillations, are shown in vector relationship in Fig. 17 $\left(C\ s\text{-vectors} = \dfrac{1}{k}\,\dfrac{M}{\frac{1}{2}\,\rho\,V^2\,b^2\,\alpha}\right)$. The $(C_2\,s_2)$-moment, due to an angular position α, lags an amount ϕ_2 behind α. The $(C_4\,s_4)$-moment, due to an angular velocity $\dot{\alpha}$, lags an amount ϕ_4 behind $-\,\dot{\alpha}$. The components along the vertical (imaginary) axis yield $\left(\dfrac{1}{k}\,G_2\,s_2 + F_4\,s_4\right)$, opposing $\dot{\alpha}$. Therefore they represent positive damping if $G_2\,s_2$ and $F_4\,s_4$ are positive, and negative damping (amplification) if $G_2\,s_2$ and $F_4\,s_4$ are negative—all in accordance with Eq. 16. The components along the horizontal (real) axis yield $\left(\dfrac{1}{k}\,F_2\,s_2 - G_4\,s_4\right)$ opposing $K_2\,\alpha$. Therefore they represent a reduction in frequency if $F_2\,s_2$ is positive, and an increase in frequency if $F_2\,s_2$ is negative, unless outweighed by the term $G_4\,s_4$—all in accordance with Eq. 17.

LIMITING AMPLITUDES AND AMPLITUDE RESPONSE CURVES

Effect of Curvature of the Static Lift Graph.—The equation of the static lift graph, straight model, with origin assumed at α_i, may be written in the general form: $C_L = A_o + A_1 \alpha + A_2 \alpha^2 + A_3 \alpha^3 + \cdots$, in which A_o is the ordinate (C_L) and A_1 the slope (s_1) at α_i. If η_o is the amplitude of vertical oscillations, and α_o the equivalent abscissa,

$$\eta = \eta_o \sin(\omega t) \qquad \alpha_o = -\omega \eta_o / V$$

$$\dot{\eta} = \omega \eta_o \cos(\omega t) \qquad \alpha = -\dot{\eta}/V = \alpha_o \cos(\omega t)$$

The work done in a cycle is $\Delta W = F_1 \int_o^{2\pi/\omega} L \dot{\eta}\, dt = -F_1 V \int_o^{2\pi/\omega} [L \alpha_o \times \cos(\omega t)]\, dt$. Substituting the expressions for L, C_L, and α, and performing the integration, $\Delta W = F_1 \left(\frac{\pi}{2} \rho V^2 b \right) \eta_o \left[A_1 \alpha_o + \frac{3}{4} A_3 \alpha^3_o + \frac{15}{24} A_5 \alpha^5_o + \cdots \right]$.

For constant slope s_1, $\Delta W = F_1 \left(\frac{\pi}{2} \rho V^2 b \right) \eta_o (s_1 \alpha_o)$. Hence, for the curving graph,

$$\text{Equivalent } s_1 = A_1 + \frac{3}{4} A_3 \alpha^2_o + \frac{15}{24} A_5 \alpha^4_o + \cdots \cdots \cdots \quad (18)$$

Equating this expression to zero determines the limiting amplitude.

If the equation of the C_L-curve (in the range traversed) is a cubic or quartic equation, or if terms in that equation of higher order than α^4 may be dropped, then (for zero structural damping) the limiting amplitude, defined by equivalent $s_1 = 0$, will be given by Eq. 18 as

$$\alpha_m = \text{limiting } \alpha_o = \sqrt{-\frac{4}{3}\frac{A_1}{A_3}} = \sqrt{-\frac{4}{3}\frac{s_1}{A_3}} \cdots \cdots \cdots \quad (19a)$$

Hence,

$$\eta_m = \text{limiting } \eta_o = \frac{V}{\omega}\sqrt{-\frac{4}{3}\frac{A_1}{A_3}} = \frac{V}{\omega}\sqrt{-\frac{4}{3}\frac{s_1}{A_3}} \cdots \cdots \quad (19b)$$

If the slope ($F_1 s_1$) at α_i is negative, representing instability, Eqs. 19 define a maximum limiting amplitude, terminating the instability range. If the slope at α_i is positive, representing stability, Eqs. 19 define a minimum limiting amplitude, beyond which potential instability commences. For real values, in either case, by Eqs. 19, A_1 ($= s_1$) and A_3 must be opposed in sign. The usual identification is a section of the graph curving away from the vertical axis. A section of the graph curving toward the vertical axis will not yield a limiting (maximum or minimum) amplitude.

By Eq. 19*b*, the limiting vertical amplitude η_m increases with V; but by Eq. 19*a*, the range α_m along the graph, defining the limiting vertical amplitude η_m, is independent of V but depends only upon the geometry of the graph.

Extending the ΔW-integration over a segment of the span, using the sine-curve wave form, and equating to zero, the limiting vertical amplitude at midsegment is

$$\eta_{ms} = \sqrt{\frac{4}{3}}\,\eta_m = \frac{4}{3}\,\frac{V}{\omega}\sqrt{-\,\frac{s_1}{A_3}} \tag{19c}$$

Eqs. 19 are in agreement with corresponding equations previously published.[23]

Limiting Amplitudes of Torsional Oscillations.—The equation of the static torque graph, straight model, with origin assumed at α_i, may be written in the general form, $C_M = A_o + A_1\,\alpha + A_2\,\alpha^2 + A_3\,\alpha^3 + \cdots$, in which A_o is the ordinate (C_M) and A_1 the slope (s_2) at α_i; and the equation of the static torque graph, curved model, with origin at $\beta = 0$, may be written similarly, $C_M = B_o + B_1\,\beta + B_2\,\beta^2 + B_3\,\beta^3 + \cdots$, in which B_o is the ordinate (C_M) and B_1 the slope (s_4) at $\beta = 0$. By the same integration as used above for vertical oscillations,

$$\text{Equivalent } s_2 = A_1 + \frac{3}{4}\,A_3\,\alpha^2_o + \frac{15}{24}\,A_5\,\alpha^4_o + \cdots \tag{20a}$$

and

$$\text{Equivalent } s_4 = B_1 + \frac{3}{4}\,B_3\,\beta^2_o + \frac{15}{24}\,B_5\,\beta^4_o + \cdots \tag{20b}$$

in which α_o is the angular amplitude and β_o is the corresponding abscissa of the β-torque graph. For zero damping (defining the limiting amplitude), by Eq. 16, $\left(\frac{1}{k}\,G_2\,s_2 + F_4\,s_4\right) = 0$. Substituting equivalent s_2 and equivalent s_4, dropping the terms of fourth and higher order, and noting that $\beta_o = k\,\alpha_o$,

$$\alpha_m = \text{limiting } \alpha_o = \sqrt{-\,\frac{4}{3}\,\frac{A_1\,G_2 + k\,B_1\,F_4}{A_3\,G_2 + k^3\,B_3\,F_4}} \tag{21a}$$

With $A_1 = s_2$, $B_1 = s_4$, $G_2/k \approx 2$, $F_4 \approx 1$, and with the negligible k^3-term dropped, Eq. 21*a* may be written:

$$\alpha_m = \text{limiting } \alpha_o \approx \sqrt{-\,\frac{4\,(s_2 + \frac{1}{2}\,s_4)}{3\,A_3}} \tag{21b}$$

Analogous to Eq. 19*c*, the limiting angular amplitude at midsegment of a span will be

$$\alpha_{ms} = \sqrt{\frac{4}{3}}\,\alpha_m \approx \frac{4}{3}\sqrt{-\,\frac{s_2 + \frac{1}{2}\,s_4}{A_3}} \tag{21c}$$

Eqs. 21 correspond to equations previously published,[24] with $\frac{1}{2}\,s_4$ now added to s_2. For $s_4 \approx 2\,s_2$, the calculated value of the limiting amplitude in torsion is thereby multiplied by $\sqrt{2}$. This supplies a factor previously missing in checking the theory against recorded oscillations and adds new confirmation of the basic validity of the theory.

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[23] "Rigidity and Aerodynamic Stability of Suspension Bridges," by D. B. Steinman, *Transactions*, ASCE, Vol. 110, 1945, p. 452, Eqs. 34*a* and 34*b*.

[24] *Ibid.*, p. 461, Eqs. 54*a* and 54*b*.

For $d/b = 0.20$, the static torque curves (Fig. 4) yield $s_2 = -0.50$, $s_4 = -1.00$, and $A_3 = +3.7$. Hence, by Eq. 21c, the limiting segment amplitude is 0.70 = 40°, which checks the reported value for the Tacoma bridge failure.

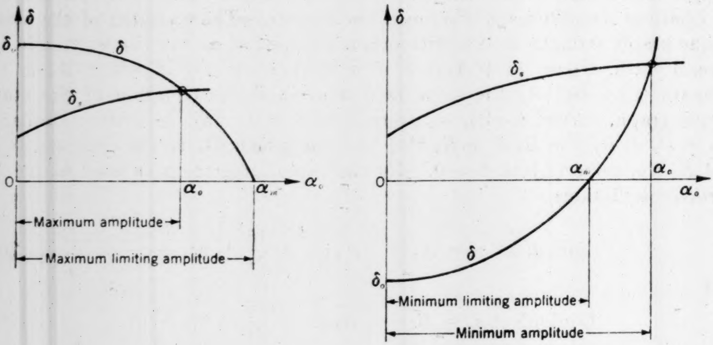


FIG. 18.—LIMITING, MAXIMUM, AND MINIMUM AMPLITUDES

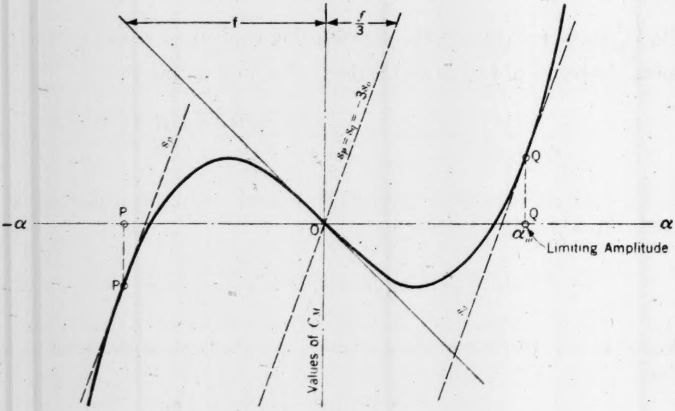


FIG. 19.—GRAPHIC METHOD FOR LIMITING AMPLITUDES

The limiting amplitudes defined by Eqs. 19 and 21 are for zero positive damping. Positive damping δ_s (structural and friction) will reduce the maximum amplitudes and raise the minimum amplitudes. This is illustrated in Fig. 18.

Graphic Methods for Limiting Amplitudes.—In any lift or torque graph, let s_o, s_p, and s_q be the respective slopes at the center O and the ends, P and Q, of any range of oscillation, and let s_e be the equivalent slope over the range PQ as affected by the curvature of the graph. Then, by Eqs. 18 and 20, very closely,

$$s_e \approx \tfrac{3}{4}\, s_o + \tfrac{1}{8}\, s_p + \tfrac{1}{8}\, s_q \ldots\ldots\ldots\ldots\ldots\ldots (22)$$

This yields a simple graphic method (Fig. 19) for plotting limiting vertical amplitudes, defined by $s_e = 0$. The two points P and Q, marking the limits of the range and spaced at equal horizontal distances α_m from 0, are given by their slopes:

$$s_p = s_q = -\,3\, s_o \qquad \text{or} \qquad \tfrac{1}{2}\,(s_p + s_q) = -\,3\, s_o \ldots\ldots\ldots (23)$$

The limiting amplitudes η_m will then be $(V/\omega)\,\alpha_m$. Fig. 19 yields a maximum limiting amplitude, and the inverted reflection of Fig. 19 would yield a minimum limiting amplitude.

The approximate value of the coefficient A_3, for substitution in Eqs. 19, may be similarly obtained from the geometry of the lift graph:

$$A_3 \approx \frac{s_p + s_q - 2\, s_o}{6\, \alpha^2_o} \ldots\ldots\ldots\ldots\ldots\ldots (24)$$

in which α_o is any convenient assumed amplitude. The same geometric relation applied to the two torque graphs yields the corresponding coefficients A_3 and B_3 for substitution in Eqs. 21.

Eqs. 19 and Eqs. 21 to 24 are approximate short cuts which may be sufficiently accurate for practical requirements. For more accurate determinations of limiting amplitudes, Eqs. 18 and 20 should be used.

Use of Slope Graphs.—Since the significant feature of static lift and torque graphs appearing in the foregoing equations (Eqs. 18 to 24) is the magnitude and variation of the slope of the graphs, a directly indicated geometric device is the construction and application of slope graphs. By plotting the slopes of any lift or torque graph as the ordinates of a new curve, the corresponding slope graph is obtained (Fig. 20). Any range below the zero axis represents instability. The four respective slope graphs for any section will be referred to as the s_1-graph, s_2-graph, s_3-graph, and s_4-graph. If the lift or torque graph is antisymmetrical, the slope graph will be symmetrical. If the lift or torque graph is a cubic equation, the slope graph will be a parabola.

The center of oscillation O is marked on the curve, at α_i for the s_1-graph and the s_2-graph, and at $\beta = 0$ for the s_3-graph and the s_4-graph.

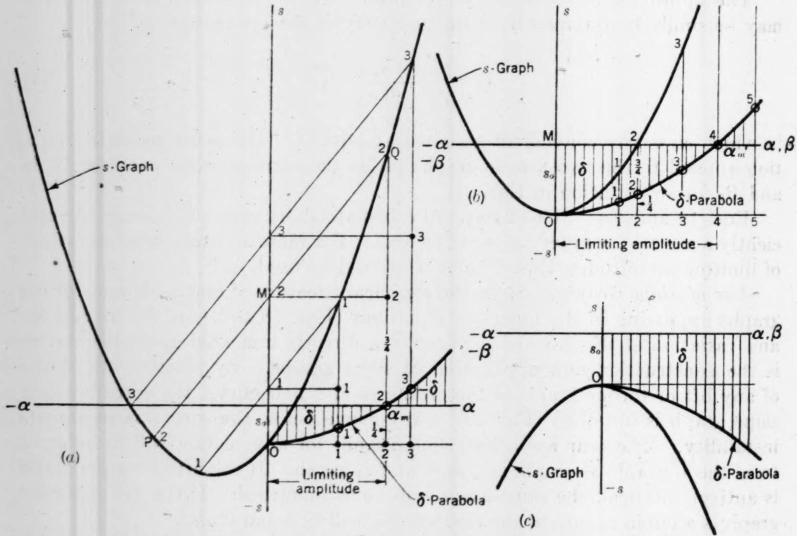
In Fig. 20(*a*), for any assumed amplitude α (or β), the terminal points P and Q are marked on the curve at equal horizontal distances $\pm\, \alpha$ (or $\pm\, \beta$) from O. The ordinates of O, P, and Q are s_o, s_p, and s_q. The midpoint M of the chord PQ is on the vertical through O, and MO is the middle ordinate of the arc PQ. By Eq. 24, $A_3 = \dfrac{1}{3}\dfrac{\overline{\mathrm{MO}}}{\alpha^2}$; and $B_3 = \dfrac{1}{3}\dfrac{\overline{\mathrm{MO}}}{\beta^2}$, yielding A_3 or B_3, positive if the chord PQ is above O. By Eq. 22, $s_e = s_o - \tfrac{1}{4}\,\overline{\mathrm{MO}}$—yielding the equivalent slope for the curving range PQ.

From the horizontal line through O, lay off the ordinates $\tfrac{1}{4}\,\overline{\mathrm{OM}}$ on the respective Q-verticals. The curve thus obtained is the s_e-graph, showing the variation of equivalent slope s_e with amplitude. Since δ is proportional to s_e, this is also the δ-amplitude graph, showing the variation of δ with amplitude. The limiting amplitude occurs where this δ-amplitude graph crosses the zero horizontal axis; this is the point of demarcation between stability and instability. It is given directly by $\overline{MO} = 4\, s_o$.

Fig. 20(*a*) shows the construction for the general case, with the center of oscillation O off center, at an angle of incidence α_i.

For the symmetrical case, with the center of oscillation O at zero and the slope curve symmetrical, the construction is simplified (Fig. 20(*b*)). The ordinates of the δ-amplitude graph (from the horizontal line through O) are simply one fourth of the corresponding ordinates of the slope graph.

Figs. 20(*a*) and 20(*b*) are drawn for an unstable range, yielding a maximum limiting amplitude. If Figs. 20(*a*) and 20(*b*) are inverted, stability and instability are reversed and the intercepted amplitudes are minimum limiting amplitudes.


s
s-Graph
3
2 Q
3
3
M 2
1
2
$\frac{3}{4}$
1
1
3
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-α
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3
P 2
1
s_o
δ
α_m
$\frac{1}{4}$
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1
2
3
(a)
Limiting amplitude
δ-Parabola
-s
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3
5
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4
5
Limiting amplitude
-s
s
-α
-β
α, β
s_o
δ
O
s-Graph
δ-Parabola
-s
(c)

FIG. 20.—CONSTRUCTION OF δ-AMPLITUDE GRAPHS FROM SLOPE GRAPHS

For all practical purposes, the δ-amplitude graph, as a rule, may be taken as a parabola; accordingly, for convenient identification, it will be referred to as the "δ-parabola." The base is the limiting amplitude (given by the foregoing formulas or graphic constructions) and the altitude is δ_o, the value of δ at zero amplitude.

In Fig. 20, the δ-values (varying with amplitude) are represented by the ordinates between the δ-parabola and the horizontal zero axis of the slope graph. If the zero axis is on the convex side of the δ-parabola (Fig. 20(*c*)) the δ-ordinates increase with amplitude and there is no limiting (maximum or minimum) amplitude. In such case, a δ-parabola below the zero axis will represent catastrophic instability of unlimited amplitude, and a δ-parabola above the zero axis (Fig. 20(c) inverted) will represent unlimited stability.

Graphic Construction of Amplitude Response Curves.—Fig. 21 shows a simple graphic procedure for plotting an amplitude response curve from a given δ-graph

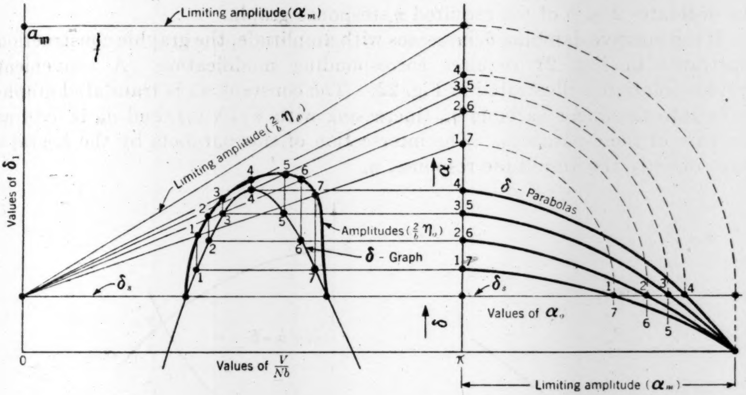

Limiting amplitude (α_m)
a_m
Values of δ_1
Limiting amplitude ($\frac{2}{b}\eta_m$)
Amplitudes ($\frac{2}{b}\eta_o$)
δ - Graph
δ - Parabolas
α_o
δ_s
δ_s
δ
Values of α_o
0
Values of $\frac{V}{Nb}$
π
Limiting amplitude (α_m)

FIG. 21.—CONSTRUCTION OF VERTICAL AMPLITUDE RESPONSE CURVE FOR CONSTANT DAMPING

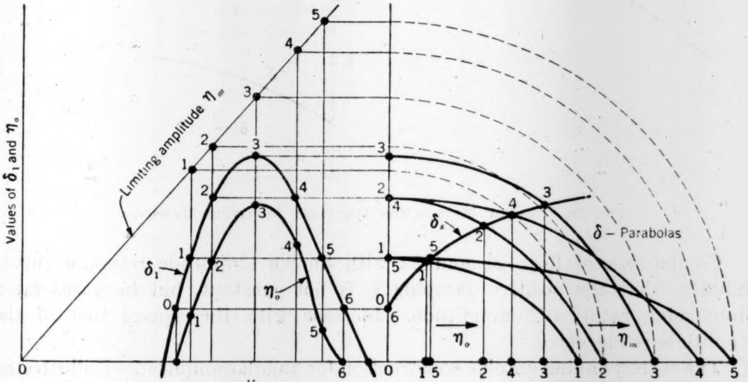

Limiting amplitude η_m
Values of δ_1 and η_o
δ_1
η_o
δ_s
δ – Parabolas
η_o
η_m
0
Values of $\frac{V}{Nb}$
Values of η

FIG. 22.—CONSTRUCTION OF VERTICAL AMPLITUDE RESPONSE CURVE FOR VARIABLE DAMPING

(for vertical oscillations). The points of the δ-graph are projected horizontally to give the altitudes of the δ-amplitude curves. These parabolic curves have a common base—namely, the limiting amplitude α_m, determined as in Fig. 19 or Fig. 20; and they may be drawn as simple parabolas or, for greater accuracy, their form (with proportional ordinates) is determined as in Fig. 20. A line representing the positive damping (δ_s) is drawn across the curves, horizontally if the positive damping is assumed constant (independent of amplitude). The points where this transversal cuts the parabolas give the respective amplitudes α_o. Using a vertical axis at $V/(N\,b) = \pi$ (or any convenient multiple of π), each ordinate α_o is multiplied geometrically by $\dfrac{V/(N\,b)}{\pi} = \dfrac{1}{k} = 2\,\dfrac{V}{\omega\,b}$ to give the ordinates $2\,\eta_o/b$ of the required η_o-response graph.

If the positive damping δ_s increases with amplitude, the graphic construction illustrated in Fig. 21 requires corresponding modification. A convenient graphic solution is illustrated in Fig. 22. The constant α_m is translated graphically into the linear variable η_m (increasing with $V/(N\,b)$), and η_m is used as the base of the δ-parabola. The intersection of this parabola by the δ_s-graph gives directly the amplitude response, η_o.

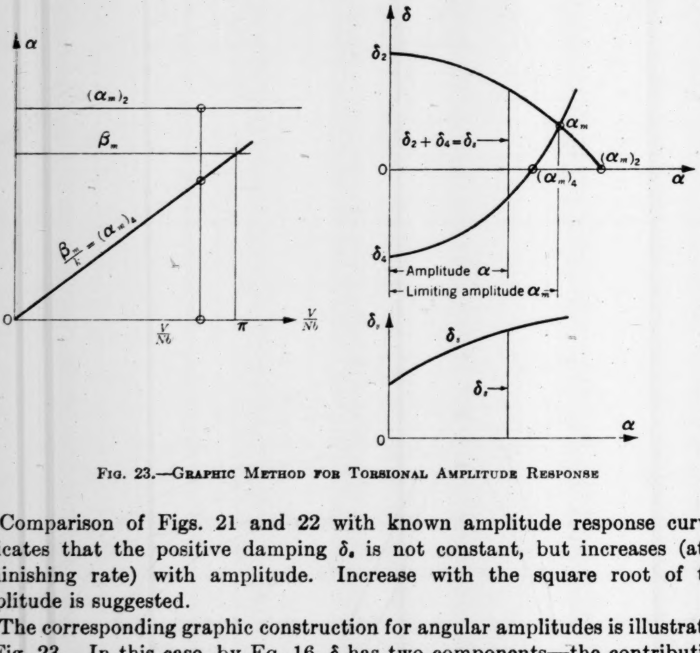


FIG. 23.—GRAPHIC METHOD FOR TORSIONAL AMPLITUDE RESPONSE

Comparison of Figs. 21 and 22 with known amplitude response curves indicates that the positive damping δ_s is not constant, but increases (at a diminishing rate) with amplitude. Increase with the square root of the amplitude is suggested.

The corresponding graphic construction for angular amplitudes is illustrated in Fig. 23. In this case, by Eq. 16, δ has two components—the contribution $G_2\,s_2$, here denoted by δ_2, and the contribution $F_4\,s_4$, denoted by δ_4—so that $\delta = \delta_2 + \delta_4$. The corresponding limiting amplitudes are the constant $(\alpha_m)_2$ and the linear varying $(\alpha_m)_4$, the latter given (Fig. 23(*a*)) by β_m/k. With $(\alpha_m)_2$ and $(\alpha_m)_4$ as the respective bases, the δ_2-parabola and δ_4-parabola are drawn additively (Fig. 23(*b*)); their intersection gives the resultant limiting amplitude α_m. The vertical intercept equal to the corresponding ordinate of the δ_s-graph determines the required amplitude response. α_o.

The amplitudes given by Figs. 18 to 23, inclusive, are for a section model, or the mean amplitudes over a segment of a span. Multiply by $\sqrt{4/3}$ to obtain the corresponding amplitude at midsegment.

Coupled Oscillations

Vertical and Torsional Oscillations, with Damping Included.—Let:

$$X_1 = \frac{4}{\mu\, s_1}\left[\left(\frac{\omega_1}{\omega}\right)^2 - 1\right] \qquad X_2 = \frac{2\, a_1}{\mu\, s_2}\left[\left(\frac{\omega_2}{\omega}\right)^2 - 1\right]$$

$$q_3 = \frac{s_3}{s_1} \qquad q_4 = \frac{s_4}{s_2}$$

$$d_1 = \frac{4}{\mu\, s_1}\, g_1 \qquad d_2 = \frac{2\, a_1}{\mu\, s_2}\, g_2$$

Then Eqs. 10 reduce to

$$\left(X_1 + i\, d_1 + \frac{i}{k}\, C_1\right)\frac{2\,\eta}{b} = \left[\frac{1}{k^2}\, C_1 - \frac{i}{k}\, q_3 \left(C_3 - \frac{\pi\, r_1}{s_3}\right)\right]\alpha \ldots (25a)$$

and

$$\left(\frac{i}{k}\, C_2\right)\frac{2\,\eta}{b} = \left(-X_2 - i\, d_2 + \frac{1}{k^2}\, C_2 - \frac{i}{k}\, q_4\, C_4\right)\alpha \ldots\ldots (25b)$$

Apply the determinant, substitute $C = F - i\, G$, and $F'_3 = F_3 - \pi\, r_1/s_3$, and separate the real and imaginary terms. The imaginary terms yield

$$X_2 = H_2 - H_1\, X_1 \ldots\ldots\ldots\ldots\ldots\ldots (26a)$$

in which $H_1 = \dfrac{J_1}{F_1 + k\, d_1}$; $J_1 = q_4\, F_4 + \dfrac{G_2}{k} + k\, d_2$; $J_2 = \dfrac{F_2}{k} - q_4\, G_4$; and

$$H_2 = \frac{\dfrac{q_3}{k}(F_2\, G_3 + F'_3\, G_2) - \dfrac{q_4}{k}(F_1\, G_4 + F_4\, G_1) + d_1\, J_2 - d_2\, G_1}{F_1 + k\, d_1}.$$

The real terms yield

$$2\, X_1 = -H_3 \pm \sqrt{H^2_3 - 4\, H_4} \ldots\ldots\ldots\ldots\ldots (26b)$$

in which $H_3 = \dfrac{G_1}{k} + \dfrac{1}{H_1}\left(\dfrac{J_2}{k} - H_2\right)$; and $H_4 = \dfrac{1}{k\, H_1}\Big[-G_1\, H_2 + \dfrac{q_4}{k}(F_1\, F_4$ $- G_1\, G_4) - \dfrac{q_3}{k}(F_2\, F'_3 - G_2\, G_3) + d_1\, J_1 + d_2\, F_1\Big]$.

Let $a_2 = a_1\,(\omega_2/\omega_1)^2$. By definition of X_1 and X_2, and from Eq. 26*a*,

$$(a_2 - a_1) = \left(\tfrac{1}{2}\, s_2\, X_2 - \tfrac{1}{4}\, a_2\, s_1\, X_1\right)\mu \ldots\ldots\ldots\ldots (27a)$$

$$\left(\frac{\omega_1}{\omega}\right)^2 = 1 + \frac{\mu\, s_1}{4}\, X_1 \ldots\ldots\ldots\ldots\ldots\ldots (27b)$$

$$\left(\frac{\omega_2}{\omega}\right)^2 = 1 + \frac{\mu\, s_2}{2\, a_1}\, X_2 \ldots\ldots\ldots\ldots\ldots\ldots (27c)$$

and

$$\left(\frac{\omega_1}{\omega}\right)^2 = \frac{2\,a_1/s_2 + 4\,H_1/s_1 + \mu\,H_2}{2\,a_2/s_2 + 4\,H_1/s_1} \qquad \text{(27d)}$$

in which the last term $\mu\,H_2$ is relatively negligible. By Eq. 27*d*, the coupled frequency ω will normally have an intermediate value between ω_1 and ω_2.

With a few trial values of k and interpolating, the values of X_1 and X_2 are found to satisfy Eq. 27*a*, and the coupled frequency ω is then given by any of the remaining Eqs. 27. The critical (flutter) velocity is then given by

$$V_c = \frac{\omega\,b}{2\,k} \qquad \text{(28)}$$

The amplitude ratio (including the phase difference ψ) is given by Eqs. 25, yielding

$$\frac{2\,\eta}{\alpha\,b} = \frac{C_1/k - i\,q_3\,(C_3 - \pi\,r_1/s_3)}{k\,X_1 + i\,C_1 + i\,k\,d_1} = \frac{1}{C_2}\,(-\,q_4\,C_4 + i\,k\,X_2 - k\,d_2) - \frac{i}{k} \qquad \text{(29)}$$

For zero damping, the foregoing solution is simplified by dropping the d_1-term and the d_2-term from the equations.

Coupled Oscillations with Damping: Numerical Example.—As shown by Figs. 14 and 16, the flat plate ($d/b = 0.02$) is a stable section against both vertical and torsional oscillations. Such sections have potential instability in coupled oscillations.

The following data are assumed:

$$\begin{array}{llllll} \omega_1 = 0.867 & a_1 = 0.625 & \mu = 0.0318 & s_1 = 5.25 & s_3 = -\,1.75 & q_3 = -\,0.33 \\ \omega_2 = 1.552 & a_2 = 2.00 & b = 60 & s_2 = 1.10 & s_4 = \,0.55 & q_4 = \,0.50 \end{array}$$

Let the structural damping (in both types of oscillations) be $\delta_s = 0.05$. Hence, $g_1 = g_2 = 0.016$, $d_1 = 0.383$, and $d_2 = 0.571$. The damping raises V_c, and therefore lowers k.

Estimating k by short-cut formulas (subsequently presented), the F-values and G-values in Table 2(*a*) are read from Figs. 11 and 12, adding $-\,\pi\,r_1/s_3 = 1.80$ to the F_3-values to obtain F'_3. The computation of X_1 and X_2 by Eqs. 26 is recorded in Table 2(*b*). To satisfy Eq. 27*a*, $X_2 - 4.77\,X_1 = 78.62$. Hence, by interpolation in Table 2,

$$k = 0.349 \qquad \begin{array}{ll} X_1 = -\,14.97 & H_1 = \,0.504 \\ X_2 = \,7.36 & H_2 = -\,0.159 \end{array}$$

By any of Eqs. 27*b*, 27*c*, and 27*d*, $\omega = 1.41$. Hence, $V_c = 121 = 83$ miles per hr. By either of Eqs. 29, the amplitude ratio is $2\,\eta/(\alpha\,b) = -\,0.546 - 0.182\,i = -\,0.58e^{0.32i}$. The angular oscillation dominates. The vertical oscillation leads by $\psi = 0.32 = 0.051$ cycle. The center of oscillation is shifted $0.58\,\frac{b}{2}$ toward the leading edge. The complete solution is self-checking when the two Eqs. 29 yield identical values of $2\,\eta/(\alpha\,b)$.

For the case of zero damping, the solution is similar, but with the d_1-term and the d_2-term dropped from the equations. Somewhat higher values of k are used as trial values ($k = 0.45$, 0.475, and 0.50). The solution yields

$$k = 0.462 \quad \begin{array}{lll} X_1 = -15.59 & H_1 = 0.347 & \omega = 1.46 \\ X_2 = 4.26 & H_2 = -1.144 & V_c = 65 \text{ miles per hr.} \end{array}$$

Again by Eqs. 29, the amplitude ratio is $2\,\eta/(\alpha\, b) = -0.314 - 0.081\, i = -0.32\, e^{0.25 i}$.

Coupled Oscillations: Simplified Approximate Formulas and Short Cuts.—Eq. 27*a* may be written in the form:

$$X_2 = A\,X_1 + B \dots\dots\dots\dots (30)$$

in which A and B are known constants of the section; thus, $A = \dfrac{a_2\, s_1}{2\, s_2}$ and $B = \dfrac{2\,(a_2 - a_1)}{\mu\, s_2}$. From Eqs. 26*a* and 30,

$$X_1 = \frac{H_2 - B}{A + H_1} \dots\dots\dots\dots (31a)$$

With H_2 generally negligible in comparison with B, Eq. 31*a* may be reduced to

$$X_1 \approx \frac{-B}{A + H_1}; \qquad X_2 \approx -H_1\, X_1 \dots\dots\dots\dots (31b)$$

Without requiring the final value of k, Eqs. 31 offer an abridged solution, yielding values of X_1 and X_2 that are fairly constant over a substantial range of values of k. With X_1 and X_2 known, the frequency ω is given by Eqs. 27. Dropping the small H_2-term, Eq. 27*d* yields, with sufficient accuracy,

$$\left(\frac{\omega}{\omega_1}\right)^2 = \frac{H_1 + A}{H_1 + \dfrac{a_1}{a_2} A} \dots\dots\dots\dots (32a)$$

and

$$\left(\frac{\omega}{\omega_2}\right)^2 = \frac{H_1 + A}{\dfrac{a_2}{a_1} H_1 + A} \dots\dots\dots\dots (32b)$$

—requiring only H_1 for the determination of ω, and yielding a fairly constant value of ω over a substantial range of values of k.

To test the practical accuracy of the abridged solutions given by Eqs. 30 to 32, they are applied to the data of the preceding numerical example (using $A = 4.77$ and $B = 78.62$) and the results are given in Table 3.

If H_3 and H_4 are known for different values of k, and with X_1 known from Eq. 31*a* or Eq. 31*b*, the true value of k is given, with sharp definition, by applying the relation (from Eq. 26*b*):

$$f\,(X_1) = X_1 + H_3 + \frac{H_4}{X_1} = 0 \dots\dots\dots\dots (33)$$

This computation is illustrated in Table 4(*a*) by application to the data of the preceding numerical example, from which the true value is found to be $k = 0.349$.

Dropping the relatively small term H_4/X_1, Eq. 33 yields, approximately,

$$X_1 \approx -H_3 \approx \frac{H_2 - J_2/k}{H_1} \quad\quad (34)$$

TABLE 2.—NUMERICAL EXAMPLE;

Ratio, k	(*a*) From Figs. 11 and 12							
	F_1	F_2	F'_3	F_4	G_1	G_2	G_3	G_4
0.30	0.970	1.015	2.690	0.935	0.170	−0.025	0.475	0.285
0.35	0.963	1.023	2.668	0.917	0.197	−0.037	0.534	0.323
0.40	0.953	1.032	2.633	0.895	0.224	−0.050	0.600	0.364

Upon substituting the expressions for H_2 and J_2, and dropping the minor terms, Eq. 34 yields

$$k^2 \approx \frac{F_1\, F_2}{X_2\,(F_1 + k\, d_1)} \quad\quad (35a)$$

which, for zero damping, reduces to

$$k^2 \approx \frac{F_2}{X_2} \quad\quad (35b)$$

Eqs. 35 offer a short cut for preliminary approximate estimates of k. For the data of the preceding examples, the results are listed in Table 4(*b*). With $\omega = 1.41$, the approximate value of $k = 0.346$ (instead of the true value of $k = 0.349$) yields $V_e = 83.5$ miles per hr (instead of the true value of 83 miles per hr).

Eqs. 35 may be written in terms of the ratio of vertical to torsional instability (or stability), δ_1/δ_2, as given by Figs. 14 and 16. By Eqs. 13 and 16, for zero structural damping,

$$H_1 = \frac{J_1}{F_1} = A\,\frac{a_1}{a_2}\,\frac{\delta_2}{\delta_1} \quad\quad (36)$$

Substituting Eq. 36 in the expression for X_2 of Eq. 31*b*, Eq. 35*b* becomes

$$k^2 \approx \frac{F_2}{B}\left(1 + \frac{a_2}{a_1}\,\frac{\delta_1}{\delta_2}\right) \quad\quad (37a)$$

With damping included,

$$k^2 \approx \frac{F_2}{B}\left(1 + \frac{a_2}{a_1}\,\frac{\delta_1}{\delta_2}\right)\frac{F}{F_1 + k\, d_1} \quad\quad (37b)$$

The added factor indicates the reduction of k (increase of V_e) by structural damping.

For the physical significance of the ratio a_1/a_2, recurring in these formulas, by definition, $\frac{a_1}{a_2} = \left(\frac{\omega_1}{\omega_2}\right)^2$. Hence, for a section with $\omega_1 = \omega_2$, the foregoing approximate formulas yield $B = 0$, $k = \infty$, and the critical velocity $V_c = 0$; such section would be unstable (for coupled oscillations) at all wind velocities. As the difference between the frequencies $(\omega_2 - \omega_1)$ is reduced, the critical velocity is lowered. Actually the lower limit of V_c is given by the discriminant of Eq. 26*b*, $H^2_3 - 4\,H_4 = 0$, or $2\,X_1 = -\,H_3$. This occurs with ω_2 slightly exceeding ω_1. If the differences between the frequencies were reversed, with ω_1 exceeding ω_2, then B would be negative and, by Eqs. 37, k would be imaginary, indicating that the section would be stable, with no critical velocity at which self-excited coupled oscillations could develop.

COUPLED OSCILLATIONS WITH DAMPING

(*b*) By Eqs. 26									Ratio, k
J_1	J_2	H_1	H_2	H_3	H_4	X_1	X_2	$X_2 - 4.77\,X_1$	
0.556	3.241	0.514	−0.031	21.71	33.80	−20.02	10.27	105.77	0.30
0.552	2.763	0.504	−0.162	16.56	25.92	−14.81	7.30	77.94	0.35
0.551	2.398	0.499	−0.269	13.12	20.22	−11.33	5.38	59.42	0.40

TABLE 3.—COUPLED OSCILLATIONS WITH DAMPING, SIMPLIFIED APPROXIMATE METHODS

Trial k	H_1	X_1, Eq. 31*b*	X_2, Eq. 31*b*	ω, Eqs. 32	H_2	X_1, Eq. 31*a*	X_2, Eq. 30	ω, Eqs. 27
0.30	0.514	−14.88	7.65	1.41	−0.031	−14.91	7.50	1.41
0.35	0.504	−14.91	7.52	1.41	−0.162	−14.94	7.36	1.41
0.40	0.499	−14.92	7.45	1.41	−0.269	−14.97	7.21	1.42
True value: 0.349	0.504	−14.94	7.36	1.41	−0.159	−14.94	7.36	1.41

TABLE 4.—COMPUTATION OF THE FREQUENCY RATIO, k

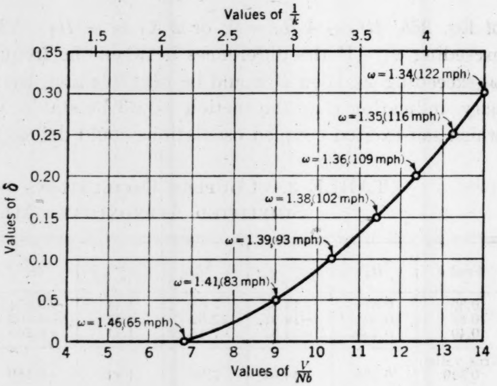
Trial k	(*a*) TRUE VALUE BY EQ. 33				(*b*) APPROXIMATE VALUE BY EQS. 35				
	H_3	H_4	X_1, Eq. 31*a*	$f(X_1)$	F_1	$F_1 + k\,d_1$	F_2	X_2, Eq. 31*b*	k, Eq. 35*a*
0.30	21.71	33.80	−14.91	4.54	0.970	1.085	1.015	7.65	0.344
0.35	16.56	25.92	−14.94	−0.11	0.963	1.097	1.023	7.50	0.346
0.40	13.12	20.22	−14.97	−3.20	0.953	1.106	1.032	7.45	0.346

Coupled Oscillations: Comparison with the Theodorsen Equations.—For the same applications to coupled oscillations, Mr. Theodorsen's equations yield lower values of k and corresponding higher values of the critical velocity V_c. Part of the difference is due to the application to a real section ($d/b = 0.02$), instead of an idealized case with $s_1 = 2\pi$ and $s_3 = \pi/2$. If, however, the Theodorsen coefficients $\left(C,\ C,\ C,\ \frac{1 - C}{2\,q_4}\right)$ are substituted for the writer's general coefficients (C_1, C_2, C_3, and C_4), and the classical values of s_1, s_2, s_3, and s_4 are used, a perfect check is obtained. The writer's equations for coupled oscillations (both exact and simplified) provide a convenient and completely generalized solution for the application of any of the theories. With the other sets of coefficients substituted, their unifying relations and identities permit a still further simplification of the writer's formulas, yielding solutions of maximum simplicity for the problem of coupled oscillations, a problem commonly regarded as recondite and formidable.

Instability Graph for Coupled Oscillations.—The critical velocity yielded by the equations for coupled oscillations is the wind velocity of zero net amplification or damping, represented by $\delta - \delta_s = 0$, or $\delta = \delta_s$. Consequently, the critical velocity computed for any given value of the (positive) structural damping δ_s is also the wind velocity that produces an equal negative damping or potential amplification ratio δ. By calculating the critical velocities for different values of δ, the instability curve for coupled oscillations is obtained.

TABLE 5.—NUMERICAL EXAMPLE, CONSTRUCTION OF INSTABILITY GRAPH

δ	k	ω	V_c	$V/(N\,b)$
0	0.462	1.46	65	6.81
0.05	0.349	1.41	83	9.02
0.10	0.304	1.39	93	10.37
0.15	0.274	1.38	102	11.48
0.20	0.254	1.36	109	12.39
0.25	0.237	1.35	116	13.28
0.30	0.223	1.34	122	14.10



As shown in the preceding section, the approximate short cut, Eq. 35*a*, yields a solution very close to the exact values. This provides a quick method for computing and plotting the instability graph (or negative damping curve) for coupled oscillations.

Coupled Oscillations: Instability Graph (Numerical Example).—For the same section ($d/b = 0.02$) and data as in the preceding examples, with trial $k = 0.30$ for the new values, Eq. 35*a* yields the values of k shown in Table 5, for the different values of δ, and Eqs. 32 yield the corresponding values of ω. Plotting these values, the instability graph for coupled oscillations will be as shown in the table.

Coupled Oscillations in Spans with Vertical and Torsional Modes Dissimilar.—The preceding analysis for coupled oscillations, derived for a unit section, is valid for a complete bridge provided the vertical and torsional wave forms are practically similar over the span or spans, with common nodes, so that the amplitude ratio η/α is uniform throughout the length. If the vertical and torsional modes differ materially, a correction is required in the analysis.

The two force equations for a unit section, such as Eqs. 10 or 25, may be represented in abbreviated form as $(A_1)\,\frac{2\,\eta}{b} + (A_2)\,\alpha = 0$; and $(B_1)\,\frac{2\,\eta}{b} + (B_2)\,\alpha = 0$—for the vertical and torsional forces, respectively.

Let $2\,\eta/b = u_1\,\phi_1\,(x)$; and $\alpha = u_2\,\phi_2\,(x)$, in which u_1 and u_2 are time functions, and ϕ_1 and ϕ_2 are the respective (dissimilar) wave forms (along the span or spans). The virtual work done by the vertical and torsional forces, respectively (in an element of time), over the spans will then be

$$\delta W = A_1\,u_1\,\delta u_1 \int_0^L \phi_1\,\phi_1\,dx + A_2\,u_2\,\delta u_1 \int_0^L \phi_1\,\phi_2\,dx = 0$$

and

$$\delta W = B_1\,u_1\,\delta u_2 \int_0^L \phi_1\,\phi_2\,dx + B_2\,u_2\,\delta u_2 \int_0^L \phi_2\,\phi_2\,dx = 0$$

Write the numerical expressions for $u_1\,\phi_1$ and $u_2\,\phi_2$ in normalized form so as to make $\int_0^L \phi_1\,\phi_1\,dx = 1$; $\int_0^L \phi_2\,\phi_2\,dx = 1$; and $\int_0^L \phi_1\,\phi_2\,dx = U$. The virtual work equations then reduce to $(A_1)\,u_1 + (A_2)\,U\,u_2 = 0$; and $(B_1)\,U\,u_1 + (B_2)\,u_2 = 0$. Substituting the expressions represented by A_1, A_2, B_1, and B_2 from Eqs. 25,

$$\left(X_1 + i\,d_1 + \frac{i}{k}\,C_1\right) u_1 = \left(\frac{1}{k^2}\,C_1 - \frac{i}{k}\,q_3\,C_3\right) U\,u_2 \ldots\ldots (38a)$$

and

$$\left(\frac{i}{k}\,C_2\right) U\,u_1 = \left(-\,X_2 - i\,d_2 + \frac{1}{k^2}\,C_2 - \frac{i}{k}\,q_4\,C_4\right) u_2 \ldots\ldots (38b)$$

Proceeding as with the solution of Eqs. 25, a duplication of Eqs. 26 to 28 is obtained, with U-terms added in the expressions for H_2 and H_4 (also in Eq. 29) as the sole modifications:

$$H_2 = \frac{\frac{q_3}{k}\,U^2\,(F_2\,G_3 + F'_3\,G_2) - \frac{q_4}{k}\,(F_1\,G_4 + F_4\,G_1) + d_1\,J_2 - d_2\,G_1 + \frac{1}{k^2}\,(1 - U^2)(F_1\,F_2 - G_1\,G_2)}{F_1 + k\,d_1} \ldots (39a)$$

and

$$H_4 = \frac{1}{k\,H_1}\left[-\,G_1\,H_2 + \frac{q_4}{k}\,(F_1\,F_4 - G_1\,G_4) - \frac{q_3}{k}\,U^2\,(F_2\,F'_3 - G_2\,G_3) + d_1\,J_1 + d_2\,F_1 + \frac{1}{k^2}\,(1 - U^2)\,(F_1\,G_2 + F_2\,G_1)\right] \ldots (39b)$$

—and Eq. 29 is replaced by

$$\frac{u_1}{u_2} = \left(\frac{C_1/k - i\,q_3\,(C_3 - \pi\,r_1/s_3)}{k\,X_1 + i\,C_1 + i\,k\,d_1} \right) U = \frac{1}{U} \left(\frac{-\,q_4\,C_4 + i\,k\,X_2 - k\,d_2}{C_2} - \frac{i}{k} \right) \quad \text{..(40)}$$

For the case in which vertical and torsional modes are similar, $U = 1$, and Eqs. 39 and 40 reduce to the simpler ones previously derived.

Coupled Oscillations with Dissimilar Modes: Numerical Example.—Let the span lengths equal $l = 2{,}800$; $l_1 = 1{,}100$; and $(L = l + 2\,l_1)$; and let the wave forms be

Vertical mode—main span: $\phi_1 = \alpha_1\,(\sin \pi\,x/l - 0.113 \sin 3\,\pi\,x/l)$
Vertical mode—side spans: $\phi_1 = \alpha_1\,(0.740 \sin \pi\,x/l_1)$
Torsional mode—main span: $\phi_2 = \alpha_2\,(\sin \pi\,x/l - 0.493 \sin 3\,\pi\,x/l)$
Torsional mode—side spans: $\phi_2 = \alpha_2\,(0.493 \sin \pi\,x/l_1)$

To normalize these expressions:

$$\int_0^L \phi_1\,\phi_1\,dx = 1; \qquad \text{hence, } \alpha_1 = \frac{0.830}{\sqrt{0.5\,l}}$$

$$\int_0^L \phi_2\,\phi_2\,dx = 1; \qquad \text{hence, } \alpha_2 = \frac{0.835}{\sqrt{0.5\,l}}$$

Hence, $U = \int_0^L \phi_1\,\phi_2\,dx = 0.932$.

For the section data of the preceding examples $(d/b = 0.02)$, with zero damping, the solution including the U-correction, using Eqs. 39 and 40, yields $k = 0.43$, $\omega = 1.46$, $V_c = 69$ miles per hr, and $u_1/u_2 = -0.34 - 0.089\,i = -0.35\,e^{0.26\,i}$.

Conclusion

General expressions have been derived for the aerodynamic forces on any oscillating section and for the opposing dynamic forces. The resulting general equations yield solutions for all problems of vertical, torsional, and coupled oscillations, including analytical determination and prediction of frequencies, critical velocities, rate of amplification, and limiting amplitudes. Maximum simplicity and practical applicability have been the governing considerations. Numerical applications, graphic methods, and simplifying short cuts have been included.

The methods of classical aerodynamic theory are limited to idealized streamline sections and are not applicable to other sections, such as practical bridge sections. In order to overcome this limitation and to secure complete generality of application to any form of section whatsoever, the unknown parameters and coefficients in the aerodynamic equations are supplied by static wind-tunnel tests on section models, using tilted straight models to represent the effects of vertical velocity and angular position, and horizontal curved models to represent the effects of angular velocity. With the use of curved models and the added consideration of the effects of varying angular position, the results of prior analysis of the problem of bridge oscillations have been extended and amplified toward greater finality and verified validity.

The coefficients representing the correction for phase lag have been derived in simple, general form from the pressure distribution graphs for the straight and curved models, by simply applying to the pressure ordinates the progressive phase difference across the width of the section. In classical aerodynamic theory, the corresponding coefficients (for the limited case of an ideal flat plate or streamline airfoil) are derived from the theoretical discontinuity or vortex sheet extending from the trailing edge of the section to infinity, so that the coefficients theoretically represent a function of the complete past history of the motion of the oscillating section. The coefficients derived by the two different concepts are not identical. The classical derivation is not applicable to nonstreamline sections, and no method of adapting it so as to fit the known behavior of nonstreamline sections is apparent. The coefficients derived by the simpler concept do fit the known behavior of nonstreamline sections. The equations of this paper, however, are written in completely general form so as to retain their usefulness and validity for any substitution of coefficients derived by other theories, existing or anticipated.

The analysis here submitted is believed to be the first generalized aerodynamic theory of oscillating sections, not limited to sections of any one type. It is also believed to constitute the aerodynamic theory of bridge oscillations brought to greater completeness and generality than any prior presentation. It is submitted to the profession in order that constructive review and discussion may further advance the solution of the problem toward definitive completeness and finality.

DISCUSSION

David J. Peery,[25] **M. ASCE.**—The writer had the privilege of discussing and checking much of the author's work during the preparation of the paper, which was completed in essentially the present form early in 1947. The author has combined extensive research into the fields of aerodynamics and vibrations with his knowledge of suspension bridges. This paper presents a reliable and practical analysis. The following discussion will further confirm the validity of using tests on static models.

An oscillating suspension bridge is under the action of elastic, inertia, damping, and aerodynamic forces. The elastic and inertia forces may be computed accurately, and the natural modes of vibration resulting from these forces have been obtained by several investigators and verified by model tests. The structural damping forces are more difficult to determine, since models do not simulate the true conditions in the prototype. Fortunately it is conservative to assume a limiting case of no damping.

The aerodynamic forces on a thin airfoil oscillating in an ideal fluid have been obtained theoretically. Theodor von Kármán, Hon. M. ASCE, and W. R. Sears[26] have shown that these forces may be resolved into three components as follows:

(*a*) Forces that would be produced if the wake had no effect, called "quasi-steady" forces;
(*b*) Forces produced by the reaction of the fluid accelerated by the motion of a solid body, or "apparent mass" forces; and
(*c*) Forces that depend on the vorticity distribution in the wake.

The forces in group (*a*) are readily obtained by wind-tunnel tests of straight and curved models under steady flow conditions. The forces in group (*b*) are in phase with the inertia forces acting on the oscillating mass of the structure, and are negligible in comparison for suspension bridges. The forces in group (*c*) may be evaluated theoretically for a thin airfoil, but a similar analysis for a suspension bridge cross section appears impossible.

The theoretical thin airfoil, shown in Fig. 24(*a*), has a lift force of $\pi \rho V^2 b \alpha$ acting at the quarter chord point. Similarly, the curved thin airfoil in Fig. 24(*b*) has a lift force of $\pi \rho V^2 b \beta/2$ acting at the midpoint. These theoretical forces differ slightly from experimental values because the airflow tends to separate from the leading edge of a thin plate in violation of the assumed condition of a rounded leading edge. For the conditions shown in Fig. 8(*b*) the theoretical pressures are negative at all points, and are symmetrically distributed as the ordinates of a semi-ellipse. The positive experimental values shown result from flow separation at the corner of the leading edge, and this condition also exists for practical bridge roadways.

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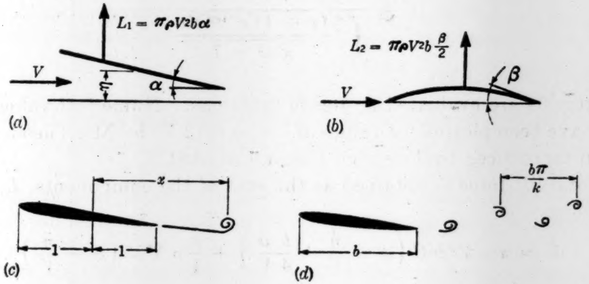
[25] Head, Dept. of Aeronautical Eng., Pennsylvania State College, State College, Pa.

[26] "Airfoil Theory for Non-Uniform Motion," by Theodor von Kármán and W. R. Sears, *Journal of the Aeronautical Sciences*, August, 1938, p. 379.

For the oscillating thin airfoil, the effective angle of attack is decreased because of the vertical velocity $\dot{\eta}$. The vertical velocity varies across the width, and has a value of $\dot{\eta} - \frac{b\,\dot{\alpha}}{4}$ at the third quarter chord point, which determines the angle of attack of the airfoil. The effective angle of attack is therefore $\alpha - \frac{\dot{\eta}}{V} + \frac{b\,\dot{\alpha}}{4\,V}$. Substituting this value for the lift force in Fig. 24(*a*) the following value is obtained:



$L_1 = \pi \rho V^2 b \alpha$
V
η
α
(a)
$L_2 = \pi \rho V^2 b \frac{\beta}{2}$
β
V
(b)
x
1
1
(c)
$\frac{b\pi}{k}$
b
(d)

Fig. 24

$$L_1 = \pi\,\rho\,V^2\,b\left(\alpha - \frac{\dot{\eta}}{V} + \frac{b\,\dot{\alpha}}{4\,V}\right)\ldots\ldots\ldots\ldots\ldots\ldots(41)$$

The effective angular velocity at the midchord point is equal to $\dot{\alpha} - \frac{\ddot{\eta}}{V}$. From the author's discussion of curved models, this is equal to $\frac{2\,V\,\beta}{b}$, with β as shown in Fig. 24(*b*). The following lift force resulting from the effective angular velocity is obtained from Fig. 24(*b*), thus:

$$L_2 = \frac{\pi}{4}\,\rho\,V\,b^2\left(\dot{\alpha} - \frac{\ddot{\eta}}{V}\right)\ldots\ldots\ldots\ldots\ldots\ldots(42)$$

The last term of Eq. 42 represents the "apparent mass" effect, and is equal to the inertia force on a cylinder of air of diameter *b*. The other terms of L_1 and L_2 are quasi-steady forces, which the author obtains from wind-tunnel tests of static models similar to those shown in Figs. 24(*a*) and 24(*b*).

When an airfoil is suddenly moved so that the lift is changed, a vortex is shed from the trailing edge, and this vortex moves downstream with the fluid as shown in Fig. 24(*c*). The lift on the airfoil does not reach its final value until this vortex has moved an infinite distance downstream, and the lift is considerably smaller when the vortex is near the airfoil. The vortex trail behind an oscillating airfoil is shown in Fig. 24(*d*), and the reduction in lift, $-L_3$, resulting from this vortex trail, may be expressed in the following form:

$$-L_3 = (1 - C)\,L_1\ldots\ldots\ldots\ldots\ldots\ldots(43)$$

The term C is evaluated by Theodore Theodorsen[2] by considering the vortex trail between the trailing edge ($x = 1$) and infinity; thus:

$$C = \frac{\displaystyle\int_1^{\infty} \frac{x\, e^{-ikx}\, dx}{\sqrt{x^2 - 1}}}{\displaystyle\int_1^{\infty} \frac{(x + 1)\, e^{-ikx}\, dx}{\sqrt{x^2 - 1}}} \qquad (44)$$

These integrals are evaluated as Bessel functions. Numerical values of $C = F + i\,G$ have been plotted for values of $k = \omega\, b/(2\, V)$ by Mr. Theodorsen, and have been reproduced by Friedrich Bleich,[27] M. ASCE.

The total lift force is obtained as the sum of the components, L_1, L_2, and L_3, as follows:

$$L = \pi\, \rho\, V^2\, b\, C \left(\alpha - \frac{\dot{\eta}}{V} + \frac{b\, \dot{\alpha}}{4\, V}\right) + \frac{\pi}{4}\, \rho\, V\, b^2 \left(\dot{\alpha} - \frac{\ddot{\eta}}{V}\right) \qquad (45)$$

Eq. 45, obtained by combining the steady flow conditions of Figs. 24(a) and 24(b) with the wake effects of Fig. 24(d), corresponds with the values obtained by Mr. Theodorsen and other investigators from considerations of the velocity potential for unsteady flow conditions. This verifies the author's procedure of using static models to obtain the quasi-steady forces.

The forces resulting from the wake vorticity will be considerably different for a bridge cross section than for an airfoil. Eqs. 43 and 44 are derived for an airfoil by assuming that the Kutta condition, in which the flow leaves the sharp trailing edge, is satisfied at any instant. For an oscillating bridge cross section with a blunt trailing edge, the trailing vortices may be shed at various points. The author has evaluated the wake effects from the pressure distribution on a static model, considering a phase lag. This procedure involves more approximations than were used in obtaining the quasi-steady forces, but appears to be the best method available for evaluating these effects.

At present, two methods are available for calculating aerodynamic forces on oscillating bridge sections. Mr. Bleich[27] uses the basic forces for a thin airfoil and superimposes an alternating force that must be determined empirically from dynamic model tests. This method applies only to open truss structures which approximate a flat plate. The author determines all forces from static model tests, and thus obtains a solution which is applicable to all cross sections, including girder-stiffened roadways.

In reviewing the present state of knowledge of suspension bridge vibration analysis, it appears that a designer who makes use of the available theory may be confident of the safety of his structure. Static models should be tested carefully. The models should span between walls of the tunnel to insure two-dimensional flow. Since flow separation effects are important, the Reynolds number for the model flow should correspond with that for the prototype.

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[27] "Dynamic Instability of Truss-Stiffened Suspension Bridges Under Wind Action," by Friedrich Bleich, *Transactions*, ASCE, Vol. 114, 1949, p. 1177.

Tests on a dynamic model of the final structure are probably advisable in order to verify the calculations of aerodynamic forces.

Since aerodynamic forces cannot be obtained exactly, some conservatism in the design for wind forces is desirable. However, it seems overconservative to revert to the heavy types of stiffening trusses which were used in the early 1900's. If such conservatism is substituted for scientific analysis, forty years of progress in suspension bridge design will have been lost.

ALEXANDER KLEMIN.[28]—By sheer virtue of necessity, aeronautics leads the field in its knowledge of aerodynamics and in its powerful methods of attacking problems of oscillation. For example, in the study of wing flutter, the aeronautical engineer considers the following factors: Mass, moment of inertia, elastic resistance to distortion, gravitational resistance to displacement, internal friction of the material, aerodynamic forces, and damping moments and their changes with vertical and angular velocities. He must make full use of the wind tunnel, in force and oscillation tests, using specially constructed models which follow the laws of dynamic similarity. When he has written down the long, complicated equations, he finds that they cannot be solved because aerodynamic forces vary as the square of the velocity, and the differential equations are no longer (alas) linear differential equations with constant coefficients. He then introduces the powerful idea of "resistance derivatives," which render the equations tractable. When attempts are made to deal with aeronautical oscillation problems by "short-cut methods," these methods fail. It is apparently essential to make a complete, recondite analysis, to take all factors into account, and to study "coupling," which is the bugbear of electrical as well as of aeronautical engineering.

From the writer's reading of the subject of bridge oscillations, he has the impression that civil engineers have tried to solve such problems by simplified methods, without sufficient aerodynamic data, and have failed.

The great merit of this paper lies in the fact that the author has boldly, and with originality, adapted aeronautical methods, both experimental and analytical, to the problem of bridge oscillations. The writer is satisfied that the methods which proved sound in aeronautics have been correctly and carefully adapted in this paper. Thus, perhaps for the first time in engineering literature, a paper has provided a sound basis for the study of bridge oscillations.

ABRAHAM SLAVIN,[29] M. ASCE.—The subject matter in this paper is essentially related to aerodynamic stability of bridges, although it is completely generalized for applicability to other types of section. The author had previously developed simple criteria and formulas for determining the aerodynamic stability of suspension bridges.[30] These simple criteria are of particular sig-

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[30] "Rigidity and Aerodynamic Stability of Suspension Bridges," by D. B. Steinman, *Transactions*, ASCE, Vol. 110, 1945, p. 439.

nificance and considerable practical value.[31,32,33] The paper under discussion is a continuance of this pioneering work and a notable contribution to engineering science. It represents creative research work in solving a new and critical problem with which the profession has been dramatically confronted. For full appreciation of the analytical and practical significance of this important contribution, it should be recognized that it is the first complete theory of aerodynamic oscillations which has the following pertinent features:

(*a*) It is not limited to streamline airfoil sections and idealized thin plates but is applicable to all bridge sections of whatever type or form, including flat plates, girder-stiffened sections, truss-stiffened sections, and all modified or combination sections;

(*b*) It is not limited to coupled oscillations (as in airplane flutter) but also covers vertical oscillations and torsional oscillations (which are the known forms of bridge instability);

(*c*) It is not limited to determining critical flutter velocity but also determines and predicts critical velocities and, in addition, negative damping, rate of amplification, limiting amplitudes, and amplitude response at all wind velocities;

(*d*) It does not require oscillating model tests, which are costly and time consuming, but instead determines all necessary constants of any section by simple static tests on small-scale section models; and

(*e*) It is independent of the known defects and inconsistencies of conventional aerodynamic theory.

Those who have sought to apply conventional airfoil theory to bridge sections have overlooked the fact that the conventional theory assumes rounded leading edges and sharp, tapering trailing edges, and any attempt to apply the theory to bridge sections or even to thin flat plates introduces material errors in the critical region in the vicinity of the leading edge. It should be evident that such reasoning, although perhaps excellent for mathematical exposition, does not apply to the structural sections in bridges. The nonstreamline sections in bridges do not fit the classical derivation, in aerodynamic theory, of the behavior of the streamline sections.

In this paper, the author has combined aerodynamic science, vibration theory, and bridge dynamics, to create a new scientific approach to the problem of bridge aerodynamics. The resulting contribution is a technical achievement that is of the highest practical usefulness.

CHARLES W. DOHN,[34] JUN. ASCE.—These studies of aerodynamic theory and stability as related to bridge oscillations, and particularly as related to suspension bridge oscillations, are stimulating. They point a way toward the design of aerodynamically stable bridges with a minimum of material for maximum stability. However, many engineers (and even those who may have frequent occasion to design such bridges) will probably feel hard pressed to

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[31] "Rigidity and Aerodynamic Stability of Suspension Bridges," by D. B. Steinman, *Transactions*, ASCE, Vol. 110, 1945, pp. 572–574.

[32] *Ibid.*, p. 575, Eq. 177.

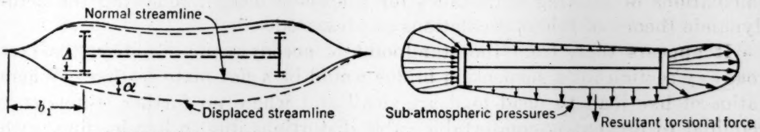
[33] *Ibid.*, p. 576, Eq. 178.

[34] Engr., New York, N. Y.

follow all the masterful mathematics involved and will look to the author for the simplifying concluding recommendations or specifications.

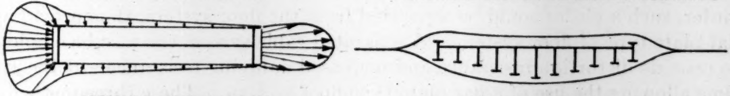
The method of predicting aerodynamic behavior as described, although largely analytical, requires tests on section models. Data are presented for a section consisting of side girders and solid roadway with a ratio of $d/b = 0.20$, and also for a section that is essentially a flat plate with a ratio of $d/b = 0.02$. The former section proves to be an unstable section and the latter section a stable one. It is probably not a widely appreciated fact that truss systems can also be aerodynamically unstable and, hence, that data demonstrating stability and unstability of truss sections would be particularly pertinent. Would data for girders with $d/b = 0.20$ correspond to truss sections with $d/b = 0.40$, as might be inferred from the author's previous paper[35] presented in 1943? The bulletin on wind-tunnel tests of suspension bridge section models by the aerodynamic laboratories of the Virginia Polytechnic Institute[36] at Blacksburg contains experimental data for a wide range of $\frac{d}{b}$-values for both girders and trusses, and also various slotted and finned sections. This bulletin contains a good explanation of the causes of aerodynamic oscillations and arrives at interesting conclusions. For roadway and girder sections, it appears that sections with $d/b < 0.075$ and 0.05 for deck girders are both torsionally and vertically stable. For greater $\frac{d}{b}$-values, lateral slots and fins are recommended to eliminate torsional and vertical instability, and many practical stable aerodynamic sections are presented.

The nature and distribution of the basic pressures which are the cause of the phenomenon of aerodynamic oscillation are illustrated in Fig. 25. Although they are not conclusive, verified, or complete by any means, they are illustrative of how a movement Δ of the **H**-section, in Fig. 25(*a*), causes a displacement of the streamlines and a change in the pressure pattern from that in Fig. 25(*b*) to that in Fig. 25(*c*), thus amplifying the movement or deflection and building



(*a*) STREAMLINES OF FLOW ABOUT AN UNSTABLE SECTION

(*c*) PRESSURE PATTERN OF AN UNSTABLE SECTION DEFLECTING DOWNWARD



(*b*) PRESSURE PATTERN OF AN UNSTABLE SECTION AT REST

(*d*) PARTIAL STREAMLINING TO PRODUCE A MORE STABLE SECTION

Fig. 25.—Airflow About Bridge Sections

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[35] "Rigidity and Aerodynamic Stability of Suspension Bridges," by D. B. Steinman, *Transactions*, ASCE, Vol. 110, 1945, p. 458.

[36] "Wind Tunnel Tests of Suspension Bridge Section Models," by F. J. Maher, D. Frederick, E. R. Estes, and D. B. Steinman, *Bulletin of the Virginia Polytechnic Inst., Eng. Experiment Station Series No. 69*, Blacksburg, Va., September, 1948.

up oscillations. For shallow girders and wide roadways the effect of deflection Δ on angle α and on the pressure pattern is proportionately less. An outside projection or cantilever increases the distance b_1 (Fig. 25(*c*)) so that any deflection will cause a relatively smaller angle α.

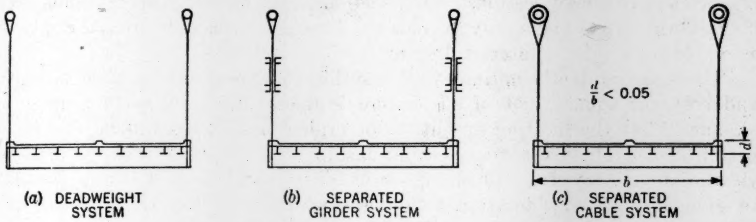
In the case of an unstable section, such as that illustrated in Figs. 25(*a*), 25(*b*), and 25(*c*), a reduction of pressure on the one surface, due to streamline displacement, is greater than any change in pressure on the other surface. The net induced force is in the direction of the movement or the deflection of the section, and oscillations are built up or amplified. The limiting amplitude of these oscillations may now be determined from wind-tunnel tests and the presented theory. The streamlines of flow at different distances from the section will be deflected in varying amounts and it is difficult to evaluate the mass of air undergoing a change of velocity and direction in a given time. Furthermore, eddy currents are formed in the wake of the section, altering the nature of the circulation of air about it; in addition, friction losses as well as turbulence are present. Hence, wind-tunnel tests are necessary for quantitative results. The study of the nature of the flow about the section, however, serves as a qualitative guide to design.

In the case of deck girders the vertical difference in pressure due to movement is accentuated by the greater change in curvature of the streamlines, and, hence, the deck girder type of section is more unstable for a given ratio d/b. Fig. 25(*d*) illustrates a way to obtain streamlining of a limited nature by varying the depth of roadway stringers to obtain stiffening action. This section followed from a direct study of streamlines and would not otherwise be apparent. It is presented as merely suggestive of a qualitative method of section design which can be checked quantitatively by wind-tunnel tests and calculations of limiting amplitudes for a specific design, following the aerodynamic theory of bridge oscillations as presented.

It appears that, from the standpoint of aerodynamic stability, the best roadway section for a suspension bridge would be a flat plate section. Where ratios of live load to dead load are small and where no further stiffening is required to prevent uncomfortable cable distortions due to live loading, such sections are also aerodynamically stable. In bridges requiring a stiffening girder, such a girder could be separated from the floor system, thus providing a flat plate type of floor system. A separated cable system can also be employed to provide distortion resistance and increased damping resistance—at the same time allowing the use of a flat plate type floor section. These three methods of obtaining a flat plate type floor section are illustrated in Fig. 26, in which Fig. 26(*a*) represents the dead-weight system; Fig. 26(*b*), the separated girder system; and Fig. 26(*c*), the separated cable system. Any combination of all three systems is possible and practicable for extremely long spans. In the interest of obtaining a smaller value of d a composite stringer should be used.

A reference list of notations used with dimensional significances, explanatory notes, and cross references would aid in the study and the application of the theory presented in this paper.

What is meant by "coupled oscillations"? It would appear that, according to Table 5, a flat plate with a ratio $d/b = 0.02$ can exhibit instability due to coupled oscillations. How serious can this be for a practical bridge section? What are the proper modifications or specifications to be followed in designing a flat plate type roadway section in order to avoid the instability caused by coupled oscillations? Evidently, from the material presented, if the frequency of vertical oscillation is greater than the frequency of torsional oscillation, stability of a flat plate section is secured against coupled oscillations. Can this relation be obtained practically, and, if so, how? Can the design of a stay system aid in increasing the frequency of vertical oscillation more than the frequency of torsional oscillation and thus assure stability against coupled oscillations with a flat plate type roadway section? Is the omission of stays in such a case advisable? Would a suspension bridge roadway section, which is stiffer in the longitudinal direction, afford more stability against coupled oscillations?



(*a*) DEADWEIGHT SYSTEM (*b*) SEPARATED GIRDER SYSTEM (*c*) SEPARATED CABLE SYSTEM

FIG. 26.—FLAT PLATE TYPE ROADWAY SECTIONS

Returning again to the H-section, as one increases *d* to obtain added stiffness the value of *d*/*b* and the aerodynamic instability of the section increase. What then would be the optimum section to use? From the standpoint of prevention of aerodynamic instability, very small values of $d/b < 0.05$ would appear best, and, for maximum resistance to aerodynamic forces, the highest practicable ratio *d*/*b* would seem best. The present tendency (and that of the immediate past) of using $\frac{d}{b}$-values approximating 0.20 for plate girders and 0.40 for trusses is evidently most dangerous and is dictated mainly by "appearance engineering." The first Tacoma (Wash.) Bridge, which was a stiffened plate girder, had a $\frac{d}{b}$-value of about 0.20. The Delaware Memorial Bridge between Delaware and New Jersey, as designed, has a $\frac{d}{b}$-value of about 0.40 with a stiffening truss. Resort to roadway slots and stays can help prevent oscillations with such ratios; however, it would be better either to increase *d* to obtain added stiffness or to increase *b* to obtain added stability, or to adopt a section other than an H-section. A relatively heavy dead load will stiffen such a bridge against undue oscillation and the engineers in charge of the design of the Delaware Bridge are relying largely on the dead-weight factor. This bridge has long expensive approaches, and an increase of from 10% to 25% in the cost of the suspended structure is almost a minor item.

Taking into account construction cost indexes, the cost per square foot from back to back of abutment anchorages for suspension bridges has constantly decreased. Present costs per square foot of deck are less than one half to one fourth of the costs of early long span suspension bridges varying according to length. This decrease is caused partly by the increased efficiency of the construction machinery and partly by the savings in design. Evidently there are practical limits in the reduction of cost. Complete tabulated data on bridges including live and dead loads, all principal structural features, clearance dimensions, roadway sections, cable sections, and special features, plus cost breakdowns, are not generally available. At present it is difficult and costly to assemble such data as interest dictates.

There are evidently optimum relationships between aerodynamic stability and cost per square foot of suspension bridges. The ability to predict in advance what the limiting amplitude of bridge oscillations will be enables a better choice of this optimum relationship. Furthermore with a broader view of the nature of the phenomenon of bridge oscillations it is now possible to give a fuller consideration to designs which eliminate or minimize the occurrence of such phenomenon.

BORIS A. BAKHMETEFF,[37] HON. M. ASCE.—In this paper, which may be considered as the crowning accomplishment of a lifetime endeavor, the author has now united the results of all his previous work into a comprehensive whole. As evidenced by the references cited, the general subject of periodical action caused by agencies of fluid mechanical origin is not new. However, it was necessary to introduce and develop certain novel concepts before the special oscillations of the type covered in the paper could be treated with the fullness and detail required for making the methods directly applicable to practical design.

The subject matter of the problem is necessarily complex and varied. It involves questions of fluid mechanics, elasticity, operational analysis, etc. The paper demonstrates once more, how the former watertight compartments between lines of specialization crumble, and what type of approach is required by present-day engineering science. Mathematical skill is essential, but by far not sufficient. A clear physical perception of the phenomena involved and their dependence on attendant circumstances is equally imperative, and then an ability, sharpened by training and experience, to discern between the essential and the secondary to work out the practical solutions based on simplified premises. The paper reflects these high attainments. The author has deeply deserved the gratitude of the profession.

Certain features in Mr. Steinman's contribution call for special emphasis, especially from the viewpoint of fluid mechanics. First, there is the clear formulation of the fact that instability arises out of "self-induced" pendulations, with the acting force engendered by the linear or angular displacements of the oscillating member and growing with such displacements. In connection therewith, there is the very simple and lucid criterion for appraising the eventual quality of a structural shape, based on the sense (positive or negative) and the magnitude of the "angular gradient" (a term which the writer prefers to that of "slope" used by the author) of the lift or moment characteristic.

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[37] Prof., Civ. Eng., Columbia Univ., New York, N. Y.

Finally, there is an experimental technique by which, in tilting a structural member, or curving it, one may determine the eventual aerodynamic response of a shape in its swinging or torsional motion, by way of a static test.

The problems involved in the paper required close cooperation between theory and some specially devised experiments. Once more this procedure, which at times has been termed "semi-empirical," has proved to be so fruitful whenever questions of fluid mechanics are concerned. A part of the experiments was conducted with some assistance, direct or indirect, by the Society's Committee on Research. As chairman, the writer feels gratified that the committee has helped in ploughing a furrow which brought such ample harvest.

There is but one suggestion which the writer would like to make. The breadth and variety of the subject matter encompassed into a short paper make the presentation very brief. For the nonspecialized engineer it is not always easy reading. Could one express the hope that the author might find the opportunity for a detailed treatment of the subject in the form of a book, which would allow the noninitiated engineer to acquire the necessary apprenticeship with greater ease and certainty?

F. J. MAHER[38] AND J. B. EADES.[39]—The paper by Mr. Steinman represents the most significant contribution to date in the field of aerodynamic behavior of suspension bridges. The derivation of the aerodynamic lift and moment equations for shapes such as bridge sections is well nigh impossible by any known theoretical approach. The correctness of Mr. Steinman's approach is substantiated by the correspondence of his results with the equations of theoretical aerodynamics. Actually the equations in the paper are in a more general form inasmuch as the coefficients may be applied to any shapes developing lift forces and moments, whether they be airfoils or bridge sections.

The ingenious use of the concept of the curved model has permitted the author to bring his earlier work to completion. The effects of angular velocity on the lift and moment equations have been reduced to quantities that can be readily determined.

The equations in the paper do not include the effect of the drag component except in so far as the lift and moment curves are corrected for drag. When the section is oscillating either vertically or torsionally the drag force experiences a periodic variation of twice the frequency of the primary oscillation as a result of the changing orientation of the section with respect to the relative wind. This variation is superimposed upon some mean value of drag and would presumably give rise to a vibration (in a horizontal plane) about some mean deflected position. A periodic variation in the forward speed of an airplane (phugoid oscillation) is comparable to this possible motion of the deck. The variable drag force should also cause some additional twisting of the deck particularly if the section is not symmetrical. The total effect of the drag force on the oscillations of typical suspension bridges may be rather small.

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[38] Associate Prof. of Applied Mechanics, Virginia Polytechnic Inst., Blacksburg, Va.

[39] Asst. Prof. of Aeronautical Eng., Virginia Polytechnic Inst., Blacksburg, Va.

The writers would like to inquire if any study has been made along these lines and if so what is the approximate magnitude of the effect of the drag force.

The development of the complex coefficients is based on the assumption that the phase lag is linear across the width. The writers believe that an expansion of the section, "Effect of Phase Lag Across the Width of the Section" (under the heading, "General Aerodynamic Equations") particularly with regard to the assumption of linearity, would be helpful. Possibly the author could clarify this point in his closure.

It is unfortunate that more test information regarding the dynamic properties of section models is not available to check the theory presented. The check of the results of Louis G. Dunn,[21] cited by the author, provides a significant correlation. The writers have made some tests in the wind tunnel at the Virginia Polytechnic Institute at Blacksburg with the view toward developing a simple method of obtaining the torsional instability graphs for section models. Basically, the method consists of oscillating the section model at a constant frequency in a variable wind stream. The power needed to keep the model oscillating at a constant frequency and amplitude is measured.

Power differences are measured from the base power required to oscillate the system without the model but with an equal moment of inertia. The logarithmic decrement or increment is then computed from the power differences. Such a curve for a flat plate is shown in Fig. 27 as compared with the curve of Fig. 16. In this case the test values were as follows: $d/b = 0.0125$; $b^2/w = 0.151$; amplitude = 2°; and angle of incidence = 0°. Mr. Steinman's results were for the section $d/b = 0.02$ and $b^2/w = 0.268$. Although the values of b^2/w and of d/b are somewhat different, the experimental curve verifies the general shape of the theoretical curve within the limits tested. Curves for the section $d/b = 0.20$ check qualitatively the results of the author with differences in the magnitude of δ and a slightly higher value of the critical value of $V/(N\,b)$ (about 2.9). The experimental method is not as yet refined and is noted here only to show a correlation in the general form of the curves.

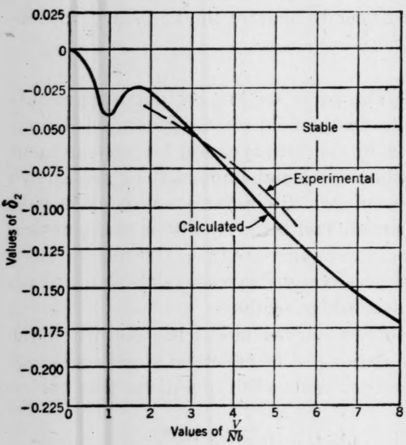

FIG. 27.—TORSIONAL STABILITY GRAPH FOR A FLAT PLATE

The section models used in obtaining the data for the author's calculations were idealized sections without flanges or floor beams and stringers. It was felt that the validity of the general conclusions would not be affected by this idealization. The correlation of the author's results with those of Mr. Dunn seem to bear out this position since Mr. Dunn's tests were made on the actual section model of the Tacoma Narrows Bridge in Washington.

Edmund Pinney.[40]—The crucial point of the paper is the "principle of relativity" as stated under the heading, "Use of Stationary Section Models: Use of Curved Models." This is the notion that the flow about a rotating body immersed in a uniform stream is equivalent to the flow about a stationary body whose form is obtained by curving the original body in a suitable manner. The reasoning employed by the author to justify this principle is not sufficiently clear to reveal its essential limitations. A careful analysis shows that it is valid in the case of laminar flow but fails when free vorticity or turbulence is present. In view of the fact that turbulence is a very prominent feature of the flow about oscillating or unstreamlined bodies such as bridges, a theory of this kind is of questionable value.

To support this principle the author mentions the work of G. A. Gourjienko.[10] This relates to the curvilinear flight of airships in which turbulence is a negligible factor. The reference to R. Kassner and H. Fingado[3] in connection with Analogy 2 is evidence against the principle of relativity, and not for it. If this principle were valid, all the terms (not just one) of the air forces of Messrs. Kassner and Fingado would be derivable from it. This is clearly not the case, because their results agree with those of Theodore Theodorsen.[2]

To show that this principle of relativity is invalid when turbulence is present, it is enough to exhibit a single example in which it fails. The simplest such example known to the writer is that of a thin airfoil of zero camber immersed in a uniform stream of velocity, V, and oscillating about its midchord point with an angle of attack $\alpha(t) = \alpha_o \cos(2\,\pi\,N\,t)$, N being the frequency of the oscillation. According to the author (see text concerning Analogy 2), the forces and moments acting on this airfoil should be the same as those acting on a stationary thin airfoil with a camber equal to $b\,\dot{\alpha}/(8\,V)$.

The lift and moment expressions for the oscillating airfoil have been derived by Messrs. Theodorsen, Küssner, and many others. They are given by the author (see Table 1, Eqs. 2, and nineteen lines after Eq. 37*b*) as

$$L = \tfrac{1}{4}\,\pi\,\rho\,b^2\,V\,\dot{\alpha} + \pi\,\rho\,b\,V^2\,C\left(\pi\,\frac{N\,b}{V}\right)\left(\alpha + \frac{b\,\dot{\alpha}}{4\,V}\right) \ldots\ldots (46a)$$

and

$$M = -\,\frac{\pi\,\rho\,b^3\,V\,\dot{\alpha}}{16} - \frac{\pi\,\rho\,b^4\,\ddot{\alpha}}{128} + \tfrac{1}{4}\,\pi\,\rho\,b^2\,V^2\,C\left(\pi\,\frac{N\,b}{V}\right)\left(\alpha + \frac{b\,\dot{\alpha}}{4\,V}\right) \ldots (46b)$$

in which $C(k)$ is given by Mr. Theodorsen as

$$C(k) = \frac{H^{(2)}{}_1\,(k)}{H^{(2)}{}_1\,(k) + i\,H^{(2)}{}_0\,(k)} \ldots\ldots\ldots\ldots\ldots\ldots (47)$$

In Eq. 47, $H^{(2)}{}_0$ and $H^{(2)}{}_1$ are Hankel functions of the second kind. Substituting the foregoing expression for camber into the formulas[41] for the lift and moment on a cambered thin airfoil:

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[40] Dept. of Math., Univ. of California, Berkeley, Calif.

[41] "General Aerodynamic Theory—Perfect Fluids," by Theodor von Kármán and J. M. Burgers, in "Aerodynamic Theory" (W. F. Durand, editor), Springer, Berlin, 1935, Vol. 2, formulas (3.1), (4.1), (7.13), (10.8), and the footnote on p. 6.

$$L = \pi \rho b V^2 \left(\alpha + \frac{b \dot{\alpha}}{4 V} \right) \ldots\ldots\ldots\ldots\ldots \text{(48a)}$$

and

$$M = \frac{\pi \rho b^2 V^2 \alpha}{4} \ldots\ldots\ldots\ldots\ldots \text{(48b)}$$

In the notation used by Theodor von Kármán, Hon. M. ASCE, and J. M. Burgers, the camber is $f/(2\,b)$. In Eq. 47, $C(k) \rightarrow 1$ as $k \rightarrow 0$, and $C(k) \rightarrow \frac{1}{2}$ as $k \rightarrow \infty$. It follows that Eqs. 46 and 48 approach each other as $N \rightarrow 0$, when turbulence also approaches zero. However, as $N \rightarrow \infty$, when turbulence becomes important, $L \rightarrow \frac{\pi \rho b^2 V \dot{\alpha}}{2}$ and $M \rightarrow -\frac{\pi \rho b^4 \ddot{\alpha}}{128}$ (the leading terms of Eqs. 46); and $L \rightarrow \frac{\pi \rho b^2 V \dot{\alpha}}{4}$ and $M = \frac{\pi \rho b^2 V^2 \alpha}{4}$ (the leading terms of Eqs. 48)—exhibiting a profound difference.

The author is primarily interested in applying his principle of relativity to the analysis of unstreamlined shapes like standard suspension bridge sections. This example of the failure of that analysis involves a much simpler streamlined section. Nevertheless the principle, if valid, must apply to both cases; since it fails in one, it cannot be relied upon in the other.

On the other hand, the inconsistency between Eqs. 46 and 48 cannot be blamed on the Kutta condition. This condition was employed in both derivations. If the principle of relativity were valid, they would have led to the same result.

Lack of a high degree of mathematical rigor in dealing with intuitively obvious physical situations may certainly be excused in a paper of this kind. However, in the paragraph preceding Eqs. 2, the author has by-passed some important details of nonobvious fundamental derivations in a manner that needs justification. For example, why should the mass m_a involved in this treatment be the mass of air lying inside the circle circumscribed about the bridge section? Why not less air, or more? Does the author envision the situation as that of a solid cylinder of air rotating in a hole in an otherwise still atmosphere? Such a flow is hydrodynamically impossible. It is true that the terms in Eqs. 2 based on this assumption also appear in Mr. Theodorsen's thin airfoil analysis, but this fact does not establish them in this more general case.

Harold A. Thomas,[42] M. ASCE.—No impartial reader can fail to appreciate the vast amount of thought which the author has devoted to the subject of aerodynamic oscillations of bridges. The essential problem solved in the paper (within a limited range of conditions) is the prediction of the wind-tunnel performance of an oscillating model of a suspension bridge floor and stiffener system from data obtained from wind-tunnel tests on natural-scale and distorted static models of the same section. The analysis makes use of tests on a series of static models which substitute curved surfaces for certain plane surfaces of the prototype, it being assumed that the performance of a straight model in a curved wind can be simulated with satisfactory accuracy by that of a curved model in a straight wind. Another assumption used in the analysis is that the instantaneous pressure pattern on an accelerating surface can be simulated with satisfactory accuracy by that on a static surface having the same orientation with respect to the apparent wind direction. This step is a first approximation, which is applicable only when the maximum velocity of every oscillating solid part is small in comparison with the wind velocity, this condition being met satisfactorily in those vibrations of actual bridges where the amplitude of oscillation and the number of oscillatory loops are small. Within the aforementioned limitations, it is not likely that competent readers will question the applicability of the author's assumptions or the correctness of his brilliant mathematical analysis.

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[42] Prof. of Civ. Eng., Carnegie Inst. of Technology, Pittsburgh, Pa.

It seems probable that the publication of this interesting and technically excellent paper will tend to influence future engineers to employ static models rather than oscillating models in studying the probable aerodynamic performance of specific bridges under design. To maintain a balanced view, the writer would like to comment on certain advantages of oscillating models.

The number of tests that have been made on oscillating models of suspension bridge floor sections in wind tunnels of reasonably large size is very limited, and it is quite possible that such tests, being intended for research as well as for control of specific designs, have been expensive and time consuming. However, the writer has had some experience with oscillating models installed in an air jet 17 in. in diameter, and from this work he can state with assurance that there is no intrinsic reason why the construction and testing of a pair of oscillating models (one for vertical oscillations and one for angular oscillations) should be more expensive or time consuming than the construction and testing of a series of natural and distorted static models of the same scale, equipped with a piezometer system. Mounting a model so that it can oscillate vertically at any desired frequencies is a comparatively simple matter. The same may be said about mounting the model on a pair of ball bearings so that it can oscillate about an axis. More complicated mountings permitting both kinds of oscillations simultaneously should not be required in routine tests for controlling design.

The oscillating models have the advantage of giving the final results instantly, without the necessity for making computations or the possibility of having the work encumbered by arguments on the reliability of various assumptions and analytical processes. The results appear in a direct visual form which is comprehensible with absolute clearness not only by engineers but also by nontechnical men interested in the financial and public welfare aspects of the project. The oscillating models require no piezometer measurements to determine surface pressures. The results obtained on oscillating models are accurate for all amplitudes and frequencies, whereas in analytical methods based on the performance of static models, uncertainty regarding the accuracy of the fundamental assumptions increases with amplitude and frequency, wind velocity remaining constant.

The foregoing remarks on practical model testing are not inconsistent with the writer's admiration of this splendid contribution to a general understanding of the aerodynamic oscillations of bridges.

Elmo G. Peterson,[43] **Jun. ASCE.**—It is certainly reasonable to base an analysis of the aerodynamic stability on the pressure distribution graphs of the section since the forces causing vertical or torsional motion of a bridge are produced mainly by differences in the pressures on top and bottom of the bridge deck. Since measurements of these pressures on oscillating models have not as yet been obtained, the proper method for correcting the pressure distribution graphs of static models is still unknown. However, the evidence presented (in the numerical examples pertaining to the Tacoma (Wash.) Narrows Bridge, under the heading, "Vertical Instability") for the validity of the proposed phase lag correction (Eqs. 4*a* and 4*b*) shows that this modification of the static pressure distribution graphs is at least of the proper order and magnitude. An earlier publication[44] by the author explains this correction for phase lag. Actual measurements of the pressures on oscillating models would provide the necessary data for confirmation or modification of the theory, but these would be very difficult to obtain with present-day pressure measuring instruments. Perhaps further proof may come only from additional oscillating tests on models.

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[43] Asst. Prof. of Civ. Eng., Oklahoma Agri. & Mech. College, Stillwater, Okla.

The writer obtained the static pressure distribution graphs for H-sections, deck sections, and through sections of plate girder bridges, using only straight models and modifying them by incorporating slots of various widths in the bridge deck alongside the plate girders.[45]

The resulting lift and moment curves for the H-sections are almost identical with those already presented.[46] The lift and moment curves for the deck sections and through sections show that these forms of cross section are very unstable but that deck slots do reduce the instability. It is possible that further modifications of the deck section will yield a stable section. There are several psychological advantages of a deck section, besides sightseeing, which warrant additional testing of this nonsymmetrical section to determine if it can be stabilized.

Acknowledgment.—The aforementioned project was sponsored jointly by the Iowa Institute of Hydraulic Research (Iowa City), The Engineering Foundation, and the Committee on Hydraulic Research of the Hydraulics Division, American Society of Civil Engineers.

William Allan,[47] M. ASCE.—In this paper new working tools for the designer of long span bridges have been developed from basic principles in the several areas of mechanics by painstaking research, intuitively inspired experimentation, clear physical perception, and mathematical skill. Since 1927, a little more than 20 years, the profession has come far from the early wind-force measurements made at Göttingen, Germany,[48] on models of bridge trusses and of trussed bridges with solid floors. There the wind directions were perpendicular to the planes of the trusses and also on the slant laterally and from above, but no displacements were observed.

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[44] "Design of Bridges Against Wind, Part III," by D. B. Steinman, *Civil Engineering*, December, 1945, pp. 559–560.

[45] "An Investigation of the Aerodynamic Stability of Bridge Sections," by Elmo G. Peterson, thesis presented to the State University of Iowa at Iowa City, Iowa, in August, 1948, in partial fulfilment of the requirements for the degree of Master of Science.

[46] "Wind Tunnel Tests Yield Aerodynamically Stable Bridge Sections," by D. B. Steinman, *Civil Engineering*, December, 1947. pp. 747, Fig. 2.

[47] Prof. of Civ. Eng. and Dean, School of Technology, The City College of New York, New York, N. Y.

[48] "Messungen von Brückenträgern" in "Ergebnisse der Aerodynamische Versuchsanstalt zu Göttingen," by L. Prandtl and A. Betz, Oldenbourg, München and Berlin, 1927.

Neither the problem of incipient stability or self-induced oscillations in hydraulics[49] nor that of flutter[11] in aeronautics has the complexity of the oscillation problem concerned with characteristic floor and stiffening member sections of suspension bridges. Mr. Steinman's treatment of stability resulting in Figs. 13 to 16 and Table 5 for selected sections is especially masterful and the results are very valuable.

It seems certain that the paper as printed is an abridgment of the original manuscript material and the writer hopes that the author may one day find it possible to give the profession a more detailed work on the subject and more data and examples on typical cross sections.

Paul Lieber.[50]—A rational and quantitative procedure for investigating the aero-elastic stability of suspension bridges is offered in this paper. The theory for nonsteady flow about an oscillating plate developed by Theodore Theodorsen, R. Kassner and H. Fingado, and others has been successfully used for predicting aircraft flutter. In such cases, however, the bodies (airfoils) are streamlined; and, for all practical purposes, the circulation is determined by the Kutta condition. The author correctly points out that the Kutta condition cannot be applied to sections used for suspension bridge construction such as an H-section, for example. Accordingly, it becomes necessary in these cases to determine some of the aerodynamic coefficients experimentally.

The author employs a concept that the effect of curvature of a section model on the flow in a static wind-tunnel test is equivalent to the effect produced by the angular velocity of the section. Thus, the nonsteady forces are obtained from static wind-tunnel tests. These forces are then introduced into the equations of motion, yielding solutions which are in good agreement with some experimental data.

One would expect viscosity effects to be an important consideration in relating wind-tunnel results obtained with stationary section models to the prototypes, especially for sections to which the Kutta condition cannot be favorably applied. The effects of the Reynolds number are not discussed in this paper, and such an investigation is advisable before wind-tunnel results obtained with section models are applied to prototypes.

In the writer's opinion this paper is an important contribution to both civil engineering and to aeronautics.

S. K. Ghaswala,[51] Assoc. M. ASCE.—The paper forms a significant contribution to the subject of aerodynamic theory of bridge oscillations, and clearly bears the mark of intense creative research and experimentation in a subject which is still in its nebular stage of growth and development. Until comparatively recently, the subject of wind resistance of bridges was relegated to a static sphere, as is evident from the fact that the conventional pressure of from 30 lb per sq ft to 50 lb per sq ft was adopted and always considered safe and justifiable. Only recently did the failures of the Tacoma Narrows Bridge in Washington (in 1940) and the Chester Bridge in Illinois (in 1944) arouse the bridge engineering profession from slumber by vividly demonstrating the aerodynamic effect of wind in the former failure and the aerostatic effect in the latter. The insufficiency of the conventional methods of wind pressure design were then fully realized and the suspension bridge was thrust into its true classification as a dynamic structure.

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[49] "Zur Theorie des Wasserschlosses bei Selbsttätig Geregelten Turbinenanlagen," by D. Thoma, Oldenbourg, München, 1910.

[50] Asst. Prof of Applied Mechanics, Polytechnic Inst. of Brooklyn, Brooklyn, N. Y.

[51] Chartered Engr., Fort Bombay, India.

The paper affords a clear insight into the subject of oscillation of bridge sections of various types. The methods of classical aerodynamics applicable to perfectly streamlined bodies only have been modified to suit practical bridge sections, and differential equations, which are believed to be the basic formulas governing this new aerodynamic theory of bridge oscillations, have been evolved. By this method, the actual wind pressure design of long suspension bridges of the future will form the acid test of the theory which, as the author claims, is the most complete and most generalized yet presented. In any event, this illuminating paper will deserve the praise of all those capable of appreciating the finer points of analytical research in an uncharted field.

Peter L. Tea.[52]—In designing a new machine as much science as is applicable is used, and also much art, consisting of judgment formed from prior experience. A few machines are built and tested under required conditions, and are then treated roughly to observe how they stand up. Much is thus learned to improve the final design for production.

This convenient process cannot be applied to bridge design. Only one bridge is to be built to suit the expected traffic and conditions at the site, and its cost is very high. Scale models are expensive; they require time to build and to test in large wind tunnels, which involve more expense. In addition, if the model shows marked instability, a new design is required and the process is repeated. Finally, to what degree of accuracy will the model reproduce the oscillations of the actual bridge? A good aerodynamic theory of bridge oscillations produced by the wind has been urgently required.

This timely paper indicates a keen sense of analysis guided by unusual intuition and ingenuity, and an enormous amount of work. Using his theory, Mr. Steinman has made checks indicating good agreement with the performance of a large number of bridges in the United States and abroad.

The procedure calls for static wind-tunnel tests on small-scale test specimens of short length—a very inexpensive operation. The pressure distribution observations are used to obtain a dimensionless form factor y as a function of x along the width b, taken as unity. The y-function could be applied to determine the lift and the torque on a rigid or quasi-rigid bridge of cross section similar to that of the test specimen. The y-function for static conditions is modulated very ingeniously by multiplying y by the phase-lag factor $e^{-i\phi}$ to obtain the shape factor (dimensionless, a function of x) for application to a bridge where there is sinusoidal motion, vertical or torsional, in one of its natural modes From the modulated function the F-curves and G-curves are obtained; and they form a most important part of Mr. Steinman's research, being applied to study the stability or instability of actual bridges (Eq. 13 and 16 and Figs. 15

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[52] Associate Prof. of Drafting, The College of the City of New York, New York, N. Y.

and 16). The writer suggests that some wind-tunnel tests be devised (if practicable), under dynamic conditions, to obtain results experimentally which could be predicted by the application of the *F*-curves and *G*-curves, for comparison and discussion.

This paper is a contribution to bridge and structural design of the first order.

Edward Adams Richardson.[53]—The far from simple aerodynamic theory of flutter has now been reduced successfully to a set of procedures applicable with some ease to practical problems of design in certain important cases. This has been done in part through the use of the pressure distribution over an airfoil in the static case, modified through a simple assumption of validity when differences in phase across the section must be considered. The method of Theodore Theodorsen[11] is based on the integration of the equivalent effect due to a trailing system of somewhat idealized vortices. In part, also, the simplification is secured through simple experiments, cheaply made, on wind-tunnel models of the cross section and on curved modifications thereof—a most important accomplishment.

As these equations have been derived by Mr. Steinman, they apply to the most common type of suspension bridge structure, involving two sets of cables, suspenders, and stiffener girders, or one set for each side of the bridge. They are capable of modification for cases in which the two substantially independent suspending systems are interconnected by transverse suspenders, discussed in an earlier paper,[54] and used in the stiffening of the Deer Isle Bridge in Maine with great success. Such a bridge tends to be free of torsional oscillations although it may exhibit slight lateral swinging with accompanying cyclic variations in angle and in effective wind velocity.

Although most bridges within the range of the given analysis will develop the types of oscillation calculable therefrom, the theory is not developed so as to show that the oscillations may occur. In cases of larger and heavier bridges, or short times of exposure to conditions favoring flutter, it is quite possible that the calculated behavior may not occur. The reasons, therefore, have been discussed previously[55] and need not be repeated. This is a characteristic of all flutter equations, however, not of those particular equations under discussion.

Certain conclusions derivable from the examples should be noted. These examples cover two sets of data: (*a*) Bridge floors with flat plate cross sections and (*b*) those with H-shaped cross sections. Group (*a*) is typical of so-called stable types of cross sections. For them, if combined torsional and vertical oscillations are possible, flutter is possible; but, if either the vertical or the torsional oscillations may be suppressed, then the other type cannot be unstable. This conclusion is consistent with the work of the late Friedrich Bleich,[56] M. ASCE, who used a flat plate bridge floor for study. Group (*b*) differs from group (*a*) by the dangerous instability which can occur not only when there are combined oscillations, but also when vertical oscillations are suppressed, and, with very low degrees of damping, when torsional oscillations are suppressed. Obviously stable types of cross sections are to be preferred. In either case, however, if the torsional oscillations can be suppressed by such methods as those previously described by Mr. Steinman,[54] very moderate damping will insure stable behavior. Hence, it is possible to build relatively slender bridges which are stable, and it becomes a matter of importance to investigate the limitations of such designs through the extension of the analysis to the transverse suspender type of bridge. Likewise, it would be desirable to apply the analysis to such a bridge as the George Washington structure, in New York, N. Y., which has been classified as intermediate between those bridges which are very stable and the other bridges which have shown important oscillations, to determine just what stability may be expected of the structure, the influences of floor type (both present and proposed), and other features.

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[53] Publications Dept., Bethlehem Steel Co., Bethlehem, Pa.

[54] "Rigidity and Aerodynamic Stability of Suspension Bridges," by D. B. Steinman, *Transactions*, ASCE, Vol. 110, 1945, p. 470.

[55] *Ibid.*, p. 481.

[56] "Report No. 10 on Oscillations of Suspension Bridges with Flat Plate Floors," by Friedrich Bleich, A.I.S.C., New York, N. Y., October, 1945.

Mr. Steinman is indeed to be commended for the work he has done in this field.

Hans H. Bleich,[57] M. ASCE.—The paper represents an attempt to obtain a solution of the important problem of aerodynamically excited vibrations in suspension bridges. The general dynamic equations of motion can be established and solved without particular difficulty, provided the air forces acting on the vibrating section are known; and the problem confronting the author was, therefore, essentially one of finding these forces.

The author has conceived a novel approach for the determination of the air forces acting on a vibrating section, using tests on static models only. Pressure distribution graphs are obtained, from which phase-correction factors C_1, C_2, C_3, and C_4 are determined by Eqs. 5. In deriving these equations the author found it necessary to make the assumption that the pressures on the oscillating sections are $y_1 \, e^{-i\phi}$ and $y_3 \, e^{-i\phi}$, in which y_1 and y_3 are the static pressure distributions. In the writer's opinion, there is no factual basis for this assumption; and, as the quantitative results of the analysis are entirely dependent on these phase-correction factors, the results appear to lack a rational basis.

In the third paragraph of the "Conclusion" it is stated that, for the limiting case of an ideal flat plate, the coefficients in the classical aerodynamic flutter theory and the coefficients determined by the theory proposed in the paper are not identical. The results of the flutter theory check very closely with experiments,[58] thus contradicting the proposed theory for an important special case. The author states:

> "The classical derivation is not applicable to nonstreamline sections, and no method of adapting it so as to fit the known behavior of nonstreamline sections is apparent. The coefficients derived by the simpler concept do fit the known behavior of nonstreamline sections. * * *"

The writer fails to see evidence to support the second part of the statement, and for the following reason: A very important point to be considered when weighing experimental evidence to support any theory is the question of whether the theory to be checked rests on a rational theoretical basis or only on theoretically unfounded assumptions. In the first case one or two examples of agreement between theory and test may be considered reasonable proof; in the second case only a series of tests covering the entire range of application and variables involved can furnish support for the theory. It would appear that the theory presented in this paper belongs to the type described as the second case. Not only is there insufficient evidence for it, but the disagreement with the exact theory supported by tests speaks strongly against it.

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[57] Associate Engr., Hardesty and Hanover, New York, N. Y.; and Lecturer in Civ. Eng., Columbia Univ., New York, N. Y.

[58] "Use of Generalized Coordinates in Flutter Analysis," by Samuel J. Loring, *Journal*, Soc. of Automotive Engrs., April, 1944, p. 113.

It is the writer's opinion that it is not at all possible, as proposed in the paper, to solve this problem using static tests as a sole basis, however desirable this procedure might be; but that only tests on oscillating section models can supply the necessary data.

A first step in this direction was made in 1946 by the late Friedrich Bleich,[27] M. ASCE, in his reports to the Advisory Board on Investigation of Suspension Bridges, Public Roads Administration, where he presented a method for treating catastrophic vibrations of truss-stiffened bridges on a rational basis. Further steps in this direction should be made.

F. B. FARQUHARSON,[69] M. ASCE.—It would be fortunate indeed if the much simplified procedure advocated in this paper could be proved to be as reliable and as universal in application as is claimed. The writer has hopefully followed the development of this method through its various stages and, although he has found many contributions which assisted his own thinking, it has yet to be shown that this approach is valid.

The most rational check of the validity of the author's predictions, through the use of static section models, would seem to arise directly from laboratory tests on dynamic models. It is the intention of the writer to offer such data for comparison with two of the author's examples. These data were taken from the record obtained from the joint study sponsored by the Washington Toll Bridge Authority and the United States Public Roads Administration where a very extensive investigation has been made on dynamic models of the full bridge as well as dynamic section models.

In Fig. 28, which is a replot of Fig. 13, it would appear that a line drawn horizontally through the appropriate value for the structural decrement δ_s would give a complete picture of the $\frac{V}{N\,b}$-range in which vertical motion would show for that condition of structural damping. Before any motion could be predicted the δ_s-line would have to fall below $\delta_1 = 0.0152$. It would fall below $\delta_1 = 0.0025$ to permit motion in the intermediate stability range and, finally, below $\delta_1 = 0.001$ for motion to show in the lower range.

No evidence has ever appeared in the laboratory which would prove the existence of the lower stability range because of the impossibility of designing a suspension for dynamic section models with $\delta_s = 0.001$ or lower. Similar reasoning would suggest that lower range motion (and probably also intermediate range motion) is also impossible in the field. The author suggests (under the heading, "Vertical Instability: Numerical Examples") that three stability

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[69] Director, Eng. Experiment Station, Univ. of Washington, Seattle, Wash.

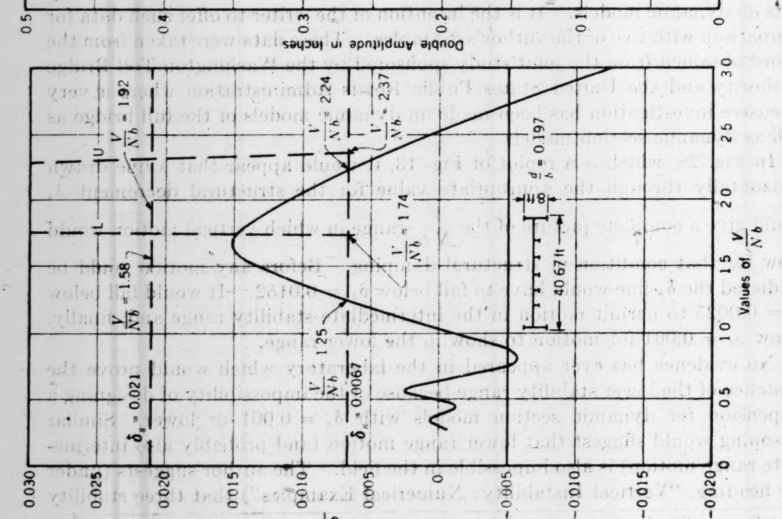

FIG. 28.—VERTICAL INSTABILITY GRAPH FOR AN H-SECTION ($d/b = 0.20$)

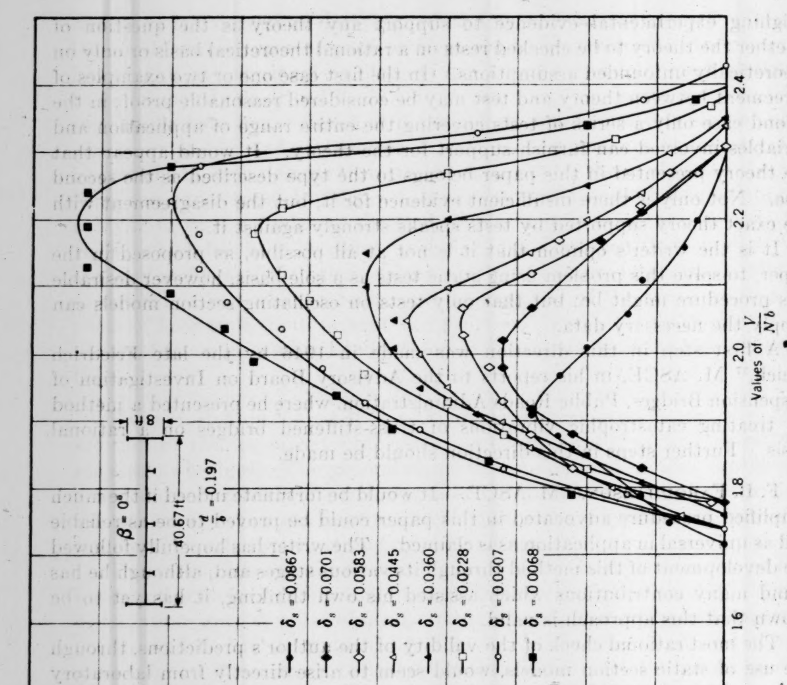

FIG. 29.—OBSERVATIONS ON A DYNAMIC SECTION MODEL, TACOMA NARROWS BRIDGE IN WASHINGTON (CLOSED DECK, NO CURB, AND MODEL SCALE = 1:50)

ranges were evident on the original Tacoma Bridge in the field and cites their boundaries. This information is evidently taken from one of the illustrations of the Carmody report[60] and is entirely unreliable for reasons already pointed out by the writer.[61,62]

Furthermore, it is quite unrealistic to suppose that structural damping involving a decrement as low as 0.0025 could ever be expected in the field on any bridge yet built. The very mild winds which made up the meager record available[61] were so vagrant in nature that the bridge continually wandered from one vertical mode to another and the lack of correlation between velocity readings and observations of the type of motion has led to many erroneous conclusions.

Laboratory tests on an exact replica of the original Tacoma Bridge, with eight different values of structural damping covering the range from $\delta_s = 0.0067$ to $\delta_s = 0.0867$, demonstrated that $\left(\frac{V}{N\,b}\right)_C$ was substantially constant over this range of damping (which was assumed to go well beyond any field possibilities) with an average $\left(\frac{V}{N\,b}\right)_C = 1.75$ (Fig. 29). Vertical motion was apparent in the upper critical region under these conditions of damping, over a range of $\frac{V}{N\,b} = 1.74$ to $\frac{V}{N\,b} = 2.37$.

If a horizontal line is drawn, in Fig. 28, for $\delta_s = 0.0067$, the author's calculations show that the instability range for this structural damping would be from $\frac{V}{N\,b} = 1.25$ to $\frac{V}{N\,b} = 2.24$ (Fig. 28). It is likewise evident that change in damping over the range shown in Fig. 29 involved a negligible (3%) change in the lower critical ratio $\frac{V}{N\,b}$ and a slightly greater (5%) change in the upper critical value of $\frac{V}{N\,b}$. On the basis of these dynamic section model tests the bounding lines in Fig. 28 for the upper critical range should be very nearly vertical, and displaced rather sharply to the right. Thus, the complete results for the eight section model tests with different damping would fit within the two vertical lines L and U if they were extended to cover the range $\delta_s = 0.0867$ which is far beyond the limitations of the author's calculations. It appears that the maximum value of δ_1 (logarithmic increment) in Fig. 28 would have to be multiplied by at least 5.8 to reach a level already demonstrated to be possible on this section.

Tests on dynamic section models at the California Institute of Technology in Pasadena are in equally poor agreement with the author's calculations although they were confined to a single value of structural damping where

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[60] "The Failure of the Tacoma Narrows Bridge," a report to the Hon. John M. Carmody, Administrator of the Federal Works Agency, Washington, D. C., March 28, 1941, by a Board of Engineers consisting of O. H. Ammann, Theodor von Kármán, and Glenn B. Woodruff, Fig. 29.

[61] "Aerodynamic Stability of Suspension Bridges," Pt. I, by F. B. Farquharson, *Bulletin No. 116*, Univ. of Washington Eng. Exp. Station, Seattle, Wash., 1950, pp. 89–99.

[62] "Rigidity and Aerodynamic Stability of Suspension Bridges," by D. B. Steinman, *Transactions*, ASCE, Vol. 110, 1945, p. 493.

$\delta_s = 0.021$. At this damping, represented by the upper horizontal line in Fig. 28, oscillations were recorded over a range of from $\frac{V}{N\,b} = 1.58$ to $\frac{V}{N\,b} = 1.92$. These tests were made on an idealized model consisting of a symmetrical section without girder flanges, sidewalk curbs, or stringers ($d/b = 0.205$). This is the same type of section on which the author has made his static section model tests, which form the foundation for his theory. Laboratory experience has demonstrated that details of the type neglected in the idealized model are often of great importance on this type of section where the actuating forces depend primarily on the distribution of vorticity in the wake.[26]

It is significant that the laboratory tests on this idealized dynamic model have indicated values for the logarithmic increment well above the maximum of $\delta_1 = 0.0152$ revealed by the author's calculations. It should also be noted that vertical motion in the intermediate instability range was observed in these tests where $\delta_s = 0.021$.

In the many full model tests run in the laboratory it has always been observed that the range between the upper and lower critical values of $\frac{V}{N\,b}$ for vertical motion was greater than that noted on the section model. Thus, for the full model of exactly the same section as was used in Fig. 29 the instability range at $\delta_s = 0.009$ was from $\frac{V}{N\,b} = 2.06$ to $\frac{V}{N\,b} = 3.0$.

It seems reasonable to the writer to assume that the author's approach may be more nearly valid for sections stiffened by trusses where the forces arising from vorticity are probably a negligible factor.

The author's characterization (under the heading, "Vertical Instability: Numerical Examples") of the lower instability range as "subharmonic" does not seem to be in accord with the mechanism of vortex excitation. As is well known,[63, 64] the relationship which governs the frequency of vortex discharge about an object immersed in a flow is given by the expression:

$$f = \frac{V\,S}{d} \qquad \text{(49)}$$

in which V is the wind velocity; d is the depth of the object (measured normal to the flow); and S is the Strouhal number characteristic of the shape of the object.

It was demonstrated at the California Institute of Technology that $S = 0.187$ (approximately) for an idealized model of the original Tacoma Bridge; and it was observed at the University of Washington in Seattle on a full model of the original bridge that the $(3 - N\,V)$-mode appeared twice—once in the intermediate range of instability and, again, at approximately twice the wind velocity, and with much increased amplitude in the high range of instability. It was also observed that the frequency was the same in each appearance and the critical velocity in the high range was 3.2 ft per sec. Thus, using $S = 0.187$, the frequency of the vortex is 3.75 cycles per sec which is close to twice the value

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[63] "Uber den Mechanismus des Flüssigheits und Luftwiderstandes," by Theodor von Kármán and H. Rubach, *Physkalische Zeitschrift*, January 15, 1912, p. 49.

[64] "Some Experimental Results on Wing Flutter," by William Bollay and Charles D. Brown, *Journal of the Aeronautical Sciences*, June, 1941, p. 317, Fig. 11.

of the recorded frequency of oscillation. For subharmonic excitation to occur[65] it is only necessary for the exciting frequency to approximate some multiple of the natural frequency of the structure.

Since V_C for the same mode in the intermediate range was approximately one half that in the high range, it is clear that this excitation was harmonic. Obviously the author has reversed the terminology in describing these ranges.

Comparison of Laboratory Data With Numerical Examples (*Under the Heading, "Torsional Instability"*).—Fig. 15 is misleading in two respects: First, an accurate rendering of Eq. 16 would cause the calculated curve to intersect the experimental curve at $\frac{V}{N\,b} = 7.15$ and $\delta_2 = 0.30$. These two curves are parallel only over a limited range. Second, the author seems to have made an error in sign in the calculation of his suggested correction factor to bring the experimental results into harmony with his calculations. Fortunately, all the data necessary to calculate the experimental curve are readily available.[21]

This model was mounted on a somewhat complicated suspension which involved a section of cable from which the model was hung by suspenders. The ends of the cables were restrained by coil springs mounted outside the tunnel. Three measurements were taken at zero wind velocity to isolate the value of δ_s as follows:

$$\delta_s = \delta_3 - \delta_2 + \delta_1 \qquad (50)$$

in which δ_1 is the decrement for springs alone; δ_2 is the decrement for springs plus section; and δ_3 is the decrement for springs plus section plus suspension. The record shows that $\delta_s = 0.0299$ and a careful recalculation of the experimental curve plotted in Fig. 15 shows it to be accurate without the suggested added correction to account for residual damping.

However, the critical value of $\frac{V}{N\,b}$ yielded by this experimental curve at zero damping is in serious error for reasons not entirely clear. Many tests run at a later date on a simplified suspension at both the University of Washington and the California Institute of Technology have revealed much lower values for $\left(\frac{V}{N\,b}\right)_C$. The author is in error in quoting the results of tests from the California Institute of Technology run in 1945 and the value quoted is also inaccurate. No tests were run in this laboratory after April, 1943, and the value of $\left(\frac{V}{N\,b}\right)_C = 2.5$ $\left(\text{for } \frac{d}{b} = 0.2\right)$ does not appear in any of the records from that institution. In the course of a series of studies on the effect of damping on the torsional mode, $\left(\frac{V}{N\,b}\right)_C = 1.9$ was reported at zero structural damping and $\frac{d}{b} = 0.2$. At $\delta_s = 0.0299$, $\left(\frac{V}{N\,b}\right)_C = 2.5$.

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[65] "Mechanical Vibrations," by J. P. Den Hartog, McGraw-Hill Book Co., Inc. New York, N. Y., 1947, pp. 448–449.

Dynamic section model tests at the University of Washington have shown that $\left(\frac{V}{N\,b}\right)_C = 2.01$, at $\delta_s = 0.0019$, and $\frac{d}{b} = 0.2$, with no evidence that tends to lend any support to the author's calculated $\left(\frac{V}{N\,b}\right)_C = 2.5$ at $\delta_s = 0$.

Mr. Steinman's dependence on static section model tests in his empirical-theoretical approach to this problem does not hold up, in the case of the structure stiffened by plate girders, in either vertical or torsional motion. The writer has not found the time to explore its feasibility on truss-stiffened structures where vorticity is probably of much smaller significance.

Extensive full model tests run in comparison with dynamic section model tests have demonstrated the validity of the dynamic section model approach to this problem; those who advocate sole reliance on static model tests have yet to prove their case.

George S. Vincent,[66] M. ASCE.—There is a significant relation between Mr. Steinman's analysis and that developed by the late Friedrich Bleich, M. ASCE, in 1945.[27] Both follow the Theodorsen analysis of flutter of a thin flat plate[2]—omitting, however, the effect of the aileron and neglecting the inertial forces of the surrounding cylinder of air. They differ in that Mr. Bleich considers only a horizontal or slightly angled wind, he uses the Theodorsen basic flat plate forces, and then he introduces, as a corrective term, an equivalent lift force derived from dynamic wind tests of a section model of the bridge. The author, on the other hand, includes a procedure for covering the effect of an initial angle of incidence and he builds up the force terms by data derived from static wind tests on normal and curved section models, thus requiring no corrective term to fit the analysis to the actual section (provided the static test data are adequate for the purpose). Furthermore, he uses the lift force and torque force terms separately to derive predictions for pure vertical and pure torsional oscillations.

The best available method for appraising the analysis lies in correlating it with the results of the three-dimensional aerodynamic model tests made at the University of Washington (in Seattle), cooperating with the Washington Toll Bridge Authority, the United States Bureau of Public Roads, and the university.

The treatment of the effective angle of incidence, α_i, must be extended for a cambered bridge. Eq. 11*f* expresses the angle, α_d, through which the structure will be rotated by the static wind forces—that is, the lift and torque acting at the midpoint of the width, when they are resisted by the over-all torsional rigidity, K_2, of the structure. If the bridge is cambered, the part of the horizontal static wind force resisted by the truss is carried to the truss supports several feet below the line of application of the wind force to the truss, resulting in an overturning moment which tilts the leeward side downward. On the full model of the new Tacoma Narrows Bridge (in Washington), with a 2,800-ft vertical curve between grades of from + 3% to − 3%, this effect of camber was about twice the effect of the action represented by Eq. 11*f* as depicted by the section model behavior, and was of opposite sign. Thus, the resultant full model pitch as determined by both factors was about equal to that of the section model but opposite in direction. It could be neglected for small angles of incidence, with prototype velocities less than 60 miles per hr, but rose from 1° to $2\frac{1}{2}$° for higher velocities and greater angles of incidence.

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[66] Prin. Highway Bridge Engr., Bureau of Public Roads, U. S. Dept. of Commerce, Structural Research Laboratory, Univ. of Washington, Seattle, Wash.

The overturning wind moment not only depresses the leeward side but also increases the tension in the leeward cable, thus increasing its spring constant without altering the mass suspended from it. This action increases its natural frequency while correspondingly decreasing the natural frequency of the windward cable, which might be expected to affect the oscillation of the bridge in the wind. A test was made to determine this point. The wind shoes were loosened, permitting the suspended structure to swing downwind and relieve the truss of all wind load since it had no wind reaction. The shoes were then fastened in the unstressed position to steady the structure and the routine tests were made. The change in behavior was insignificant.

The author indicates that the curves for δ, the logarithmic increment due to wind action, are parabolic when plotted against the amplitude. A considerable number of these curves has been plotted directly from aerodynamic tests on both full models and section models and for sections subject to pure vertical oscillation and those subject to flutter, the structural damping having been determined by other tests and eliminated. Some of the curves are approximately parabolic but more of them are practically straight or have pronounced reverse curvature. For the same section and angle of incidence they often show a progressive change as the wind velocity is increased.

The numerical examples illustrating coupled oscillation (flutter) have more significance than is implied by the statement that the data are assumed. With the exception of the s-values and the q-values all the assumed data are identical to the properties of the prototype of the full model of the Tacoma Narrows Bridge as used and identified by Mr. Bleich in his illustrative problem.[67] The vertical and torsional frequencies are the same and the assumed equations for the wave forms are those calculated by Mr. Bleich for that structure. (It should be noted that the author's value of b is twice that of Mr. Bleich and his value of μ is $\frac{2}{\pi}$ times that of Mr. Bleich.)

Both analyses treat the bridge as a thin, flat plate which assumes that the truss members have a negligible effect on the air forces. Mr. Bleich's solution for the critical velocity agrees quite well with the curve from the full model tests.[68] The velocity computed by Mr. Steinman is about half as great and corresponds to the test results for an angle of incidence of + 3° (the sign convention for the angle of incidence being the reverse of that used by Mr. Bleich[68]). As the author indicates, the effect of an angle of incidence is to determine the point on the curves at which the slopes are taken. Figs. 5 and 6, from which the slopes are taken for the examples, show constant slopes between the angles of ± 5° or 6°; hence, throughout this range of the angle of incidence the solution would be the same as indicated—that is, 69 miles per hr.

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[67] "Dynamic Instability of Truss-Stiffened Suspension Bridges Under Wind Action," by Friedrich Bleich, *Transactions*, ASCE, Vol. 114, 1949, p. 1202.

[68] *Ibid.*, p. 1208, Fig. 12.

For this numerical example (the new Tacoma Narrows Bridge but without slots or bottom laterals) it appears that the flat plate analysis will predict the flutter velocity quite well without correction, possibly because of the skeleton construction of the sidewalk stringer at the leading edge. Other truss-stiffened sections tested have required the corrective term.[69]

Mr. Bleich's analysis has the advantage that the corrective term is introduced at the end and used in only the final steps, whereas the author's experimental values, which provide the adaptation to the section under investigation, are subjected to several successive processes. In each of these processes simplifying approximations are made, such as the idealized expressions for the slopes of the lift and torque curves and the use of the "δ-parabolas." The effects of these are probably cumulative. Furthermore, the static models are idealized, as evidenced by the perfect symmetry of the lift and torque graphs for positive and negative angles of incidence. Most tests on models of actual bridge sections show that these curves are not symmetrical but are materially modified by the influence of the stringers and, sometimes, the curbs, and that they are affected by the girder flanges. A few such curves have been published.[70]

Another handicap is the reliance on the static model tests for the air force terms. If the lift and torque graphs have pronounced negative slopes it may be expected that the structure will oscillate and that, with increasing wind velocity, the amplitude will increase indefinitely as is recognized by Eq. 19*b*. This tendency is explained by the "negative slope theory" proposed by J. P. Den Hartog as an explanation of the "galloping" of transmission lines.[71] It does not explain the successive restricted modes, each of which prevails over only a limited range of velocity. The phase functions, applied to the static forces, are also not entirely adequate. The vortex action, not revealed by the static tests, superimposes upon the static pressure pattern a small periodic variation which, when resonant with a natural frequency of the structure, assumes an importance far beyond its relative strength. Furthermore, as clearly indicated by smoke stream tests, the vortex action becomes much stronger once oscillation begins and its frequency is controlled by the motion of the structure over a limited velocity range.

Perhaps one reason for underestimating the importance of the vortex is that, on streamlined sections, the vortex is shed from the trailing edge and exerts no continuing force on the body. On the other hand, the vortex is shed from an abrupt corner of the leading edge of a bridge section and sharply modifies the velocity pattern and the pressure on the structure as it moves across. Vortex action is not so vigorous on truss sections because of shallower depths in the vicinity of the deck, and, therefore, the flat plate analysis gives fair predictions of the critical velocity of the flutter characterizing such sections. Vortex action lowers the critical velocity.

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[69] "Dynamic Instability of Truss-Stiffened Suspension Bridges Under Wind Action," by Friedrich Bleich, *Transactions*, ASCE, Vol. 114, 1949, pp. 1214–1218.

[70] "Aerodynamic Stability of Suspension Bridges with Special Reference to the Tacoma Narrows Bridge," Pt. I, Investigations Prior to October, 1941, by F. B. Farquharson, *Bulletin No. 116*, Univ. of Washington Eng. Experiment Station, Seattle, Wash., Chapter 7, pp. 60–68.

[71] "Mechanical Vibrations," by J. P. Den Hartog, McGraw-Hill Book Company, Inc., New York, N. Y 1947, Article No. 59.

The writer recommends the use of the flutter analysis for the preliminary examination of proposed designs and as a method of surveying the probable influences of alternative dimensions, weight distribution, and elastic properties. For the most reliable investigation of the aerodynamic behavior of the preferred design (aside from costly full model tests) he advocates dynamic wind-tunnel tests on section models mounted on free springs to produce the proper vertical and torsional frequencies. These tests will reveal, quite reliably, the critical velocity of any catastrophic oscillation and will yield data on energy transfer (as measured by the logarithmic increment) which can be applied to the prototype by a single integration process. This must be used with the best available analysis of the structural damping of the bridge and with due consideration of the velocity, uniformity, and steadiness of the wind at the site.

There is strong evidence that at least one important suspension bridge would undergo objectionable oscillation except for the favorable effect of the terrain on the wind. It is not likely that this is the only case of the kind. At the present stage of knowledge there is some risk in assuming that a proposed design will be satisfactory simply because it resembles a satisfactory existing bridge.

D. B. Steinman,[72] M. ASCE.—To all who have contributed discussions, the writer is deeply grateful. The preparation for this work necessarily included an exhaustive analytical and critical study of the ponderous literature of airplane flutter theory. After checking and collating the involved mathematical analyses of the various writers on the subject, the writer was forced to the conclusion that those theories do not offer a directly applicable solution of the problem of bridge oscillations, and that existing aerodynamic theory is inadequate for a solution of the problem. New creative thinking was necessary, with a courageous departure from the limitations of prior conventional theory.

The mathematical derivation of the conventionally accepted theory for airplane flutter is so abstract, involved, and indirect that the physical picture is lost from view, and fallacies of assumptions and gaps in the physical reasoning may be buried under the mountains of mathematics. Few understand the derivation, and still fewer have even tried to check the analysis. Nevertheless, proud reference to the theory has become a cult, and any challenge or departure is attacked as heresy.

The Concept of Curved Models.—The outstanding new concept beyond the writer's previous publications on the subject is the use of curved section models. This was an inspiration. It supplied the missing link in the treatment of angular oscillations. It formed a perfect parallel to the accepted use of tilted section models to represent vertical oscillations. It was a clearly logical extension of the same physical and mathematical thinking. The concept of curved section models represents a relativity principle. It corresponds to a transformation of coordinates, including space and time, to yield a "curvature of space."

The addition of curved section models to the writer's analysis supplied the previously missing terms for complete correspondence with established aerodynamic theory for airfoils. It also supplied the missing factors for complete correspondence between the writer's formulas and the known data from large-scale wind-tunnel tests on bridge models. In addition, the writer made an independent check by comparing the established theoretical formulas for curved thin plates with the corresponding established formulas for flat plates in angular oscillation. The concept of curved section models has been checked, verified, and confirmed in a hundred different ways. The writer is thoroughly convinced of its basic validity.

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[72] Cons. Engr., New York, N. Y.

Because the idea seemed so startlingly new, the writer expected discouraging criticism. It required courage to present it to the profession. This was done in a paper[73] before the Iowa Hydraulics Conference in 1947. Instead of meeting with adverse criticism, the reception was most encouraging. The writer, however, was somewhat disappointed to learn that the idea was not entirely new.[10] He subsequently found that the concept is well known and established in the more modern literature of airfoil analysis, under the name of "dynamic camber"; under this name, the curved thin plate analogy is used and accepted as a short cut for writing the formulas for airfoils in angular oscillation.[74] The writer was the first to conceive and apply the idea as an actual test procedure in the study of airplane flutter and bridge oscillations.

The curved model concept, as applied to an idealized thin plate, checks identically the "quasi-steady force terms"[26] in the theoretical formulas for oscillating airfoils. No claim is made that the curved model yields all the terms in the complete theoretical formulas; the centrifugal force term (easily added) and the acceleration term (negligible) are written separately. The criticism that the curved model does not duplicate all the terms of the complete theoretical formulas represents a lack of understanding of the concept and of the underlying theory. The sole purpose of the static models, both tilted and curved, is to supply the terms that cannot be written by theory. In the application to bridge oscillations, the acceleration terms are written by established aerodynamic theory for thin plates, with simplified extension of the derivation to other sections. Since these acceleration terms are negligible and are, in fact, neglected, challenging the mathematical rigor of their evaluation is hairsplitting criticism.

Furthermore, it is not claimed that the static model tests give the forces corrected for "phase lag." In the terminology of conventional aerodynamics, this correction is identified with "vorticity distribution in the wake." The correction for phase lag or vorticity, whichever name is preferred, is represented by writing the correction factor as a coefficient C in all of the quasi-steady force terms in the formulas.

Vorticity.—In the interests of clear physical thinking, a common misconception needs correction. That misconception is represented by the use of such phrases as "vortex buffeting" or "forces produced by vorticity." These phrases represent a mathematical device, not a physical action. In the repeated use of these phrases, the fact that they are only a mathematical device has been forgotten. The oscillations of a section, even in the case of the forced vibrations of a cylinder, are not caused or produced "by the vortices." The vortices in the wake are merely counters, markers, or footprints providing a convenient physical and mathematical "trail" from which the circulation about the section and the consequent forces acting on the section may be inferred, formulated, and computed. The limitations of classical aerodynamics necessitate this backhanded approach. From footprints, a crime may be reconstructed. From the "vortex trail," the forces acting on a section and their relative phase may be deduced. The vortices in the wake do not exert a force on the section. Their work was finished at the instant when they were shed into the wake.

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[73] "Problems of Aerodynamic and Hydrodynamic Stability," by D. B. Steinman, *Proceedings*, 3d Hydraulics Conference, *Bulletin No. 31*, State Univ. of Iowa Studies in Eng., Iowa City, Iowa, 1947.

[74] "An Investigation of the Conditions for the Occurrence of Flutter in Aircraft and the Development of Criteria for the Prediction and Elimination of Such Flutter," by Albert E. Lombard, Jr., thesis presented to the California Inst. of Technology at Pasadena, Calif., in 1939, in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

Another common error, also associated with the concept of vortex buffeting and arising from the same failure to visualize the physical picture underlying the mathematics, is the notion that the point of shedding of the vortex is the point of application of the related lift force. There is no direct relation between the two. The lift force is determined by the circulation, and the change in circulation is determined only by the strength of the released vortex and in no way by the location of the point of shedding. In a streamline airfoil, the vortices are shed at the trailing edge, but the corresponding lift force acts at or near the forward quarter point. In fact, by conventional aerodynamic theory, the shedding and trailing vortices do not produce any shift of the resultant lift force from its fixed position at the forward quarter point. In a truss or girder bridge section, vortices may be shed at the leading edge, but that does not mean that the corresponding lift force is applied at the leading edge. Any such crude assumption is wholly unscientific and would vitiate an analysis.

The Concept of Phase Lag.—In the interests of clarity, the writer prefers to emphasize the concept of phase lag. Although few may have recognized the fact, the coefficients C in the formulas of the conventional flutter theory are really phase-lag factors. That is why the coefficients C have the value of unity at zero phase lag $\left(\text{infinite } \frac{V}{N\,b}\right)$. That is also why the "apparent mass" terms, representing instantaneous inertia forces, do not have the phase-lag factor C applied to them. Cloudy references to "turbulence" and "vorticity distribution" tend to obscure the true relationship.

In comparison with ice coated wires, or other sections of negligible width (compared with V), the new feature introduced with wider sections is the time required for the fluid flow, or for any disturbance in the fluid flow, to traverse the width of the section. This introduces the new factor—phase lag. A flow disturbance, initiated at the leading edge, takes time to traverse the width and encounters a progressively increasing difference of phase as it traverses the oscillating section. As different points of the width are reached, different stages of the cycle of oscillation are encountered, including differences of velocity and even differences of direction of motion. To cover all cases, the general expression for energy input per cycle is $\Delta W = \pi\, P\, a \cos \phi$, in which P is the force; a is the amplitude; and ϕ is the phase difference between amplifying force and velocity of displacement. The multiplier $\cos \phi$ is the correction factor in multiplying two vectors differing in phase by the angle ϕ, illustrated by the "power factor" applied to the product of volts by amperes in alternating current.

The writer wishes to re-emphasize the importance of his application of the phase-lag factor to the rational aerodynamic analysis of oscillating sections.

Comparison with Conventional Flutter Theory.—Among all the heartwarming comments on this paper, there were only three adverse discussions. Mr. Pinney challenges the writer's concept of curved models by seeking to prove that the resulting formulas for a thin flat plate are not consistent with the formulas of conventionally accepted flutter theory. In this attempt Mr. Pinney has unwittingly made a number of mathematical errors and omissions. These errors are material. In his Eq. 48*a*, the correction for centrifugal force is missing, and in Eq. 48*b*, the angular acceleration term is missing, also the centrifugal force term (which yields zero M only for $C = 1$). In his equation (for $C = \frac{1}{2}$) derived from Eq. 46*a*, the numerical cofficient is wrong and the angle term is missing. In his equation derived from Eq. 46*b*, the angle term and the angular velocity term are missing. In Eqs. 48*a* and 48*b*, the coefficint C should be inserted at all the "quasi-static" terms. If this is done, together with correction of the other cited errors, Eqs. 46 and 48 will be completely identical. Moreover, Eqs. 46 and 48 cannot be expected to remain identical when $C = \frac{1}{2}$ is substituted in the one and $C = 1$ is substituted in the other. The same value of C must be used in the two pairs of formulas for any valid comparison. Confusing references to turbulence should not be allowed to obscure this obvious truth. If these corrections are made, Mr. Pinney's argument vanishes.

The best way to show the consistency of the writer's theory of bridge oscillations with the accepted conventional theory of airplane flutter is by direct application to thin flat plates. This comparison will also bring out the essential differences or points of departure.

For the limiting case of a thin flat plate ($d/b = 0$), or a streamline airfoil, by aerodynamic theory: $s_1 = 2\,\pi$; $s_2 = \pi/2$; $s_3 = -\pi$; and $s_4 = 0$. With these values, Eqs. 2 reduce to

$$L = \pi\,\rho\,V^2\,b\left[C_1\left(\alpha - \frac{\dot{\eta}}{V}\right) + (1 + C_3)\left(\frac{b\,\dot{\alpha}}{4\,V}\right) - \frac{b\,\ddot{\eta}}{4\,V^2}\right] \ldots (51a)$$

and

$$M = \frac{\pi}{4}\,\rho\,V^2\,b^2\left[C_2\left(\alpha - \frac{\dot{\eta}}{V}\right) - \frac{4}{\pi}\,(C_4\,s_4)\left(\frac{b\,\dot{\alpha}}{4\,V}\right) - \frac{b^2\,\ddot{\alpha}}{32\,V^2}\right] \ldots (51b)$$

Although $s_4 = 0$ for this case, $C_4\,s_4$ is not zero (except at zero k). To avoid this indeterminacy, $C_4\,s_4$ may be resolved into two component contributions (which cancel at zero k). Since $s_1 = 2\,\pi$ and $s_3 = -\,\pi$, the wind direction of zero lift on the cambered model is at $\alpha = \beta/2$. For this condition $C_L = s_1\,\alpha + s_3\,\beta = 0$, and $C_M = s_2\,\alpha + s_4\,\beta = \frac{\pi}{4}\,\beta$, represented by a couple. For such condition of constant zero lift (hence, constant zero "circulation" and constant zero strength of trailing vortex wake), aerodynamic theory yields (or assumes) zero phase lag, or a phase-lag factor of unity. Now, bringing the wind up to the horizontal (subtracting $\alpha = \beta/2$) will add a lift force ($C_L = -\,C_1\,s_1\,\alpha = -\,C_1\,\pi\,\beta$) acting downward at or near the quarter point, which produces a moment represented by $C_M = - C_2 s_2 \alpha = - \frac{\pi}{4} C_2 \beta$. The total torque on the section in horizontal wind will then be represented by $C_M = \frac{\pi}{4} (1 - C_2) \beta$. Thus, $C_4 s_4$ has been resolved into $\frac{\pi}{4} (1 - C_2)$. To complete the picture, the cambered model correction for centrifugal force is added, represented by $C_L = - \pi \beta$. The total lift due to $\dot{\alpha}$ will then be represented by $C_L = - \pi (1 + C_1) \beta$, replacing $C_L = - \pi (1 + C_3) \beta$.

With these substitutions, Eqs. 51 reduce to

$$L = \pi \rho V^2 b \left[C_1 \left(\alpha - \frac{\dot{\eta}}{V} \right) + (1 + C_1) \left(\frac{b \dot{\alpha}}{4 V} \right) - \frac{b \ddot{\eta}}{4 V^2} \right] \ldots (52a)$$

and

$$M = \frac{\pi}{4} \rho V^2 b^2 \left[C_2 \left(\alpha - \frac{\dot{\eta}}{V} \right) - (1 - C_2) \left(\frac{b \dot{\alpha}}{4 V} \right) - \frac{b^2 \ddot{\alpha}}{32 V^2} \right] \ldots (52b)$$

Conventional aerodynamic theory (by Theodore Theodorsen,[2] R. Kassner and H. Fingado,[3] and subsequent writers) yields the same formulas, with C written instead of C_1 and C_2. For the limiting case of infinite $\frac{V}{N b}$ (or zero k), for which all C-factors are unity, Eqs. 51 and 52 check Mr. Theodorsen's equations identically.

The foregoing method of writing Eqs. 52 from Eqs. 51 depends upon the validity of accepted aerodynamic theory. An extension of the same method to the general case (deeper sections) can be used to eliminate C_3 and C_4 from the general formulas of this paper, leaving only C_1 and C_2. The writer has not done this, because he is not convinced of the complete validity of conventional aerodynamic theory.

In Mr. Theodorsen's analysis, $C = F - i\,G$, a complex function of k, similar to but not identical with, C_1, C_2, C_3, and C_4 of Eqs. 5. The coefficient C represents the phase correction and is derived from the circulation corresponding to the "discontinuity" or vortex sheet, of simple harmonic variation of strength, extending from the trailing edge of the section to infinity. A peculiar feature of the theory is that all lift resultants affected by the multiplier C are invariably applied at the forward quarter point, whether due to angular position, vertical velocity, or angular velocity.

For an ideal flat plate or streamline airfoil, the static effect of α (also representing $- \dot{\eta}/V$) is an upward lift resultant at the forward quarter point. The phase correction, by Mr. Theodorsen's analysis, multiplies this lift resultant by C, without changing its point of application. By Eqs. 51 and 52, the phase correction multiplies this lift force by C_1 and its moment by C_2, representing an indicated shift in its center of pressure. Since the effect of phase difference across the section alters the pressure distribution, it would appear that the C-coefficients should not be the same for lift and for moment but should normally be different, as represented by C_1 and C_2.

Although the writer's F-functions and G-functions are not identical with those introduced by Mr. Theodorsen, certain governing features are common to both. At the limit of high $\frac{V}{N\,b}$ (or low k), corresponding to zero phase difference, all F-functions approach a common limit of $F = 1$, and all G functions approach a common limit of $G = 0$. At the other extreme of low $\frac{V}{N\,b}$ (or high k), the G-functions have a common limit ($G = 0$), but the F-functions have different limits ($F = \frac{1}{2}$, versus F_1, F_2, F_3, and $F_4 = 0$). For this condition of infinite phase difference or infinite frequency, one would expect that the aerodynamic effect should be vanishing, as represented by $F = 0$. At intermediate values of $\frac{V}{N\,b}$ or k, the two sets of F-functions and G-functions show different laws of variation. The sinusoidal variations of the writer's F-functions and G-functions are more consistent with the known sinusoidal curves of instability response graphs.

It should be noted that the Theodorsen equations are intended to apply only to streamline airfoils with rounded leading edge and tapered trailing edge. It is a violation of the inherent assumptions of Mr. Theodorsen's derivation to apply his equations to a thin flat plate ($d/b = 0$). For deeper nonstreamline sections, the Theodorsen equations are admittedly inapplicable.

According to Theodor von Kármán, Hon. M. ASCE, and J. M. Burgers,[9] a weak point in the idealized aerodynamic theory is that it involves a rough approximation in dealing with the leading edge. Even in the case of an infinitely thin airfoil, the "stagnation point" is not at the leading edge, as assumed, but is some distance along the airfoil. The induced flow velocity turns out to have infinite values at the leading edge, and the neglect of these induced velocities against V results in a material error. For example, the actual pressure distribution graph for a cambered airfoil, as represented by Fig. 8(b), is not a symmetrical curve (ellipse) as derived by classical theory, and therefore does not yield zero moment.

By using actual test data as a starting point, instead of relying wholly on idealized aerodynamic theory of known imperfections, the writer's formulas should represent the closer approach to reality and to generalized validity.

For H-sections and other practical bridge sections, the problem of predicting the aerodynamic forces mathematically appears to be far beyond the resources of conventional aerodynamic theory in its present development. The semi-empirical method as presented in the paper is offered as a practical engineering solution. Unique flexibility in application to all sections (and varying angles of incidence) has been secured by writing the slopes (s_1, s_2, s_3, and s_4) as parameters in the equations; and a simple, practical method, scientifically logical and of indicated validity, for writing the phase-correction coefficients (C_1, C_2, C_3, and C_4) for any section, without limitation, is supplied by Eqs. 5. Classical theory offers no possible method for writing these important coefficients for the general case.

When an engineering problem urgently needs a solution, engineers cannot leave it unsolved merely because existing, established theory is incapable of solving it.

Linearity of Phase Lag.—Messrs. Maher and Eades invite an expansion of the topic of phase lag, particularly with regard to the assumption of linearity.

The following notes, excerpted from a more extensive analysis by the writer, may be helpful.

An instantaneous (differential) change in the position or motion of a body produces a change in the flow of the surrounding fluid. This change of flow is not propagated instantaneously, but with a finite velocity V of the fluid. As stated by Messrs. von Kármán and Burgers,[9]

> "The action produced by a force is carried along by the general motion of the fluid with the velocity V * * *. In the picture of an airfoil moving through air at rest, the vortices remain at the spots where they are formed."

Accordingly, an aerodynamic disturbance (change of flow) traverses the width b of a section with the flow velocity V, or in the total time $t_1 = b/V$. For an oscillating section, the change of phase per second is $\omega = 2\,\pi\,N$. Hence, the total phase difference encountered by the aerodynamic disturbance as it traverses the width of the section is

$$\phi_1 = \omega\, t_1 = \omega \frac{b}{V} = 2\,\pi \frac{N\,b}{V} = 2k \ldots\ldots\ldots\ldots\ldots (53)$$

As the flow disturbance traverses the width of the section with the velocity V, it will traverse any fraction x of the width b in the time $t = x\,b/V$. Hence, the phase difference encountered by the disturbance as it traverses the width $x\,b$ is

$$\phi = \omega\, t = \omega \frac{x\,b}{V} = x\,\phi_1 = 2\,k\,x \ldots\ldots\ldots\ldots\ldots (54)$$

Note that the flow disturbance traverses the section with linear rate of propagation from the leading edge to the trailing edge. It starts at or near the leading edge at the instant of the differential change in the position or motion of the section. Consequently, at any fraction x of the width of the section, ϕ is the phase lag of the flow disturbance relative to the causative displacement of the section. The local flow disturbance, in turn, instantaneously determines the local change of pressure. Accordingly, the local change of pressure will follow the same harmonic cycle as the oscillations of the section, but with a phase lag of ϕ between them. By Eq. 54, the phase lag ϕ is proportional to x. It is zero at the leading edge and increases in straight-line variation to a maximum at the trailing edge.

The phase-lag factor is $\cos\phi$ for the real terms and $\sin\phi$ for the imaginary terms. These are the multipliers of the quasi-steady pressures along the width of the section, represented by the ordinates y of the pressure distribution graph, as applied in Eqs. 5. In mathematical language, the two multipliers are combined in their equivalent, $e^{-i\phi}$, as a single multiplier.

Classical hydrodynamics (or aerodynamics) has been devoted primarily to the study of steady states, and the time lag or phase lag arising in nonsteady states is a comparatively new problem.

Those who have studied the conventional theory of airplane flutter may be surprised to learn that the coefficient C in Mr. Theodorsen's formulas represents a phase correction of linear variation across the section but with that variation (paradoxically) reversed in direction. This fact is not apparent from the formulas. A closer mathematical analysis of the theory reveals that the phase factor for the individual ordinates (corresponding to the writer's phase factor $e^{-i\phi}$) is given by the linear relation:

$$C_x = x + (1 - x)\,C \qquad (55)$$

in which (using Mr. Theodorsen's notation) x is $+1$ at the trailing edge, zero at the midsection, and -1 at the leading edge. Accordingly, the phase-correction factor C_x is unity at the trailing edge (a consequence of the Kutta condition), C at the midsection, and $(2\,C - 1)$ at the leading edge, with a straight line joining these three values and all intermediate values. The overall phase factor C is simply the value at the midsection, or the average value for the entire section. From these values of C_x, the phase lag ϕ is zero at the trailing edge and increases toward the leading edge. This is exactly the reverse of the direction of variation one would expect. In this case, moreover, it is not the phase lag but the phase-lag factor that shows straight-line variation, where trigonometric or exponential variation would be expected. These anomalies remain to be explained.

The Reynolds Number.—Mr. Lieber properly raises the question of viscosity effects, represented by the Reynolds number. This would apply to oscillating models as well as static models. Wind-tunnel tests at the University of Washington, in Seattle, in which given section models were subjected to the widest possible range of variation of wind velocities, yielded practically identical lift and torque graphs at all wind velocities. The effect of the wide variation in the Reynolds numbers was found to be negligible. In the wind-tunnel tests at the Virginia Polytechnic Institute in Blacksburg, which supplied the graphs used by the writer, maximum wind velocites were used in order to minimize any deviation due to the Reynolds number.

Oscillating Model Tests.—Mr. Thomas comments on the advantages of oscillating models. It would certainly be a most happy consummation if someone could develop quick, inexpensive methods of making oscillating model tests that would be sufficiently accurate and that would yield all the required information. There remain, however, several problems yet to be solved before small-scale oscillating models will supply the answer. These unsolved problems include improved methods of pivoting and suspension for studying separately vertical, angular, and coupled oscillations; elimination of friction; elimination or complete measured control of structural damping; methods of carrying the tests down to the lower velocity ratios; methods of frictionless recording of the oscillations; and methods of measuring logarithmic increment or decrement (negative or positive damping). Even with the most elaborate and expensive equipment for oscillating model tests, both in the United States and abroad, results have been obtained that were misleading or incomplete, and significant relations have been missed that were revealed by the writer's pioneering theoretical analysis.

Even if simplified and perfected laboratory techniques for oscillating model tests become available, a comprehensive unifying scientific theory is still needed for correlation, control, guidance, and prediction. Aerodynamic theory was not scrapped when wind-tunnel testing was developed.

Coupled Oscillations.—In response to one of the questions asked by Mr. Dohn, the following notes on coupled oscillations may be helpful.

A section that has both vertical and angular stability over the entire wind velocity range, or over any substantial part of that range, may nevertheless be subject to instability in the form of combined vertical and angular oscillations, termed "coupled oscillations." In the case of airplane surfaces or combinations of airplane surfaces, otherwise aerodynamically stable, coupled oscillations may occur, producing the serious phenomenon of "flutter," which may attain destructive amplitude.

As far as is known, no coupled oscillations of actual bridges have ever been identified and recorded. Coupled oscillations have, however, been observed in wind-tunnel tests on bridge models, also in small demonstration models. Since coupled oscillations of bridge sections are a possible form of instability, the bridge engineer has to take such possibility into account.

Other questions asked by Mr. Dohn, although of real interest, are beyond the scope and province of the paper, and space limitations do not permit an adequate answer.

Effective Angle of Incidence.—The writer, calling attention for the first time to the difference between apparent angle of incidence and the true effective angle of incidence, stated that his formulas (Eqs. 11) covering this correction were written for a section model and would require appropriate modification for a bridge or a scale model of a bridge. Mr. Vincent is correct in noting that the additional moment due to camber would need to be considered in such amplification of the formulas.

The δ-Parabolas.—The writer's "δ-parabolas" are based on the assumption at Eqs. 19 that the static lift or torque graph is represented by a cubic equation. If the lift or torque graph differs from a cubic, the δ-amplitude curves given by the construction of Fig. 20 would not be parabolas; and, if the variation from a cubic is substantial, the simplified graphic solutions shown in Figs. 19 and 20 would need to be modified. The graphic short cuts and examples given were intended merely to be illustrative. The main objective was to demonstrate the writer's thesis that amplitudes are determined by the curvature of the static lift and torque graphs.

Contrary to the inference in Mr. Vincent's discussion, the δ-parabolas and the related graphic short cuts (Figs. 19 and 20) are not an integral part of the writer's theory and do not affect its basic accuracy. They were added merely as related items of interest and of suggested usefulness. They certainly do not affect the analysis for critical velocities and instability response in vertical, torsional, and coupled oscillations. The writer has concentrated on deriving and predicting the instability response curves because they are of the most vital significance in studying the potential instability of a structure. They are basic characteristic curves, unaffected by changes in stiffness, damping, or modes of oscillation. Amplitude response curves are of secondary value, and can be computed or plotted by various indicated methods from the basic instability response curves. Oscillating model tests yield incomplete information unless they give instability response curves. A hundred different amplitude response curves (for different modes and different damping values) are required to give the information that is supplied by a single instability response curve.

The section models used in the static wind-tunnel tests for this paper were simple sections (thin plate and symmetrical H), without stringers, curbs, or flanges. Instead of inverting or reversing the model in the wind tunnel to cancel deviations from perfect symmetry, the raw wind-tunnel graphs were inverted and reversed, yielding the smooth, symmetrical, averaged curves passing through the origin. It is realized, of course, that unsymmetrical and modified sections will yield unsymmetrical and modified wind-tunnel curves. The numerical examples will be correspondingly modified, but the theory and its basic formulas are not affected.

The Negative Slope Theory.—Mr. Vincent states that the "'negative slope theory' * * * does not explain the successive restricted modes, each of which prevails over only a limited range of velocity." That is exactly what the writer's theory does explain (see Figs. 13 and 15 and other publications of the writer[75,76,77]). The writer's extension of the "negative slope theory" from its original limited application to narrow sections (such as ice coated transmission lines) to the more general and more complex case of wider sections (such as airfoils and bridge sections) introduced all the new aspects of the theory—notably: Torsional oscillations, phase difference, ranges of instability for "stable" sections, and successive ranges of limited instability (in addition to the range of catastrophic instability) for unstable sections. Since any given range of limited instability is expressed in terms of $\frac{V}{N\,b}$, an increase in wind velocity requires an increase in the frequency N for continued instability in the same range, and that is why successive restricted modes are evoked, each of which prevails over only the corresponding limited range of wind velocity. No other known theory explains and predicts all these observed phenomena and relationships, and no other known theory offers a quantitative basis for their evaluation.

Vortex Action.—Mr. Vincent makes the interesting suggestion that "vortex action" may contribute a superimposed effect. The writer has made an independent study, with original contributions,[73] covering "vortex action" in producing forced vibrations of cylindrical sections (sections of zero static lift). The essential characteristic feature is the appearance of the alternate vortex trail, with corresponding periodic alternations of lift force, even when the section is held stationary. To the writer's knowledge, tests have not revealed this characteristic feature of vortex action in static model tests of bridge sections. Alternating vortices observed in the case of an oscillating section model are a different phenomenon. They are explained by the oscillations, and are not the cause of the oscillations; the vortices then correspond to the known

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[75] "Rigidity and Aerodynamic Stability of Suspension Bridges," by D. B. Steinman, *Transactions*, ASCE, Vol. 110, 1945, p. 548, Fig. 26.

[76] "Simple Model Tests Predict Aeroydnamic Characteristics of Bridges," by D. B. Steinman, *Civil Engineering*, January, 1947, Figs. 4, 5, 6, and 7 and February, 1947, Figs. 12 and 14.

[77] "Problems of Aerodynamic and Hydrodynamic Stability," by D. B. Steinman, *Proceedings*, 3d Hydraulics Conference, *Bulletin No. 31*, State Univ. of Iowa Studies in Eng., Iowa City, Iowa, 1947, Fig. 11.

alternations of lift force on the oscillating section, and the period of the vortex discharge is determined by the period of the oscillations. Whether there is, in addition, an independent vortex action (of independently determined period) superimposed upon the known aerodynamic forces, requires further research. The present indications are that any such superimposed independent vortex action is imperceptible or negligible in effect. It may contribute a refinement in the theory, but it certainly does not offer an independent theory explaining the known phenomena of bridge oscillations.

Limitations of the Conventional Flutter Analysis.—Mr. Vincent advocates the use of the Theodorsen flutter analysis for the preliminary examination of proposed bridge designs. Aside from the inaccuracies and inconsistencies involved in applying the conventional flutter analysis to non-airfoil sections, such procedure would completely miss the investigation of potential vertical and torsional oscillations. All it would give would be a value (of questionable accuracy) for critical velocity for coupled oscillations, with no information on the intensity of potential amplification (negative damping) and potential amplitude response for vertical, torsional, and coupled oscillations. It would also give no information on the low velocity ranges of limited instability. After all, the challenging problem is that of vertical and torsional oscillations, whereas coupled oscillations, in the case of bridges, are still hypothetical and unrecorded.

Influence of Terrain.—The writer believes that much valuable time and effort have been wasted in seeking some mysterious influence of the terrain in the investigations of aerodynamic effects. All that need to be known in reference to any bridge exposure are the potential velocity, direction, and inclination of the wind. Wind gusts are only one of the many possible causes of initiating small oscillations, but wind gusts will not explain the amplification that constitutes instability. It is the steady component of the wind that produces continuing and amplified oscillations.

The writer has stressed the significance of wind inclination in his prior published writings, and he is in full agreement with the last paragraph of Mr. Vincent's discussion.

Comparison with Oscillating Model Tests.—Mr. Farquharson challenges the validity of the writer's analysis by emphasizing quantitative deviations between selected results from various oscillating model tests and the values plotted in the writer's numerical examples. The comparison is not valid for a number of reasons. In the first place, the sections used in the oscillating model tests were not the same as those used in the writer's numerical examples; there were material differences in both form and proportions. The cross section Mr. Farquharson has added in Fig. 28 is incorrect for the graph above it and is therefore misleading. He states, elsewhere in his discussion, that these differences in the cross section are of "great importance." In addition, the writer's numerical examples used assumed or approximate data (including roughly approximate preliminary values of the slopes s_1, s_2, s_3, and s_4), and were intended merely for qualitative and illustrative use and not as the basis for any definitive critical test of validity. Moreover, different oscillating model tests made by Mr. Farquharson at different times and under different conditions have yielded varying results, and tests made by Louis G. Dunn and others at the California Institute of Technology in Pasadena have yielded still different results. By selecting data from such varying range, almost any kind of case can be made. By the same reasoning, anyone who was so inclined could easily challenge the validity of Mr. Farquharson's oscillating model tests.

In the California tests cited by Mr. Farquharson, using a section of the same form as the writer's and differing only in the $\frac{d}{b}$-ratio, much closer agreement with the writer's computed values was obtained. The range of the observed oscillations indicates maximum instability approximately at $\frac{V}{N\,b} = 1.75$ compared with the writer's value (Fig. 13) of $\frac{V}{N\,b} = 1.60$. In the actual graph, this displacement would be relatively small. The small difference in the $\frac{d}{b}$-ratio is sufficient and in the right direction to explain the disparity.

Had Mr. Farquharson presented an instability response graph (as distinguished from an amplitude response graph) it would be directly comparable with the writer's instability graphs (Figs. 13 and 15). Such direct comparison would provide a direct check of the basic critical velocities (at zero damping) as computed by the writer. Instead, Mr. Farquharson gives critical velocities (for the principal range only) at various high damping values. As a result, an important error is introduced since all of his values of structural damping are too high. They need to be corrected by subtracting the atmospheric damping in still air. The writer's formulas and his graphs include automatic correction for atmospheric damping; but Mr. Farquharson's damping values, obtained by oscillating the model in still air, include atmospheric damping and high friction damping as well as structural damping. All his horizontal lines need to be lowered an undetermined amount, and this will modify the comparisons he has made and the conclusions he has drawn.

Mr. Farquharson declares that dynamic model tests have indicated values for the logarithmic increment well above the maximum of $\delta_1 = 0.0152$ shown in Fig. 13, and cites observed vertical motion at $\delta_s = 0.021$ in the California tests and much higher values in his own tests. There are two errors in this comparison. In the first place, he bases his conclusion on his values of δ_s, and all the cited values of δ_s need to be corrected downward an unknown amount, as explained herein. In the second place, the ordinates of any instability graph depend upon the arbitrarily assumed value of the width factor b^2/w (representing the nondimensional density-mass factor). All the ordinates may be increased or diminished in any desired ratio by increasing or diminishing the assumed value of b^2/w in the same ratio. Without giving the values of b^2/w he used, in comparison with the value assumed by the writer in the numerical example, Mr. Farquharson's comparison is invalidated.

The form of the amplitude response graphs obtained by Mr. Farquharson (shown in Fig. 29) constitutes a truly remarkable check of the writer's theory as represented in the amplitude response graph derived in Fig. 22. No other known theory will yield these results.

Lower Instability Ranges.—Mr. Farquharson agrees that his oscillating model, by its limitations, is incapable of revealing or disproving the lower instability ranges given by the writer's theory and shown in Fig. 13. Consequently, it cannot be used to establish the existence or nonexistence of such lower instability ranges. He admits that the California tests showed the second or intermediate range, although he could not get that range on his model because of its excessive structural damping (0.0068 and higher), including friction damping.

Mr. Farquharson then adds that structural damping values as low as 0.0025 could never be expected on any bridge yet built. On the contrary, the actual structural damping value on the original Tacoma Bridge was as low as 0.002, or lower. The girders and roadway contributed only 1.5% of the total stiffness; consequently, their structural damping value is multiplied by the low factor of 0.015. The remainder of the stiffness was given by the gravitational resistance of the cables, and such nonstress contribution adds nothing to the structural damping. The observed persistent oscillations of the bridge in various modes at low wind velocities are further proof of the low damping value.

With the principal instability range and the second instability range conceded, all considerations of mathematical and physical continuity point to the logic of a series of diminishing instability ranges converging toward the origin. The trigonometric integrals that are necessarily involved yield such a tapering sinusoidal graph. This is one place where the writer's theory reveals relationships that are beyond the limitations of oscillating model tests.

Contrary to Mr. Farquharson's surmise, the writer did not derive the lower instability ranges from the plotted diagram in the Carmody report (the official report of the Tacoma Bridge failure). The originally published diagram in that report[60] is unscientific. It confuses and obscures the true picture by showing observed oscillations plotted on velocities V, whereas they should be plotted on velocity ratios $\frac{V}{N\,b}$. This would make the mode of oscillation immaterial, thereby yielding cumulative massed data from all the modes. The writer, following this principle, obtained the instability ranges by plotting all the recorded observations from the official log, after reducing each observation to its value of $\frac{V}{N\,b}$. When the observations were so plotted, the successive critical ranges were clearly revealed. Subsequently, Mr. Farquharson reported a calibration correction for the anemometer used in the observations,[61] which modifies the critical ranges previously revealed but does not nullify them. Instability ranges at low $\frac{V}{N \cdot b}$ remain confirmed by the recorded observations of the higher modes at low wind velocities within the calibration range.

Subharmonic Oscillations.—The writer used the appellation "subharmonic," in quotes and in parentheses, merely to facilitate identification by those who had applied that designation. The writer holds no brief for the correctness of that term. In fact, he had long ago discarded the implied concept of forced vibrations due to vortex action as having any useful validity for the bridge application.

Incidentally, a bridge is a linear system with respect to oscillations, and (contrary to Mr. Farquharson's conclusion) it is impossible for a linear system to be excited to amplified oscillations at ½ (or any other submultiple) of the frequency of the disturbance.[65] This basic fact would seem to eliminate vortex action as the explanation of the phenomena of bridge oscillations.

Torsional Instability Graphs.—Mr. Farquharson's comment on the torsional instability graphs in Fig. 15 is hairsplitting. Even if he is correct in noting that the two parallel lines are only approximately parallel and would meet somewhere outside the figure, the agreement of the two curves in Fig. 15 still constitutes a remarkable check of the writer's theory.

That Mr. Dunn's experimental graph in Fig. 15 shows too high a value for the critical velocity is established.[22] The writer's conclusion was not based on the laboratory data (which would not have revealed the error), but was derived from internal evidence in Mr. Dunn's instability graphs as published in the official report.[21] For different values of the spring constant and of the density-mass ratio, the graphs showed different critical velocity ratios, ranging from 3.3 to 5.1, whereas the values should have been identical. The only consistent explanation was that, through some unknown experimental error, the plotted zero line had been displaced upward from the true zero line at which the different graphs (having different slopes) would meet at a common point (allowing for minor experimental errors). Since the plotted zero line was determined by subtractive elimination of the structural damping (lowering the zero line), the only reasonable conclusion was that the subtractive correction had been too small, and that some contribution to the total damping (possibly the nonlinear friction damping) had been missed or overlooked by the investigators. The graph needed to be raised relative to the zero line, thereby reducing the indicated value of the critical velocity ratio. The shift derived by the writer reduces the critical ratio $\frac{V}{N\,b}$ from Mr. Dunn's value of 3.3 (ranging to 5.1 in his other tests) to a lower corrected value of 2.5. Mr. Farquharson agrees that this shift is in the right direction, but indicates that it does not go far enough, since he now cites values as low as 1.9 at zero damping, 2.01 at low damping, and 2.5 at $\delta_s = 0.03$ (which would be a lower figure after subtracting atmospheric damping). Earlier reported tests gave values of 2.6 to 2.8 for the critical velocity ratio for the Tacoma section. The variation in the reported results of oscillating model tests by different experimenters at different times, with their respective values of critical velocity ratio ranging from 5.1 down to 1.9, is far greater than the variation of the writer's computed value of 2.5 from the values now cited by Mr. Farquharson. The difference in section ratios is in the right direction to explain the variation.

Mr. Farquharson does not challenge the shape of the writer's torsional instability graph shown in Fig. 15; nor does he offer a corresponding instability graph, obtained experimentally, for direct comparison.

A Scientific Check of the Theory.—The writer would welcome an open-minded, cooperative, scientific check of his theory by oscillating model tests. Only by tests on identical sections, under identical conditions, can a scientifically valid comparison be made. Such tests would either confirm the validity of the writer's theory, or indicate the direction of study for any correction or further refinement. Mr. Farquharson agrees, in the opening sentence of his discussion, that such consummation "would be fortunate indeed."

Testing of Theories.—There is a habit of blindly accepting a mathematical analysis, without checking the derivation and investigating its limitations, merely because the final formula appears to check certain experiments; and there is a corresponding habit of blindly rejecting a mathematical analysis, however scientific and rational, merely because the final formula does not appear to check certain experiments. Both of these procedures are unscientific. The accepted theory may contain inconsistencies, anomalies, and limitations that are not revealed by the experimental tests to which it has been applied. The other theory may have been tested unscientifically or inaccurately, or a single link in the theory may require modification, and premature rejection may mean the loss of a scientific contribution to progress.

It is true that the Theodorsen flutter theory has been checked by experiments on streamline airfoils, but that does not necessarily prove that the theory and its derivation are perfect. Certain anomalies and inconsistencies remain unexplained. Moreover, that theory cannot be applied to nonairfoil sections—not even to thin flat plates—without violating the underlying assumptions implicit in the analysis.

The writer's theory has not been tested by application to streamline airfoil sections. It may check such tests as well as the Theodorsen theory, even though the phase-correction coefficients are not identical in the two theories. In the airfoil application, the velocity V is relatively high and the width b is relatively narrow, so that the total phase difference across the section is relatively small. On account of the smallness of the total phase difference across the section, any inconsistencies or imperfections in the Theodorsen theory may not be revealed. Moreover, at high values of $\frac{V}{N\,b}$, the two theories are in virtual agreement and, at the limit, the two theories are identical.

The real test of the writer's theory is in the application to bridge sections, for which the theory was written and to which the Theodorsen theory is admittedly inapplicable. This is a far more difficult and complex problem, beyond the resources of classical aerodynamics. The writer's theory is the only analysis that explains all the known phenomena of bridge oscillations, the only analysis that shows remarkable qualitative confirmation, and the only analysis that offers a quantitative solution for all phases of the problem. The scientific attitude demands fair consideration and testing before condemnation.

Critical Review.—The writer has not claimed finality of perfection for his theory, nor has he asked blind acceptance. He has submitted it to the profession for critical analysis, constructive review, and scientific checking. The fact that it contains new ideas and departs from conventional theories that have become sacrosanct is not a rational reason for prejudiced attack or blind rejection. That is not the way to make progress.

As stated in the paper, Eqs. 2 are the pivotal formulas of the analysis presented. They are the equations giving the air forces acting on any oscillating section. They are new, and represent a contribution of basic importance. To the writer's knowledge, the problem of writing a general formula for the air forces acting on oscillating sections has never been solved before. Derived with absolute mathematical rigor, the basic validity of Eqs. 2 cannot be successfully challenged. No assumptions, approximations, or short cuts are involved in their derivation. If the writer had ended his paper at the establishment of Eqs. 2, that contribution alone would have justified publication as a scientific advance and as a basis for further progress. In order to serve maximum usefulness, however, the writer decided to give the remainder of his analysis.

The pivotal formulas (Eqs. 2), written with complete generality, contain four coefficients, C_1, C_2, C_3, and C_4, the phase-correction factors (approaching the value of unity at the limit). These four coefficients are given rationally and scientifically by Eqs. 5. The basic validity of Eqs. 5 cannot be challenged successfully. They involve the use of the phase factor $e^{-i\varphi}$. That use of the phase factor $e^{-i\varphi}$ is not an arbitrary assumption, as asserted by Mr. Bleich. It is rigorously established in vibration mechanics.

All the essential formulas that follow in the paper are of unquestioned rigor and validity. They are so written as to remain unchanged if any desired modification or substitution is made for the F-values and G-values given by Eqs. 5.

The writer has also extended and applied the formulas to unsymmetrical sections (with laterally eccentric center of gravity or stiffness axis, or both), with strikingly interesting results checking the known behavior of oscillating models, previously unexplained. Space limitation has prevented the inclusion of this material.

The value of the writer's contribution, subject only to possible future refinement, is trustfully submitted to the judgment of the profession.

Acknowledgments.—The writer wishes to express his heartfelt gratitude to all who gave him encouragement and cooperation at a time when he felt the loneliness of his task. In particular he wishes to record his indebtedness to Messrs. Maher, Perry, Peterson, Bakhmeteff, and Tea for personal services, encouragement, and valued cooperation in making required tests. Hunter Rouse and Joseph W. Barker, Members, ASCE, and Harry D. Belock, respectively, made available the facilities of: The Iowa Institute of Hydraulic Research in Iowa City, Iowa; the Virginia Polytechnic Institute through the Research Corporation; and the Reeves Instrument Corporation. Messrs. Allan, Klemin, Thomas, Lieber, Slavin, Richardson, Ghaswala, Dohn, and others named herein are thanked for their generous discussions.

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ADVANCES IN SEWAGE TREATMENT IN THE DECADE ENDING WITH THE YEAR 1949

REPORT OF THE COMMITTEE OF THE SANITARY ENGINEERING DIVISION ON SEWERAGE AND SEWAGE TREATMENT

This report of the Committee on Sewerage and Sewage Treatment covers the years 1948 and 1949, and, in addition, at the request of officers of the Sanitary Engineering Division, the 10-year period from 1940 to 1949, a period of war activities followed by inflation. As yet a vigorous readjustment of prices has not occurred. However, the shortage of steel and cement and other materials has largely disappeared. There is still a considerable backlog of delayed civilian work, such as sewers, sewage treatment works, and garbage and waste disposal works, which was shelved because of high construction costs and the hope that lower prices would shortly prevail.

The economic situation, as exemplified by the *Engineering News-Record* construction index, has greatly increased the over-all cost of all construction work and worried both municipal authorities and engineers, because old estimates, upon which bond issues were voted, do not suffice under present cost conditions.

LEGISLATIVE DEVELOPMENT

Sewage Research.—The annual review of the literature on sewage and waste treatment and stream pollution by the Committee on Research of the Federation of Sewage Works Associations (1)(2)[1] continues to summarize the art each year from the research, laboratory, and operating standpoints, as it has done each year since 1933. The number of articles reviewed pertaining to trade wastes increased from 45 in 1939 to 110 in 1948. The National Council for Stream Improvement, Incorporated (3), is studying problems of disposal from de-inking waste paper, strawboard and other board manufacture, and various types of pulp and paper processes. This council was organized in 1943 and has grown in membership and number of projects undertaken. The appropriations under Public Law No. 845 concerning research total $1,000,000 for 1949–1950, at 41 institutions, on 97 topics.

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[1] Numerals in parentheses, thus: (1), refer to corresponding items in the Bibliography (see Appendix).

United States Public Health Service.—In a countrywide survey (4), the U. S. Public Health Service finds a need for a sanitation program estimated to cost $7,834,581,000 as of June, 1946. The major part of the program is for water and sewerage works, to benefit 79,000,000 people. Under Public Law No. 845, the Service is proceeding to enter upon its water pollution control activities. President Truman has appointed six nongovernment members of the Water Pollution Control Advisory Board (5).

Federal Legislation on Stream Pollution.—The long protracted struggle for federal stream pollution control finally culminated in the passage of Public Law No. 845 (6), which was enacted on June 30, 1948. This is known as the "Water Pollution Control Act," and provides for water pollution control activities in the Public Health Service of the Federal Security Agency and in the Federal Works Agency. Before the act had been fully implemented and tested, a bill (7) was introduced by Congressman T. C. Crook (Indiana) which would seriously modify the provisions of the act. Under Public Law No. 845 no action can be taken through the courts in water pollution control matters unless the pollution originating in one state brought about objectionable conditions in another state, and no action can be taken by the United States Attorney General in any state unless it is approved by the water pollution agency of the state in which the pollution originated. The amended bill (7) would remove the so-called states-rights provisions and abolish the provision for public hearings. Apparently the effect of these amendments is to concentrate control in the U. S. Public Health Service and make it difficult for the full value of cooperative state programs to be realized.

Other bills introduced in the Eighty-first Congress, First Session, were:

House of Representatives Bill No. 5089, by J. W. Byrnes (Wisconsin), would allow amounts paid for industrial waste treatment works to be amortized at an accelerated rate for income tax purposes. (A similar bill is House of Representatives Bill No. 5091, by T. Chatham (North Carolina)).

House of Representatives Bill No. 4825, by T. M. Hand (New Jersey), would amend the Oil Pollution Act of 1924 to include all parts of the sea within the United States and all inland waters navigable in fact and amend or strike out certain minor clauses.

State Legislation.—There is still confusion in the various states on pollution control in administration and lack of adequate laws (8). In 1949, 28 state legislatures enacted or considered some form of abatement legislation. Of these, probably 23 have taken action.

WATER POLLUTION CONTROL

Compacts and Other Interstate Agreements.—On June 30, 1948, the fifth interstate pollution control compact in the United States (9) went into effect between Indiana, West Virginia, Pennsylvania, Illinois, New York, Ohio, Kentucky, and Virginia, replacing an informal agreement made in 1928 between the same eight states and Maryland, Tennessee, and North Carolina. This compact is administered by representatives of these states as the Ohio

River Valley Water Sanitation Commission. The Interstate Sanitation Commission (1936), the Interstate Commission on the Delaware River Basin (1936), the Interstate Commission on the Potomac River Basin (1941), and the New England Interstate Water Pollution Control Commission (1947) were formed by earlier compacts. The Ohio compact covers the greatest number of states, the greatest drainage area, and the largest population. Three interstate pollution control agreements are in force as of September 1, 1948, known as the Great Lakes Drainage Basin Sanitation Agreement, the Upper Mississippi River Drainage Basin Sanitation Agreement, and the Interstate Sanitation Committee (North Dakota, South Dakota, and Minnesota).

In the agencies created by interstate compact, two instruments appear to be evolving: (*a*) Control by commission order and specifications of minimum treatment requirements, including the zoning of the waters; and (*b*) control by coordinated investigation, advice, and promotion of uniform legislative practices, leaving to the individual states the exercise of enforcement. The Interstate Sanitation Commission adopts the first policy; and the New England Interstate Water Pollution Control Commission, the second.

Ohio River Valley Water Sanitation Commission.—The Ohio River Valley Water Sanitation Commission in its first annual report (10) describes the project and new sources of industrial pollution from brine and soda-ash production. Among the important problems is industrial waste control, particularly phenol waste. Included are the text of the compact, enabling legislation, a statement of policies, and the Cincinnati (Ohio) Pool hearing report.

Standards for Ohio River.—The Ohio River Valley Water Sanitation Commission has suggested (10) substantially complete removal of settleable solids, not less than 45% removal of total suspended matter, and a 65% reduction of B.O.D. (biochemical oxygen demand) in organic wastes discharged into the Cincinnati Pool at low flow—but not less than 35%, when conditions permit, provided a minimum content of 4 ppm of D.O. (dissolved oxygen) can be maintained below Cincinnati.

Pollution Surveys.—Two pollution surveys are of interest, namely, those of Allegheny County, Pennsylvania, and the International Joint Commission.

Allegheny County.—In Allegheny County the sanitary authority (11) now comprises 102 communities and 30 industries, including more than 85% of the population of the area. In 1945 with a sewered population of 1,342,000, the daily discharge of sewage from 33 main outlets exceeded 200 mgd. Concentration limits are indicated for toxic and deleterious wastes in the composite sewage of the treatment plant influent (12). These limits are based on the effect of the wastes on treatment. All acid iron wastes should be excluded from the system.

International Joint Commission.—The interim report (13) made to the International Joint Commission by the Board of Technical Advisers relative to the Pollution of International Boundary Waters covers practically the same area as did the 1914 report (14)(15). The boundary waters of the Saint Clair River, Lake Saint Clair, Detroit River, and Saint Marys River between Michigan and Ontario, Canada, are extensively used for a variety of purposes. Since 1913 industrial wastes have become the major problem in the United

States, amounting to 1,204 mgd, and in Canada amounting to 374 mgd. About 90% of the population concerned is in the United States. Among the significant industrial wastes are phenols, cyanides, ammonium compounds, oils, and ether soluble material. The B.O.D. is important in the River Rouge, where anaerobic nuisance occurs. The total suspended solids discharged by industry in the United States total 1,707,400 lb daily in the United States and 668,300 lb daily in Canada. The pollution has affected municipal water supplies, injures fish, wild fowl, and aquatic life, and interferes with the recreational use of boundary waters. Various industries are working to treat their wastes.

The estimated costs of remedial measures are:

	Municipal	Industrial
United States	$68,000,000	$13,000,000
Canada	14,000,000	3,000,000
Total	$82,000,000	$16,000,000

The advisory board recommended that the Commission maintain contact with the pollution control problem, formulate a program, and work for the cooperation of all agencies concerned.

M. LeBosquet, Jr. (16), describes the 1913 conditions, the extent of the 1946–1948 survey, and certain objectives of the International Joint Commission (17), such as maintaining in effluents a pH-value between 5.5 and 10.6, iron (as Fe) less than 17 ppm, an odor of 8 or less in boundary waters caused by added material, and oil 15 ppm or less. A tentative value for phenol of 10 ppb (parts per billion) or less in effluents is still under consideration.

PROJECTS

Notable Projects Nearing Completion.—Among the large projects placed under construction during the postwar period are those at New York, N. Y., Boston, Mass., Los Angeles, Calif., Philadelphia, Pa., and the West-Southwest Works of The Sanitary District of Chicago (Ill.).

Beginning in 1934, New York City has built seven sewage treatment works, now treating about 40% of the total sewage of the city. Under construction are two major sewage treatment works: Owls Head (capacity 165 mgd), with ten contracts totaling $15,000,000; and Hunts Point (120 mgd), with six contracts totaling $15,600,000. The construction of the Twenty-sixth Ward Plant (60 mgd) was interrupted by the war but is now being completed by contracts totaling $5,600,000. These plants are all a part of the New York City sewage treatment program (18), costing $200,000,000. On December 16, 1948, by a consent order of the Interstate Sanitation Commission, New York City is obligated to a construction program of $78,000,000 by the end of 1953, and to complete by 1959 the treatment of all sewage of the city.

The Nut Island Plant (19) at Boston should be in operation during 1951, for a flow of 112 mgd and for 900,000 people. The plant includes comminutors, grit chambers, aeration (20 min), and sludge digestion.

The Los Angeles project (20) is being built substantially as recommended by Metcalf and Eddy in 1944 (21). The submarine outfall (discharging 1 mile offshore) was put in service on September 2, 1949. The headworks, effluent outfall connections, and sewage disinfection plant are completed. Pre-areation tanks, primary settling tanks, and digestion units are nearing completion, but unfinished sludge-handling facilities may postpone operation until the end of 1950.

The Philadelphia program (22) includes three plants. To finance the program, a sewer rental tax was inaugurated to service bonds sold without regard to the constitutional limits fixed for general revenue. This tax was first collected in 1946 after 5 years of litigation. The program comprises three plants, each serving about one third of the city. Construction of the new Northeast plant began in May, 1947, and operation is scheduled by the middle of 1950, serving 669,000 population with a flow of 91 mgd. The design as of 1970 is based on a population of 876,000 and on a flow of 125 mgd. The plant includes primary settling tanks (1.65 hours); aeration tanks (2.5 hours with air at 0.3 cu ft per gal); final tanks (2 hours); and sludge digestion tanks (8 units 110 ft in diameter and 32 ft deep). Digested sludge will be lagooned during 6 years to 8 years before deciding the ultimate method of sludge disposal. Because of the difference in the condition of the Delaware River, a lesser degree of treatment is required at the Southeast and Southwest plants.

The West-Southwest Works of The Sanitary District of Chicago combine the West Side Works (Imhoff tanks) and the Southwest Works (activated sludge tanks), with additional equipment and tanks to treat the West Side effluent mixed with Southwest sewage by the activated sludge process. The design capacity provides for treating a 900-mgd average flow of sewage and 500 tons per day of dry solids, with dewatering and heat drying of solids. The pumps and air blowers are driven by steam turbines. The aeration tanks were completed early in November, 1949; but, because of the coal strike, operation was postponed until December. Tests during the summer of 1949 indicate that the mixture of Imhoff effluent with the strong Southwest sewage will be easier to treat than the Southwest sewage alone.

Reports.—A number of reports on projects, proposed or on which construction is about to begin, have reached the committee, concerning the Allegheny County Sanitary Authority in Pennsylvania; the Blackstone Valley Sewer District in Rhode Island; the East Bay cities in California; Seattle, Wash.; Tacoma, Wash.; Portland, Ore.; and Wilmington, Del.

Allegheny County.—The sewage disposal project of the Allegheny County Sanitary Authority (11) includes 129 municipal subdivisions, in an area of 749.5 sq miles, with a 1945 population of 1,342,000. The industrial and commercial flow is estimated to exceed the domestic flow of sewage. A central plant will be located on the Ohio River, including primary treatment and pre-aeration, with chlorination, further to reduce the oxygen demand and to retard the rate of oxygen depletion, during critical low-water periods, attaining a removal of at least 40% of the B.O.D. The program includes 248 miles of intercepting sewers, receiving up to 250% of the average flow. The project is to be financed by revenue bonds, serviced by rates ranging from 7¢ to 18¢ per 1,000 gal, according to the amount of flow per quarter. The normal domestic sewage is estimated per capita at 120 gal per day, with an addition of 115 gal per day for infiltration and 115 gal per day for storm water, making a total of 350 gal per day for rate making.

Blackstone Valley.—The Blackstone Valley Sewer District (23) lies in Rhode Island and includes the cities of Pawtucket and Central Falls and three towns, with a municipal area of 75 sq miles. The water consumption in 1947 averaged 16 mgd and private supplies provided 35 mgd. The combined sewage and industrial wastes total about 50 mgd. About 13 mgd of industrial wastes can be discharged directly to the streams. Six treatment plants are proposed. The largest (Bucklin Point) serves 158,000 people; will account for 47 mgd as of 1975, with a total equivalent population of 400,000; and includes pre-aeration tanks, sedimentation, and chlorination—inasmuch as the effluent discharges into Narragansett Bay and affects shellfish laying as well as recreational use of the waters. The entire project is estimated to cost about $15,662,000. Construction started in 1949.

East Bay Municipal Utility Districts.—The sewage disposal project (24) for Special District No. 1, comprising the cities of Oakland, Berkeley, Alameda, Piedmont, Albany, and Emeryville in California, was authorized by the voters in November, 1946, with a bond issue of $23,500,000. Construction began on February 24, 1948. In general, the project (25) consists of three large intercepting sewers, an outfall sewer, and a primary treatment plant. About 6.5 miles of the intercepting sewers is completed, and the rest of the system will be under contract early in 1950.

Seattle.—On September 25, 1948, Abel Wolman, M. ASCE, made a report on sewage disposal to the City of Seattle (26), recommending the continued use of the waters of Puget Sound for disposal by dilution, the extension of various sewer outlets to deep water with multiple outlets, and a ban on construction of combined sewers. He questioned the standards of the Washington State Pollution Control Commission (which provide that B. Coli (bacteria, coliform group) should not exceed 50 per 100 ml (most probable number). The State Pollution Control Commission replied in November, 1948, stating (27) that pouring raw sewage into the nearest watercourse is a violation of the state pollution control law and indicating that "* * * the minimum acceptable degree of treatment for the sewage and industrial wastes of the City of Seattle will be primary treatment." The commission contends that it is difficult to predict what conditions will result from the method of disposal by dilution in deep water and that Mr. Wolman is of the same opinion.

Tacoma.—At Tacoma C. S. Seabrook, and Richard G. Tyler, M. ASCE, report (28) that the sewage originating in the northerly part of Tacoma and discharging along the southwesterly shore line of Commencement Bay will require no treatment. Because of adequate currents, the extension of existing outfalls to about 40 ft below mean low water will insure satisfactory dilution and eliminate deposit of solids along the shore. The sewage originating in the southerly part of the city and currently discharging into the city waterway and the Puyallup River should receive primary treatment, comprising primary sedimentation with flocculation, the removal and destruction of all grease and floating material, the chlorination of both sludge and effluent, and the pumping of the chlorinated (undigested) sludge into the bay about 2,500 ft from shore at a depth of about 200 ft. Messrs. Seabrook and Tyler propose to eliminate sludge digestion. Mr. Tyler notes that it is difficult to chlorinate sewage sludge adequately unless it is quite fluid and well disintegrated. He states that it is obviously most economical to chlorinate the raw sewage, yet he approves the proposed treatment.

Portland.—Construction began in 1948 on the project at Portland (29), with a 17,000-ft tunnel, carrying the sewage flow from the Willamette River watershed to the Columbia River. The fully digested sludge is to be discharged through the outfall sewer with the settling tank effluent, into the Columbia River, which is believed to furnish ample dilution.

Wilmington.—The problem at Wilmington is described in a recent report (30) on collecting the sewage and industrial flow and discharging the effluent from a treatment works into the Delaware River. Wilmington had an area of 16 sq miles and a population of 120,000 in 1948. The proposed treatment works (40 mgd) include sedimentation with separate sludge digestion, in earth lagoons. The water supply averages around 17 mgd (domestic use averaging 50 gal per capita per day). The interceptors are planned for three times the dry weather flow.

Among the other projects which have advanced during the postwar period are those at San Francisco, Calif., Honolulu, Hawaiian Islands, and Oklahoma City, Okla.

Enlargement of Existing Plants.—Among the existing plants which have been rehabilitated or enlarged in the postwar period are those for Baltimore, Md., and the County Sanitation Districts of Los Angeles County in California.

Baltimore.—The Baltimore sewage works (31) are situated on a 547-acre tract of land. The original works went into operation in 1911, with primary settling tanks, separate unheated sludge digestion tanks, and 30 acres of trickling filters. Humus tanks were installed later.

In March, 1935, a new stage of operation began, with mechanically cleaned screens and settling tanks. Additional heated sludge digestion tanks went into service as of December 31, 1947 (making 2,326,100 cu ft of capacity). Of thirteen units, three are heated (554,700 cu ft) and ten are unheated (1,771,400 cu ft). Digested sludge is elutriated with trickling filter effluent and dewatered on vacuum filters, the cake being deposited around the plant.

The capacity of the eighteen acres of trickling filters (32) has been increased by supplementary rotary distributors, from a loading of 1,960 persons per acre-ft to 4,400 persons per acre-ft.

From 1940 to 1948 the population increased from 806,900 to 956,700; the flow increased from 88.3 mgd to 132.1 mgd (109 gal per capita per day to 138.1 gal per capita per day); the B.O.D. of the sewage declined from 266 ppm to 202 ppm; and the suspended solids increased from 151 ppm to 194 ppm. The improvements have increased the plant capacity (exclusive of the activated sludge unit) from 100 mgd to 140 mgd in average flow, and the plant can process peak flows somewhat greater than 200 mgd.

County Sanitation Districts of Los Angeles County.—A M Rawn, M. ASCE, reports (33) that the County Sanitation Districts of Los Angeles County, with a sewage flow exceeding 100 mgd, have expanded their system by the addition of four new districts, and are enlarging their facilities to provide a capacity of 280 mgd by the end of 1950. The outfall facilities from the plant to the ocean are being paralleled with a second tunnel, to be completed by 1952. The local sewage treatment plants at Pasadena, Covina, El Monte, and Long Beach in California have been superseded by the works of the districts.

Ocean Disposal.—Ocean disposal on the Pacific Coast is described by the late A. K. Warren, M. ASCE, and Mr. Rawn (34*a*); and ocean disposal on the Atlantic Coast, by A. D. Weston, M. ASCE (34*b*).

Pacific Seaboard.—On the Pacific Coast south of Point Conception in California, Los Angeles is the principal center of population, with a discharge into the open ocean. A high-rate activated sludge plant is under construction. San Diego, Calif., discharges the effluent of a primary treatment plant (with 2-hour pre-aeration) into San Diego Bay. North of Point Conception, sewage disposal is confined chiefly to bay waters, and treatment is being installed on San Francisco Bay by the East Bay cities and by parts of San Francisco. In Southern California, the citizens are aware of the value of the coastal recreational waters. The gross pollution of the bathing beaches in the vicinity of Los Angeles finally led the California State Board of Health to forbid bathing along 10 miles of beach and to force Los Angeles by court action to take steps to remove the pollution.

On the coast of Washington, Seattle and Tacoma are on the arms of Puget Sound. In Oregon, Portland is arranging to discharge sewage effluent and digested sludge into the Columbia River, which has a flow ranging from 75,000 cu ft per sec to 150,000 cu ft per sec (35*a*). Most of the coastal towns are relatively small, but the pollution from pulp mills in some locations is great (36*a*).

Atlantic Seaboard.—The general coast line from the Maine–New Brunswick boundary to the Texas-Mexico boundary covers some 3,574 miles, with variations in tide and atmospheric temperatures. The major part of the population discharging sewage into the ocean along the Atlantic Seaboard (34*b*) is concentrated in a few states with large metropolitan areas. In 1938, out of a total population of 19,302,000, no treatment was furnished for 11,558,000. The principal centers were in Massachusetts, New York City, New Jersey, Pennsylvania, and Maryland. Along the Atlantic seaboard in recent years (37), there has been activity in clearing the coastal waters from pollution in Maine, New Hampshire, Massachusetts, Rhode Island, Connecticut, New York, New Jersey, Pennsylvania, Maryland, Virginia, North Carolina, South Carolina, and Florida.

Disposal in the Great Lakes Area.—In the Great Lakes area (38), there has been activity in controlling pollution for several years. In New York, Buffalo completed its sewage works in September, 1939. Detroit, Mich., is planning to enlarge its existing plant. In Ohio, Cleveland is studying improving its Southerly Works and Toledo is planning secondary treatment. Indiana has been cleaning up a number of rivers, in particular, the Calumet River (Grand and Little). The Sanitary District of Chicago is approaching completion of its vast program of sewage treatment.

Sewage Treatment

Increase in Sewage Treatment Plants.—The activity in construction of sewage treatment works is summarized by the United States Public Health Service, as follows:

Year	No. of plants	Estimated population served
1940	5,580	40,617,000
1945	5,786	46,865,000
1948	6,058	48,698,000

In 1948, of the total number of plants, 2,984 offered primary treatment and 2,917 complete treatment.

Sewage Treatment Works at Military Installations.—The Sub-Committee on Sewage Treatment in Military Installations of the Committee on Sanitary Engineering, National Research Council (39), contacted 135 sewage treatment works out of 399 at military installations and analyzed the results from 133. In addition 25 Navy sewage treatment works were included. In part, the summary and conclusions indicated that (40) the sewage tended to: (*a*) Be more concentrated than domestic sewage; (*b*) contain a higher proportion of volatile solids in proportion to total solids; (*c*) contain a higher proportion of grease; and (*d*) exhibit larger and more rapid variations in diurnal flow. The installations usually contained screens and comminutors. Grit chambers were omitted in new installations, except in locations on sandy soil. Both rectangular and circular settling tanks were used, with longer detention periods than in municipal practice and inferior results, probably due to nonuniformity of flow.

TABLE 1.—Trickling Filters at Military Sewage Treatment Plants

No. of plants	Group designation	Range of B.O.D.[a] loading (lb per acre-ft per day)	Filter and second settling B.O.D.[a] effluent (ppm)	Over-all B.O.D.[a] reduction (%)
4	Deep, no recirculation	253– 429	86.9	92.2
4	Deep, recirculation	245-1,170	85.7	90.8
8	Two stage	192– 930[b]	66.5[b]	90.2
4	Shallow, recirculation	792–2,320	81.8	88.1
4	Deep, recirculation	1,190–1,950	78.7	87.5
3	Deep, no recirculation	720–2,130	75.7	82.8
3	Deep, recirculation	2,530–5,750	70.3	78.3
4	Shallow, recirculation	2,370–8,250	70.1	74.8

[a] Biochemical oxygen demand. [b] Second stage.

More than one half of all the military sewage treatment plants contained trickling filters—34 out of 300 being selected for study. Comparisons are given on single-stage, deep, no-recirculation filters; single-stage, deep filters, with recirculation; and two-stage shallow filters with recirculation. The results are summarized in Table 1.

None of the filters loaded with more than 1,000 lb per acre-ft produced nitrification. Recirculation improved performance and permitted increased loading.

About 50 contact aeration plants were built at Army cantonments and a few were built for the Navy. Under favorable conditions of loading and sewage concentration, good effluents were obtained with over-all removals of B.O.D. averaging from 80% to 95% in the plants studied. With stale sewage or aeration troubles, effluents were unsatisfactory and odors were intense. The results at 5 plants show a removal of B.O.D. ranging from 60.4% to 85.8%, using from 1.35 cu ft per gal of sewage to 4.21 cu ft per gal of sewage (40*a*).

> "The difficulties of operation of contact aerators, high maintenance labor requirements, and more than occasional odor nuisances indicate that contact aerators are less desirable for use in military camps than are trickling filters. No superiority of contact aerators over activated sludge has been demonstrated."

Design, Construction, and Maintenance

Effect of Economic Cycle on Design.—Following World War I, a sudden rise occurred in construction costs, which quickly declined to a fairly even level for a number of years, rising in 1929 and 1930, and then declining during the panic years of 1932–1933–1934. Since the end of World War II, construction costs have steadily risen. It is not clear whether the peak has been reached by January, 1950. Although it is now apparent that plants built in 1934–1935 were installed at bargain prices compared to costs today, the impact of inflation on operating costs of sewage treatment works has varied with the type. For instance, in the field of sludge disposal, the operating costs for dewatering and heat drying of sludge have risen greatly since 1941, because of the increased cost of labor, coal, ferric chloride, and other supplies—whereas the operating cost of sludge digestion increased but slightly, only the labor factor being involved. The activated sludge plants practicing sludge digestion, followed by disposal at sea, are much freer from the effects of inflation than are those using dewatering on vacuum filters and heat drying for sale as fertilizer. Another factor is the value of by-products, if such are produced. During the war years, grease recovery was emphasized and the value increased. The same grease today is a drug on the market. The price of heat-dried activated sludge in commerce f.o.b. (free on board cars in carload loadings) the plant following the end of World War II rose somewhat above the war ceilings. As competing organics were released from other uses, the price of heat-dried activated sludge declined. It is evident that the designer of sewage treatment works should endeavor to forecast the future and develop a plant design which will produce plants as free as possible from the influence of fluctuations in the cost of operation.

Mechanization of Sewage Works.—Since 1920 there has been a decided trend towards mechanization in sewage treatment works. Sludge thickeners of the rotary or straight-line type were among the first mechanical equipment installed. With the advent of the activated sludge process and the large installation at Milwaukee, Wis., the use of mechanical equipment became more general, with the introduction of air compressors (centrifugal or positive types) and sludge filters (rotating vacuum type), together with rotary driers on a large scale. Around 1935, the development of the flash drier for sludge drying and the adaptation of the multiple hearth for the incineration of sludge opened a new era, which was slowed down by World War II. Although such a development may be useful in large installations for municipalities having populations of more than 100,000, depending on load conditions, it may be unwise for towns with populations of less than 50,000. For example, at East Chicago, Ind. (1940 population, 54,637), a treatment works went into service in 1947, fully equipped with vacuum filters and an incinerator. The equipment has been idle ever since because it is cheaper to lagoon the sludge.

The North Side Works (Chicago), built in 1923–1928, are completely electrified, and contain 370 electric motors totaling 21,000 hp. The West-Southwest Works, now nearing completion, will have more than 1,150 motors totaling 13,000 hp. The larger units, such as sewage pumps and air blowers, are driven by steam turbines. This mechanization introduces a problem of continuous inspection and maintenance.

Maintenance of Equipment.—The conditions during World War II seriously impaired civilian maintenance of equipment and structures in sewage pumping stations and treatment works. This difficulty continued through the period from 1944 to 1947, resulting in delayed maintenance. In 1948, conditions improved; and, in 1949, general rehabilitation began in earnest. In a series of staff articles from October, 1943, to November, 1949 (41), Morris M. Cohn, M. ASCE, covers the maintenance of a wide range of devices in sewage works.

Sewer Construction—Large Interceptors.—In the construction of large intercepting sewers, tunneling methods have been improved, and the amount of labor has been reduced. In suitable soil, the use of air spades is quite general. A common cross section is an inverted half ellipse, as this permits a flat bottom for convenient use during construction. In tunnel work, for the support of the excavation, steel liner plates are favored, braced by steel ribs. To prevent movement of the soil, a practice has developed of jacking the ribs and liner plates to a firm bearing against the excavated soil, thus preventing surface settlement. In placing concrete, pipes from 8 in. to 12 in. in diameter are driven down at intervals from 700 ft to 1,000 ft apart, and extended inside the tunnel. Through these pipes the concrete may be pumped into place by concrete pumps or air ejectors. This procedure results in a hard dense concrete. After the concrete has set, cement grout is pumped in between the earth and the liner plate.

Air Conditioning Flow.—The growth of air conditioning flow is illustrated by the experience of Chicago since 1932 (42). In the entire city, by the end of 1949 the number of installations had increased from 205 to 6,544 and the tonnage of refrigeration from 18,000 to 165,750. In 1948 the peak demand for air conditioning required 270 mgd, of which 105 mgd originated in the downtown area. The maximum day pumpage of the water works in 1948 was 1,293 mgd, compared with a yearly average pumpage of 972 mgd. Thus, at the peak of warm weather, the air conditioning flow may rise to approximately 21% of the total water pumpage. In Milwaukee (43), the amount of clear water from air conditioning plants has led to the construction of additional storm sewers or direct discharge to the nearest river. In 1948, 7.5 mgd of water from cooling systems was diverted from the sewers leading to the treatment works. About 100 air conditioning and industrial cooling systems are in use. In 1948, J. L. Ferebee, M. ASCE, estimated the combined flow of air conditioning and condenser water at 13 mgd, compared with an average sewage flow of 137.4 mgd, or about 10%.

Fine Screens.—Some 30 years ago or 40 years ago, fine screens (under $\frac{1}{8}$-in. opening) were considered suitable for treatment of raw sewage where ample dilution was available. In the 1945 *Engineering News-Record* inventory (44), 77 screening plants were listed. However, the U. S. Public Health Service inventory of 1945 showed 47 screening installations, including 2 to protect trickling filters and 1 ahead of activated sludge. In 1949, the committee finds 45 fine screens in use, of which 23 give the sole treatment to raw sewage; 5 are followed by activated sludge; and 17 are followed by tanks or trickling filters. Where screening is the sole treatment, chlorine is applied in 11 cases, either seasonally or the year around.

In California, fine screening as a minimum treatment is barred by the State Department of Public Health (45). In New Jersey, fine screening was barred on November 1, 1927 (46). In New York State, the Niagara Falls sewage (47) is treated by a fine screen, in operation since June 1, 1939, of the revolving inclined disk type, with $\frac{1}{32}$-in. slots. In 1947, 2.7 cu ft of wet screenings was removed per million gal, containing 30.2 lb of dry solids. Of the fine screens in use on municipal sewage, the Reinsch-Wurl type and the Dorr type of screens are the major survivors.

OPERATION

Sedimentation.—Where primary treatment only is needed, rectangular sedimentation tanks have been built on a large scale at the Minneapolis–Saint Paul Sanitary District in Minnesota, and later, at Detroit. The Minneapolis-Saint Paul tanks (48) are built in two batteries of 3 units each, the units being 56 ft wide by 290 ft long, with an average water depth of 15.5 ft. The detention period at 134 mgd is 2 hours. The Detroit tanks (49) are each 112 ft wide by 272 ft 8 in. long, with 238 ft of weir (128 sq ft of tank area per foot of weir). These units are designed for a minimum flow of 22.25 mgd and a maximum flow of 89 mgd. In the period from July 1, 1948, to June 30, 1949, with an average sewage flow of 394 mgd, the average displacement was 1.2 hours. The reduction in suspended solids was 52.4%; and, in 5-day B.O.D., 43.8%. The sludge removed contained 90.23% water.

Pre-Aeration.—The first scientific appraisal of the effect of aeration was made by William M. Black and Earle B. Phelps (50) in 1910. J. D. Walker, Assoc. M. ASCE (51), urges the use of pre-aeration to increase the treatability of sewage, with periods of from 15 min to 90 min. C. P. Gunson (52) notes that 15-min pre-aeration with 0.3 cu ft of air per gallon is helpful at Denver, Colo., followed by 15-min flocculation ahead of the clarifiers.

Pre-Aeration with Return of Sludge.—There is also another type of pre-aeration, in which a short period of aeration in the activated sludge process is used ahead of biological filters. Probably the first plant of this type was at Birmingham, England, where a short-period, activated sludge plant was installed (53) ahead of a trickling filter, based on experimental work in 1920–1921.

Since 1925, the treatment has been built up to cover the entire flow at the Minworth works at Birmingham.

At Decatur, Ill., William D. Hatfield, Assoc. M. ASCE (54), reported the results of a 2-year (1925–1926) testing station, using a 2.5-hour aeration period, which reduced the B.O.D. of settled sewage by from 30% to 40% and enabled the trickling filters to be dosed at three times the previous rate with unaerated sewage. A pre-aeration unit was completed for the entire plant and placed in operation in January, 1928. The sewage was aerated from 1.85 hours to 2.66 hours. The return flow from the clarifiers ranged from 5% to 20% of the sewage flow (55).

Self-Flocculation.—To improve the settling qualities of sewage, self-flocculation has been tested in the laboratory (56) and in the plant (57). Periods suggested by A. J. Fischer, Assoc. M. ASCE, and A. Hillman (57) are from 30 min to 40 min for flocculation, followed by 1.5 hours to 2.0 hours in the clarifier. Under favorable conditions, the clarification of sewage can be improved by preflocculation without the use of chemicals. According to the U. S. Public Health Service Census, 175 plants were so equipped in 1943. In 1942 the Dorr Company alone listed 67 installations. The second report of this committee (58*a*) shows an increase in removal of suspended solids at the West Side Plant (Chicago) by from 5.6% to 6.5% and at Minneapolis–Saint Paul, by 4.6%.

Diffuser Plates.—In its earlier reports (35*b*)(36*b*)(58*b*)(59*a*), the committee has discussed the behavior of diffuser plates and the maintenance problems involved. The investigations at the Southwest Works (Chicago) have continued, both on diffusers and on air cleaners. In the United States, some operators still prefer a diffuser of low permeability (rating under 25 cu ft per sq ft per min). Others are considering plates having a rating of from 80 cu ft per min to 120 cu ft per min. Unless the air is properly cleaned and a suitable rate of air passage maintained, trouble apparently may develop with any type of porous diffuser. A special committee of the Federation of Sewage Works Associations has prepared a manual on the subject, which is now being processed for publication (Norval E. Anderson, M. ASCE, chairman).

Types of Diffuser Media.—Since the original experiments at Chicago, Indianapolis (Ind.), and Milwaukee, the majority of diffused air activated sludge plants have utilized flat plates of porous media for the diffusion of the air. Originally, in 1919, the only plate available was the "Filtros." Since 1928, "Carborundum" and "Alundum" have been offered, with plates of greater porosity than the "Filtros." For experimental purposes, several other materials have been suggested. In 1939, porous tubes mounted on pipe headers were suggested and used in smaller plants, the headers being fixed in position. One of the first large installations was at Gary, Ind., in 1939. Such porous tubes apparently clog more rapidly than plates, according to the tests of The Sanitary District of Chicago.

Aeration Details.—The design of diffuser installations has received attention in recent years, from various aspects. The experimental work of the Milwaukee Sewerage Commission led the late T. C. Hatton, M. ASCE, to install a ridge and furrow system in the original Milwaukee plant in 1926. Investigations of C. H. Hurd, M. ASCE, at Indianapolis, and of The Sanitary District of Chicago indicated the greater efficiency of spiral flow aeration, as installed at Indianapolis in 1925, and at the North Side Works in Chicago in 1928. In tests prior to construction at Chicago, various plate arrangements were studied, chiefly with reference to the development of suitable velocities across the bottom of the tank. This research led to the adoption of two rows of plate boxes, with aeration plates extending 5.5 ft from the adjoining wall. Subsequently, various widths of channel and numbers of rows of plates were tried at the Calumet and the Southwest Works. Frank C. Roe (60) has recently suggested the so-called wide band air diffusion system in an effort to widen the zone of air supply, by using porous diffuser tubes at right angles to the tank side walls. Mr. Anderson, at Chicago, has varied the procedure by setting shorter plate holders, connected to a pipe header and placed normal to the channel wall. This arrangement is apparently economical, providing any desired bank of aeration desired, and easy control of air distribution to the plates.

The manual of the Federation of Sewage Works Associations indicates that a wide band gives slower circulating velocities and greater bubble contact. Hence, for a weak sewage, when minimum velocity of circulation determines the air requirement, a narrow diffuser band near the side of the tank would be desirable; and, for a normal or strong sewage, the diffusers should be placed in a wide band.

Chicago Sanitary District.—The Sanitary District of Chicago began an intensive study of air diffusers on a small scale in 1941 because of clogging difficulties at its Southwest Works after 2 years of operation and, in 1942, an intensive study on small groups of plates and tubes of various permeabilities to determine relative clogging rates. The investigation was expanded until by the end of 1945 each of the eight aeration tanks in battery A was completely fitted with a different diffuser system and the eight tanks were operated in a parallel test for determining both the relative clogging rate and air economy of the respective systems. Each tank is 136 ft by 434 ft, arranged in four passes, making the flow channels 34 ft wide, center to center of the walls, and 1,736 ft long, with a water depth of 15 ft over the diffuser plates. Each tank has a nominal treating capacity of 32 mgd. Mr. Anderson (61)(61*a*) states that the tests showed no difference in the air economy of diffuser plates within the permeability range of from 40 to 120, and that the air economy of diffuser plates improved after a prior service of about 1 year.

Air economy improved with greater number of diffusers, about as follows:

No. of rows of plates	Plate ratio	Relative air requirement (%)
4	1:10	100
2	1:20	104
1	1:40	112

Porous tube diffusers have about the same air economy as a like number of horizontal plates, if operated at the same submergence. Any possible power saving by reduced submergence is offset by the greater quantity of air required. Vertical porous plates show less air economy than do horizontal plates. Slotted tube diffusers are considerably inferior to porous diffusers in air economy. So-called precision tubes, on incomplete tests, apparently clog faster than do porous tubes.

A plate supplied with clean air should pass from 12,000,000 cu ft of air to 15,000,000 cu ft of air. With the more porous diffusers, the rate of air supplied per square foot of plate surface has increased in practice from 1 cu ft per min to 6 cu ft per min in certain cases. Apparently the life of a porous diffuser before clogging to its permissible limit is determined mainly by the total amount of air passed, rather than by the length of time in service. At the Bowery Bay Plant in New York City, diffuser plates have passed about 12,000,000 cu ft of air in 7 years at a rate of 4 cu ft per min. At the North Side Plant in Chicago, the plates after 21 years of service have passed 11,000,000 cu ft of air at a rate of 1 cu ft per min.

Adequate air filters are therefore highly important to remove dust from the air supply. Apparently plates operating on air containing 0.5 mg of dust per 1,000 cu ft of air have operated five times as long as those operating on 2.5 mg of dust per 1,000 cu ft of air. The tests indicate that the clogging rate of porous diffusers is inversely proportional to the permeability rating.

The results of tests as to relative air economy are briefly summarized in Table 2, prepared by Mr. Anderson.

TABLE 2.—SUMMARY OF RELATIVE AIR AND POWER REQUIREMENTS FOR EQUAL DISSOLVED OXYGEN IN AERATION TANKS 32.75 FT WIDE BY 15.5 FT DEEP WITH VARIOUS DIFFUSER SYSTEMS AT SOUTHWEST SEWAGE TREATMENT WORKS, CHICAGO, ILL.

Diffuser system[a]	Diffusers in tank	Depth to diffusers (ft)	Relative Air (%)		Relative Power[c] (%)	
			New diffusers	In service[b]	New diffusers	In service[b]
4 rows of porous plates	5,640	15.0	110	100[d]	110	100[d]
2 rows of porous plates	2,820	15.0	115	104	115	104
1 row of porous plates	1,410	15.0	124	112	124	112
Porous plates in containers[e]	2,720	15.1	115	104	116	105
Porous tubes (swings)	2,552	13.5	124	112	115	104
Vertical porous plates	2,720	13.0	147	....	132	....
Slotted brass tubes	1,410	15.0	170	170	170	170

[a] Rate of air application was about 12 cu ft per min per 1,000 cu ft of tank for 4 rows of porous plates. For other systems, it was proportionately higher. [b] Diffusers in service for a year or more. [c] Making allowance for difference in air pressure due to depth of submergence. [d] Arbitrarily taken as 100% for comparison. [e] In cast-iron containers, normal to the wall.

In the battery C addition to the West-Southwest plant, 80-permeability porous plates are to be installed in 7 tanks and 120-permeability plates in 1 tank. The plate holders are placed normal to the tank walls, projecting out 8 ft 5 in. A plate ratio of 1:11.7, with an operating rate of about 3 cu ft per min, was computed to give the lowest total annual cost in this installation. In the influent half of the tanks, there are 27% more plates, to secure more uniform life, since diffusers clog more rapidly in the influent half of the aeration tank.

Air Cleaning Devices.—Equipment for air cleaning is not uniform in activated sludge plants. With dirty air (9.0 mg per 1,000 cu ft) at Southwest Works in Chicago, a new air filtering system is being installed, consisting of bag filters with loose asbestos fiber lining for primary filtering and automatic electronic filters for secondary filtering (installed in the fall of 1947). This system is expected to reduce the dust content to less than 0.5 mg per 1,000 cu ft.

In New York City (62) at three plants, traveling oil screens are in service, followed by a 10-ply treated cellular air mat. With adequate maintenance, this equipment has proved satisfactory.

Air Diffusion Problems.—Much of the difficulty with diffusers has been caused by the failure of the designer to install efficient and suitable devices for removing dust from the air used. Very few data are available on the dust in the atmosphere at a sewage works. The common types of clogging are: (*a*) Underneath, from dirty air or particles of rust from the air supply system; (*b*) surface, from sludge solids settling on the plate—or from a chemical deposit, such as calcium carbonate (from tannery lime wastes) or ferric oxide (from acid pickling liquor) or from biological growths in the plate or on the surface; and (*c*) internal, from moisture carried into the plate structure during humid summer weather.

Maintenance Problems of Diffuser Plates.—The Wards Island plant, in New York City, went into service in 1937 and now handles 220 mgd instead of 180 mgd, as designed. Various methods for cleaning the diffuser plates have been tried, such as brushing and treating with acid, caustic soda, and ammonia, as well as sandblasting, with temporary success. Spalling the plates by a traveling oxyacetylene torch (63) gave temporary relief in the summer and fall of 1947, but its limitations are not fully known. The plates at the Bowery Bay plant, in service since March, 1942, became seriously clogged and were cleaned by oxyacetylene spalling in 1949. The most serious diffuser clogging at the New York plants has been found to be caused by impregnation of the upper layer of the diffuser plate with fine silt and sand.

At Peoria, Ill., diffuser plates have been sandblasted (64) on the underside, about once in 2 years, by removing the end plates in a container and pulling a special nozzle back and forth beneath the plates. The plates at Peoria have passed about 11,000,000 cu ft of air in 14 years. At Milwaukee, Mr. Ferebee reports that sandblasting the upper surface of Filtros plates has worked satisfactorily and is preferable to spalling by a torch.

ACTIVATED SLUDGE

Activated Sludge.—The activated sludge process in the early stages in the United States was used by cities on inland waters, where local conditions required a high degree of treatment, as at Chicago, Cleveland, Indianapolis, Houston (Tex.), and Milwaukee. Since 1930, the installation of high-rate activated sludge plants has increased, as at the seaboard cities of Boston, Los Angeles, Philadelphia, and New York, to produce lower grade effluents at lower cost. This process involves not only a shorter aeration period, but also low solids in the aeration tanks and a low air ratio. There has also been a marked trend towards effluents containing a small amount of nitrates, with consequent air economy. Other modifications, such as step aeration, have been favored at New York. These methods give the same results as the conventional treatment. Aeration of sewage alone without return of sludge will produce a medium grade of effluent, as shown at Indianapolis (36*c*), where part of the works is operated as an aeration plant without sludge return.

In 1939, Robert T. Regester, M. ASCE (65), discussed the problems and trends in the activated sludge process. In general, the objectives he indicated have been sought: (*a*) More consistent performance under variable conditions; (*b*) economies in construction and operation; and (*c*) better technique for process control and better facility of operation. In 1949, Richard H. Gould, M. ASCE (66), summarized the status of the activated sludge process and recommended certain elements for the improvement of design or operation, such as: (*a*) Sedimentation of raw sewage before aeration (in use at North Side, Chicago, in 1928); (*b*) closer control of sludge age in the aeration system (used by many operators by holding low solids in mixed liquor); (*c*) concentration of excess activated sludge in separate tanks (in use at Calumet Works, Chicago, in 1936 and at Southwest Works, Chicago, in 1941); (*d*) and (*e*) relocation of effluent troughs and sludge removal outlets in final settling tanks to take advantage of density currents (discovered by S. I. Zack, M. ASCE, at Chicago in 1931); (*f*) correlation of air supply with abstraction of sewage impurities; (*g*) regularized reconditioning of air diffusion media; (*h*) employment of step aeration to reduce tanking, improve control, or handle greater-than-design flows; and (*i*) utilization of high-rate activated sludge where complete treatment is not required.

Various observers have endeavored to compare operating data on diffused air activated sludge plants since the Report of the Committee on Sewage Disposal, Engineering Section, American Public Health Association (67). Recently, W. E. Stanley, M. ASCE, and J. F. Berberich (68) analyzed operating data from 27 activated sludge plants by plotting four major factors against B.O.D. removal—that is, B.O.D. loading, aeration period, solids concentration in the aeration tank, and air supplied per gallon of sewage. The results obtained are not sufficient to establish any real basis of design factors, probably because of difference in sewage and character of effluent.

Short-Period Aeration.—Short-period aeration (sometimes called modified or high-rate aeration) was first tried experimentally at Milwaukee in 1915 (36*c*) with periods of from 1 hour to 5 hours. Aeration tests were made from November, 1934, to December, 1937, on batteries (one third of the plant) of the North Side Works (Chicago) with periods of from 3.0 hours to 5.0 hours and, later (from April, 1936, to July, 1937), experimentally at the West Side Works (Chicago) with periods of from 1.0 hour to 2.9 hours (36*d*). Under the conditions for the disposal of effluent in Chicago, the procedure was not adopted. In New York City, however, the procedure proved attractive. In 1943, at Jamaica, N. Y. (36*e*), high-rate aeration (from 2.29 hours to 4.88 hours) was tried, using 0.42 cu ft of air per gal (0.34 cu ft for aerating sewage and 0.08 cu ft for air lifts).

At the Bowery Bay plant (36*f*), a comparison was made from 1942 to 1944 of step aeration and high-rate aeration. High-rate aeration produced a decreased volume of sludge, with a heavier digested sludge of a lower volatile content and a lesser quantity. In New York City, in practicing high-rate activated sludge treatment, low concentrations of solids are carried in the aeration tanks and the sludge age is kept under from 8 hours to 12 hours. At the Jamaica Sewage Treatment Plant, high-rate activated sludge has been used for 3 years and is the approved method of operation for a plant originally designed for step aeration and conventional activated sludge. In 1948, the average operation results for 40-mgd flow with 2-hour aeration, using 0.39 cu ft of air per gal, reduced the suspended solids 80% and the B.O.D. 75%. The thickened sludge to digesters contained 6.49% of solids.

The new Owls Head plant (New York City)(160 mgd) is designed for a 2-hour aeration period, based on the average sewage flow. At the Rockaway plant (15 mgd) an aeration period of slightly more than 2 hours is provided.

Step Aeration.—Step aeration is a modification of the activated sludge process (69), whereby the sewage flow can be introduced in regulated amounts at multiple points throughout the flow of the returned activated sludge through the tanks. Provision has been made in the design of the Tallmans Island, Bowery Bay, Jamaica, Twenty-sixth Ward, and Hunts Point plants in New York City for step aeration. All have facilities for receiving sewage in any or all four passes, except Tallmans Island, which cannot receive sewage in the third pass. The Wards Island plant was remodeled in 1949 to provide step aeration, with the expectation of increasing the capacity of the plant from 180 mgd to 240 mgd. This will permit adding settled sewage to the first, second, or fourth passes. In operation, lower solids will be carried in the aeration tank effluent.

Operating results at Bowery Bay (70) from March 7, 1947, to September 30, 1947, indicate 1-hour sedimentation followed by 2.5-hour aeration and 1.7-hour final settling, with a flow of 40 mgd and 25% return sludge. Using 0.65 cu ft of air per gal, around 90% of the B.O.D. was removed. In the method of operation, the return sludge was aerated for 6.8 hours, then half the primary effluent flow was in contact with sludge and aerated 1.6 hours, and half was in contact with sludge and aerated 0.6 hour. During the process, 590 cu ft of air was used per pound of B.O.D. removed. In the effluent the D.O. averaged 4.3 ppm; and the B.O.D., 14.5 ppm. The basic control was "sludge age"—that is, the average number of days the sewage solids remain under aeration. Mathematically, the sludge age is calculated by dividing the weight of suspended solids under aeration by the daily suspended solids load imposed on the aeration system.

Use of Oxygen in Activated Sludge Process.—Based on laboratory studies on a small scale (14.9 cu cm per min to 56.7 cu cm per min) and the use of oxygen in place of air, Daniel A. Okun, Assoc. M. ASCE (71), claims that greater efficiency can be obtained by removal of B.O.D. in smaller volume aeration tanks and that bulking, if it occurs, does not impair bioprecipitation. F. W. Mohlman notes that more data on a larger scale are required to substantiate Mr. Okun's claims, and particularly tests in parallel with air in one aerator and oxygen in another.

Aeration Details.—Tapered aeration has been adopted in a limited way. Although 25% return sludge in the aeration tanks is commonly taken as a basis of design, many plants are able to return 50%. The size of blower units in the larger plants has now risen to 130,000 cu ft per min at a pressure of 8.5 lb per sq in. At the West–Southwest Works (Chicago), the blower installation comprises three units at 70,000 cu ft per min each, one at 115,000 cu ft per min, and two units at 130,000 cu ft per min each, designed for a maximum pressure of 8.5 lb per sq in., all driven by steam turbines.

Return Sludge Pumps.—In large-scale activated sludge plants, centrifugal pumps have been used to return the sludge, as in the North Side and Calumet Works (Chicago), Wards Island (New York), and the Easterly Works at Cleveland. The Southwest Works (Chicago) are equipped with air lifts in place of pumps for returning activated sludge. This procedure was adopted for three reasons: (*a*) The difficulty of conducting sludge into and out of a common pumping station and the very large conduits which would otherwise have been required; (*b*) the problem of securing efficient pumps of proper characteristics; and (*c*) the lower cost of the air-lift installation. Two lifts each 16 in. in diameter were provided for each settling tank, one for ordinary flows, and the other to provide additional capacity for a maximum rate of 45% of the average sewage flow. In New York City, the plants at Tallmans Island (40 mgd), Bowery Bay (40 mgd), and Jamaica (65 mgd) also have air lifts for return sludge.

Plant Design.—In the West-Southwest Works (Chicago), the aeration tanks are designed for 37.5 mgd and the final settling tanks for 12.5 mgd each. Since this is the largest plant which has yet been built, it is probable that these final settling tanks are as large as any circular tanks yet projected (126 ft in diameter) in an activated sludge plant. The need of utility tunnels for air mains and general piping has been shown. In the Chicago plants (Calumet and Southwest), grit chambers were found necessary and have been installed at Calumet. Very few new plants are now being equipped with fine screens, preliminary settling tanks being preferred. In New York, the use of fine screens at the Jamaica plant and at the Twenty-sixth Ward plant has been discontinued. However, Milwaukee has used $\frac{3}{32}$-in. slotted screens since 1926, and Indianapolis has used its $\frac{1}{16}$-in. by 2-in. slotted screens since 1925.

SLUDGE PROCESSING

Final Settling Tanks (Activated Sludge).—With the exception of the Jamaica plant (tanks, 120 ft in diameter), the other four New York City plants have rectangular final settling tanks, varying in length from 93.42 ft to 179 ft, and in liquid depth from 12 ft to 14 ft. The development of final settling tanks was reviewed in the second progress report (58*c*) of this committee. Whether circular tanks are used, as at Chicago, or rectangular tanks, as at New York City, it is evident that activated sludge in final settling tanks behaves differently from raw sewage solids, because of density currents. Mr. Anderson (61) believes the sludge drawoff in rectangular tanks should be at the inlet end, whereas Mr. Gould prefers it at the outlet end. In the Southwest Works (Chicago), Mr. Anderson has used circular settling tanks 126 ft in diameter, with a weir trough 98 ft in diameter, carried on cantilever brackets. The sludge drawoff is at the center of the tank. D. W. Townsend, M. ASCE, states (61*b*) that at Milwaukee a depth of 15 ft is required, although in some plants where the ratio of peak flow to average flow is lower, he has used 12 ft.

Sludge Concentration.—For various reasons, in disposing of liquid sludge, concentration of the solids has been attempted to lessen the volume. Sometimes concentration is used prior to pumping through a pipe line (as at Chicago, Baltimore, and Cleveland), or to reduce the volume prior to barging to sea (as at Elizabeth, N. J., and New York City), or prior to digestion or filtration. The methods adopted may be physical, by settling, or chemical, such as by the addition of a coagulant or by elutriation. General data on types of tanks, dimensions, and operating results were given in the third committee report (36*g*). At the North Side Works (Chicago), the results in 1945–1948 were very similar to those for 1942–1944. Likewise, at the Southwest Works, results in 1945–1946 were very similar to those for 1944.

Control of Bulking by Use of Digested Sludge.—Digested sludge and digester overflow were tried at Peoria by L. S. Kraus (72) to control bulking activated sludge with a monthly plant efficiency of 95% removal of B.O.D. and suspended solids. The digested sludge is first converted to activated sludge in a sludge treatment aeration tank for 24 hours. Apparently no more air is required with about three and one-half times the solids in the system than normally used, and a better effluent results. The Kraus procedure is used at Peoria and Decatur and is incorporated in the Northeast plant (Philadelphia). C. Preston Witcher (73) also reports using a similar procedure at Ann Arbor, Mich., to maintain the sludge in better condition.

Mechanical Aerators.—For the smaller municipal and institutional plants, mechanical aerators are increasing in popularity. There are probably in excess of 131 installations in the United States. A number of types are available, with a central rotating device. Douglas E. Dreier, Jun. ASCE (74), has compiled data on 29 installations in the United States.

Effect of Digestion on Sewage Sludge.—Mr. Rawn and E. J. Candell (75) have investigated the effects of anaerobic digestion on sewage sludge and suggest that, wherever sludge banks will not be formed, either 11-day-old or 60-day-old digested sludge may be disposed of by dilution in the clarifier effluent, if conditions permit disposal of the latter with safety. Apparently additional studies are desirable.

Sludge Lagoons.—The review by the Committee on Sewage Disposal of the Engineering Section of the American Public Health Association (76) shows a growing utilization of lagoons in the United States, either as a temporary method of sludge disposal or as a semipermanent method. The largest installation is at the Southwest Works (Chicago), with a capacity of 2,530 acre-ft, allowing for 51.2 cu ft per capita. For the proper operation of lagoons, with a freedom from odor, some pretreatment of the sludge in heat-controlled digestion tanks is desirable.

Ponding of industrial wastes, often referred to as "lagooning," has been practiced for many years. This method is particularly applicable where the industrial operations are seasonal (77). For beet sugar wastes, ponds were used prior to 1919. In the chemical industry, the sludge from settling tanks at the Semet-Solvay Works at Syracuse, N. Y., was pumped to lagoons around 1895. Of recent years, the disposal of cannery wastes in lagoons (78) has been stimulated by the use of sodium nitrate, suggested by N. H. Sanborn in 1940. Lagoons for ponding sulfite waters are in use in Maine on the Androscoggin River. At Creighton, Pa., to impound from 300 tons to 400 tons daily of grinding abrasive, the Pittsburgh Plate Glass Company has installed a pipe line 1 mile long, through which the thickened wastes (from 20% to 30% solid) are pumped to a pond (700 ft by 1,600 ft in area) created by damming a valley (79).

TABLE 3.—DIGESTION PLANTS IN THE UNITED STATES IN DECEMBER, 1945

Type	Heated	Unheated	Total
Municipal..........	869	710	1,579
Institutional........	22	98	120
Total..........	891	802	1,699

Sludge Digestion.—The U. S. Public Health Census, as of December, 1945, lists separate digestion plants as given in Table 3.

The municipal heated tanks include three of the 2-story or "clari-gester" type and the unheated include 5 municipal and 2 institutional installations. Included under heated digestion are 111 stage digestion installations.

The Committee of the Sanitary Engineering Division, ASCE, on Sludge Digestion in its final report (80) summarized standard practice in separate sludge digestion as of 1938. The committee indicated that, instead of stating the capacity of the sludge tank in cubic feet per capita, it should be stated in terms of the weight of solids in the sludge removed daily from the sewage. In 1938, S. A. Greeley, M. ASCE (34*c*), recommended as a yardstick of design instead of "cubic feet per capita," the "pounds of suspended solids to be delivered into the digestion tank per cubic foot of digestion tank capacity per month," and indicated the then current rates of from 2 lb per cu ft per month to 3 lb per cu ft per month for relatively complete digestion, and rates as high as 4.5 lb for partial digestion. In 9 cities, he cited actual rates from operating data ranging from 0.80 lb per cu ft per month to 3.26 lb per cu ft per month, and from 2.03 cu ft per capita to 4.45 cu ft per capita. Recently, R. S. Rankin, M. ASCE, suggested (81) that a more desirable yardstick is pounds of volatile solids per cubic foot per month, and further advocated measuring sludge loadings on a time of detention basis, as determined by the displacement of the raw sludge feed. Long detention periods in the digesters give the greatest freedom from supernatant troubles and a denser sludge. In 14 plants examined by Mr. Rankin, the detention periods ranged from 11 days to 72 days. The loading in pounds per cubic foot per month for total solids ranged from 1.6 to 10.0; and, for volatile matter, from 1.2 to 8.6. The highest loading was for the 11-day period.

At Boston, with ultimate disposal of the digested sludge into the ocean, heat-controlled digestion has been adopted, with loadings of 2.2 lb of total solids per cubic foot per month and of 1.67 lb of volatile solids per cubic foot per month, with approximately a 40-day detention period. The capacity is approximately 1.4 cu ft per capita. At Portland, Ore., the capacity for sludge digestion is about 2 cu ft per capita, with loadings of 2.70 lb of total solids, or 1.8 lb of volatile matter per cubic foot per month and a 39-day detention period. Recently, the California State Board of Health permitted the East Bay Municipal Utility District at Oakland to install the first 2 of 4 digesters, with an initial loading of 3.78 lb of solids per cubic foot per month for the maximum month, and an ultimate loading of 5.5 lb of solids per cubic foot per month in 1970, with the understanding that the adequacy will be judged by actual results of operation. The sludge is derived from settling only.

Mr. Greeley currently states that the loads on sludge tanks should be judged with due regard to the characteristics of the incoming sludge and the methods at the particular plant for disposing of the digested or partly digested material. In his opinion, major weight should be given first to the load in terms of pounds of dry suspended solids in the sludge per cubic foot of tank capacity per month; second, to the available liquid displacement period; and, third, to the load in pounds of volatile solids per cubic foot per month. In his opinion, the minimum liquid displacement for sludge from preliminary settling tanks may be from 12 days to 15 days, where density in the withdrawn sludge is not important. On the other hand, at New York City, where the excess activated sludge is handled, a period of 25 days is preferred, to obtain a denser sludge for barging to sea.

The design basis 15 years ago was in general 2.75 lb of total suspended solids per cubic foot per month, although at Springfield, Ill., loads of more than 3 lb were handled. In recent years, loadings as high as 5.34 lb per cubic foot per month have been attained on sedimentation sludge. In the Hunts Point (New York City) project, an annual average was used for both preliminary and final sludge from 3.6 lb per cu ft per month to 4.8 lb per cu ft per month in heated digestion tanks, with a total loading in all sludge tanks of 1.8 lb per cu ft per month.

The size of large unit digesters has increased from 85 ft in diameter in 1930–1931 to 115 ft in diameter in 1949. Apparently the largest in operation is 105 ft in diameter at Detroit.

Since 1938, the number of separate heated-digestion tanks in the United States has increased, chiefly in the north. As a rule, digestion temperatures around 95° F are preferred, although many digesters are maintained at 85° F in the colder climates. The collection of gas by floating covers has grown in favor. Through 1949, floating covers had been installed in 813 plants, including 235 war installations.

The design of heated tanks has been improved, with better control of heat loss. Circulation by pumping or mixing by a suitable mechanism is favored. A variety of heating devices was described in the fourth report of the committee (35c).

Stage Digestion.—Two-stage digestion has become popular, with the primary tank equipped with devices for heating the digesting sludge and accomplishing the major part of the digestion, and the secondary tank used mainly for storage under relatively quiescent conditions (80). In the renovation of the Baltimore sewage treatment works, the original separate unheated-digestion tanks (31) have been gradually scrapped. As of December 31, 1947, there were three heated-digestion tanks, with a working capacity of 554,700 cu ft, and ten unheated open tanks, with a working capacity of 1,771,400 cu ft, making a total capacity of 2,326,100 cu ft, or approximately 2.46 cu ft per capita. At New York City, in the (activated sludge) Hunts Point Sewage Treatment Works now under construction, 19-day displacement is provided in four heated-digestion tanks, and 26-day displacement in four secondary digestion tanks, based on annual average quantities.

At the County Sanitation Districts of Los Angeles County, the four-stage digestion of settled sludge as practiced by Mr. Rawn, the late A. Perry Banta, M. ASCE, and Richard Pomeroy (82) is still unique. It produces a digested sludge of good quality with a loading rate in excess of 8 lb of dry solids per cubic foot per month. Mr. Rawn reports a volatile content in the digested sludge of slightly more than 51%, and the following periods of digestion:

Days	Return sludge (%)
9.4	50 (minimum)
11.4	25 (maximum)

Gas Utilization.—Gas utilization began in the United States about 1913 at Atlanta, Ga., for lighting, heating, and laboratory purposes. In 1926, at Plainfield, N. J., J. R. Downes installed two small gas-driven, air-driven (each 15-hp) air compressors. Later, in 1928, at Charlotte, N. C., two gas engines were installed, one (225 hp) on a blower, the other (55 hp) on a pump, followed in 1932 at Springfield, Ill., by a 180-hp engine driving a blower. Since 1926, installations have been made in about 213 municipalities of 271 gas engine units, ranging from 15 hp to 1,700 hp, for driving pumps, air compressors, and electric generators. New York City alone has 8 plants with 39 engine units installed, with a total horsepower of 40,170. The more recent engines are 1,450-hp dual-fuel units, which are actually gas engines operating on the diesel cycle, using a pilot injection of diesel fuel oil for ignition purposes. This type of engine can operate fully on diesel fuel if necessary, or on gas. Automatic controls adjust the amount of diesel fuel necessary to meet the gas flow and maintain full power output. Nine 1,688-hp dual-fuel diesel engines are being installed at the Los Angeles–Hyperion plant. At the Nut Island plant (Boston) two 820-hp dual-fuel diesel and one 840-hp gas engine are being provided. At both Los Angeles and Nut Island, power will be entirely derived from within the plant.

Experience at Cleveland, Chicago, and New York City indicates the impracticability of selling excess sludge gas to the gas utility.

Elutriation.—A. L. Genter, M. ASCE, reports 15 elutriation plants in operation and 14 under construction in the United States and Canada. Furthermore plans are completed for about 22 other plants. The enlarged Richmond–Sunset Plant in San Francisco was completed and went into operation in 1948, with an extra vacuum filter. The novel method of elutriating the digested sludge between and after both stages of digestion, the saving in digestion space, and reduction in chemical consumption, are worthwhile. In January, 1948, in cleaning out the secondary digester at Marion, Ind., in which experiments were conducted in 1943, a deposit of heavy sludge 10 ft deep in the 65-ft-diameter tank was discovered, with an average solids content of about 18.5%, and a volatile content of 37.9%. Thickening digested sludge prior to filtration was effected at Marion (83), by pumping the sludge from the bottom of a primary digester into a secondary digester through a 6-in. pipe line, with the admixture of cold quarry water, two volumes of water to one of sludge. Sludge concentrations increased from 5.88% or 6.65% to 8.65% or 9.94%. Mr. Genter finds that all installations filtering elutriated digested solids show chemical savings. At Springfield, Mass., J. R. McDonald uses a mixture of alum and ferric chloride for conditioning the elutriated sludge, in amounts of 69% alum and 31% ferric chloride, and reports in 1947 and 1948 that the average amount of the combination used was but 2% of the dry solids. In the new Virginia plants at Newport News and Portsmouth, about 2% ferric chloride is used on elutriated digested solids.

Sludge Dewatering by Vacuum Filters.—During the two decades prior to 1940, extensive experimental work was conducted at Milwaukee, Chicago, Baltimore, and elsewhere on the vacuum filtration of various types of sewage sludge. This research was followed by the installation of vacuum filters at a number of sewage plants, both large and small. In the period from 1940 to 1949, further experiments were made on sludge filtration in the United States and England, and approximately 97 sewage plants were equipped with vacuum filters in the United States. The largest installation in the world is now under construction at Chicago, with 98 filters, and a total of 55,860 sq ft in filter area.

Mr. Genter has described (84) the principles and factors which affect vacuum filtration of sludge and also the operating control (85). The design and construction of vacuum filters have been improved and operating methods devised to prolong the life of filter cloths. Ferric chloride is the approved conditioner, either with or without lime. The life of canton flannel or cotton cloths varies from 72 hours to 900 hours, whereas woolen cloths last from 400 hours to 2,500 hours at different installations. In one case, nylon is reported to have lasted 2,000 hours.

TABLE 4.—HEAT-DRYING AND INCINERATION INSTALLATIONS IN THE UNITED STATES (1950)

Type of drier	Incineration	Heat-dried fertilizer	Under construction
Flash	5	9	4
Multiple hearth	35	1	6
Rotary	....	2	....
Vacuum spray	1	....	....
Total	41	12	10

Dewatering and Heat-Drying Sludge.—A summary of heat-drying and incineration installations (to 1950) is given in Table 4.

In 1949, flash driers were under construction at the West-Southwest Works (Chicago), San Diego, Los Angeles, and Fond du Lac, Wis. In 1948–1949, multiple hearth units were under design or construction at Fall River, Mass., Toronto, Ont., Canada, Akron, Ohio, Ann Arbor, Mich., Providence, R. I., and Bridgeport, Conn. The units at Toronto and Ann Arbor are being equipped with flues, dampers, and burners, so that the unit can be used as an incinerator or a drier, as at Dayton, Ohio.

Trickling Filters.—In 1945, the U. S. Public Health Service reported 1,428 conventional trickling filters in municipal service and 208 in institutional service, a total of 1,636 plants. In its third progress report (36*h*), this committee commented on trickling filter practice. Mr. Dreier (86) has reviewed the experience in the operation of 31 standard trickling filter plants, serving from 3,000 to 325,000 in population.

The development of high-rate trickling filters and their design and use were outlined (87) by Mr. Greeley. During World War II, about 90 high-rate filters were installed at army posts, and in 1942 there were approximately 90 municipal installations. In 1945, the U. S. Public Health Service reported 128 municipal and 18 institutional installations, a total of 146 plants. Recirculation clearly increases the removal of B.O.D., but little nitrification is expected. The subcommittee on sewage treatment of the National Research Council (40*b*) holds that: "None of the filters loaded over 1,000 lb per acre-ft. produced nitrification." According to the B.O.D. loading, in the installations cited, from 770 lb to 13,780 lb in B.O.D. per acre-ft per day efficiencies ranged from 81% to 31%

Oxidation Ponds.—The value of oxidation ponds (88) was first discovered at Santa Rosa, Calif., in 1924. The installations (77*a*) have been principally in California and Texas in climates with sunshine throughout the year and freedom from freezing weather. Pretreatment is important to remove solids, which would otherwise settle to the bottom of the ponds. In California there are 26 ponds for populations ranging from 450 to 18,400, and the present design practice is to permit a loading of 20 lb in B.O.D. per acre-ft per day in general, although 30 lb has been used in warm central valleys. The depth varies from 1 ft to 5 ft, although from 3 ft to 5 ft is considered an optimum. Pond effluents are usually supersaturated with dissolved oxygen, except when overloaded. When at least 30-day detention is provided, the oxidation pond effluent may be used in any way that a well-oxidized effluent can be used. Bacterial reductions are reported equal to those by chlorination. In Texas, the State Department of Health reports about 100 municipalities using oxidation ponds as secondary treatment or as holding tanks in connection with land disposal of effluent in broad irrigation. Loadings are permitted up to 50 lb in B.O.D. per acre of surface area, with an effective depth not greater than 3 ft.

CHEMICAL TREATMENT

Chlorination.—The 1945 *Engineering News-Record* survey (44) shows 1,313 sewage treatment works equipped with chlorination apparatus, compared with 921 works so equipped in 1938. Chlorine has four general classes of use: (*a*) As an aid in disinfection; (*b*) as an adjunct to primary treatment; (*c*) as an adjunct to secondary treatment; and (*d*) as an adjunct to sludge disposal.

The application of chlorine in the field of sewage treatment was covered in 1934 by the Committee on Sewage Disposal of the American Public Health Association (89). This was supplemented in 1938 by L. H. Enslow, Assoc. M. ASCE (34*d*), and by the first four reports of this committee. At the end of 1949, chlorine was recognized (90) in sewage treatment as a useful adjunct to primary treatment in controlling odors, relieving plant overload, removing grease, improving clarification, and reducing bacteria. Chlorine has also been recognized as helpful in secondary treatment in controlling odors from trickling filters, filter pooling, and filter flies. In sludge disposal, chlorine has been used to control foaming and to treat supernatant liquor and waste activated sludge.

The value of chlorine in reducing the organic load from effluents before discharge to streams is still controversial, as the action seems to be mainly one of delayed decomposition rather than one of destruction of organic matter. The evidence on control of bulking in activated sludge plants also is unconvincing. In addition the weight of evidence is that chlorine is not of much value in cleaning clogged diffuser plates. Chlorine is not an active agent in thickening sludge (91), but may be helpful in some situations in conditioning the sludge to prevent septicity.

Chlorination of Sewage Effluents.—The use of chlorine in reducing the bacterial content in sewage treatment works effluent is increasing because it helps to protect bathing beaches or water supplies. At Detroit (92), in the 8-year period from 1941 to 1949, the average use was 5,638,800 lb per yr, with a maximum annual use of 7,828,000 lb per yr. The maximum rate of daily application was approximately 35,000 lb. At Los Angeles, in connection with screening, about 36,000 lb per day are used (93). Buffalo (94), Cleveland (95), and plants of the North Shore Sanitary District north of The Sanitary District of Chicago also use considerable amounts of chlorine for the protection of the water supply and bathing beaches. At 7 of the New York City treatment plants, the effluent is chlorinated for the protection of bathing beaches during the summer recreational months, using annually about 700 tons of chlorine.

Chlorine Use on Industrial Wastes.—According to G. M. Ridenour, Assoc. M. ASCE (96), the alkali-chlorine treatment of metal plating wastes to remove cyanides and heavy metals is increasing in popularity because of economical and operating advantages. Laboratory tests indicate that the cyanides can be reduced below 1 ppm and that 16,000 lb of treated waste can be tolerated per million gallons of sewage in the activated sludge process, if evenly distributed over a 24-hour period.

Chlorine Production.—Chlorine production in the United States (97) has risen from 40 tons per day in 5 plants in 1900 to more than 5,000 tons per day in 58 plants in 1949. Since 1940, the capacity has more than doubled. The plants are chiefly east of the Mississippi River, where cheap salt, cheap power, and an ample supply of water are available, and in proximity to markets for chlorine or chlorine products. The price per ton in tank cars f.o.b. works was $35.00 per ton from 1939 to 1946, rising to $40.00 per ton in 1947, and to $50.00 per ton in 1949. Since 1940, the price of chlorine advanced only 37%, as compared with 100% for the commodity price index. By 1950, installed chlorine production capacity in the United States is expected to be about 5,300 tons per day, having risen from 2,000 tons per day in 1940.

Chemical Treatment of Sewage.—The furor over chemical precipitation current between 1930 and 1933 gradually subsided by 1940 as the various procedures experienced the test of unbiased appraisal. The Committee on Sewage Disposal of The American Public Health Association (98) summarized the historical development of the chemical treatment of sewage and the status of the art as of 1935. At that time in the United States there were only 30 plants which could be classified as chemical treatment plants, although 7 others used chemicals as an emergency measure. By 1943, according to the U. S. Public Health Service, there were 202 plants; but, of these, 12 did not use any chemicals, although they possessed some equipment. In 1945, the *Engineering News-Record* inventory (44) indicated 158 plants. In 1949, as far as the committee can ascertain, there were only 21 plants using chemicals continuously, and 16 using chemicals in a regular season. In addition, at least 12 plants are equipped to use chemicals.

In 1938, Wellington Donaldson, M. ASCE (34*e*), reviewed the status of chemical treatment of sewage and noted that 86 chemical precipitation plants were in operation and 14 were under construction. Of the plants then listed, 60 proposed to use chemicals seasonally. Chemicals were in use at 8 plants for odor control and at 14 plants for sludge conditioning. Among the plants then listed were Denver (chemical treatment abandoned); St. Charles, Ill. (Putnam process abandoned); Minneapolis–Saint Paul (never used except for test purposes); Winston-Salem, N. C. (Landreth direct oxidation still in service). The remaining installations are mostly small. With the exception of the Scott-Darcey process (a method of preparing ferric chloride at the plant site from scrap iron and chlorine), the proprietary processes are less numerous than formerly. Most of the 13 "processes" listed by Mr. Mohlman (99) in 1933 have disappeared through failure to find application in practical full scale plants.

From time to time, the Guggenheim process has been revived in various forms and offered for test. Of the early plants, New Britain, Conn. (9 mgd), went into service on March 1, 1937, and is still operating (100). The plant at Anderson, Ind. (8 mgd), went into service in June, 1940, and has operated with sulfate of alumina in recent years. The East Chicago plant (20 mgd), designed as a Guggenheim plant, went into service in 1947, using sulfate of alumina almost continuously as the coagulant. With from 3.5 hours to 4.7 hours of aeration and from 0.4 cu ft of air per gal to 0.83 cu ft of air per gal, and return of sludge, the East Chicago plant can apparently be classed as a high-rate activated sludge plant. The excess sludge is lagooned. The vacuum filters and multiple hearth incinerator stand idle.

The theory of chemical coagulation and the development of controls for various procedures were investigated very thoroughly in the period from 1935 to 1949. In more recent years, little additional information has come to light, although various workers are still seeking to discover new or more efficient chemicals.

Undoubtedly there is a place for chemicals in the field of sewage treatment to relieve overloaded units temporarily while planning or constructing extensions of plant. In New Jersey, where intermittent sand filters are prevalent, the use of chemical precipitation is advantageous because regulations of the New Jersey State Board of Health permit a loading of 400,000 gal per acre per day with chemical effluent, as compared with a loading of 150,000 gal per acre per day following plain sedimentation. Chemical treatment has proved helpful where trade waste is received and where waste is pretreated at the source particularly if acid wastes are discharged, which need neutralization. In sewage disposal, sodium nitrate has been used for odor control in the stream receiving the plant effluent (101). In the treatment of cannery wastes, sodium nitrate has been applied to lagooned wastes (102).

Industrial Wastes

In the last 10 to 15 years there has been a remarkable increase of interest in the treatment of industrial wastes. Industrial waste conferences, such as those held at Purdue University, in Lafayette, Ind., have been of value. The American Petroleum Institute (API) has studied certain phases of handling oil refining wastes, particularly the design of oil separators. For several years the American Steel and Iron Institute investigated handling waste acid pickle liquor. The National Council for Stream Improvements, Incorporated, is still actively studying paper wastes. Even with all this active research, stream pollution problems are still being solved by the use of well-known principles, and not because of recent research. Witness the control of stream pollution on the Androscoggin River in Maine, as described elsewhere, where reduction in the production of sulfite liquor, stream regulation, storage of industrial wastes with letdowns at suitable seasons, and the use of sodium nitrate in emergencies seem to have brought relief. In the oil refinery, better knowledge is needed of the character of effluents from various processes, and the sources of polluting material. The most modern separators for recovering oil are described (103) by W. B. Hart, but little is said on the subject of other taste-producing substances, such as phenol. If refineries are working to eliminate such materials from their effluents, little is revealed.

In the literature on industrial wastes in the last 10 years, the annual review by the Committee on Research of the Federation of Sewage Works Associations (1)(2) and the symposium of the American Chemical Society (104) are outstanding. In the United States, Edward F. Eldridge, M. ASCE (105), has written the only book covering a wide field. In Great Britain, B. A. Southgate has published a compilation on industrial wastes (106). Reports of the industrial waste conferences at Purdue University (107)(108)(109) contain useful information for a wide range of industries. The API has issued a number of bulletins, one of which describes the API separator (110), which functions to separate free oil from water. Probably the largest separator is at the refinery of the Standard Oil Company (Indiana) at Whiting, Ind. (handling 120 mgd). This is operated in parallel with an old separator (80 mgd). The combined effluent contains about 200 bbl of oil daily. However, such devices will not break emulsions. The treating and breaking of emulsions are under study, as well as the removal of tastes and odors. The disposal of caustic wastes requires attention. In the smaller refineries, new equipment can be installed, with new processes, to obviate discharge of troublesome material.

Treatment and Disposal of Waste Soluble Oils.—In general, soluble oils are homogenous compositions containing mineral oil, pure oil, or other hydrocarbon oils; emulsifying agents such as potassium or sodium soaps or sulfonated oils and an indeterminate amount of water. When mixed with water, a very fine stable emulsion is formed. This is widely used in machining operations as a coolant and lubricant. The soluble oil may be used in dilutions ranging from 1 in 10 to 1 in 100. Ordinarily, the dilution is about 1 to 20. The emulsion is frequently used over and over, but eventually goes to the drain and is undesirable in a stream or sewage plant. Mr. Eldridge and R. W. Purdy (111) indicate methods for breaking down the oil emulsion to recover oil, and for removing the residual oil and B.O.D. The procedure at a ball-bearing plant at Sandusky, Ohio (112), is described by Harry D. Unwin.

Anaerobic Digestion of Industrial Wastes.—Anaerobic fermentation with production of methane (113) has proved advantageous and economical for treating manufacturing wastes containing from 0.5% to 2.5% of fermentable material. At lower concentrations, aerobic methods are frequently more economical; and, at concentrations above the upper limit, evaporation with the recovery of stock feed is often profitable. Loadings of 0.2 lb of organic matter per cubic foot of fermenter volume per day are used in several plants handling distillery, yeast, and malt wastes. Strawboard wastes were digested at high rates at temperatures between 25° C and 37° C (114), with an agitator. A new development was the continuous return of sludge (similar to an activated sludge plant flow diagram). This proved more effective than the old type with intermittent return. The digestion of wool scouring wastes was studied at Lebanon, Tenn., by M. T. Singleton, M. ASCE (115), in a pilot plant operated over a 3-week period with a loading of 0.06 lb of volatile matter per cubic foot per day. The gas yield was from 8.6 cu ft per lb to 14 cu ft per lb of volatile matter. Reductions of 65% in volatile matter and 90% in grease were obtained.

Effect of Industrial Wastes on Sewage Works Operations.—Stuart E. Coburn describes (116) the headaches from treatment of combined industrial wastes and sewage, such as high temperature dye wastes and soluble oils. Metals and wastes from metallurgical and metal working plants are deleterious to sewage treatment, particularly iron, copper, chromium, zinc, and cyanides. Acid iron pickling wastes have caused trouble at Cleveland and Fostoria, Ohio, Chicago, and Worcester, Mass. Zinc has retarded sludge digestion at Fostoria, Elyria, Ohio, and London, Ont., Canada. Mercury and zinc have inhibited digestion at North Tarrytown, N. Y. (117). Chromium up to 25 ppm can be handled, but 40 ppm is disturbing. At Rahway, N. J., penicillin wastes upset sludge digestion. Most authorities agree that gasoline, mineral acids, and cyanides should be excluded from sewers. Whey has proved (118) troublesome in activated sludge treatment at Marion, Ind.

Other industrial wastes which affect the operation of sewage treatment works are listed in the second, third, and fourth progress reports of this committee (58)(36)(35).

Effect of Detergents on Sewage Treatment.—Willem Rudolfs, M. ASCE, Raymond Manganelli, and Isiah Gellman (119) point out the increase in use of surface active agents, from 27,960,000 lb in 1941 to 184,442,000 lb in 1945, and an estimated 100,000,000 lb in 1949—although these agents represent somewhat less than 30% of the total soap production in 1948. Soap production has tended to decline. In general, replacement of soap by detergents reduces the strength of the sewage. The effect of detergents on sewage plant units is not uniform. Slightly acid detergents tend to inhibit grease removal. If alkaline, they may aid grease removal. In general, in strong sewage, detergents increase the effluent B.O.D. in settling tanks, and have no material effect on digestion. The effect on the operation of the activated sludge process is threefold—causing foaming, floc agglomeration, and carry-over of suspended matter. Some detergents cause more foam than others. Increased B.O.D. in the effluent from settling tanks may affect secondary treatment devices. Detergents may not be toxic to biologic activities but indirectly may affect sewage treatment plant operation, as shown principally in (*a*) decreased efficiency of settling tanks, with consequent increased load on trickling filters; (*b*) foaming, floc agglomeration, and carry-over in activated sludge units; and (*c*) reduction in gas production from sludge digestion.

Atomic Energy.—In its fourth progress report (35*d*), this committee cited various references on possible wastes and the effect of the procedure on cooling water used in atomic bomb research. The hazards of atomic plants appear to be as follows (120): The disposal of radioactive wastes arising from—

> "* * * (1) operation of a chain-reacting pile for production of plutonium; (2) production of radio isotopes; (3) production of atomic energy; (4) distribution of radio isotopes for experimental purposes * * *"

—creates a problem, especially as some of the isotopes may have a life of 24,000 years.

Disposal may be made by: (*a*) Dilution in streams or other bodies of water; (*b*) mixing of isotopes with more stable isotopes; (*c*) storage until they decay into stable isotopes; and (*d*) concentration, with disposal by storage or by using concentrated solution to make concrete blocks which could be dumped into oceans.

Apparently there is possibility of danger to the atmosphere, soil, and surface streams (121). Arthur E. Gorman, M. ASCE, and Mr. Wolman (122) describe the possible waste problems connected with the production and use of radioactive materials. Dry radioactive wastes may be buried. Liquid wastes may be lagooned. Mr. Gorman (123) outlines the problem of safe disposal of radioactive liquid, gaseous, and solid wastes, and indicates the permissible maximum limits of contamination in air and water.

Disposal of Sludge and Sewage

Use of Reclaimed Sewage.—The industrial use of sewage effluents was discussed in the fourth progress report of this committee (35*e*). The largest use is at Baltimore (124), where in 1947, 25 mgd was sold by the city, of which 10 mgd was activated sludge effluent. The rate schedule is given in the fourth progress report of the committee (35*e*).

Use of Sludge As Fertilizing Material.—From 1934 to 1940, Milwaukee and Houston were the only two sewage treatment works in the United States selling heat-dried activated sludge.

In 1941, The Sanitary District of Chicago entered the field on the bases of sales in bulk carload lots, f.o.b. cars, at the sewage treatment works in Chicago.

During World War II, the price in bulk was controlled by the Office of Price Administration ceiling, as given in Table 5.

Since the removal of ceilings, the bulk price in Chicago rose to a peak in 1947 of $3.94 per unit of ammonia and $0.40 per unit of available phosphoric acid (APA) and has since dropped to $2.40 per unit of ammonia and $0.40 per unit of APA. The annual average bulk price per ton rose to $25.25 in 1947 and declined to $14.13 in 1949. At Houston, the price dropped to about $2.50 per unit of ammonia in 1948–1949, with an average price in 1949 of $16.87 per ton, f.o.b. plant. The average analysis was NH_3, 6.17% and P_2O_5, 3.29%.

TABLE 5.—PRICES OF HEAT-DRIED ACTIVATED SLUDGE DURING WORLD WAR II

City	Ceiling, per Unit	
	Ammonia	APA[a]
Milwaukee, Wis.,....	$3.00	$0.40
Chicago, Ill.,........	2.80	0.40
Houston, Tex.,.......	2.60	0.40

[a] Available phosphoric acid.

At San Diego, a relatively low-grade heat-dried digested sludge was sold from July 1, 1946, to March 10, 1949, for $20.00 per ton, in 100-lb sacks, f.o.b. the plant. The contractor supplied the sacks. The sludge analyzes as NH_3, 2.95%; P_2O_5, 2.5%; and K_2O, 0.13%.

The Los Angeles County Sanitation Districts continue to sell air-dried digested sludge, delivered wet on the drying beds, to the Kellogg Supply Company at a net price of about $2.00 per dry ton.

Dual Disposal of Garbage with Sewage.—There are three methods of dual disposal of garbage with sewage: (*a*) By installing household garbage grinders and discharging the ground material, mixed with water, into the sewer; (*b*) by installing central municipally operated stations for grinding garbage which is hauled, ground, and dumped into the sewer; and (*c*) by hauling garbage to the sewage treatment works, where it is ground and discharged either into the raw sewage or into the digestion tanks. The installation of household grinders progressed slowly from its inception around 1921. Until the beginning of World War II, only some 55,000 installations were made. Since the end of the war, a number of manufacturers have offered household grinders, so that their use may become more general. The fourth progress report (35*f*) presents the status of the procedure. Eventually the propaganda may die down and reason and economics may prevail. In the larger cities, it is obvious that the percentage of habitations with grinders will always be low, because it is difficult to force every householder to buy a unit.

A recent review by the Committee on Sewerage and Sewage Disposal (125) reports that at least 7 large cities prohibit garbage grinders. At least 17 do not approve their installation in hotels, restaurants, and clubs. Municipal garbage grinding is reported at Goshen, N. Y., Lansing, Mich., Marion, Ind., and Midland, Mich. Apparently food wastes, properly ground, will flow in a well-designed sewer and not produce any problem from the standpoint of volume (126).

If used extensively, grinders may increase (127) the load on sludge handling facilities, in an amount ranging from 0.07 lb per capita per day to 0.11 lb per capita per day, or around 55% of the total, to 300%. Also, grinders may build up the load on the oxidizing units approximately 30%. The New Jersey State Board of Health warns against the extensive use of grinders, because of overloaded sewage treatment plants and possible damage to stream pollution control. Seven states are apprehensive, and eleven states are not concerned. Complete separation of garbage and rubbish is necessary and hard to obtain. Eggshells have caused trouble at Houston (128).

In Indiana, cities operating sewage treatment works on a revenue bond basis are authorized to issue revenue bonds to finance the installation of kitchen garbage grinders if adequate sewage treatment facilities exist (129). At least one city (Jasper, Ind.) is taking steps to adopt kitchen grinders on a citywide basis (1940 population, 5,041).

Lansing has been operating its original sewage and garbage plant for more than 10 years, handling all the city garbage. Additions to this plant are now under construction, which will more than double its capacity and add about 1,088,000 cu ft of sludge digestion volume to the original 292,000 cu ft, making a total of 1,380,000 cu ft for an estimated future population of 125,000, or about 11 cu ft per capita. The sewage treatment is activated sludge.

Problem of Nitrogen and Phosphorus in Sewage Disposal.—The effect of sewage effluents on aquatic plants was reviewed in the fourth progress report (35*g*). It is evident the problem of sewage disposal in some instances is entering a new phase, in which attention is shifting from the B.O.D. of the effluent and oxygen balances in the usual sense to the results of fertilization, increased biological productivity of receiving waters, and attendant problems. The problem is still unsolved because there is no known method of reducing the nitrogen in sewage by more than 50% or 55%. Furthermore, phosphorus is also an element which must be considered, as Clair N. Sawyer (130) has noted that nuisance conditions can be expected with a concentration in excess of 0.01 ppm. Apparently there is no known feasible method for removing phosphorus.

Sewer Service Charges.—With problems of financing during the economic depression, followed by other difficulties in the period of inflation after World War II, sewer revenue bonds have grown in favor as a method of financing construction, and sewer rental charges have become more popular for raising funds to cover operation and maintenance. According to the *Engineering News-Record* (131), 432 cities with a population of more than 5,000 use sewer rentals. Four methods are popular, based on: (*a*) Metered water consumption; (*b*) percentage of water bills; (*c*) number of plumbing fixtures in houses; and (*d*) number of sewer connections. Among the larger cities, Akron (132) has recently adopted sewer service charges. The Akron ordinance (January 16, 1947) provides for quarterly billing and an additional charge for industrial wastes, based on B.O.D. and suspended solids in excess of the average sewage received at the sewage treatment works and for hydrogen-ion content above or below pH6. The charges are available to pay the cost of operation and maintenance and fixed charges.

Charges for Handling Industrial Wastes in Municipal Plants.—With the growth of industry many municipalities are faced with the prospect of developing a rate structure by which an equitable charge can be made for the disposal of liquid industrial wastes. Such wastes frequently are more difficult to treat alone than is domestic sewage. Sometimes they are so large in volume or of such a character that additional disposal plant facilities are required. The solution of the problem may depend on the statutes under which the municipality operates. Edmund B. Besselievre, M. ASCE (133), outlines the situation from a general standpoint, indicating that a charge should be based on the amount of suspended solids, B.O.D. contributed, and the volume of the sewage, and that the charge should include a proportionate part of the fixed charges as well as the operating costs. One method suggested is based on the industry paying only for the excess strength contributed. Another method is to charge for the entire amount of suspended solids and B.O.D. contributed, making a charge for each regardless of the strength of the city sewage. New Brunswick, N. J. (134), has adopted the second alternative, making a charge of $22.00 per million gal, $5.00 per ton of solids, and $5.00 per 100 lb of chlorine demand. Buffalo (135) has adopted the first method, basing the charge on the excess of chlorine demand and of suspended solids over that in normal sewage. The Sanitary District of Chicago has proposed a charge under a special statute based on an exemption of 10,000 gal per day and the cost of handling the excess of volume, suspended matter, and B.O.D. The rates suggested are $8.03 per million gal of flow; $8.07 per ton of solids; and $2.80 per 1,000 lb of B.O.D., less 10% for prompt payment. There is a deduction of $0.67 for each employee per year. In many instances in the United States, by direct negotiation, industry has contributed to the cost of construction and agreed to pay an annual charge thereafter for maintenance and operation.

LITIGATION

Water Pollution Suit—Illinois Versus Indiana et Al.—At the present time a pollution case is before the United States Supreme Court (No. 11, original), brought by the State of Illinois against the State of Indiana, the City of Hammond, the City of East Chicago, the City of Gary, and the City of Whiting, to restrain the state and its agencies (including 20 industrial establishments) from polluting Lake Michigan, the source of drinking water for Chicago and surrounding territory in northern Indiana and Illinois. The court appointed a Special Master, Luther E. Smith, of Saint Louis, Mo. (35*h*)(36*i*). The case is somewhat unique in that, although no testimony has been taken, much has been accomplished. The total cost of the improvements required amounts to about $15,000,000, chiefly for controlling industrial wastes. The largest single expenditure was by the Standard Oil Company of Indiana, in the amount of $6,000,000 for oil separators and for collecting the sanitary sewage in its plant and delivering it to the sewers of Whiting for treatment by Hammond. The only municipal work required cost $90,000 for revising sewers in Whiting. The program is expected to be completed by the summer of 1950.

Pollution of the Androscoggin River.—For a number of years the Androscoggin River in Maine has been seriously polluted by paper mill wastes (136). This river drains a watershed of 3,470 sq miles, with a monthly flow at Rumford averaging from 2,212 cu ft per sec to 8,552 cu ft per sec during the years 1928 to 1941, inclusive, with the lower flows in the period from August to October, averaging 2,446 cu ft per sec. The natural flow is regulated by storages of

225 billion gal above Runford and of 28 billion gal below. The sewered population totals 108,700, mostly below Rumford. In 1941 the equivalent population of the industries (97% of the wastes being from pulp and paper mills, of which 87% are sulfite wastes) totaled 2,411,500. Of this total, 65% is located in or near Berlin, N. H. Large amounts of hydrogen sulfide were generated in the vicinity of Lewiston and Auburn in Maine and created a public nuisance. With a flow around 2,100 cu ft per sec, the time required for flow from Berlin to Lewiston (a distance of 112 miles) is around 16.25 days. E. Sherman Chase, M. ASCE, holds that a reasonable criterion for pollution control is the maintenance of D.O. in the river water at all times.

The general relief measures considered were: Sewage treatment, industrial waste treatment, storage of industrial wastes, and storage of dilution water. Sewage treatment for the municipalities was found too costly compared to the benefits derived. The use of the soda process by the Oxford Company was helpful, with the recovery of fiber by save-alls. The lagoon capacity required for storage of industrial wastes totaled 255.5 million gal. Estimates indicated that storage during the period from June to October, inclusive, could be discharged in the other months of the year. During storage for several weeks the digester liquors lose a substantial proportion of their B.O.D. The river is well regulated for power purposes, and the letdown for power coincides closely with the need for dilution. Consequently, the construction of storage for diluting water was not as advocated.

The Attorney General of Maine began legal proceedings in May, 1942. Various remedial measures were tried, which did not afford complete relief in a low summer flow in 1948.

In December, 1948, a new court decree appointed W. H. Lawrence as administrator and provided for weekly samplings and analyses from May 1 to October 15, each year; specified the criteria for the control of the industrial waste letdown in relation to river flow; and required the use of at least 1,000 tons of soluble nitrate in the river at critical periods, when ordered by Mr. Lawrence. The production of sulfite pulp was to be allocated by the mills by themselves.

Miscellaneous Processes

A survey of the field of sewage treatment discloses a number of processes which are disappearing from the practice of the art:

Landreth Direct-Oxidation Process.—Apparently there is still one plant in operation at Winston-Salem. This process primarily employed excess lime precipitation, with a little electrical current. The report of F. A. Dallyn, George A. Johnson, and A. W. Delaporte (137) on tests at Toronto, and the report of Metcalf & Eddy (138) in 1926 to Allentown, Pa., indicated the lack of value in the procedure.

Putnam Process.—The Putnam process was installed in at least 4 places, mostly small towns, since 1933. In the largest town, Saint Charles, Ill., the process has been abandoned. It included chemical precipitation, the resulting sludge being retorted and burned to a char for use as a precipitant. The claim that the distillates can be sold as insecticides is unsubstantiated.

Laughlin Process.—The Laughlin process proposed chemical precipitation of sewage, with the addition of paper pulp. The essentials were covered by an early British patent (139). As far as is known, the first plant in the United States was installed at Dearborn, Mich., in 1931–1932. The second plant was installed at Perth Amboy, N. J., in 1935. At both places the use of paper pulp was quickly abandoned, but chemical precipitation continued and the effluent passed through a magnetite filter.

Guggenheim Process.—The Guggenheim process, when originally tested by The Sanitary District of Chicago in a pilot plant at the North Side, involved the use of ferric sulfate for chemical precipitation and the reclamation of ammonia nitrogen by filtration through a zeolite. The Chicago estimates of the cost of construction and operation were higher than those for activated sludge. The process was then modified to a precipitation by ferric sulfate in connection with short-period aeration and return of sludge. This process was installed by several of the smaller municipalities (New Britain, Anderson, East Chicago, and Essex Fells, N. J.), and several small industries. Anderson is reported to be using sulfate of alumina. At East Chicago, on a weak sewage, sulfate of alumina is added (from 187 lb per million gal to 965 lb per million gal). Essex Fells is reported to be using sulfate of alumina. New Britain is reported to be using 160 lb of copperas and 29.5 lb of chlorine per million gal.

Mallory Process.—The Mallory process as disclosed by various patent claims involves the use of mathematical formulas derived from operating data obtained from various activated sludge plants. As far as is known, neither the process nor its equations are in use in any operating municipal plant.

Contact Aerators.—Contact aeration has intrigued various investigators since G. E. Waring, Jr. In the United States, the Hays process attained prominence in army camps during World War I. The Hays patent (United States No. 1,991,896) was issued on February 19, 1935. The process apparently was anticipated by Karl Imhoff, M. ASCE, H. W. Clark, and others. The only Kessener brush aerator in the United States is at an institutional plant in New Jersey. Various types of contactors have been proposed by J. Dohman, A. T. Maltby, and others. The Imhoff contactor aerator was tried at the Calumet Works (Chicago) in 1927, but did not prove satisfactory. All these types of contactors have failed because the surface area of activated sludge is so much greater than the area of active sludge attached to fixed or revolving surfaces. Such devices frequently give rise to odors and produce an inferior effluent.

Magnetite Filters.—Magnetite filters were introduced by W. C. Laughlin at Dearborn, in 1939, and later were installed at Denver, the Minneapolis–Saint Paul Sanitary District, Cleveland, Perth Amboy, Plainfield, and elsewhere. As far as the committee can learn, their use at Denver, Minneapolis–Saint Paul, Cleveland, and Plainfield has been discontinued, owing to the difficulty and cost of operation and the failure to produce improved effluents.

Centrifugal Dewatering of Sludge.—The use of centrifugal machines to dewater sludge appears to have stopped. In the plant at Columbus, Ohio, where centrifugals were installed for concentrating sludge, the equipment has been removed. In general, the cost of centrifuging sewage sludge has been such that the procedure proved uneconomical and the removal of solids insufficient.

Although it is possible to build a machine to run at a speed which will produce a thickened sludge, approaching a cake from a vacuum filter, it is not economical because of low capacity and the solids in the effluent.

Respectfully submitted,

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Committee on Sewerage and Sewage Treatment, January 19, 1950

APPENDIX. BIBLIOGRAPHY OF REFERENCES

(1) **"A Critical Review of the Literature of 1947 on Sewage and Waste Treatment and Stream Pollution,"** by Committee on Research, Federation of Sewage Works Assns., *Sewage Works Journal*, March, 1948, pp. 224–272.

(2) **"A Critical Review of the Literature of 1948 on Sewage and Waste Treatment and Stream Pollution,"** by Committee on Research, Federation of Sewage Works Assns., *ibid.*, March, 1949, pp. 228–265.

(3) **"Activities of the National Council for Stream Improvement, Inc.,"** by Harry W. Gehm, *Water & Sewage Works*, July, 1946, p. 289.

(4) **"Nation-wide Inventory of Sanitation Needs,"** *Supplement No. 204*, Public Health Repts., U. S. Public Health Service, Washington, D. C., April, 1948.

(5) *Engineering News-Record*, February 24, 1949, p. 14.

(6) **"Water Pollution Control Act,"** Public Law No. 845 (originally Barkley-Taft Bill, Senate Bill No. 418), 80th Cong., 2d Session, 1948, Chapter 758.

(7) **"A Bill to Amend the Water Pollution Control Act, etc.,"** by T. C. Crook, House of Representatives Bill No. 4568, 81st Cong., 1st Session, 1949.

(8) **"Pollution Abatement Laws Present Confusing Picture,"** by A. J. Fox, *Engineering News-Record*, September 15, 1949, p. 23.

(9) **"Ohio River Compact and Other Interstate Agreements,"** by Kenneth S. Watson, *Journal*, A.W.W.A., January, 1949, pp. 18–22.

(10) **"First Annual Report, 1948–1949,"** Ohio River Valley Water Sanitation Comm., Cincinnati, Ohio, November 15, 1949.

(11) **"Sewage Disposal Program of the Allegheny County Sanitary Authority,"** by J. F. Laboon, *Sewage Works Journal*, March, 1949, pp. 197–214.

(12) **"Report on the Proposed Collection and Treatment of Municipal Sewage and Industrial Wastes,"** by Allegheny County San. Authority, Pittsburgh, Pa., January, 1948.

(13) **"Pollution of Boundary Waters,"** interim rept. made to the International Joint Comm. by the Board of Technical Advisers, U. S. Govt. Printing Office, Washington, D. C., March, 1949.

(14) **"Pollution of Boundary Waters: Including Report of the Sanitary Experts,"** progress rept. of the International Joint Comm., U. S. Govt. Printing, Office, Washington, D. C., January 16, 1914.

(15) **"On the Pollution of Boundary Waters Reference,"** final rept. of the International Joint Comm., U. S. Govt. Printing Office, Washington, D. C., 1918.

(16) **"International Joint Commission Study of the Detroit River,"** by M. LeBosquet, Jr., *Sewage Works Journal*, May, 1949, pp. 525–533.

(17) **"United States-Canadian Boundary Waters Pollution Control,"** *Public Health Reports*, U. S. Public Health Service, Washington, D. C., September 17, 1948, pp. 1238–1240.

(18) **"$200 Million Program Protects World's Greatest Harbor,"** by Frederick H. Zurmuhlen, *Sewage Works Engineering*, October, 1949, p. 496.

(19) **"Disposal Plant to Alleviate Boston Harbor Nuisance,"** by Karl R. Kennison, *Civil Engineering*, Pt. 1, June, 1946, pp. 254–257 and Pt. II, July, 1946, pp. 304–307.

(20) **"Hyperion Plant to Clean Up So. California Beaches,"** *Sewage Works Engineering*, July, 1947, pp. 362–365.

(21) **"Sewage Disposal for Los Angeles and Associated Communities,"** rept. to the Board of Public Works, City of Los Angeles, Calif., by Metcalf & Eddy, Los Angeles, Calif., April 25, 1944.

(22) **"Philadelphia Sustains Pace in Sewage Plant Construction,"** by Samuel S. Baxter, *Sewage Works Engineering*, October, 1949, p. 500.

(23) **"The Proposed Blackstone Valley Sewage and Wastes Collection System and Treatment Works,"** by E. Sherman Chase, *Journal*, Boston Soc. of Civ. Engrs., April, 1949, pp. 165–191.

(24) **"Interceptors Come First in California Sewage Plan,"** *Engineering News-Record*, September 15, 1949, pp. 38–41.

(25) **"East Bay Cities Plan Plants to Cost $23,000,000,"** by Darrell A. Root, *Sewage Works Engineering*, July, 1947, pp. 366–369.

(26) **"City of Seattle—Report on Sewage Disposal,"** by Abel Wolman, Seattle, Wash., September 25, 1948.

(27) **"Seattle Sewage Treatment Problem,"** by Pollution Control Comm., State of Washington, Olympia, Wash., November, 1948.

(28) **"Pre-Design Report on the Proposed Sewage Treatment Plant for the City of Tacoma, Washington,"** by C. S. Seabrook and Richard G. Tyler, Tacoma, Wash., 1949.

(29) **"Portland's Digested Sludge to Be Discharged Into River,"** by John W. Cunningham, *Sewage Works Engineering*, July, 1947, p. 357.

(30) **"Report to the Street and Sewer Department on Sewage Treatment for the City of Wilmington, Delaware,"** by Whitman, Requardt and Associates, Wilmington, Del., September, 1948.

(31) **"The Operation of the Back River Sewage Works, Baltimore, Md.,"** by C. E. Keefer, *Sewage Works Journal*, May, 1949, pp. 554–576.

(32) **"Mechanical-Equipment Installation Add 40 Percent to Sewage Plant Capacity,"** by C. E. Keefer, *Engineering News-Record*, October 20, 1949, pp. 32–35.

(33) **"Los Angeles County Increases Capacity from 4 to 200 M.G.D.,"** by A M Rawn, *Sewage Works Engineering*, July, 1947, p. 370.

(34) **"Modern Sewage Disposal,"** The Federation of Sewage Work Assns., New York, N. Y., 1938. (*a*) Chapter XVIII. (*b*) Chapter XIX. (*c*) Chapter III, pp. 28–46. (*d*) Chapter VIII, pp. 98–109. (*e*) Chapter VII, pp. 85–97.

(35) **"Advances in Sewage Treatment and Present Status of the Art: Fourth Progress Report of the Committee of the Sanitary Engineering Division on Sewerage and Sewage Treatment,"** *Proceedings*, ASCE, October, 1948, p. 1315. (*a*) p. 1320. (*b*) pp. 1324–1326. (*c*) p. 1329. (*d*) p. 1345. (*e*) pp. 1345–1346. (*f*) pp. 1347–1349. (*g*) pp. 1350–1353. (*h*) pp. 1353–1354.

(36) **"Advances in Sewage Treatment and Present Status of the Art: Third Progress Report of the Committee of the Sanitary Engineering Division on Sewerage and Sewage Treatment,"** *ibid.*, June, 1946, p. 799. (*a*) p. 803. (*b*) pp. 817–819. (*c*) p. 811. (*d*) p. 813, Tables 5 and 6. (*e*) p. 814, Table 8. (*f*) p. 815, Table 9. (*g*) pp. 823–829. (*h*) pp. 821–822. (*i*) p. 845.

(37) **"Sewage and Wastes Treatment on the Atlantic Seaboard,"** *Sewage Works Engineering*, October, 1949, pp. 479–484.

(38) **"Water Pollution Under Study in Great Lakes Area,"** *ibid.*, October, 1948, p. 520.

(39) **"Sewage Treatment at Military Installations,"** rept. of the Subcommittee on Sewage Treatment in Military Installations, Committee on San. Eng., National Research Council, Div. of Medical Science, *Sewage Works Journal*, September, 1946, pp. 791–1030.

(40) **"Sewage Treatment at Military Installations—Summary and Conclusions,"** NRC Subcommittee on Sewage Treatment, *ibid.*, January, 1948, pp. 52–95. (*a*) p. 87. (*b*) p. 83.

(41) **"Plant Operating Experiences,"** by Morris M. Cohn, *Sewage Works Engineering*, installments XXIII to LIV, June, 1946, to November, 1949.

(42) **"Air Conditioning As a Problem of Water Distribution and Disposal,"** by L. D. Gayton, read before Illinois Chapter, Am. Soc. of Heating and Ventilating Engrs., April 11, 1949 (mimeographed).

(43) **"Air Conditioning Water Affects Sewers,"** *Water & Sewerage Works*, July, 1949, p. 263.

(44) **"Inventory of Sewage Disposal Facilities,"** *Engineering News-Record*, November 14, 1946, pp. 112–113.

(45) **"Developments in Sewage Disposal for San Francisco,"** by Clyde C. Kennedy, *Sewage Works Journal*, November, 1947, pp. 963–979.

(46) **"Control of Pollution in New Jersey,"** by H. P. Croft and S. A. Kowalchik, *Public Health News*, Trenton, N. J., December, 1947, p. 6.

(47) **"Sewage Treatment Report, Niagara Falls, N. Y.,"** by E. J. Smith, Buffalo, N. Y.

(48) **"Sixteenth Report of the Minneapolis-Saint Paul Sanitary District for the Year 1948,"** Minneapolis, Minn., p. 27.

(49) **"The Detroit Sewage Treatment Plant,"** by Arthur B. Morrill, *Sewage Works Journal,* July, 1939, pp. 609–617.

(50) **"Report of Colonel William M. Black and Professor Earle B. Phelps Concerning the Location of Sewer Outlets and the Discharge of Sewage Into New York Harbor,"** Board of Estimate and Apportionment, New York, N. Y., March 23, 1911.

(51) **"Preaeration Increases Sewage Treatability,"** by J. D. Walker, *Sewage Works Journal,* January, 1948, pp. 115–119.

(52) **"Preaerated Raw Sewage Recirculation Increases Primary Sedimentation Efficiency at Denver,"** by C. P. Gunson, *ibid.,* November, 1946, pp. 1121–1126.

(53) **"The Works of the Birmingham, Tame and Rea District Drainage Board,"** by Herbert C. Whitehead, Birmingham, England, July, 1937.

(54) **"Partial Aeration of Strong Sewage with Activated Sludge,"** by William D. Hatfield, *Industrial and Engineering Chemistry,* August, 1928, pp. 832–835.

(55) **"Operation of the Pre-Aeration Plant at Decatur, Illinois,"** by William D. Hatfield, *Sewage Works Journal,* October, 1931, pp. 621–635.

(56) **"A Critical Review of the Literature of 1940 on Sewage and Waste Treatment and Stream Pollution: Sedimentation,"** Committee on Research, Federation of Sewage Works Assns., *ibid.,* March, 1941, pp. 193–194.

(57) **"Improved Sewage Clarification by Pre-flocculation Without Chemicals,"** by A. J. Fischer and A. Hillman, *ibid.,* March, 1940, pp. 280–306.

(58) **"Advances in Sewage Treatment and Present Status of the Art: Second Progress Report of the Committee of the Sanitary Engineering Division on Sewerage and Sewage Treatment,"** *Proceedings,* ASCE, April, 1944, p. 457. (*a*) pp. 490–492. (*b*) pp. 479–480. (*c*) pp. 475–478.

(59) **"Advances in Sewage Treatment and Present Status of the Art: First Progress Report of the Committee of the Sanitary Engineering Division on Sewerage and Sewage Treatment,"** *ibid.,* November, 1942, p. 1559. (*a*) pp. 1563–1564.

(60) **"The 'Wide Band' Air Diffusion System,"** by Frank C. Roe, *Sewage Works Journal,* November, 1948, pp. 1054–1058.

(61) **"Design of Final Settling Tanks for Activated Sludge,"** by Norval E. Anderson, *ibid.,* January, 1945, pp. 50–63. (*a*) pp. 63–65. (*b*) July, 1945, pp. 803–808.

(62) **"Air Diffusion Problems at Activated Sludge Plants,"** by Lloyd R. Setter, *Water & Sewage Works,* December, 1948, pp. 450–456.

(63) **"Diffuser Plates Cleaned by Oxy-Acetylene Flames,"** *ibid.,* November, 1949, p. 447.

(64) **"Cleaning Air Diffuser Plates by Sandblasting,"** by John R. Longley, *ibid.,* October, 1948, pp. 363–365.

(65) **"Problems and Trends in Activated Sludge Practice,"** by Robert T. Regester, *Transactions,* ASCE, Vol. 106, 1941, p. 158.

(66) **"New Horizons for Activated Sludge,"** by Richard H. Gould, *Engineering News-Record,* September 1, 1949, pp. 180–182.

(67) **"The Operation and Control of Activated Sludge Sewage Treatment Works,"** rept. of the Committee on Sewage Disposal, Am. Public Health Assn., Public Health Eng. Section, *Sewage Works Journal*, January, 1942, pp. 3–69.

(68) **"B.O.D. Loadings in Activated Sludge Plants,"** by W. E. Stanley and J. F. Berberich, *Journal*, Boston Soc. of Civ. Engrs., July, 1948, pp. 236–254.

(69) United States Patent 2,337,384, granted to R. H. Gould, Washington, D. C., December 21, 1943.

(70) **"Practical Results of Step Aeration,"** by Wilbur N. Torpey, *Sewage Works Journal*, September, 1948, pp. 781–788.

(71) **"System of Bio-Precipitation of Organic Matter from Sewage,"** by Daniel A. Okun, *ibid.*, September, 1949, pp. 763–794.

(72) **"The Use of Digested Sludge and Digester Overflow to Control Bulking Activated Sludge,"** by L. S. Kraus, *ibid.*, November, 1945, pp. 1177–1190.

(73) **"Annual Operation Report of the Ann Arbor (Mich.) Sewage Treatment Plant for the Year 1946,"** by C. Preston Witcher, *ibid.*, September, 1948, pp. 914–915.

(74) **"Summary of Experience in Mechanical Activated Sludge Plant Operation,"** by Douglas E. Dreier, *ibid.*, January, 1945, pp. 101–126.

(75) **"Some Effects of Anaerobic Digestion on Sewage Sludge,"** by A M Rawn and E. J. Candell, *Transactions*, ASCE, Vol. 115, 1950, pp. 181–193.

(76) **"Sludge Lagoons,"** rept. of the Committee on Sewage Disposal, Eng. Section, Am. Public Health Assn., *Sewage Works Journal*, September, 1948, pp. 817–831.

(77) **"Oxidation Ponds,"** rept. of the Committee on Sewage Disposal, Eng. Section, Am. Public Health Assn., *ibid.*, November, 1948, pp. 1025–1031, especially 1028–1031. (*a*) pp. 1025–1028.

(78) **"Cannery Waste Disposal Lagoons: A Summary of Cooperative Work Conducted by Health Departments of Several States Comprising the Upper Mississippi River Basin Sanitation Agreement and the National Canners Association,"** by L. F. Warrick, T. F. Wisniewski, and N. H. Sanborn, 1945.

(79) **"Pittsburgh Plate Disposal System Advances Clear Stream Program,"** *Press Release 2327*, Pittsburgh Plate Glass Co., Pittsburgh, Pa., 1949.

(80) **"Standard Practice in Separate Sludge Digestion: Final Report of the Committee of the Sanitary Engineering Division on Sludge Digestion,"** *Transactions*, ASCE, Vol. 103, 1938, p. 1662.

(81) **"Digester Capacity Requirements,"** by R. S. Rankin, *Sewage Works Journal*, May, 1948, pp. 478–489.

(82) **"Multiple-Stage Sewage Sludge Digestion,"** by A M Rawn, A. Perry Banta, and Richard Pomeroy, *Transactions*, ASCE, Vol. 104, 1939, pp. 93–132.

(83) **"Sludge Concentration and Disposal at Marion, Indiana,"** by David Backmeyer, *Sewage Works Journal*, November, 1943, pp. 1201–1207.

(84) **"Principles and Factors Influencing Vacuum Filtration of Sludge,"** by A. L. Genter, *Sewage Works Jaurnal*, November, 1941, pp. 1164–1208.

(85) **"Computing Coagulant Requirements in Sludge Conditioning,"** by A. L. Genter, *Transactions*, ASCE, Vol. 111, 1946, p. 635.

(86) **"Experience in the Operation of Standard Trickling Filters,"** by Douglas E. Dreier, *Water & Sewage Works*, May, 1949, Reference and Data Section, pp. R-147-R-155.

(87) **"Considerations and Procedures in the Design of High Rate Trickling Filters,"** by Samuel A. Greeley, *Sewage Works Journal*, September, 1948, pp. 789–805.

(88) **"Sewage Oxidation Ponds—Performance, Operation and Design,"** by D. H. Caldwell, *ibid.*, May, 1946, pp. 433–458.

(89) **"Chlorination in Sewage Treatment,"** rept. of Committee on Sewage Disposal, Am. Public Health Assn., *Technical Reprint No. 464*, Wallace & Tiernan Co., Inc., Newark, N. J. 1933.

(90) **"Chlorination of Sewage,"** by N. S. Chamberlin, *Water & Sewage Works*, May, 1949, Reference and Data Section, pp. R-137-R-142.

(91) **"Waste Activated Sludge Thickening at Hagerstown, Md.,"** by Don K. Croft, *Sewage Works Journal*, September, 1949, pp. 892–896.

(92) **Annual Reports of the Department of Water Supply, City of Detroit, Mich., for the Fiscal Years Ended June 30, 1942 to June 30, 1949,** Detroit, Mich.

(93) **"Chlorination of Los Angeles Sewage at Hyperion Plant,"** by W. A. Schneider, *Water & Sewage Works*, February, 1948, pp. 77–78.

(94) **"Annual Report of the Buffalo Sewer Authority, 1947–1948,"** Buffalo, N. Y., 1949, p. 26.

(95) **"Report on the Administration, Operation, and Maintenance of the Division of Sewage Disposal, Cleveland, Ohio, for the Year 1948,"** by Joseph W. Ellms, Cleveland, Ohio, 1949.

(96) **"Laboratory Studies on Effect of Alkali-Chlorinated Cyanide, Case Hardening, Copper and Zinc Plating Wastes on Aerobic and Anaerobic Sewage Treatment Processes,"** by G. M. Ridenour, *Sewage Works Journal*, November, 1948, pp. 1059–1073.

(97) **"The Chlor Alkali Industry in the United States,"** by R. L. Murray, *Industrial and Engineering Chemistry*, October, 1949, pp. 2155–2164.

(98) **"Chemical Treatment of Sewage,"** rept. of Committee on Sewage Disposal, Am. Public Health Assn., *Sewage Works Journal*, November, 1935, pp. 997–1108.

(99) **"Recent Advances in the Chemical Treatment of Sewage,"** by F. W. Mohlman, *American Journal of Public Health*, January, 1934, pp. 25–31.

(100) **"Annual Report of the Sewage Disposal Commission of the City of New Britain, Connecticut, for the Year Ending March 31, 1949,"** by J. R. Szymanski, New Britain, Conn., 1949.

(101) **"Treatment of Industrial Wastes from Paper Mills and Tannery on Neponset River,"** by Almon L. Fales, *Industrial and Engineering Chemistry*, March, 1929, pp. 216–231.

(102) **"Nitrate Treatment of Cannery Wastes,"** by N. H. Sanborn, *The Canner*, March 22, 1941, pp. 12–14.

(103) **"Industrial Waste Disposal for Petroleum Refineries and Allied Plants,"** by W. B. Hart, *Petroleum Processing*, January, 1946 through September, 1947 (reprint of 20 articles), Cleveland, Ohio, 1947.

(104) **"Industrial Wastes,"** *Industrial and Engineering Chemistry*, May, 1947, pp. 560–682.

(105) **"Industrial Waste Treatment Practice,"** by Edward F. Eldridge, McGraw-Hill Book Co., Inc., New York, N. Y., 1942.

(106) **"Treatment & Disposal of Industrial Waste Waters,"** by B. A. Southgate, H. M. Stationery Office, London, 1948.

(107) *Proceedings*, 1st Industrial Waste Conference, Purdue Univ., Lafayette, Ind., 1944.

(108) *Proceedings*, 2d Industrial Waste Conference, Purdue Univ., Lafayette, Ind., 1946.

(109) *Proceedings*, 3d Industrial Waste Conference, Purdue Univ., Lafayette, Ind., 1948.

(110) **"Manual on Disposal of Refinery Wastes, Section I, Waste Water Containing Oil,"** Am. Petroleum Inst., 4th Ed., New York, N. Y., August, 1949.

(111) **"Treatment and Disposal of Waste Soluble Oils,"** by Edward F. Eldridge and R. W. Purdy, *Sewage Works Journal*, September, 1948, pp. 849–856.

(112) **"Metallurgical Plant Wastes and Their Treatment,"** by Harry D. Unwin, *Water & Sewage Works*, October, 1949, pp. 399–405.

(113) **"Design and Loading Factors for Anaerobic Fermentation Plants,"** by A. M. Buswell, Am. Chemical Soc., Atlantic City, N. J., meeting, September 18–23, 1949 (abstract).

(114) **"A Novel High Rate Anaerobic Waste Treatment System,"** by Harry W. Gehm and Philip F. Morgan, *Sewage Works Journal*, September, 1949, pp. 851–858.

(115) **"Experiments on Anaerobic Digestion of Wool Scouring Wastes,"** by M. T. Singleton, *ibid.*, March, 1949, pp. 286–293.

(116) **"Headaches from Treatment of Combined Industrial Wastes and Sewage,"** by Stuart E. Coburn, *Journal* Boston Soc. of Civ. Engrs., January, 1949, pp. 64–77.

(117) **"Industrial Wastes Inhibit Digestion of Sludge,"** by Thomas M. Riddick, *American City*, May, 1948, pp. 90–92.

(118) **"Disposal of Cheese Whey by Digestion,"** by David Backmeyer, *Sewage Works Journal*, November, 1948, pp. 1115–1118.

(119) **"Effect of Certain Detergents on Sewage Treatment,"** by Willem Rudolfs, Raymond Manganelli, and Isiah Gellman, *ibid.*, July, 1949, pp. 605–612.

(120) **"Atom-plant Waste Disposal Poses Hazardous Problems,"** *Engineering News-Record*, April 22, 1948, p. 1.

(121) **"Nuclear Fission Developments Pose Sanitary-Engineering Challenge,"** *ibid.*, October 14, 1948, pp. 91–92.

(122) **"Nuclear Fission Operations and the Sanitary Engineer,"** by Arthur E. Gorman and Abel Wolman, *Sewage Works Journal*, January, 1949, pp. 63–74.

(123) **"Safe Disposal of Radioactive Wastes,"** by A. E. Gorman, *Civil Engineering*, March, 1949, pp. 29–32.

(124) **"Industrial Water Supply From Processed Sewage Treatment Plant Effluent at Baltimore, Md.,"** by Abel Wolman, *Sewage Works Journal*, January, 1948, pp. 15–23.

(125) **"Kitchen Garbage Grinders and Their Relation to the Public Works Field,"** *American City*, February, 1949, p. 117.

(126) **"Effects of Ground Garbage on Sewage Treatment Processes,"** by Willem Rudolfs, *Sewage Works Journal*, November, 1946, pp. 1144–1150.

(127) **"Ground Garbage—Its Effect Upon the Sewer System and Sewage Treatment Plant,"** by S. L. Tolman, *ibid.*, May, 1947, pp. 441–463.

(128) **"Sewer Velocities Required for Kitchen-Ground Wastes,"** by Kenneth W. Cosens and Eric J. Hanemann, *American City*, January, 1949, pp. 102–103.

(129) Senate Bill No. 147, State of Indiana, Acts of 1949, Chapter 87, Indianapolis, Ind., 1949.

(130) **"Fertilization of Lakes by Agricultural and Urban Drainage,"** by Clair N. Sawyer, *Journal*, New England Water Works Assn., June, 1947, pp. 109–127.

(131) **"Use of Sewer Rentals Is Increasing,"** *Engineering News-Record*, May 13, 1948, p. 699.

(132) **"Akron Adopts Sewer Charges to Finance Plant Operation,"** by W. R. LaDue, *Sewage Works Engineering*, February, 1948, p. 66.

(133) **"What is a Fair Charge for Handling Industrial Wastes in Municipal Plants?"** by Edmund B. Besselievre, *Engineering News-Record*, June 14, 1945, pp. 825–827.

(134) **"City Bases Sewage Charge on Industrial Waste Characteristics,"** *American City*, April, 1944, pp. 57 and 77.

(135) **"Special Sewer Service Charges for Industrial Wastes,"** by G. E. Symons and F. W. Crane, *Water Works & Sewerage*, March, 1944, pp. 113–115.

(136) **"Pollution of the Androscoggin River by Industrial Wastes and Control Measures Thereof,"** by E. Sherman Chase, *Journal*, Boston Soc. of Civ. Engrs., July, 1949, pp. 357–381.

(137) **"Direct Oxidation Tests at Toronto,"** by F. A. Dallyn, George A. Johnson, and A. V. Delaporte, *Canadian Engineer*, February 19, 1924, pp. 243–254.

(138) **"Working Tests Show Direct Oxidation Not Suitable for Sewage of Allentown, Pa.,"** by Metcalf & Eddy, *Engineering News-Record*, September 9, 1926, pp. 422–424.

(139) British Patent No. 3,252, granted to J. Bock, London, A.D. 1883.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852

TRANSACTIONS

Paper No. 2422

THE SOCIETY AND THE PROFESSION

ADDRESS AT THE ANNUAL CONVENTION, TORONTO, ONTARIO, CANADA, JULY 12, 1950

BY ERNEST E. HOWARD,[1] PRESIDENT, ASCE

In the early years of our Society the By-Laws contained a provision that at the Annual Convention the President should deliver an address giving a summary of engineering progress during the preceding year. It is fortunate for the audience—and for me—that present By-Laws require only that the President shall make an address. For it is obvious that any address which could mention—even briefly—every engineering project on which the past year has witnessed important progress would occupy hours of time.

Indeed, several years before the turn of the century President Benjamin M. Harrod declined the assignment, saying the task had become so increasingly difficult with the field of engineering so extended, with duties so specialized and exacting, that no completely satisfactory account could be rendered by any single participant in the movement. The obvious impossibility of any such summary today is eloquent evidence of the increased impact of engineering upon our way of life. Approximate census figures indicate that since 1850 our population has increased about seven times; the number of engineers, about twenty-four times. These figures may not adequately reflect the great number of engineers in administrative positions in industry.

The publications of our own and of other engineering societies, and the voluminous and excellent technical engineering press—in addition to the requirements of our By-Laws—make that old summary superfluous. I have concluded therefore to offer a few comments on special current activities of our Society; and, from the vantage point of this mid-century year, to look back over the path our profession has traversed during the past century and possibly to look somewhat ahead. If you should say that crystal gazing is not consistent for an engineer, I would remind you that we are constantly in the business of visualizing in the present things that will materialize in the future.

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CURRENT ACTIVITIES

This has been a year of rather exceptional activities on the part of members of the Board of Direction and of many of the Committees. Probably few members who have not served in such capacities realize the amount of time required for Society affairs. These services I commend for the thanks and appreciation of the members.

Time does not permit a review of current work of all committees, but some activities of rather special character should be mentioned. Appreciation is due to the Technical Divisions for the way in which they have taken on the heavy burden of editing the papers for *Proceedings* and *Transactions*. These are no casual tasks but require hours of close concentration. The same may be said for the continuing efforts of the Committee on Constitution and By-Laws. That laborious work is not yet completed.

Particularly should be mentioned the work of the Committee on National Water Policy. Some seventy members of the Society have been devoting their time and talents to this work, in common with a limited number of members from other societies represented in Engineers Joint Council (EJC), for the work has been carried on under an EJC Water Policy Panel. The report of this committee issued last month is a significant and statesmanlike document certain to prove a valuable contribution to engineering science and to our nation's welfare.

This year has seen the adoption by the Society of extensive revisions to the Constitution. We are now in a period of some adjustment resulting from a change of fiscal year dates, etc., and the shift from one directorship from Zone 1 to Zone 4. The transitional period should not adversely affect Society operations although it involves modified terms for some officers. Following a pattern initiated in California, several areas have developed regional organizations and have met in regional conferences. These appear to be highly favored by our members and more such regional activities, readily provided for under the revised Constitution, may be expected.

PROBLEMS OF A GROWING SOCIETY

The Society continues to grow in numbers, and the abilities and qualifications of applicants appear to be of high order. We now number more than 29,000. I regret that I cannot report that our financial condition improves with increased numbers. Because of the low average of dues received compared to the amount spent per member, we have the paradox of our financial condition worsening as our membership increases.

We have faced and now face the condition of which President R. E. Dougherty warned in his annual message of 1948. He urged increased dues which would in some measure offset the reduced value of the dollar and warned that without increased dues there would be further loss of efficiency in the functioning of the Society. Following the decision of our membership not to increase dues, the Board was obliged to pare to the minimum in every activity in order to maintain a balanced budget. (There are still in the United States some who believe in the soundness of a balanced budget—even when it involves self-denial.) This inevitably has reduced the efficiency, and the extent, of our functioning. It is my personal conviction that we should at an early date provide increased funds by increased dues—and I would apply the increase principally to increased publications, to increased publicity for the profession, and to support of membership activities.

The publication of our *Transactions*, the permanent history of the development of engineering accomplishment, has in no measure kept pace with the increase in numbers of our members, with the volume of work done, nor with the importance and intricacy of that work—which can rightly be termed the progress of the profession. Statistics are readily available for only the past thirty-seven years. In 1913 we had 7,000 members; we now have more than four times that number. The amount of engineering construction accomplished during the past year is nearly four times that of 1913. Of the complexity and scientific importance of this engineering work of the past year as compared with that of 1913, I leave you to judge. We did not last year publish four times as many pages in *Transactions* as we did in 1913—nor three times, nor twice as many pages. We published only 60% of the number of pages published in 1913. Of that showing we cannot be proud. Our vast membership certainly includes many men who can contribute valuable additions to permanent engineering literature. We should be publishing several times as many pages in *Transactions* as we did forty years ago; and we should publish without aggravating delays.

Other activities and functions of the Society have been limited and reduced because of the stringent economies which must be observed if we are to live within our income. We are a realistic group; we know well the reduced purchasing power of the dollar in every field of activity. I hope the time may not be far distant when our members will conclude to increase dues and to increase activities. One of our older members recently called attention to the fact that dues paid by members of all grades are at greatly lower percentages of their average incomes than the percentages of earlier years.

PROFESSIONAL ISSUES ALSO VITAL

It seems well once in a while to pause in our never ending rush of activities to survey our profession, to review something of its problems and its accomplishments, and to re-examine the relation of the engineer to his fellow man, to his profession, and to the world about him. While we may only be looking at old scenes and repeating old convictions, we may derive some new inspiration from the accomplishments of those who went before and some new vision for the future. We ought occasionally to turn our thoughts to the philosophy, the romance, and the poetry that give savor to our work.

There is much in which we can and should take pride in our Society and our profession, and many reasons for optimism as to the future of both. This optimism is well warranted from a study of the origins and developments of our Society from its early records and publications, for these reveal a continuity of unwavering purpose to develop engineering into a learned profession second to none in its ideals and purposes, with confidence that this could be done through the Society. One finds there much of introspection and serious appraisal. The founders were soberly cognizant of the problems of developing the profession, and our leaders through the years continued to reflect upon the full meaning and significance of the profession in the social order.

Some of the conditions and problems our predecessors faced no longer disturb us. Others, about which we are still concerned, are much less severe. Problems of professional recognition, of standards of ethics, of a professional group preponderantly of employees, of a profession composed at once of employers and employees, and many another which they recognized still command our attention. However, we have progressed, we are progressing; what has been done warrants high optimism for the future.

In 1868 President W. J. McAlpine, who declared engineering to be "the noblest profession of them all," said:

> "We of the older school have fought a long and severe battle to bring the profession up to its present position. The rising school must continue the fight with the same earnestness * * *."

That challenge has been and is being met, although we have not yet met the demand of another early president that the education of the engineer should include as a prerequisite the full college course required of other professions.

Place of the Engineer in Organized Society

Emerson characterized thinking man as in three great phases: The study of the world about us,—the sciences; the study of man's history and thoughts,—the humanities; and action, or the translation and application of this thinking into usable results. This "action" is the work of the engineer.

You will observe that action is a part of thinking. This is not easy. Thinking is such hard work that we often shrink from it; it is so much easier to consult our prejudices. I believe it was Jane Taylor who wrote:

"How few think justly of the thinking few,
How many never think who think they do."

You doubtless have heard of the woman about to be anesthetized for an operation who asked the doctor how long it would be after she regained consciousness before she could think clearly. The doctor hesitated a moment and then said: "I'm afraid you're counting too much on this anesthetic." We well know the need for much clear thinking in the confusion of our day. Engineers have contributed and can contribute to the common good the kind of thinking that Lucretius in "de Rerum Natura" prescribes:

"Then be it ours with steady mind to grasp,
The purport of the skies—the law behind,
The wandering courses of the sun and moon."

It has become almost trite to refer to this as the engineering age. However, engineers should take pride in their accomplishments and satisfaction that they have become so widely recognized. As a release from their inhibitions they should on occasion boast a little about their contribution to society. Plans are now in progress for a great exhibition to be held in Chicago, Ill., in 1952, in commemoration of the centenary of our Society, to illustrate and demonstrate the accomplishments of engineering in one hundred years. There is much to show, and many a story to tell.

ADVANCE GAINS RECOGNITION

We all know how increasingly during the past two centuries life has become conditioned by engineering. We are constantly surrounded by the work of the engineer. Food, shelter, transportation, and communication,—every element of our life has been affected. Even the curl of the hair has become permanent by the work of the engineer. I suppose women would still be curling their hair on their fingers if some engineer had not devised that electrical octopus under which they used to writhe for beauty's sake,—but I understand that function has now been taken over by the chemical engineer. We can well paraphrase the well-known Christopher Wren epitaph and say: "If you would see the works of the engineer, look about you."

We need not today echo the old complaint that engineering is not recognized. The age of engineering may not yet have become the age of the engineer, but the engineer back of the engineering is coming more and more to be understood. There has been notable change and improvement in the last four decades—say, beginning with World War I. The contributions of engineering to the prosecution of both world wars were almost universally recognized. The operations of state highway departments during these decades have done much to familiarize the average citizen with the engineer's work. Our profession should take pride in the universally high accomplishment of these governmental agencies, probably the first to be principally patterned and modeled along engineering concepts, and almost without exception operated in accordance with the engineer's high ideals of unswerving integrity.

I am convinced, too, that there is marked improvement in the recognition of the profession—and of the individual engineer. One category where much work remains to be done is with our friends of the legal profession. There are lawyers who so little understand the professional character of engineering as to demand surety bonds to guarantee professional services. I asked one such if he would provide a surety bond to guarantee winning a lawsuit! We still need to make the engineer and engineering as a profession,—as well as the works of engineering, known to our fellow men.

Recognition by the public! Why the public has moved in and taken our vocabulary. We had barely inured ourselves to the word engineer as noun, adjective, verb, and adverb being applied to anything and everything, when the language of the street seized our word "blueprint." Nowadays everything has to reach a "blueprint" state, whether affairs of state or a matrimonial venture. The pattern of life and the way to heaven has to be "blueprinted." At least the public realizes that "blueprints" have to do with orderly planning, and that they represent the work of engineers.

ENGINEERING; SCIENCE APPLIED FOR PEOPLE'S USE

Thus we have engineering—in its broad sense, the application of science for the use of mankind. The engineer finds his place in organized society between the sciences and the humanities. With his pragmatic philosophy of useful accomplishment he does not aspire to the high name of the pure scientist. He may well, for instance, mathematically be interested in a limited number of functions applied through a narrow range of variables—a very low ideal from the standpoint of a real mathematician. Neither can he join with the scientist who spurned the suggestion that he was working for the good of humanity, who said: "Humanity is merely a scrofulous itch on the face of a minor planet."

The engineer does not regard humanity as a "scrofulous itch" because his work is for humanity. It is for people that buildings and highways and water supplies and bridges are built. All is for the service and convenience of his fellow man. He responds to the challenge of the soaring chorus in Handel's "Messiah": "Make the crooked places straight, make the rough places plain, make straight in the desert a highway * * *." To this work for humanity nothing that science can provide or philosophy can offer can go amiss.

The engineer looks to the scientist just as Jason, seeking to harness brazen-hoofed wild bulls for the plowing of wide furrows for sowing dragon's teeth, had need of the secret charms from the golden casket of Medea, "* * * some for healing, some for slaying." So the engineer looks to the scientist for the secrets that provide new materials, new processes, and new powers. Medea could not do the plowing, but she could supply the secrets so that Jason could do it.

Engineering is not merely science applied; it is science applied with art. Because of the way his work is carried on, the engineer is sometimes regarded as a taciturn and prosaic realist, devoid of sensitive appreciations. The engineer is in fact a creative artist, an artist who drives his imagination with the reins of mathematics. With this artistry there must be combined a genius for invention, a gift for visualization, and a sound understanding of the immutable laws that govern the actuality. For the criteria of the engineer's art are not the canons of some transient school of thought, the opinions of critics of the day, but the fundamental laws of nature. These are not man made nor are they subject to speciously artful technicalities. The engineer seeks to know these laws and to adjust his structures to them.

The engineer artist is not limited to a small canvas. He makes his huge characters on the face of a continent. His imaginings are crystallized in far-flung structures. He is not confined to a single medium but his visions find final expression in earth and wood, in stone and steel, and in marble and bronze; in water brought to the thirsty; in rest for the weary; in light for darkness; in strength for the weak; in some release from fear; and in the gain of that inch of time that man through the ages has sought.

My characterization of the philosophy of the engineer as a philosophy of pragmatism should not be interpreted to imply any failure of the engineer to recognize the deep significance of all the spiritual aspects of man's emotional adjustments to the world in which he lives, to the great universe that envelops him, or to the unknowable mysteries of life before which the most learned must walk humbly. That he is a realist does not mean that he is a materialist.

The Engineer, a Citizen

Of late years the question has been often advanced as to whether engineers, in addition to all these works, and aside from their professional engagements, do their part as citizens. C. R. Young, M. ASCE, past-president of the Engineering Institute of Canada, recently more specifically asked whether engineers were doing their part to "* * * make fast the moorings of democracy." Here too I am appreciative of present accomplishments and optimistic for the future. It seems to me evident that engineers are not behind other professions—where personal gain is not involved—in taking an active part in civic activities. The peripatetic character of much engineering employment tends to make it difficult for the young engineer to put his roots down in a community and to become a part of it; but, with awakened interest in the problems of citizenship characteristic of so many of our Local Sections, I am confident the younger engineers coming on will become more articulate and more active than the older generation.

This is well, for with their objective realism, engineers are deeply concerned with the creeping paralysis which threatens to sap the very foundations upon which this country has been able to achieve the highest standards of living for the most people that the world has ever known. This was not the achievement of self-appointed economic planners from above, but came of multitudes of ambitious, independent, and self-reliant men from below. Those forefathers who laid the foundations for our greatness did not look for the ease of a guaranteed security, but for the rugged opportunity to meet the experiences of life and to derive honor from them. They founded this country in search for liberty—liberty from government. That liberty, that freedom, was bought at too great a price to be frittered away by indifference or to be lost by greed.

There can be no question that the engineer can, and should, contribute to the solution of some of these political and social problems which, to some extent, are the by-products of his engineering accomplishments.

THE FUTURE—UNLIMITED

In summary of it all, the task of the engineer is to conquer, subdue, and control nature for the benefit, comfort, and convenience of his fellow man. Nature is no loving mother but a cold, ruthless, impersonal tyrant ever ready to destroy man. To survive, man must protect himself from storm, from heat, and from cold; must make the earth produce wheat and corn instead of thorns and thistles; must develop improved conditions and materials better for his use than nature can offer.

Many years past I condensed into a single paragraph in epigraphic form, placed as an inscription on a bridge, thoughts gleaned from a chapter in Buckle's *History of Civilization*. These lines I venture here to quote again, for they seem to me both a challenge and a declaration of faith, a creed, for the engineer:

> "Formerly the richest countries were those in which nature was most bountiful: now the richest countries are those in which man is most active. For in our age of the world if nature is parsimonious, we know how to compensate for her deficiencies—our engineers can correct the error and remedy the evil. For the powers of nature, notwithstanding their apparent magnitude, are limited and stationary;—but the powers of man, so far as experience and analogy can guide us, are unlimited."

The powers of man are unlimited! You will recall that poet of the desert, contemplating the star-strewn skies, pondering man's relation to the universe, concluded with a like noble conception of the powers of man:

> "When I consider the heavens, the work of Thy fingers, the moon and the stars which Thou hast ordained, what is man that Thou art mindful of him or the son of man that Thou visiteth him. Thou hast made him a little lower than the angels, Thou hast crowned him with glory and honor, Thou madest him to have dominion over the work of Thy hands; Thou hast put all things under his feet."

The glory and honor of man is that he can have dominion over the world; and of the engineer, that he can make that dominion effective for the comfort and convenience of his fellows. The powers of nature are limited. The stars in their courses fought neither for nor against Sisera; nor have their courses changed since Deborah sang. Carlyle speaks of "Orion with his celestial hounds in their leash of sidereal fire, forever and forever coursing the Pleiades"—forever and forever keeping the same path. The whirling planets of the illimitable universe plunging through space must forever follow their same orbits. However, man can choose his own orbit; the powers of man are unlimited!

Here then is both the challenge and the assurance of the future to our profession. Should we fail to achieve our possibilities we must say with Cassius "The fault, dear Brutus, is not in our stars but in ourselves * * *." To that challenge we may well reply in words of Ulysses:

> "Some work of noble note may yet be done,
> Not unbecoming men that strove with Gods.
> * * * Come, my friends,
> 'Tis not too late to make a better world.
> Push off, and sitting well in order smite
> The sounding furrows: for my purpose holds
> To sail beyond the sunset * * *
> That which we are, we are
> One equal temper of heroic hearts
> Made weak by time and fate, but strong in will
> To strive, to seek, to find, and not to yield."

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